Schwinger Terms in Fermion Electrodynamics*

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(Received 23 June 1970)

We investigate the c-number Schwinger term and a possible operator Schwinger term in fermion electrodynamics, to the lowest nontrivial orders in perturbation theory. The results of our direct method of calculation agree with results previously obtained from the Bjorken-Johnson-Low high-energy limit. In particular, we find no evidence for the existence of an operator Schwinger term. The structure of our results helps us to understand the calculations by the split-point method of Schwinger, according to which there seems to be an operator Schwinger term. In the examples considered, we discover an analogy between our direct perturbation-theory calculations, which are performed in momentum space, and the split-point calculations, which are performed in coordinate space. Interchanging the equal-time limit with the phase-space integration is analogous to interchanging the equal-time limit with the split-point limit. Pursuing the analogy, we conclude that the split-point definition of the current probably is consistent with perturbation theory, and that the apparent discrepancies are due to the improper interchange of the equal-time limit with the split-point limit in previous calculations by the split-point method.

I. INTRODUCTION

In the last five years, remarkable results have been obtained in the study of the commutation relations of hadronic charges. This approach allows us to understand significant aspects of hadron dynamics, despite our ignorance of the precise nature of hadron interactions. The crucial conjecture, which Gell-Mann abstracted from Lagrangian field theory, is that a large class of interactions (e.g., velocity-independent interactions) may break the symmetry but leave the equal-time commutators of the charges unchanged.[1]

While charge commutators contain much dynamical information, the commutators of the corresponding local currents are certainly much richer in dynamical content. However, unlike the charge commutators, the local current commutators may depend heavily on the detailed nature of hadron interactions. Thus, even if quarks are the principal constituents of the currents, we cannot expect the commutators of the free-quark model to be even a rough facsimile of the true commutators. Worse, there is no guarantee that equal-time current commutators even exist in a useful and well-defined sense.[2]

One approach to these difficulties is to return to the original source of inspiration: Lagrangian field theory. But it has been known for 15 years that the canonical formalism of Lagrangian field theory is not equal to the task of calculating current commutators.[3] This deficiency is most evident for theories, such as quantum electrodynamics, in which the current is a product of two fermion fields. Considering the extraordinary success of quantum electrodynamics, the existence of such a fundamental difficulty may seem surprising. Of course, the real lesson is a reminder that the success of electrodynamics is due to perturbation-theory calculations. Since there is no rigorous mathematical exposition of the quantum field theory of electromagnetism, the connection between the canonical formalism and the calculational rules of perturbation theory can only be regarded as heuristic. Thus we cannot be unduly surprised to discover shortcomings in the canonical formalism.

Since for electrodynamics perturbation theory is a well-defined and very successful set of rules for calculating physical quantities, we choose to investigate the equal-time commutator of the electric current in perturbation theory. In particular, we will examine the Schwinger terms, which appear in the commutator of the time component with the space component. We calculate matrix elements of the Schwinger terms directly: by inserting intermediate states, evaluating the current matrix elements in perturbation theory, and carrying out the spin sums and phase-space integrals.

Previous investigations of fermion electrodynamics leave us with a confusing picture of the nature of the Schwinger terms. Contradictory results have been obtained concerning the possible existence of an operator Schwinger term bilinear in the photon field. The existence of this operator is indicated by perturbation-theory calculations[2,4] which use the point-splitting technique of Schwinger.[3] On the other hand, perturbation-theory calculations using the BJL (Bjorken-Johnson-Low)[5] high-energy theorem indicate that the operator does not exist.[4,6]

The calculations which we present below contain compelling evidence that previous point-splitting calculations are incorrect. In the examples considered, we discover an intriguing analogy between our direct perturbation-theory calculations, which are performed

* Work supported in part by the National Science Foundation.
† NSF Predoctoral Fellow.
[1] S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968).
[2] R. A. Brandt, Phys. Rev. **166**, 1795 (1968).
[3] T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) **14**, 396 (1955); J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).
[4] D. G. Boulware and R. Jackiw, Phys. Rev. **186**, 1442 (1969).
[5] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966); K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 74 (1966).
[6] T. Nagylaki, Phys. Rev. **158**, 1534 (1967); D. Boulware and J. Herbert, Phys. Rev. D **2**, 1055 (1970).

in momentum space, and the split-point calculations, which are performed in coordinate space. The sharing of three-momentum between the electrons and positrons in the intermediate states is found to be analogous to the spatial separation of ψ and $\bar{\psi}$ in Schwinger's definition of the current. Interchanging the equal-time limit and the phase-space integration is analogous to interchanging the equal-time limit with the split-point limit. Pursuing the analogy, we learn that the usual point-splitting calculations evidently fail because of an improper interchange of limits. In fact, because of this improper interchange, some point-splitting calculations (of higher-derivative Schwinger terms) are not even invariant under spatial translation.

It should be understood that our results provide no evidence against the appropriateness of defining the current as the limit of the product of fermion fields at separated spatial points. To the contrary, the results strongly suggest that this definition is consistent with perturbation theory. Rather, criticism is directed solely at the unjustified interchange of the equal-time limit and the split-point limit. In the examples considered below, an analogous interchange in momentum space accounts for the omission of finite and well-defined contributions. The controversy concerning the possible operator Schwinger term is explained by the fact that the omitted contributions precisely cancel the contributions which are correctly calculated.

