PHYSICAL REVIEW D VOLUME 2, NUMBER 12 15 DECEMBER 1970

Kuo Transformation in Linear and Nonlinear Chiral-Symmetry Theories*

SUSUMU OKUBO
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 25 May 1970)

Model Hamiltonians with $(3,3^*)\oplus(3^*,3)$-violating interaction in the theory of Gell-Mann, Oakes, and Renner have been investigated for both cases of linear and nonlinear realizations of the $SW(3)$ theory in connection with the Kuo transformation. For the linear case, all mass formulas remain unchanged under the transformation, but not for the nonlinear case. Also, scattering amplitudes are found not to be invariant in general under the Kuo transformation for the nonlinear case. Physical interpretations of these puzzling facts are discussed in detail.

I. KUO TRANSFORMATION

In a recent paper,[1] Kuo made a very interesting observation that in the model[2] of Gell-Mann, Oakes, and Renner (GMOR), the Hamiltonian density

$$H(x;\,\epsilon,g)\equiv H_0(x)+\epsilon_0 S^{(0)}(x)+\epsilon_8 S^{(8)}(x)+gM^{(8)}(x) \tag{1.1}$$

can be transformed by an operator V into a new form

$$\begin{aligned}\bar{H}(x;\,\epsilon,g)&\\ &\equiv VH(x;\,\epsilon,g)V^{-1}=H(x;\,\bar{\epsilon},\bar{g})\\ &=H_0(x)+\bar{\epsilon}_0 S^{(0)}(x)+\bar{\epsilon}_8 S^{(8)}(x)+\bar{g}M^{(8)}(x)\,,\end{aligned} \tag{1.2}$$

with

$$\begin{aligned}\bar{\epsilon}_0&=-\tfrac{1}{3}\epsilon_0-\tfrac{2}{3}\sqrt{2}\epsilon_8\,,\\ \bar{\epsilon}_8&=-\tfrac{2}{3}\sqrt{2}\epsilon_0+\tfrac{1}{3}\epsilon_8\,,\\ \bar{g}&=g\,.\end{aligned} \tag{1.3}$$

In Eq. (1.1), $H_0(x)$ is the $SW(3)$-invariant part, and $S^{(0)}(x)$ and $S^{(8)}(x)$ represent[3] the scalar portion of $(3^*,3)\oplus(3,3^*)$-type violation of the $SW(3)$ group while $M^{(8)}(x)$ is the eighth component of a scalar octet $M^{(\alpha)}(x)$ belonging to the $(1,8)\oplus(8,1)$ representation. The operator V is formally unitary and is given by

$$V=\exp[\pm\tfrac{3}{2}\pi i(Y-Y_5)]\,, \tag{1.4}$$

where Y and Y_5 are the hypercharge operator and its chiral counterpart, respectively.

As we have remarked elsewhere,[4] V also changes the parity operator P into

$$\bar{P}=VPV^{-1}=P\exp(3\pi iY)\,, \tag{1.5}$$

where we utilized the identity[5]

$$\exp(3\pi iY_5)=\exp(3\pi iY)\,. \tag{1.6}$$

Now, let us set

$$H(\epsilon,g)\equiv\int d^3x\,H(x;\,\epsilon,g) \tag{1.7}$$

and suppose that the Hamiltonian $H(\epsilon,g)$ has an eigenvalue $E_n(\epsilon,g)$ with eigenstate $|n,(\epsilon,g)\rangle$, i.e.,

$$H(\epsilon,g)|n,(\epsilon,g)\rangle=E_n(\epsilon,g)|n,(\epsilon,g)\rangle\,, \tag{1.8}$$

where n designates the state vector. Then, Eq. (1.2) implies that we must have

$$H(\bar{\epsilon},\bar{g})|\bar{n},(\bar{\epsilon},\bar{g})\rangle=E_n(\epsilon,g)|\bar{n},(\bar{\epsilon},\bar{g})\rangle\,, \tag{1.9}$$

where $|\bar{n},(\bar{\epsilon},\bar{g})\rangle$ is defined by

$$|\bar{n},(\bar{\epsilon},\bar{g})\rangle=V|n,(\epsilon,g)\rangle\,. \tag{1.10}$$

Notice first that if the hypercharge of the state $|n,(\epsilon,g)\rangle$ is an odd (even) integer, then the new state $|\bar{n},(\bar{\epsilon},\bar{g})\rangle$ must have the opposite (same) parity in comparison to $|n,(\epsilon,g)\rangle$ because of Eq. (1.5). Therefore, if $|n,(\epsilon,g)\rangle$ is a pion state, then the transformed state $|\bar{n},(\bar{\epsilon},\bar{g})\rangle$ should also be a pion state of the new Hamiltonian $H(\bar{\epsilon},\bar{g})$. However, if $|n,(\epsilon,g)\rangle$ corresponds to one kaon state, then $|\bar{n},(\bar{\epsilon},\bar{g})\rangle$ would give a new state corresponding to a 0^+ scalar κ meson with respect to the new Hamiltonian. At any rate, if one can formally change the label n, ϵ, g into $\bar{n}$, $\bar{\epsilon}$, $\bar{g}$ in Eq. (1.8) and if we compare the result with Eq. (1.9), one must have an identity $E_{\bar{n}}(\bar{\epsilon},\bar{g})=E_n(\epsilon,g)$. Especially, this would imply that the pion mass m_π or the η mass m_η would be invariant under the transformation (1.3) while the kaon mass m_K and the κ mass m_κ would be interchanged under Eq. (1.3). Indeed, this is the essence of a claim made by Kuo as far as the pion mass is concerned.

We have proved[6,7] elsewhere on the basis of rather general assumptions that for $g=0$, m_π^2 and m_K^2 are approximately given by the formulas

$$\begin{aligned}m_\pi^2&=[\epsilon_0+(1/\sqrt{2})\epsilon_8]K\,,\\ m_K^2&=[\epsilon_0-(1/2\sqrt{2})\epsilon_8]K\,,\end{aligned} \tag{1.11}$$

where K is a constant independent of ϵ_0 and ϵ_8. It is now obvious that our mass formula (1.11) is not invariant under Eq. (1.3). Indeed, m_π^2 changes its sign under the transformation. Upon this observation, Kuo claimed that our mass formula, Eq. (1.11), which was originally derived in a perturbative fashion by GMOR cannot be correct. How can we reconcile this? In reality, as we

* Work supported in part by the U. S. Atomic Energy Commission.

[1] T. K. Kuo, Nuovo Cimento Letters **3**, 803 (1970); Phys. Rev. D **2**, 342 (1970).

[2] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968). See also, S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968).