Previous investigations of Schwinger terms in perturbation theory[4,6] have made use of the BJL high-energy theorem.[5] There has been much discussion recently of "the failure of the BJL theorem in perturbation theory," meaning that equal-time commutators calculated from the BJL theorem and perturbation theory disagree with the equal-time commutators obtained from the canonical formalism.[7] Of course, such disagreements are not necessarily evidence against the validity of the BJL method, since, as we have already remarked, the canonical formalism is notoriously unreliable for calculating equal-time commutators. The discrepancies might well be just another sign of the inadequacy of the canonical formalism.

One motivation for the calculations presented here is to see, in some rather delicate examples, to what extent such discrepancies are a consequence of the BJL theorem or of perturbation theory. In other words, we want to test the high-energy theorem purely within the context of perturbation theory. A simple sufficient condition for the BJL theorem is uniform and absolute convergence of the Low representation as $q_0 \to \infty$. But this condition is certainly not satisfied (even after regularization) in the first example considered below; we have not examined it for the second example. An additional complication in some applications of the BJL theorem is the necessity to regularize the amplitude by subtracting a possibly infinite polynomial in q. (Regularization is necessary in the BJL calculation of both examples considered below.) Formally, regularization should not effect the coefficient of q_0^{-1}; but, especially when the discarded polynomial is infinite, formal considerations are not totally reassuring. In the study of Schwinger terms, if nowhere else, it is the better part of valor to be suspicious of formal arguments.

We have checked the BJL theorem in two examples, by calculating matrix elements of commutators directly in perturbation theory, without resorting to any regularization procedures. In both cases, the results are consistent with the BJL theorem. The labor necessary for the direct calculations heightens our appreciation of the elegance of the BJL method. However, the structure of the direct calculations is richer and more instructive, providing the clue to an understanding of the point-splitting calculations.

Two technical aspects of the direct perturbation-theory calculations should be stressed. First, it is essential to treat the equal-time commutator as the limit of the unequal-time commutator. *Even when the total result is finite, there may be individual singular contributions present which must be carefully evaluated.* Simply calculating at equal times is analogous to the fallacy of using the argument of Goto and Imamura[3] when the spectral integral diverges. In the calculations presented below, it would be equivalent to bringing a limit inside a divergent integral. The result in the cases below would be the omission of finite, well-defined contributions. These are also the contributions which account for the discrepancy with the point-splitting calculations.

The second technical remark is that we have avoided calculations in which the matrix elements contain closed loops. Thus the individual matrix elements are finite and gauge-invariant, and we need not concern ourselves with ambiguities which might arise from renormalization or regularization. All singularities are a consequence of the integration over the phase space of the intermediate states. In this way we avoid some difficulties encountered by Brandt and Kim,[8] who performed direct perturbative calculations of charge commutators in a theory of quarks and scalar mesons.[9]

In Sec. II we consider the c-number Schwinger term to lowest order in perturbation theory (i.e., for a free-fermion field theory). In Sec. III we consider the possible operator Schwinger term. In Sec. IV we discuss the relationship of our results to the point-splitting calculations.

II. c-NUMBER SCHWINGER TERMS

In this section we evaluate the c-number Schwinger term in fermion electrodynamics to the lowest order in

[7] R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 975 (1969); S. L. Adler and W. K. Tung, *ibid.* **22**, 978 (1969).

[8] R. A. Brandt and Y. S. Kim, Phys. Rev. **161**, 1473 (1967).

[9] The scalar-meson theory has tadpoles, which do not appear in electrodynamics because of Furry's theorem.

perturbation theory. But first we will derive a general representation for the c-number Schwinger term, which reduces to the spectral representation of Goto and Imamura when the latter is valid.

Consider $C(\mathbf{q},t)$, the three-dimensional Fourier transform of the vacuum expectation value of the unequal-time commutator. Insert a complete set of intermediate states and use translation invariance. The result is

$$C(\mathbf{q},t)\equiv\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle[j^0(\mathbf{x},t),j^i(0)]\rangle_0$$
$$=\sum_n e^{-iE_nt}\langle 0|j^0|n(\mathbf{q})\rangle\langle n(\mathbf{q})|j^i|0\rangle - e^{iE_nt}\langle 0|j^i|n(-\mathbf{q})\rangle\langle n(-\mathbf{q})|j^0|0\rangle, \quad (1)$$

where n represents a complete set of quantum numbers except for the total three-momentum, which has already been integrated to obtain (1). E_n is the energy of the state $|n(\mathbf{q})\rangle$, with total three-momentum $\mathbf{q}$.

The current is conserved, so

$$(p_A{}^\mu-p_B{}^\mu)\langle A|j_\mu|B\rangle=0. \quad (2)$$

Substitute (2) into (1) and use either parity or time-reversal invariance in the cross term of (1).[10] The result is a simple, general representation for the c-number Schwinger terms

$$C(\mathbf{q},0)=2q^i\lim_{t\to 0}\sum_n\frac{\cos E_nt}{E_n}|\langle 0|j^i|n(\mathbf{q})\rangle|^2. \quad (3)$$

If the spectral representation of Goto and Imamura is valid, i.e., convergent, then it is easy to show that the sum in (3) converges absolutely. In this case, the limit commutes with the sum and (3) becomes

$$C(\mathbf{q},0)=2q^i\sum_n\frac{1}{E_n}|\langle 0|j^i|n(\mathbf{q})\rangle|^2. \quad (4)$$

Furthermore, under these hypotheses, the sum in (4) is a Lorentz scalar. Choosing the rest frame for the intermediate states and rewriting the expression in configuration space, we find the usual expression for the c-number Schwinger term:

$$\langle[j^0(\mathbf{x},0),j^i(0)]\rangle_0 = -i\left\{2\sum_n\frac{1}{m_n}|\langle 0|j^i|n(\mathbf{0})\rangle|^2\right\}\frac{\partial}{\partial x_i}\delta(\mathbf{x}). \quad (5)$$

The quantity in brackets in (5) is just the spectral integral written in an unfamiliar form.