[3] The notation is the same as used in S. Okubo and V. S. Mathur, Phys. Rev. Letters **23**, 1412 (1969); Phys. Rev. D **1**, 2046 (1970).

[4] S. Okubo and V. S. Mathur, Phys. Rev. D **2**, 394 (1970). This paper is concerned with consequences of the Kuo transformation and hereafter will be referred to as (I).

[5] As we remarked in (I), a simple way to prove this is to notice that the operator $\frac{3}{2}(Y-Y_5)$ has only integral eigenvalues 0, ±1, ±2, $\cdots$ in the $SW(3)$ theory.

[6] S. Okubo and V. S. Mathur, Ref. 3; and also V. S. Mathur, S. Okubo, and J. Subba Rao, Phys. Rev. D **1**, 2058 (1970). These papers are concerned with consequences of the $W(3)$ theory and will be referred to hereafter as (II).

[7] V. S. Mathur and S. Okubo, Phys. Rev. D **1**, 3468 (1970). This paper is an investigation of the $SW(3)$ theory and will be referred to hereafter as (III).

have already explained elsewhere,[4] there is no contradiction between the mass formula, Eq. (1.11), and the existence of the Kuo transformation. In this section, we shall give a simplified account of the resolution of this dilemma. First, we observe the fact that the κ meson as well as the opposite partner of the nucleon must exist together with the kaon and the nucleon if Kuo's argument is correct. However, we know that this need not be correct for the nonlinear realization of the chiral symmetry although it is correct for the case of the linear realization of the same group. Since the nonlinear realization is intimately related to the question whether the Goldstone zero-mass bosons should emerge in the exact $SW(3)$ limit, the whole problem is actually connected to the existence of Goldstone bosons. As we shall see shortly, Kuo's argument is valid and m_π^2 should be invariant under Eq. (1.3) if we have no Goldstone bosons, i.e., if the vacuum is invariant under the full $SW(3)$ group in the exact chiral symmetry limit. However, there is no reason for its validity when zero-mass bosons appear in the theory. In the latter case, the transformation operator V is in general a very singular operator, and is very likely not implementable as a unitary operator in the mathematical sense. Indeed, it will be an operator bringing one Hilbert space into another disjoint space. Examples of such operators are well known in various models; for example, in the model of Nambu and Jona-Lasinio.[8]

To simplify our argument, let us consider the special case where we set $\epsilon_0=\epsilon_8=0$. Since the remaining parameter g is invariant under the Kuo transformation, let us write the Hamiltonian simply as H. Then H is invariant under the Kuo transformation V, as we see from Eqs. (1.2) and (1.3). Suppose that V is a unitary operator in a given Hilbert space. Then, writing $|n,(\epsilon=0,g)\rangle$ simply as $|n\rangle$, we find that both states $|n\rangle$ and $|\bar{n}\rangle$ belonging to the same Hilbert space must have the same eigenvalues with respect to the same Hamiltonian H since we have set $\epsilon_0=\epsilon_8=0$. Moreover, if there are no zero-mass particles in our theory, we find especially that both the kaon and the κ meson must coexist and have the same mass. However, one cannot prove the existence of another scalar meson δ with $T=1$ and $Y=0$ in this fashion. Now, we appeal to the continuity argument with respect to the coupling parameter g. In the exact $SU(3)$ limit $g=0$, we expect to have a scalar $SU(3)$ multiplet, say, an octet to which the κ belongs. Thus, for $g=0$, the theory will predict the parity-doublet structure for mesons as well as for baryons. For the case $g\neq 0$, we appeal to the continuity argument by changing continuously the value of g from its $SU(3)$ value $g=0$ to a general nonzero value, and we should still maintain the parity-doublet structure of particle multiplets. As we noted already, we need not have the existence of scalar mesons or the parity-doublet partner of the nucleon if we utilize a nonlinear realization of the chiral group. Therefore, we conclude that for such cases, either there is a zero-mass particle (or particles) in theory, or the operator V cannot be well defined mathematically in a given Hilbert space, or both. In the second case, the operator V must represent a formal operation[8] transforming one Hilbert space into another one.

[8] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).

For the general case with nonzero ϵ_0 and ϵ_8, the argument remains essentially the same as we used in (I). Another interesting point is that there exists a possible singularity of physical quantities as functions of ϵ_0 and ϵ_8 at $\epsilon_0=-\sqrt{2}\epsilon_8$, where the $SW(2)$ subgroup becomes exact, with the resulting zero-mass Goldstone pions. However, since this argument has been fully described in (I), we shall not go into detail here.

The purpose of this paper is to demonstrate in some detail the behavior described above for both the linear and the nonlinear realization of the chiral group. Especially, we shall investigate why the mass formulas fail to be invariant under the transformation, Eq. (1.3). In order to simplify the matter, we shall set $g=0$ identically hereafter unless otherwise so stated. Also, it is convenient to introduce a parameter a given by

$$a=\frac{1}{\sqrt{2}}\frac{\epsilon_8}{\epsilon_0}. \tag{1.12}$$

Then, transformation (1.3) is now rewritten as

$$\begin{aligned}\epsilon_0\to\bar{\epsilon}_0&=-\tfrac{1}{3}(1+4a)\epsilon_0,\\ a\to\bar{a}&=\frac{2-a}{1+4a}.\end{aligned} \tag{1.13}$$

Alternatively, transformation (1.3) can be brought into a diagonal form in terms of new variables

$$\begin{aligned}u&=\frac{1}{\sqrt{6}}(1+a)\epsilon_0=\frac{1}{\sqrt{6}}\left(\epsilon_0+\frac{1}{\sqrt{2}}\epsilon_8\right),\\ v&=\frac{1}{\sqrt{6}}(1-2a)\epsilon_0=\frac{1}{\sqrt{6}}(\epsilon_0-\sqrt{2}\epsilon_8).\end{aligned} \tag{1.14}$$

Indeed, the transformation property of u and v under Eq. (1.3) is simply given by

$$\begin{aligned}u\to\bar{u}&=-u,\\ v\to\bar{v}&=+v.\end{aligned} \tag{1.15}$$

Therefore, if we demand, as Kuo does, that a physical quantity such as pion mass m_π should be invariant under (1.3), then it must be a function of v and u^2. However, we will not use these variables u and v hereafter.

II. MASS FORMULAS IN LINEAR REPRESENTATION

As we emphasized in Sec. I, the implications of the Kuo transformation and the mass formula would be very much different in the linear and nonlinear realizations of chiral symmetry.