[10] In this step, the cross term undergoes a crucial sign change, because the space and time components transform with a relative minus sign under space or time inversion. Consequently, the cross term adds to the direct term instead of cancelling. This is a simple way to see what is special about the commutator of a time component with a space component.

In fermion electrodynamics, to the lowest order in perturbation theory, the spectral integral diverges quadratically, so that (4) and (5) are not justified. In this case, we must examine the representation (3) with greater care, and we cannot interchange the limit and the sum. The only possible intermediate state in (3) is an electron-positron pair,

$$|n(\mathbf{q})\rangle=|e_s{}^-(\mathbf{p}),e_{s'}{}^+(\mathbf{q}-\mathbf{p})\rangle. \quad (6)$$

After substituting (6) into (3) and performing the sum on spins, the result is[11]

$$C(\mathbf{q},t)=\frac{2}{(2\pi)^3}\int d^3p\,\cos[t(p_0+p_0')]\left(\frac{q^i-p^i}{p_0'}+\frac{p^i}{p_0}\right), \quad (7)$$

where $p_0=(m^2+\mathbf{p}^2)^{1/2}$, $p_0'=[m^2+(\mathbf{p}-\mathbf{q})^2]^{1/2}$.

The next step is to expand (7) in powers of $\mathbf{q}$. We use the binomial expansion for p_0', the trigonometric identity

$$\cos t(p_0+p_0')=\cos 2p_0t\,\cos t(p_0'-p_0) -\sin 2p_0t\,\sin t(p_0'-p_0),$$

and the Taylor's series for $\cos t(p_0'-p_0)$ and $\sin t(p_0'-p_0)$. The result, up to the third order in $\mathbf{q}$, is ($p\equiv|\mathbf{p}|$)

$$C(\mathbf{q},t)=\frac{q^i}{\pi^2}\int_0^\infty dp\,\cos 2p_0t\,\frac{p^2}{p_0}\left(1-\frac{1}{3}\frac{p^2}{p_0^2}\right) -\frac{q^i\mathbf{q}^2}{2\pi^2}\int_0^\infty dp\,\cos 2p_0t\,\frac{p^2m^4}{p_0^7}$$
$$-t^2\frac{q^i\mathbf{q}^2}{2\pi^2}\int_0^\infty dp\,\cos 2p_0t\,\frac{p^4}{p_0^4}\left(\tfrac{1}{3}p_0-\frac{1}{5}\frac{p^2}{p_0}\right)$$
$$-t\frac{q^i\mathbf{q}^2}{\pi^2}\int_0^\infty dp\,\sin 2p_0t\left(\frac{1}{2}\frac{p^2}{p_0^2}-\frac{5}{6}\frac{p^4}{p_0^4}+\frac{2}{5}\frac{p^6}{p_0^6}\right)+O(\mathbf{q}^5). \quad (8)$$

The third and fourth terms in (8) are finite and well defined as $t\to 0$, but we would have overlooked them if we had calculated directly at $t=0$ [or, equivalently, if we had improperly interchanged the limit and the sum in (3)].

We now consider the four terms in Eq. (8). The first term is the well-known, quadratically divergent, single-derivative Schwinger term. We may easily calculate its leading behavior by neglecting the electron mass, so that $p=p_0$. We then find $-(1/6\pi^2)q^i(1/t^2)$ for the first term.[12] In addition there is a less singular contribution depending on the mass, which we will not bother to calculate.

[11] Equation (7) is most easily verified by first undoing the step of Eq. (2).

[12] M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge U. P., London, 1958).

The next three terms contribute to the third-derivative Schwinger term, which was first calculated by Brandt.[2] The second integral converges uniformly, so we may take the limit inside the integral. The integral is found to be 2/15, so that the second term contributes $-(1/15\pi^2)q^i\mathbf{q}^2$, plus terms which vanish as $t\to 0$.

In the third integral, we are only interested in the leading divergence, which is proportional to $1/t^2$. Again we may ignore the electron mass. The resulting Fourier transform is well defined in the sense of distribution theory,[12] and for the third term we find $(1/60\pi^2)q^i\mathbf{q}^2$, plus terms proportional to t. Similarly, for the fourth term the result is $-(1/30\pi^2)q^i\mathbf{q}^2+O(t)$. Terms of higher order in $\mathbf{q}$ all vanish as $t\to 0$. Combining these results, we find

$$C(\mathbf{q},t)=-\frac{1}{6}\frac{q^i}{\pi^2}\frac{1}{t^2}-\frac{1}{12\pi^2}q^i\mathbf{q}^2+O(t,m)\,, \qquad (9)$$

or, symbolically in configuration space,

$$\langle[j^0(\mathbf{x},0),j^i(0)]\rangle_0=\infty\,\partial^i\delta(\mathbf{x})+(i/12\pi^2)\partial^i\Delta\delta(\mathbf{x}). \qquad (10)$$

As we have already remarked, (10) was first derived by Brandt,[2] and has also been obtained by Boulware and Jackiw using the BJL high-energy theorem.[4]

In deriving (9), we have defined the equal-time commutator as the limit as t approaches zero of the unequal-time commutator $A(t)B(0)-B(0)A(t)$. Of course, there are many possible ways of going to the equal-time limit. One alternative limit procedure,

$$[A(0),B(0)]_{\rm BJL}=\lim_{t\to 0+}[A(t)B(0)-B(0)A(-t)], \qquad (11)$$

is shown by Johnson and Low[5] to correspond to the equal-time commutator given by the BJL high-energy theorem. We have repeated the calculation of the free-fermion Schwinger term by the direct method illustrated in Eqs. (1)–(9), using definition (11) instead of the unequal-time commutator. The result differs from (9) by a term with support at equal times: $-(1/3\pi^2)q^i\delta'(t)$. As $t\to 0$, such a term has no well-defined meaning. It might appear in the BJL high-energy limit as a noncovariant seagull, proportional to q^0q^i.