Let us consider a $(3,3^*)\oplus(3^*,3)$ representation of the $SW(3)$ group where we have the two tensors $M_b{}^{a'}$ and $M_{b'}{}^a$ satisfying the self-conjugate condition,[9]

$$M_b{}^{a'}(x)=[M_{a'}{}^b(x)]^\dagger \quad (a,\,b=1,\,2,\,3). \tag{2.1}$$

We use the convention[9] that the primed and unprimed indices refer to those of $SU^{(-)}(3)$ and $SU^{(+)}(3)$ groups, respectively. Since the parity operation interchanges two spaces, we define the scalar nonet $\sigma_b{}^a(x)$ and the pseudoscalar nonet $\pi_b{}^a(b)$ by

$$\begin{aligned}\sigma_b{}^a(x)&=\tfrac{1}{2}[M_{b'}{}^a(x)+M_b{}^{a'}(x)],\\ \pi_b{}^a(x)&=-\tfrac{1}{2}i[M_{b'}{}^a(x)-M_b{}^{a'}(x)].\end{aligned} \tag{2.2}$$

It is sometimes more convenient to use the matrix notation, so we introduce 3×3 matrices M and $M^\dagger$ by

$$M_{ba}=M_{b'}{}^a,\quad (M^\dagger)_{ba}=M_b{}^{a'} \tag{2.3}$$

and σ and π by

$$\sigma_{ba}=\sigma_b{}^a,\quad \pi_{ba}=\pi_b{}^a. \tag{2.4}$$

Because of the Hermiticity conditions (2.1), $M^\dagger$ is indeed the Hermitian conjugate of M and the matrices σ and π are Hermitian. Moreover, we have

$$\begin{aligned}M(x)&=\sigma(x)+i\pi(x),\\ M^\dagger(x)&=\sigma(x)-i\pi(x).\end{aligned} \tag{2.5}$$

Also, it is convenient to define $\sigma_\alpha(x)$ and $\pi_\alpha(x)$ $(\alpha=0,\,1,\,\ldots,\,8)$ by the expansion

$$\begin{aligned}\sigma(x)&=\tfrac{1}{2}\sum_{\alpha=0}^{8}\sigma_\alpha(x)\lambda_\alpha,\\ \pi(x)&=\tfrac{1}{2}\sum_{\alpha=0}^{8}\pi_\alpha(x)\lambda_\alpha,\end{aligned} \tag{2.6}$$

where λ_α $(\alpha=0,\,1,\,\ldots,\,8)$ are the standard Hermitian 3×3 matrices with $\lambda_0=\sqrt{\tfrac{2}{3}}$. One can solve Eq. (2.6) to get

$$\begin{aligned}\sigma_\alpha(x)&=\mathrm{Tr}[\lambda_\alpha\sigma(x)],\\ \pi_\alpha(x)&=\mathrm{Tr}[\lambda_\alpha\pi(x)],\end{aligned} \tag{2.7}$$

if we use $\mathrm{Tr}(\lambda_\alpha\lambda_\beta)=2\delta_{\alpha\beta}$.

Since we want to maintain the structure of Eq. (1.1) with $g=0$, we consider a model Lagrangian

$$\mathcal{L}(x)=\mathcal{L}_0(x)+\mathcal{L}_1(x), \tag{2.8}$$

where

$$\begin{aligned}\mathcal{L}_0(x)&=-Z\,\mathrm{Tr}[\partial_\mu M^\dagger(x)\partial_\mu M(x)]-\mu_0{}^2\,\mathrm{Tr}[M^\dagger(x)M(x)]\\ &=-\tfrac{1}{2}Z\sum_{\alpha=0}^{8}\{[\partial_\mu\sigma_\alpha(x)]^2+[\partial_\mu\pi_\alpha(x)]^2\}\\ &\quad-\tfrac{1}{2}\mu_0{}^2\sum_{\alpha=0}^{8}\{[\sigma_\alpha(x)]^2+[\pi_\alpha(x)]^2\}\end{aligned} \tag{2.9}$$

is the free Lagrangian. An arbitrary constant Z will be suitably chosen later. The interaction Lagrangian $\mathcal{L}_1(x)$ must have the form (1.1) with $g=0$, i.e.,

$$\mathcal{L}_1(x)=-\epsilon_0S^{(0)}(x)-\epsilon_8S^{(8)}(x). \tag{2.10}$$

To find a suitable candidate for the form of $S^{(\alpha)}(x)$, let us define two 3×3 matrices by

$$\begin{aligned}R_{b'}{}^a(x)&=\epsilon^{acd}\epsilon_{b'e'f'}M_c{}^{e'}(x)M_d{}^{f'}(x),\\ R_b{}^{a'}(x)&=\epsilon^{a'c'd'}\epsilon_{bef}M_{c'}{}^e(x)M_{d'}{}^f(x)=[R_{a'}{}^b(x)]^\dagger\end{aligned} \tag{2.11}$$

and furthermore set

$$\begin{aligned}u_b{}^a(x)&\equiv\tfrac{1}{2}[R_{b'}{}^a(x)+R_b{}^{a'}(x)]\equiv\tfrac{1}{2}\sum_{\alpha=0}^{8}(\lambda_\alpha)_b{}^au_\alpha(x),\\ v_b{}^a(x)&\equiv\frac{1}{2i}[R_{b'}{}^a(x)-R_b{}^{a'}(x)]\equiv\tfrac{1}{2}\sum_{\alpha=0}^{8}(\lambda_\alpha)_b{}^av_\alpha(x).\end{aligned} \tag{2.12}$$

It is not difficult to show that $u_\alpha(x)$ and $v_\alpha(x)$ are given by

$$\begin{aligned}u_\alpha(x)=&(\sqrt{\tfrac{3}{2}})\delta_{\alpha,0}[3\sigma_0(x)\sigma_0(x)-3\pi_0(x)\pi_0(x)\\ &-\sum_{\beta=0}^{8}\sigma_\beta(x)\sigma_\beta(x)+\sum_{\beta=0}^{8}\pi_\beta(x)\pi_\beta(x)]\\ &-(\sqrt{6})[\sigma_0(x)\sigma_\alpha(x)-\pi_0(x)\pi_\alpha(x)]\\ &+\sum_{\beta,\gamma=0}^{8}d_{\alpha\beta\gamma}[\sigma_\beta(x)\sigma_\gamma(x)-\pi_\beta(x)\pi_\gamma(x)],\\ v_\alpha(x)=&-(\sqrt{6})\delta_{\alpha,0}[3\sigma_0(x)\pi_0(x)-\sum_{\beta=0}^{8}\sigma_\beta(x)\pi_\beta(x)]\\ &+(\sqrt{6})[\pi_0(x)\sigma_\alpha(x)+\sigma_0(x)\pi_\alpha(x)]\\ &-2\sum_{\beta,\gamma=0}^{8}d_{\alpha\beta\gamma}\sigma_\beta(x)\pi_\beta(x),\end{aligned} \tag{2.13}$$

where $\delta_{\alpha,0}$ is equal to the unity for $\alpha=0$ and zero for $\alpha\neq0$. After these preparations, we shall define $S^{(\alpha)}(x)$ and $P^{(\alpha)}(x)$ by

$$\begin{aligned}S^{(\alpha)}(x)&=u_\alpha(x)+A\sigma_\alpha(x),\\ P^{(\alpha)}(x)&=v_\alpha(x)+A\pi_\alpha(x),\end{aligned} \tag{2.14}$$

where A is an arbitrary constant. We notice that $R_b{}^{a'}(x)$ and $R_{b'}{}^a(x)$ belong to a $(3,3^*)\oplus(3^*,3)$ representation of the $SW(3)$ group, but not that of the $W(3)$ group, since the completely antisymmetric tensors ϵ^{abc}, ϵ_{abc}, etc., are invariant only under the $SW(3)$ group. Hence, $S^{(\alpha)}(x)$ and $P^{(\alpha)}(x)$ defined by Eq. (2.14) belong to a $(3,3^*)\oplus(3^*,3)$ representation of the $SW(3)$ but not of the $W(3)$ group. An exception to this statement arises when we have $A\to\infty$, so that the second terms in Eq. (2.14) dominate the first ones. We shall come back to this point later.

[9] R. E. Marshak, N. Mukunda, and S. Okubo, Phys. Rev. **137**, B698 (1965): R. E. Marshak, S. Okubo, and J. Wojtaszek, Phys. Rev. Letters **15**, 463 (1965); Y. Hara, Phys. Rev. **139**, B134 (1965).

Now, the Kuo transformation is represented as a transformation on M by

$$M(x)\rightarrow\bar{M}(x)=UM(x)\,,\quad M^{\dagger}(x)\rightarrow\bar{M}^{\dagger}(x)=M^{\dagger}(x)U^{\dagger}\,,\tag{2.15}$$

where the 3×3 matrix U is simply given by

$$U=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}=\exp(\pm\sqrt{3}\pi i\lambda_8)\,.\tag{2.16}$$

This induces the following transformations among $\sigma_\alpha(x)$ and $\pi_\alpha(x)$:

$$\begin{aligned}
\sigma_0(x)&\rightarrow\bar{\sigma}_0(x)=-\tfrac{1}{3}\sigma_0(x)-\tfrac{2}{3}\sqrt{2}\sigma_8(x)\,,\\
\pi_0(x)&\rightarrow\bar{\pi}_0(x)=-\tfrac{1}{3}\pi_0(x)-\tfrac{2}{3}\sqrt{2}\pi_8(x)\,,\\
\sigma_8(x)&\rightarrow\bar{\sigma}_8(x)=-\tfrac{2}{3}\sqrt{2}\sigma_0(x)+\tfrac{1}{3}\sigma_8(x)\,,\\
\pi_8(x)&\rightarrow\bar{\pi}_8(x)=-\tfrac{2}{3}\sqrt{2}\pi_0(x)+\tfrac{1}{3}\pi_8(x)\,,\\
\sigma_\alpha(x)&\rightarrow\bar{\sigma}_\alpha(x)=-\sigma_\alpha(x)\quad(\alpha=1,2,3)\,,\\
\pi_\alpha(x)&\rightarrow\bar{\pi}_\alpha(x)=-\pi_\alpha(x)\quad(\alpha=1,2,3)\,,\\
\sigma_4(x)\pm i\sigma_5(x)&\rightarrow\bar{\sigma}_4(x)\pm i\bar{\sigma}_5(x)=\pm i[\pi_4(x)\pm i\pi_5(x)]\,,\\
\pi_4(x)\pm i\pi_5(x)&\rightarrow\bar{\pi}_4(x)\pm i\bar{\pi}_5(x)=\mp i[\sigma_4(x)\pm i\sigma_5(x)]\,,\\
\sigma_6(x)\pm i\sigma_7(x)&\rightarrow\bar{\sigma}_6(x)\pm i\bar{\sigma}_7(x)=\pm i[\pi_6(x)\pm i\pi_7(x)]\,,\\
\pi_6(x)\pm i\pi_7(x)&\rightarrow\bar{\pi}_6(x)\pm i\bar{\pi}_7(x)=\mp i[\sigma_6(x)\pm i\sigma_7(x)]\,.
\end{aligned}\tag{2.17}$$

We remark that $S^{(\alpha)}(x)$ and $P^{(\alpha)}(x)$ transform exactly in the same way under the Kuo transformation by replacing $\sigma_\alpha(x)$ and $\pi_\alpha(x)$ with $S^{(\alpha)}(x)$ and $P^{(\alpha)}(x)$, respectively, in Eq. (2.17).

As we noted in Sec. I, we see indeed that the Kuo transformation interchanges particles with odd hypercharge into particles with the opposite parity while particles with zero hypercharge will transform among themselves without changing the parity.

Our formulation is applicable to both cases of linear and nonlinear realizations of the $SW(3)$ group. For the linear representation, we regard both fields $\sigma_\alpha(x)$ and $\pi_\alpha(x)$ to be kinematically independent, while for the case of nonlinear realization we impose an additional constraint $M^\dagger(x)M(x)=1$ so that $\sigma_\alpha(x)$ and $\pi_\alpha(x)$ are not independent of each other.

In this section we investigate the case of the linear realization. The consequences of the nonlinear realization will be given in Sec. III. For the linear case, it is convenient to choose $Z=1$ in the free Lagrangian $\mathfrak{L}_0(x)$ so that $(\partial/\partial x_0)\pi_\alpha(x)$ and $(\partial/\partial x_0)\sigma_\alpha(x)$ are canonical conjugates of $\pi_\alpha(x)$ and $\sigma_\alpha(x)$, respectively.