III. OPERATOR SCHWINGER TERM

In this section we will consider the possibility that there is an operator Schwinger term bilinear in the photon field. We will calculate to the lowest nontrivial perturbative order in fermion electrodynamics. In particular, we will consider the limit as $t\to 0$ of the quantity

$$Q(\mathbf{q},t)\equiv\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle 0|[j^0(\mathbf{x},t),j^i(0)]|\gamma\gamma\rangle. \qquad (12)$$

For simplicity, we have chosen identical photons in (12). To further simplify the calculation, we introduce a small photon mass λ, and we let the photons be at rest, with zero angular momentum along the i axis. We will calculate to zeroth order in the mass λ.

Strictly speaking, we are using a theory with a massive vector-meson "gluon." However, Wilson has found that the leading singularities of operator products in perturbation theory do not depend on the masses,[13] so that we may expect the same operator Schwinger term for the gluon and the photon. Furthermore, the point-splitting calculations are identical for photon and gluon. In any case, the issue is not crucial, since we are chiefly interested in studying an example, and we are not terribly concerned with whether our results apply only in the gluon model or also in electrodynamics. The choices we have made are necessary to make the calculation feasible; in the most general configuration, there would be 40 traces of eight terms each to evaluate and combine.

Next, we insert a complete set of intermediate states into the commutator in (12). To the lowest nontrivial order, there are only connected contributions from electron-positron pairs and semiconnected contributions from states of an electron-positron pair plus one or two photons.[14] Inserting these states into (12) and using translation invariance, the result is

$$\begin{aligned}Q(\mathbf{q},t)=\sum_n\, & e^{-iE_nt}\langle 0|j^0|n(\mathbf{q})\rangle\langle n(\mathbf{q})|j^i|\gamma\gamma\rangle\\ &+2e^{-i(E_n+\lambda)t}\langle 0|j^0|n(\mathbf{q})\gamma\rangle\langle n(\mathbf{q})|j^i|\gamma\rangle\\ &+e^{-i(E_n+2\lambda)t}\langle 0|j^0|n(\mathbf{q})\gamma\gamma\rangle\langle n(\mathbf{q})|j^i|0\rangle\\ &-e^{i(E_n-2\lambda)t}\langle 0|j^i|n(-\mathbf{q})\rangle\langle n(-\mathbf{q})|j^0|\gamma\gamma\rangle\\ &-2e^{i(E_n-\lambda)t}\langle 0|j^i|n(-\mathbf{q})\gamma\rangle\langle n(-\mathbf{q})|j^0|\gamma\rangle\\ &-e^{iE_nt}\langle 0|j^i|n(-\mathbf{q})\gamma\gamma\rangle\langle n(-\mathbf{q})|j^0|0\rangle, \qquad (13)\end{aligned}$$

where

$$|n(\mathbf{q})\rangle=|e_s^-(\mathbf{p}),e_{s'}^+(\mathbf{q}-\mathbf{p})\rangle \qquad (14)$$

and E_n is the energy of $|n(\mathbf{q})\rangle$. Using time-reversal invariance, the last three lines of (13) may be rewritten as

$$\begin{aligned}&e^{i(E_n-2\lambda)t}\langle\gamma\gamma|j^0|n(\mathbf{q})\rangle\langle n(\mathbf{q})|j^i|0\rangle\\ &\quad+2e^{i(E_n-\lambda)t}\langle\gamma|j^0|n(\mathbf{q})\rangle\langle n(\mathbf{q})\gamma|j^i|0\rangle\\ &\quad\quad+e^{iE_nt}\langle 0|j^0|n(\mathbf{q})\rangle\langle n(\mathbf{q})\gamma\gamma|j^i|0\rangle. \qquad (15)\end{aligned}$$

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[13] K. Wilson, Phys. Rev. **179**, 1499 (1969).

[14] For an introduction to the menagerie of disconnected contributions, see Appendix C of Ref. 1, or see M. Chanowitz, Phys. Letters **31B**, 374 (1970).

Now define the quantities $F_i(\lambda)$:

$$F_1(\lambda)=\sum_{s,s'}\langle 0|j^0|n(\mathbf{q})\rangle\langle n(\mathbf{q})|j^i|\gamma\gamma\rangle,$$

$$F_2(\lambda)=\sum_{s,s'}\langle 0|j^0|n(\mathbf{q})\gamma\rangle\langle n(\mathbf{q})|j^i|\gamma\rangle, \tag{16}$$

$$F_3(\lambda)=\sum_{s,s'}\langle 0|j^0|n(\mathbf{q})\gamma\gamma\rangle\langle n(\mathbf{q})|j^i|0\rangle.$$

[Of course, $F_i(\lambda)$ is also a function of $\mathbf{q}$ and $\mathbf{p}$, and we have merely suppressed this dependence in our notation.] Decompose $F_i(\lambda)$ into parts even and odd in λ,

$$F_i(\lambda)=F_i{}^e(\lambda)+F_i{}^o(\lambda). \tag{17}$$

Applying crossing symmetry to the terms in (15) and substituting (15)–(17) into (13), we find the compact expression

$$Q(\mathbf{q},t)=2\int\frac{d^3p}{(2\pi)^3}\cos E_nt\,\{F_1{}^e+2e^{-i\lambda t}F_2{}^e+e^{-2i\lambda t}F_3{}^e\}$$
$$-2i\int\frac{d^3p}{(2\pi)^3}\sin E_nt\,(F_1{}^o+2e^{-i\lambda t}F_2{}^o+e^{-2i\lambda t}F_3{}^o). \tag{18}$$

The calculation of the $F_i(\lambda)$ is straightforward. There are ten sets of Feynman diagrams, which yield ten traces of eight terms each. Since the diagrams contain no loops, the traces are finite and each F_i is explicitly gauge invariant. The traces are further simplified and combined by manipulations using charge conjugation, time reversal, etc. The traces which remain after these simplifications are easily evaluated because of the choices we made following Eq. (12), and with the further stipulation that $\mathbf{q}$ be in the i direction, i.e., $|\mathbf{q}|=q^i$. We omit the details and simply present the results, to first order in $\mathbf{q}$ and up to zeroth order in λ:

$$F_1(\lambda)=-q^i\frac{W}{p_0{}^5\lambda^2}\left[-2p_i{}^2+\lambda\left(-\frac{5}{2}\frac{p_i{}^2}{p_0}+\frac{p_0}{2}\right)+\lambda^2\left(\frac{1}{2}-\frac{5}{2}\frac{p_i{}^2}{p_0{}^2}\right)\right], \tag{19a}$$

$$F_2(\lambda)=+q^i\frac{W}{p_0{}^5\lambda^2}\left[-2p_i{}^2+\lambda\frac{p_i{}^2}{p_0}-\lambda^2\frac{p_i{}^2}{p_0{}^2}\right], \tag{19b}$$

$$F_3(\lambda)=-q^i\frac{W}{p_0{}^5\lambda^2}\left[-2p_i{}^2+\lambda\left(\frac{9}{2}\frac{p_i{}^2}{p_0}-\frac{p_0}{2}\right)+\lambda^2\left(1-7\frac{p_i{}^2}{p_0{}^2}\right)\right]. \tag{19c}$$

The quantity W in (19) is defined by

$$W=2m^2+2\mathbf{p}^2-2p_i{}^2. \tag{19$'$}$$

We are not taking the trouble to investigate higher orders in $\mathbf{q}$, because dimensional considerations[13] indicate that a higher-derivative operator Schwinger term is not possible to this order in perturbation theory.

Notice that the F_i are individually infrared divergent as $\lambda\to 0$. We will see that the total expression for Q, Eq. (18), is free of such divergences. But the infrared-divergent terms in (19) *are* responsible for finite contributions to (18), because in (18) they are multiplied by the phase factors $e^{-i\lambda t}$ and $e^{-2i\lambda t}$. Consequently, the presence of the infrared divergences in (19) require that we take care in examining the equal-time limit. The finite terms to which they give rise would be overlooked if we had calculated directly at $t=0$ or, equivalently, if we had cavalierly brought the limit $t\to 0$ inside the integral in (18).

Next we substitute (19) into (18) and perform the indicated integrations. We denote by $Q_i{}^{e,o}$ the contribution to Q from $F_i{}^{e,o}$. Then, for instance, for $Q_1{}^e$ we must evaluate

$$Q_1{}^e(\mathbf{q},t)=\frac{-2q^i}{\lambda^2}\int\frac{d^3p}{(2\pi)^3}\frac{2m^2+2\mathbf{p}^2-2p_i{}^2}{p_0{}^5}\times\left[-2p_i{}^2+\lambda^2\left(\frac{1}{2}-\frac{5}{2}\frac{p_i{}^2}{p_0{}^2}\right)\right]\cos 2p_0t. \tag{20}$$

After performing the angular integrations, we have

$$Q_1{}^e(\mathbf{q},t)=\frac{2q^i}{\lambda^2\pi^2}\int_m^\infty dp_0\cos 2p_0t\times|\mathbf{p}|\left(\frac{4}{15}+\frac{2}{15}\frac{m^2}{p_0{}^2}-\frac{2}{5}\frac{m^4}{p_0{}^4}-\frac{1}{2}\frac{\lambda^2m^4}{p_0{}^6}\right). \tag{21}$$

The third and fourth terms in the parentheses are finite as $t\to 0$, and they are easily evaluated. The first two terms give rise to quadratic and logarithmic divergences as $t\to 0$, and beneath these divergences they also contain terms which are finite as $t\to 0$. These finite terms can be recovered by expanding $|\mathbf{p}|$ in powers of $m^2/p_0{}^2$. This is not necessary, however, because we will see that all these terms cancel exactly in the final expression for Q. Consequently, it is sufficient for our purposes to record here only the leading singularity of the first two terms along with the finite contributions of the third and fourth terms. We have then

$$Q_i{}^e(\mathbf{q},t)=-\frac{2}{15\pi^2}q^i\left[\frac{1}{\lambda^2t^2}+\frac{2m^2}{\lambda^2}(\ln|t|+C')+2\frac{m^2}{\lambda^2}+1\right]+O(t), \tag{22}$$

where C' is an ambiguous constant which arises in the Fourier transform of p^{-1}.[12]

Evaluating $Q_1{}^o$ using the same methods, we find

$$Q_1{}^o(\mathbf{q},t)=O(t)\,, \tag{23}$$

that is, that $Q_1{}^o$ is well defined and vanishes as $t\to 0$.