Defining the vector current $V_\mu{}^{(\alpha)}(x)$ and axial-vector current $A_\mu{}^{(\alpha)}(x)$ $(\alpha=0,1,\ldots,8)$ by

$$\begin{aligned}
V_\mu{}^{(\alpha)}(x)&=-\sum_{\beta,\gamma=0}^{8}f_{\alpha\beta\gamma}\left[\sigma_\beta(x)\frac{\partial}{\partial x_\mu}\sigma_\gamma(x)+\pi_\beta(x)\frac{\partial}{\partial x_\mu}\pi_\gamma(x)\right],\\
A_\mu{}^{(\alpha)}(x)&=\sum_{\beta,\gamma=0}^{8}d_{\alpha\beta\gamma}\left[-\sigma_\beta(x)\frac{\partial}{\partial x_\mu}\pi_\alpha(x)+\pi_\beta(x)\frac{\partial}{\partial x_\mu}\sigma_\gamma(x)\right],
\end{aligned}\tag{2.18}$$

we can check the validity of the algebra of currents:

$$\begin{aligned}
[V_4{}^{(\alpha)}(x),V_4{}^{(\beta)}(y)]&=[A_4{}^{(\alpha)}(x),A_4{}^{(\beta)}(y)]\\
&=-f_{\alpha\beta\gamma}V_4{}^{(\gamma)}(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\
[V_4{}^{(\alpha)}(x),A_4{}^{(\beta)}(y)]&=-f_{\alpha\beta\gamma}A_4{}^{(\gamma)}(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})
\end{aligned}\tag{2.19}$$

for all α, $\beta=0,1,\ldots,8$ at $x_0=y_0$. Similarly, we can prove

$$\begin{aligned}
[V_4{}^{(\alpha)}(x),S^{(\beta)}(y)]&=-f_{\alpha\beta\gamma}S^{(\gamma)}(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\
[V_4{}^{(\alpha)}(x),P^{(\beta)}(y)]&=-f_{\alpha\beta\gamma}P^{(\gamma)}(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\
[A_4{}^{(\alpha)}(x),S^{(\beta)}(y)]&=-d_{\alpha\beta\gamma}P^{(\gamma)}(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\
[A_4{}^{(\alpha)}(x),P^{(\beta)}(y)]&=+d_{\alpha\beta\gamma}S^{(\gamma)}(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,
\end{aligned}\tag{2.20}$$

for $\alpha=1,\ldots,8$ and $\beta=0,1,\ldots,8$ at $x_0=y_0$. Notice that we have to exclude the case $\alpha=0$ in the third and fourth lines of Eq. (2.20) since $S^{(\alpha)}(x)$ and $P^{(\alpha)}(x)$ $(\alpha=0,1,\ldots,8)$ belong to the $(3,3^*)\oplus(3^*,3)$ representation of the $SW(3)$ group but not of the $W(3)$ group. The direct proof of this fact as well as these commutation relations is a bit involved and we shall not prove them here.

Since our Lagrangian is quadratic in the fields $\sigma_\alpha(x)$ and $\pi_\alpha(x)$, one can easily calculate the masses of the respective particles by

$$\begin{aligned}
m_\pi{}^2&=\mu_0{}^2+(2/\sqrt{6})(1-2a)\epsilon_0\,,\\
m_K{}^2&=\mu_0{}^2+(2/\sqrt{6})(1+a)\epsilon_0\,,\\
m_8{}^2&=\mu_0{}^2+(2/\sqrt{6})(1+2a)\epsilon_0\,,\\
m_0{}^2&=\mu_0{}^2-(4/\sqrt{6})\epsilon_0\,,\\
m_T{}^2&=(2/\sqrt{6})a\epsilon_0\,,\\
m_\delta{}^2&=\mu_0{}^2-(2/\sqrt{6})(1-2a)\epsilon_0\,,\\
m_\kappa{}^2&=\mu_0{}^2-(2/\sqrt{6})(1+a)\epsilon_0\,,\\
\bar{m}_8{}^2&=\mu_0{}^2-(2/\sqrt{6})(1+2a)\epsilon_0\,,\\
\bar{m}_0{}^2&=\mu_0{}^2+(4/\sqrt{6})\epsilon_0\,,\\
\bar{m}_T{}^2&=-m_T{}^2\,,
\end{aligned}\tag{2.21}$$

where $m_T{}^2$ $(\bar{m}_T{}^2)$ is the squared transition mass between π_0 (σ_0) and π_8 (σ_8), and $m_\delta{}^2$, $m_\kappa{}^2$, $\bar{m}_8{}^2$, $\bar{m}_0{}^2$, $\bar{m}_T{}^2$ are the squared masses of the scalar nonet $\sigma_\alpha(x)$.

To diagonalize the mass matrix involving π_8 and π_0, we have to introduce the physical fields $\eta(x)$ and $X(x)$ by

$$\begin{aligned}\eta(x)&=\cos\theta\,\pi_8(x)-\sin\theta\,\pi_0(x)\,,\\ X(x)&=\sin\theta\,\pi_8(x)+\cos\theta\,\pi_0(x)\,.\end{aligned}\tag{2.22}$$

Then, the standard procedure gives us

$$\begin{aligned}m_\eta{}^2&=\mu_0{}^2-\frac{1}{\sqrt{6}}(1-2a)\epsilon_0+\frac{1}{\sqrt{2}}(3+4a+4a^2)^{1/2}\epsilon_0\,,\\ m_X{}^2&=\mu_0{}^2-\frac{1}{\sqrt{6}}(1-2a)\epsilon_0-\frac{1}{\sqrt{2}}(3+4a+4a^2)^{1/2}\epsilon_0\,,\end{aligned}\tag{2.23}$$

with the mixing angle

$$\tan(2\theta)=-2\sqrt{2}a/(3+2a)\,.\tag{2.24}$$

In principle, one can interchange the formulas for $m_\eta{}^2$ and $m_X{}^2$ in Eq. (2.23). However, we defined the η field so that in the exact $SU(3)$ limit $a=0$, we should have $m_\eta=m_\pi=m_K$, i.e., η, π, and K from a pure $SU(3)$ octet at $a=0$. Similarly in the chimeral[10] $SU(3)$ limit $a=2$, we have $m_\kappa=m_\pi=m_X$ with $\tan\theta=2\sqrt{2}$, i.e., π, κ, and X belong to a pure chimeral octet. Also, X remains to be a pure $SU(3)$ singlet at $a=0$ while it is the η which becomes a chimeral $SU(3)$ singlet at $a=2$.

Eliminating the unknown parameters $\mu_0{}^2$, ϵ_0, and a, we obtain mass formulas,

$$\begin{aligned}&m_K{}^2-m_\pi{}^2=m_\delta{}^2-m_\kappa{}^2\,,\\ &(m_\eta{}^2-m_8{}^2)(m_X{}^2-m_8{}^2)=-(2/9)(m_K{}^2-m_\pi{}^2)^2\,,\\ &(\bar{m}_\eta{}^2-\bar{m}_8{}^2)(\bar{m}_X{}^2-\bar{m}_8{}^2)=-(2/9)(m_\kappa{}^2-m_\delta{}^2)^2\,,\end{aligned}\tag{2.25}$$

where

$$m_8{}^2=\tfrac{1}{3}(4m_K{}^2-m_\pi{}^2),\quad \bar{m}_8{}^2=\tfrac{1}{3}(4_\kappa{}^2-m_\delta{}^2)\,.$$

These mass formulas agree with those derived in Ref. 9, but they are not so well satisfied experimentally. Hence the present model is interesting only from the theoretical viewpoint.