For $Q_2{}^e$, we have

$$Q_2{}^e(\mathbf{q},t)=4e^{-i\lambda t}\int\frac{d^3p}{(2\pi)^3}\cos(2p_0t)F_2{}^e(\lambda)\,, \tag{24}$$

which, after integrating as in (20) through (22), becomes

$$Q_2{}^e(\mathbf{q},t)=-\frac{2q^i}{15\pi^2}e^{-i\lambda t}\Big[-\frac{2}{\lambda^2t^2}-4\frac{m^2}{\lambda^2}(\ln|t|+C') -4\frac{m^2}{\lambda^2}+4(\ln|t|+C)-\frac{2}{15}\Big]. \tag{25}$$

Now we expand the phase factor

$$e^{-i\lambda t}=1-i\lambda t-\tfrac{1}{2}\lambda^2t^2+\ldots$$

and obtain

$$Q_2{}^e(\mathbf{q},t)=-\frac{2q^i}{15\pi^2}\Big[-\frac{2}{\lambda^2t^2}+\frac{2i}{\lambda t}-4\frac{m^2}{\lambda^2}(\ln|t|+C') -4\frac{m^2}{\lambda^2}-4(\ln|t|+C)+(1)-\frac{2}{15}\Big]+O(t). \tag{26}$$

Similarly, for $Q_2{}^o$ we obtain

$$Q_2{}^o(\mathbf{q},t)=-\frac{4i}{15\pi^2}e^{-i\lambda t}\frac{q^i}{\lambda t} =-\frac{2q^i}{15\pi^2}\Big[\frac{2i}{\lambda t}+(2)\Big]+O(t). \tag{27}$$

For $Q_3{}^{e,o}$ we have, in the same way,

$$\begin{aligned}Q_3{}^e(\mathbf{q},t)&=-\frac{2q^i}{15\pi^2}e^{-2i\lambda t}\Big[\frac{1}{\lambda^2t^2}+2\frac{m^2}{\lambda^2}(\ln|t|+C') +2\frac{m^2}{\lambda^2}+4(\ln|t|+C)+\frac{32}{15}\Big]\\ &=-\frac{2q^i}{15\pi^2}\Big[\frac{1}{\lambda^2t^2}-\frac{2i}{\lambda t}+2\frac{m^2}{\lambda^2}(\ln|t|+C') +2\frac{m^2}{\lambda^2}+4(\ln|t|+C)-(2)+\frac{32}{15}\Big]+O(t)\,,\end{aligned} \tag{28}$$

$$Q_3{}^o(\mathbf{q},t)=\frac{4i}{15\pi^2}e^{-2i\lambda t}\frac{q^i}{\lambda t} =-\frac{2q^i}{15\pi^2}\Big[-\frac{2i}{\lambda t}-(4)\Big]+O(t). \tag{29}$$

In recording the final expressions for the $Q_i{}^{e,o}$, we have bracketed with parentheses the finite terms which are due to the product of the phase factors $e^{-i\lambda t}$ and $e^{-2i\lambda t}$ with the infrared-divergent terms in F_2 and F_3. These are the finite contributions which would have been omitted if we had been careless with the equal-time limit.

Combining Eqs. (22), (23), and (26)–(29), we find[15] that the sum is of order t, i.e.,

$$\lim_{t\to 0}Q(\mathbf{q},t)=0\,. \tag{30}$$

Thus we find no evidence for the existence of the operator Schwinger term. This result has previously been obtained from the BJL theorem.[4,6] We have also obtained the same result using the definition of the equal-time commutator given in Eq. (11).

Notice that if we had improperly interchanged the equal-time limit and the integration in Eq. (18), we would have obtained the result $Q(\mathbf{q},0)=-6q^i/15\pi^2$, and we would have incorrectly concluded that there is an operator Schwinger term.

IV. CONCERNING METHOD OF SPLIT POINTS

In this section we will show that the results previously obtained in split-point calculations of the c-number and operator Schwinger terms[2,4] correspond precisely to the results we would have obtained in Secs. II and III if we had improperly interchanged the equal-time limit and the phase-space integrations. By "split-point calculations," we refer specifically to calculations in which the currents are defined as the limit of a product of two fermion fields at *spatially* separated points, and in which the spatial limit is performed in a three-dimensionally symmetric way, so that, for instance, $\lim_{\epsilon\to 0}(\epsilon^i\epsilon^j/\epsilon^2)=\frac{1}{3}g^{ij}$.

Let us first briefly review the method of split points. The underlying observation is that the naive fermion current $\bar\psi(x)\Gamma\psi(x)$, being the product of two unbounded operators, is so singular an object that it is meaningless without further definition, and that this is the reason for the failure of the canonical formalism. Using the definition $j^\mu(x)=\bar\psi(x)\gamma^\mu\bar\psi(x)$ and the canonical equal-time anticommutation relations of the fermion fields, we easily obtain the result $[j^0(\mathbf{x},0),j^i(0)]=0$, which is contradicted by the more fundamental considerations of Sec. II. Schwinger[3] suggested that the current might meaningfully be defined as the limit

$$j^\mu(x)=\lim_{\epsilon\to 0}j_\epsilon{}^\mu(x)=\lim_{\epsilon\to 0}\bar\psi(x+\epsilon)\gamma^\mu\psi(x)\,, \tag{31}$$

where ϵ is a purely spatial vector. He showed in a free-

[15] We should point out that the expressions $(m^2/\lambda^2)(\ln|t|+C')$ in the $Q_i{}^e$ arise from identical expressions in the $F_i{}^e$. Thus there is no ambiguity about the cancellation of the constant C', since even in the theory of distributions the integral of zero is zero. The same remark applies to $(\ln|t|+C)$ which occurs in $Q_2{}^e$ and $Q_3{}^e$.

field theory that (31) implies that $[j^0(\mathbf{x},0),j^i(0)]$ is nonvanishing and quadratically divergent, in qualitative agreement with the results of Sec. II.

We will tentatively accept definition (31) and examine its consequences for the calculation of equal-time current commutators. But first, more generally, let us just suppose that there does exist some definition of the current as an operator-valued function defined at each space-time point x. Then the equal-time current commutator is most naturally defined by

$$[j^\mu(\mathbf{x},0),j^\nu(0)]=\lim_{x_0\to 0}[j^\mu(\mathbf{x},x_0),j^\nu(0)] \qquad (32)$$

if the limit exists. [Equation (32) is written in a shorthand form. The limit should be understood in the sense of weak convergence. For a more careful mathematical exposition of the content of this paragraph, the reader is referred to Sec. 2 of the paper of Brandt.[2]] If the commutator at equal times is completely regular and nonsingular, then the definition (32) is correct but superfluous; if, on the other hand, there are possible ambiguities due to singular contributions, then (32) is the natural way to resolve the ambiguities, provided of course the limit exists. If now we further use the definition of the current given in (31), we find that the equal-time commutator is defined by

$$[j^\mu(\mathbf{x},0),j^\nu(0)]=\lim_{x_0\to 0}\lim_{\epsilon,\eta\to 0}[j_\eta{}^\mu(\mathbf{x},x_0),j_\epsilon{}^\nu(0)],$$
$$\epsilon^0=\eta^0=0, \qquad (33)$$

where we further stipulate that the spatial limits be taken in a three-dimensionally symmetric manner, in order to ensure the proper behavior of the commutator under three-dimensional rotations.

We strongly suspect that definition (33) is consistent with perturbation theory, and that the previously noted discrepancies[4] are due to the improper interchange of the limits in (33). In practice, the limits are interchanged because it greatly facilitates the calculations, since one can then make use of the equal-time anticommutation relations of the fermion fields. The calculations at unequal times would be considerably less simple.

In previous calculations of Schwinger terms by the point-splitting method,[2–4] the limit $x_0\to 0$ is taken first, and in addition the spatial limit in the time component of the current is presumed to be smooth. One then has

$$[j^0(\mathbf{x},0),j^i(0)]=\lim_{\epsilon\to 0}[\bar\psi(\mathbf{x},0)\gamma^0\psi(\mathbf{x},0),\bar\psi(\epsilon)\gamma^i\psi(0)] \qquad (34)$$

and, after an elementary calculation using the canonical equal-time anticommutation relations, the final result is

$$[j^0(\mathbf{x},0),j^i(0)]$$
$$=-\lim_{\epsilon\to 0} j_\epsilon{}^i(0)[\delta(\mathbf{x}-\boldsymbol{\epsilon})-\delta(\mathbf{x})]$$
$$=\lim_{\epsilon\to 0} j_\epsilon{}^i(0)[(\boldsymbol{\epsilon}\cdot\nabla)\delta(\mathbf{x})+\tfrac{1}{6}(\boldsymbol{\epsilon}\cdot\nabla)^3\delta(\mathbf{x})+\cdots]. \qquad (35)$$

To compute the c-number Schwinger term to lowest order in perturbation theory, we evaluate $\langle j_\epsilon{}^i(0)\rangle_0$ to lowest order. We find $\langle j_\epsilon{}^i(0)\rangle_0=2i\epsilon^i/\pi^2\epsilon^4$, so that

$$\langle[j^0(\mathbf{x},0),j^i(0)]\rangle_0$$
$$=\frac{2i}{\pi^2}\lim_{\epsilon\to 0}\left[\frac{\epsilon^i\epsilon^j}{\epsilon^4}\partial^j\delta(\mathbf{x})+\frac{1}{6}\frac{\epsilon^i\epsilon^j\epsilon^k\epsilon^m}{\epsilon^4}\times\partial^j\partial^k\partial^m\delta(\mathbf{x})+\cdots\right]. \qquad (36)$$

Equation (36) is just the result of Brandt's split-point calculation,[2] corresponding to his Eq. (4.58). Taking the limit in a three-dimensionally symmetric manner, we find

$$\langle[j^0(\mathbf{x},0),j^i(0)]\rangle_0=\infty\,\partial^i\delta(\mathbf{x})+(i/15\pi^2)\partial^i\Delta\delta(\mathbf{x}), \qquad (37)$$

which differs from our perturbation-theory result, Eq. (10), by the coefficient of the third-derivative term.[16]

But suppose in Sec. II that we had taken the equal-time limit inside the integral of Eq. (7). Equation (7) would then have implied

$$C(\mathbf{q},0)=\frac{2}{(2\pi)^3}\int d^3p\left(\frac{q^i-p^i}{p_0'}+\frac{p^i}{p_0}\right) \qquad (38)$$

and, instead of (8), we would have found

$$C(\mathbf{q},0)=\frac{q^i}{\pi^2}\int_0^\infty dp\,\frac{p^2}{p_0}\left(1-\frac{1}{3}\frac{p^2}{p_0{}^2}\right)-\frac{q^i\mathbf{q}^2}{2\pi^2}m^4\int_0^\infty\frac{p^2dp}{p_0{}^7}, \qquad (39)$$

i.e., we would have overlooked the last two terms of (8). From (39), we would have obtained precisely the result of Eq. (37).