Now let us investigate consequences of the Kuo transformation, Eq. (1.3). We see immediately that the masses $m_\pi{}^2$ and $m_\delta{}^2$ remain invariant while $m_K{}^2$ and $m_\kappa{}^2$ are interchanged under the transformation in conformity with what we stated in Sec. I. However, the case of $m_\eta{}^2$ and $m_X{}^2$ is a bit more complicated. One can easily check that if $1+4a<0$, both $m_\eta{}^2$ and $m_X{}^2$ remain invariant under Eq. (1.3), but for the case $1+4a>0$, $m_\eta{}^2$ and $m_X{}^2$ are interchanged. Such behavior, of course, does not conflict with the argument stated in Sec. I. It illustrates the point that we must be nevertheless careful when we have a problem of two energy states with the same quantum numbers.

It is interesting to observe that both $m_\eta{}^2$ and $m_X{}^2$ have a branch cut singularity at $a=\frac{1}{2}(-1\pm\sqrt{2}i)$ as functions of the parameter a. Although its physical meaning is not clear, it implies that the $SU(3)$ perturbation is possible only for $|a|<\frac{1}{2}\sqrt{3}$. Second, the mixing angle θ transforms under the Kuo transformation by the formula

$$\tan(2\bar{\theta})=-\frac{8+7\sqrt{2}\tan(2\theta)}{7\sqrt{2}-8\tan(2\theta)}\,,\tag{2.26}$$

which is equivalent to $\theta+\bar{\theta}=\tan^{-1}(2\sqrt{2})$.

In conclusion of this section, it may be worthwhile mentioning a case in which we set $\epsilon_8=0$ but $\epsilon_0\neq0$, and $g\neq0$ in Eq. (1.1). Choosing $M^{(8)}(x)$ as

$$M^{(8)}(x)=d_{8\beta\gamma}[\sigma_\beta(x)\sigma_\gamma(x)+\pi_\beta(x)\pi_\gamma(x)]\,,\tag{2.27}$$

which belongs to the $(1,8)\oplus(8,1)$ representation, one can similarly diagonalize the Lagrangian (1.1) to obtain the following mass formulas:

$$\begin{aligned}&m_K{}^2-m_\pi{}^2=m_\kappa{}^2-m_\delta{}^2\,,\\ &(m_\eta{}^2-m_8{}^2)(m_X{}^2-m_8{}^2)=-(8/9)(m_K{}^2-m_\pi{}^2)^2\,,\\ &(\bar{m}_\eta{}^2-\bar{m}_8{}^2)(\bar{m}_X{}^2-\bar{m}_8{}^2)=-(8/9)(m_\kappa{}^2-m_\delta{}^2)^2\,,\end{aligned}\tag{2.28}$$

again in agreement with the results of Ref. 9. Notice that the change of sign on the right-hand side of the first equation occurs in comparison to the corresponding formula in Eq. (2.25). The first relation in Eq. (2.28) has been also derived by Matsuda and Oneda[11] by means of the algebra of currents. We may remark that these mass formulas, Eq. (2.28), appear to be reasonably well satisfied experimentally and that the second and third equations of Eq. (2.28) are known as Schwinger's mass formula in the literature.

III. NONLINEAR REALIZATION

In this section, we restrict ourselves again only to the case $g=0$, corresponding to the GMOR model.

As we mentioned, the first part of the formulation of Sec. II holds valid for nonlinear realization. The important difference is now we impose a restriction[12] that the 3×3 matrix $M(x)$ satisfies the constraint

$$M^\dagger(x)M(x)=1\,.\tag{3.1}$$

Therefore, $\sigma_\alpha(x)$ and $\pi_\alpha(x)$ defined in Sec. II are no longer kinematically independent. Because of this, it is more convenient to use other specific representation for $M(x)$ satisfying Eq. (3.1). We shall here consider two cases which are often used in the literature:

$$M(x)=\exp[2if\varphi(x)]\,,\tag{3.2}$$

$$M(x)=\frac{1+if\varphi(x)}{1-if\varphi(x)}\,,\tag{3.3}$$

where f is a constant and $\varphi(x)$ represents a Hermitian 3×3 pseudoscalar-meson field. Both Eqs. (3.2) and (3.3) can be expanded as

$$M(x)=1+2if\varphi(x)-2f^2\varphi^2(x)+O(\varphi^3)\,.\tag{3.4}$$

[10] For the definition of the chimeral $SU(3)$ group, see (II) and (III).

[11] S. Matsuda and S. Oneda, Phys. Rev. **179**, 1301 (1969).

[12] E.g., J. A. Cronin, Phys. Rev. **161**, 1483 (1967).

For a while, all we utilize are Eqs. (3.1) and (3.4) and many resulting consequences are valid for much wider classes of representations other than Eqs. (3.2) and (3.3).

It is convenient to choose the arbitrary constant Z in $\mathcal{L}_0(x)$ to be $Z^{-1}=8f^2$. Setting

$$\varphi(x)=\frac{1}{\sqrt{2}}\sum_{\alpha=0}^{8}\varphi_\alpha(x)\lambda_\alpha\,,\qquad \varphi_\alpha(x)=\frac{1}{\sqrt{2}}\mathrm{Tr}[\varphi(x)\lambda_\alpha]\,, \tag{3.5}$$

one finds now

$$\mathcal{L}_0(x)=\tfrac{1}{2}\sum_{\alpha=0}^{8}(\partial_\mu\varphi_\alpha(x))(\partial_\mu\varphi_\alpha(x))+\tfrac{3}{2}\mu_0{}^2+O(\varphi^3)\,. \tag{3.6}$$

Similarly, one computes

$$\sigma_\alpha(x)=(\sqrt{6})\delta_{\alpha,0}-2f^2\sum_{\beta,\gamma=0}^{8}d_{\alpha\beta\gamma}\varphi_\beta(x)\varphi_\gamma(x)+O(\varphi^3)\,, \tag{3.7}$$
$$\pi_\alpha(x)=2\sqrt{2}f\varphi_\alpha(x)+O(\varphi^2)\,,$$

where $\sigma_\alpha(x)$ and $\pi_\alpha(x)$ are defined as in Sec. II. Hence, we obtain

$$\begin{aligned}\mathcal{L}_1(x)=&-(\sqrt{6})\epsilon_0(2+A)+(\sqrt{\tfrac{2}{3}})2f^2\epsilon_0(8+A)\varphi_0(x)\varphi_0(x)\\&-(\sqrt{\tfrac{2}{3}})2f^2\epsilon_0 2\sqrt{2}a(4-A)\varphi_0(x)\varphi_8(x)\\&+(\sqrt{\tfrac{2}{3}})2f^2\epsilon_0(2+A)\{(1+a)\sum_{\beta=1}^{3}\varphi_\beta(x)\varphi_\beta(x)\\&+(1-\tfrac{1}{2}a)\sum_{\beta=4}^{7}\varphi_\beta(x)\varphi_\beta(x)+(1-a)\varphi_8(x)\varphi_8(x)\}\\&+O(\varphi^3)\,.\end{aligned} \tag{3.8}$$

Therefore, if the linearization procedure ordinarily used is correct and if we identify the coefficients of quadratic terms as mass terms, we obtain

$$\begin{aligned}m_\pi{}^2&=-(\sqrt{\tfrac{2}{3}})4f^2\epsilon_0(2+A)(1+a)\,,\\m_K{}^2&=-(\sqrt{\tfrac{2}{3}})4f^2\epsilon_0(2+A)(1-\tfrac{1}{2}a)\,,\\m_8{}^2&=-(\sqrt{\tfrac{2}{3}})4f^2\epsilon_0(2+A)(1-a)\,,\\m_0{}^2&=-(\sqrt{\tfrac{2}{3}})4f^2\epsilon_0(8+A)\,,\\m_T{}^2&=(\sqrt{\tfrac{2}{3}})2f^2 2\sqrt{2}\epsilon_0(4-A)a\,.\end{aligned} \tag{3.9}$$

Notice that in this case we have no scalar mesons, and that the masses are no longer invariant under the Kuo transformation [Eq. (1.3)].

Before we explore this phenomenon more carefully, let us define parameters ξ_0 and ξ_8 by

$$\xi_0=\langle 0|S^{(0)}(0)|0\rangle\,,\qquad \xi_8=\langle 0|S^{(8)}(0)|0\rangle\,, \tag{3.10}$$

as in (II) and (III).[6,7] Again, if we use the linearization method and neglect all higher-order terms in the expansion of φ, one immediately sees

$$\xi_0=(\sqrt{6})(2+A)\,,\qquad \xi_8=0\,. \tag{3.11}$$

Therefore, defining b and γ by

$$b=\xi_8/\sqrt{2}\xi_0\,,\qquad \gamma=-\tfrac{2}{3}\epsilon_0\xi_0\,, \tag{3.12}$$

as in papers (II) and (III), this leads to

$$b=0\,,\qquad \gamma=-(\sqrt{\tfrac{2}{3}})2\epsilon_0(2+A)\,. \tag{3.13}$$

Thus, our mass formulas can be rewritten as

$$\begin{aligned}m_\pi{}^2&=2f^2\gamma(1+a)\,,\\m_K{}^2&=2f^2\gamma(1-\tfrac{1}{2}a)\,,\\m_8{}^2&=2f^2\gamma(1-a)\,,\\m_0{}^2&=2f^2\gamma\frac{8+A}{2+A}\,,\\m_T{}^2&=-2f^2\gamma\frac{4-A}{2+A}\sqrt{2}a\,.\end{aligned} \tag{3.14}$$

These formulas for $m_\pi{}^2$ and $m_K{}^2$ agree exactly with the results obtained in papers (II) and (III) and with GMOR. Also, comparing them with the mass formulas given there, we identify f^2 by

$$f_\pi{}^2=f_K{}^2=1/f^2\,, \tag{3.15}$$

where f_π and f_K are the leptonic decay constants of π^+ and K^+, respectively. The identity $f_\pi=f_K$ is expected here since we have $b=0$ [see (II) and (III)].

However, a word of caution is necessary for the validity of our approximation. As we emphasized in (II) and (III), the fact that we have $b=0$ automatically implies $a=0$, i.e., the exact validity of the $SU(3)$ group. This shows that somehow our linearization procedure cannot be exactly correct. Presumably, the only thing we should conclude from our approximation is that b is likely to be a small number. A better way is to give up the linearization procedure using the new technique of nonpolynomial field theory extensively studied by Salam and Strathdee.[13] However, it is still too difficult to compute the relevant quantities and we have nothing more to say about it.

Since we get $m_8{}^2<0$ for $a>1$ and $m_\pi{}^2<0$ for $a<-1$, we should restrict ourselves to the domain $-1<a<1$. Outside this region, ghosts states would appear and the positivity condition of the Hilbert space which we utilized so heavily in (II) and (III) will not be possible any longer. This restriction for the domain $-1<a<1$ is consistent with results of papers (II) and (III) if we discard the interval $1<a<2$.

[13] A. Salam and J. Strathdee, Phys. Rev. D **1**, 3296 (1970); R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **187**, 1999 (1969).

Returning to the original problem, one can consistently set $\varphi_0(x)=0$ only for the case of Eq. (3.2), i.e., $M=\exp[2\pi if\varphi(x)]$, as has been remarked by Cronin.[12] For other cases, we have to retain $\varphi_0(x)$ as an independent field. Hence, except for the case of Eq. (3.2), we have to diagonalize the mass matrix for the $\varphi_0-\varphi_8$ sector as in Sec. II. Then, one obtains

$$\begin{aligned} m_\eta^2&=\gamma f^2\frac{1}{2+A}[2(A+5)-(2+A)a-D^{1/2}],\\ m_X^2&=\gamma f^2\frac{1}{2+A}[2(A+5)-(2+A)a+D^{1/2}], \end{aligned} \tag{3.16}$$

where D is given by

$$D=36+12(2+A)a+(9A^2-60A+132)a^2. \tag{3.17}$$

The mixing angle θ is determined from the relation

$$\tan(2\theta)=\frac{-2\sqrt{2}(4-A)a}{6+(A+2)a}. \tag{3.18}$$

We chose the negative sign in front of $D^{1/2}$ in the formula m_η^2 so that at the exact $SU(3)$ limit $a=0$, we have $m_\eta=m_K=m_\pi$.

As one can easily check, our masses m_π, m_K, m_η, and m_X are no longer invariant under the Kuo transformation, Eq. (1.3). We observe also that m_η^2 and m_X^2 have in general square branch cuts at

$$a=\frac{-2(2+A)\pm4\sqrt{2}(4-A)i}{3A^2-20A+44} \tag{3.19}$$

as a function of a except for cases $A=4$ and $A=\infty$. Although the physical significance of the cut is not clear, its presence implies that the $SU(3)$ perturbation with respect to a must be restricted to

$$|a|<\frac{2(9A^2-28A+36)^{1/2}}{3A^2-20A+44}, \tag{3.20}$$

except for the cases $A=4$ and $A=\infty$. Notice that for $A=4$, we have $\tan\theta=0$, i.e., we have no η-X mixing problem.