In their calculation of the c-number Schwinger term by the point-splitting method, Boulware and Jackiw do not obtain Eq. (37). Instead of the coefficient $i/15\pi^2$, they find $i/60\pi^2$, smaller by a factor 4. The explanation is that after the interchange of limits, (33) is not even

[16] The reader of Ref. 2 may be confused by the fact that in Eq. (1.11) Brandt gives $i/12\pi^2$ as the coefficient of the third-derivative c-number term and of the operator term. Brandt's split-point calculations with three-dimensional averaging would give $i/15\pi^2$ for both, but he has normalized the coefficient of $i/12\pi^2$ in order to agree with his independent calculation of the third-derivative term, which is carried out by careful examination of the free-field spectral representation.

translationally invariant. The source of the discrepancy is that Brandt uses our definition (31), i.e., $j_\epsilon^\mu(x) = \bar\psi(x+\epsilon)\gamma^\mu\psi(x)$, while Boulware and Jackiw use $j_\epsilon^\mu(x) = \bar\psi(x+\frac{1}{2}\epsilon)\gamma^\mu\psi(x-\frac{1}{2}\epsilon)$. By repeating the derivation of (35), it is easy to see that this is the source of the discrepancy. In Sec. II we would have had an analogous ambiguity if we had incorrectly deduced Eq. (38) from Eq. (7). The form of (38) is a consequence of how the momenta are shared by the electron and the positron in Eq. (6). If, instead of Eq. (6), we had defined $|n(\mathbf{q})\rangle = |e_s^-(\mathbf{p}+\frac{1}{2}\mathbf{q})e_{s'}^+(\frac{1}{2}\mathbf{q}-\mathbf{p})\rangle$, then instead of Eq. (38) we would have had

$$C(\mathbf{q},0) = \frac{2}{(2\pi)^3}\int d^3p \left\{ \frac{\frac{1}{2}q^i - p^i}{[m^2+(\frac{1}{2}\mathbf{q}-\mathbf{p})^2]^{1/2}} + \frac{p^i + \frac{1}{2}q^i}{[m^2+(\mathbf{p}+\frac{1}{2}\mathbf{q})^2]^{1/2}} \right\}. \quad (40)$$

From Eq. (40), we would have obtained $(i/60\pi^2)\partial^i\Delta\delta(\mathbf{x})$, in agreement with Boulware and Jackiw.

The ambiguity thus viewed from momentum space is very familiar. To get from (38) to (40) we have shifted the origin of integration, by letting $\mathbf{p} \to \mathbf{p}+\frac{1}{2}\mathbf{q}$. But it is well known that such a shift is improper for a divergent integral. The calculation in Sec. II does not suffer from this disease, because $C(\mathbf{q},t)$ is finite for $t \neq 0$, so that we may shift the origin as we please. Shifts of the origin in (7) only distribute the coefficient $i/12\pi^2$ of (10) differently among the last three terms of (8). The source of the ambiguity in (38)–(40) is precisely the loss of the last two terms in (8).

Let us next consider the possible operator Schwinger term, which would be bilinear in the photon field. To investigate this term using Eq. (35), one must evaluate $\langle 0|j_\epsilon^i|\gamma_1\gamma_2\rangle$ to the lowest nontrivial order in perturbation theory. Both Brandt and Boulware and Jackiw find that there is an operator Schwinger term. They obtain the same result, in this case, because first-derivative Schwinger terms are not effected by the ambiguity which we have just discussed. Using a three-dimensionally symmetric limit[16] as before, the operator Schwinger term they report may be written in the form

$$(i/15\pi^2)(2A^i\mathbf{A}\cdot\nabla + \mathbf{A}\cdot\mathbf{A}\partial^i)\delta(\mathbf{x}), \quad (41)$$

where A is the photon field, and the operator products are understood to be normal ordered. If we substitute (41) into Eq. (12), we find that $Q(\mathbf{q},0) = -6q^i/15\pi^2$. As we remarked at the end of Sec. III, this is just the result we would have obtained in our direct perturbation-theory calculation if we had taken the limit inside the integral in Eq. (18).

We have therefore seen in these two examples a remarkable correspondence between the point-splitting calculation and the direct perturbative calculations which would have resulted if sufficient care had not been taken with the equal-time limit. It has previously been conjectured that the point-splitting calculations are one way to define equal-time commutators, and that perturbation theory offers another, different way.[4] On the basis of the evidence presented here, which is compelling though circumstantial, we would instead conjecture that the split-point definition given in Eq. (33) is consistent with perturbation theory (at least, to the lowest nontrivial order), and the discrepancies—such as the supposed operator Schwinger term—are a consequence of improper interchange of the limits in (33). This conjecture can best be tested by direct calculations based on Eq. (33).[17]

ACKNOWLEDGMENTS

I am grateful to D. Yennie for his interest in this calculation and for his helpful suggestions. I would also like to thank R. Isaacs, T. Kinoshita, H. Terazawa, and K. Wilson for the discussions I have had with them.

[17] We have just learned that for the c-number case such a direct calculation has been carried out by P. Otterson, J. Math. Phys. **10**, 1525 (1969). Otterson's results are consistent with our conjecture. He finds that the correct result, Eq. (10), can be obtained only if the limits in (33) are not interchanged.