We may remark that the present mass formula for m_η^2 is very much different from that proposed in (III). This is partly expected since the discussion of the parabolic boundary $1-a-b+3ab=0$ in the interval $1<a<2$ was a very important clue in (III), a fact which we cannot utilize in the present context. We see that we get $m_0^2=0$ in the exact $SW(3)$ limit in the present model and hence that our model may not be realistic. Also, the formula, Eq. (3.16), is based upon the assumption that we have nontrivial $\varphi_0(x)$. For the case $M=\exp[2if\varphi(x)]$, we could have set $\varphi_0(x)=0$ from the beginning and we would not have any complication due to the mixing at all.

From Eq. (3.14), we find $m_\pi^2=0$ at $a=-1$ and $m_K^2=0$ at $a=2$ in agreement with results of (II) and (III). However, as we emphasized already, our mass formulas are not invariant under the Kuo transformation in contrast to the linear case discussed in Sec. II. In view of the argument given in Sec. I, this is of course expected since we now have zero-mass Goldstone bosons in the exact $SW(3)$ limit. It is nevertheless interesting to explore the mechanism behind this phenomenon. To this end, let us write $M=M(\varphi)$ in order to show an explicit dependence of M upon $\varphi(x)$. Now, the Kuo transformation is represented as a mapping

$$\varphi(x)\to\bar\varphi(x) \tag{3.21}$$

such that

$$M(\bar\varphi)=UM(\varphi), \tag{3.22}$$

where the 3×3 matrix U is given by Eq. (2.16). Then, writing our Hamiltonian as $H(\varphi,\epsilon)$, one can easily check the identity

$$H(\varphi,\epsilon)=H(\bar\varphi,\bar\epsilon), \tag{3.23}$$

where $\bar\epsilon_0$ and $\bar\epsilon_8$ are expressed in terms of ϵ_0 and ϵ_8 by Eq. (1.3). Equations (3.21)–(3.23) are now another way of expressing the essence of the Kuo transformation. Therefore, if we use $\bar\varphi(x)$ instead of $\varphi(x)$ as the fundamental fields and if we accept the linearization procedure with respect to $\bar\varphi(x)$ rather than $\varphi(x)$, then we would obtain another set of mass formulas for m_π^2, m_K^2, etc., where we replace ϵ_0 and a by $\bar\epsilon_0$ and $\bar a$ in Eqs. (3.14) and (3.16). At first glance, this is a very puzzling fact, as we get two independent solutions for masses of particles simply by choosing two different sets of canonical variables. The reason for this peculiarity is that the transformation $\varphi\to\bar\varphi$ is singular and mathematically not well defined, so that the resulting operator V defined in Sec. I seems to correspond to the formal operator which brings one Hilbert space into another disjoint one. In other words, a theory in which the canonical field is given by $\varphi(x)$ represents an entirely different world from the one where the fundamental field is specified by $\bar\varphi(x)$. Hence we have no conflict since the two worlds described by $\varphi(x)$ and $\bar\varphi(x)$, respectively, are entirely disjoint. Also, as we emphasized in (I), the choice between $\varphi(x)$ and $\bar\varphi(x)$ as the fundamental canonical field variable is essentially due to our convention of choosing which of the ordinary and chimeral $SU(3)$ symmetries should become a good symmetry of the vacuum in the exact $SW(3)$ limit. Ordinarily, we accept the $SU(3)$ symmetry at $a=0$ to be a good symmetry. In that case, the chimeral $SU(3)$ group at $a=2$ will be attained by the emergence of the Goldstone kaon. On the other hand, if we choose the chimeral symmetry to be a good symmetry, the ordinary $SU(3)$ will become a bad symmetry with a zero-mass Goldstone κ meson. At any rate, in the first case we have to adopt $\varphi(x)$ as the relevant field while for the second case it is $\bar\varphi(x)$ rather than $\varphi(x)$ that is to be used as the fundamental variable. However, the interchange of the ordinary and chimeral $SU(3)$ is simply a change of conventional label without any fundamental importance. Nevertheless, the best criterion for selecting the correct case is probably to compute the vacuum energy and accept the case giving the lower vacuum energy.[14] In the present model, the vacuum energy per unit volume is computed to be $E_0=-\frac{3}{2}\gamma$ and $\bar{E}_0=-\frac{3}{2}\bar{\gamma}$, respectively, for the two cases, apart from a common constant term. Hence, if $\gamma>\bar{\gamma}$, we should select $\varphi(x)$ as the canonical field of the theory. On the other hand, if $\gamma<\bar{\gamma}$, we should adopt $\bar{\varphi}(x)$ instead.

Now, let us show explicitly how the transition $\varphi(x)\to\bar{\varphi}(x)$ is effected. This is sensitively dependent upon the explicit forms assumed for $M(\varphi)$. Hence we consider as two examples Eqs. (3.2) and (3.3). First, consider Eq. (3.2), i.e., $M(\varphi)=\exp(2if\varphi)$, so that we have to solve

$$\exp[2if\bar{\varphi}(x)]=U\exp[2if\varphi(x)]$$

in view of Eq. (3.22). Noting that $U=\exp(\pm i\sqrt{3}\pi\lambda_8)$, and using the standard Baker-Hausdorff formula $\exp A\,\exp B=\exp(A+B+\frac{1}{2}[A,B]+\cdots)$, we can formally solve this problem by expanding

$$\bar{\varphi}(x)=\pm(\sqrt{3}\pi/2f)\lambda_8+O(\varphi)$$

in a power series in φ, neglecting the question of the convergence of the expansion. Setting

$$\varphi_\alpha(x)=(1/\sqrt{2})\mathrm{Tr}(\varphi(x)\lambda_\alpha),$$
$$\bar{\varphi}_\alpha(x)=(1/\sqrt{2})\mathrm{Tr}(\bar{\varphi}(x)\lambda_\alpha),$$

one finds after some calculations that

$$\begin{aligned}
\bar{\varphi}_0(x)&\equiv\varphi_0(x),\\
\bar{\varphi}_1(x)&=\varphi_1(x)\mp\tfrac{3}{4}\pi\sqrt{2}f[\varphi_4(x)\varphi_6(x)+\varphi_5(x)\varphi_7(x)]+O(\varphi^3),\\
\bar{\varphi}_2(x)&=\varphi_2(x)\mp\tfrac{3}{4}\pi\sqrt{2}f[\varphi_5(x)\varphi_6(x)-\varphi_4(x)\varphi_7(x)]+O(\varphi^3),\\
\bar{\varphi}_3(x)&=\varphi_3(x)\mp\tfrac{3}{8}\pi\sqrt{2}f[\varphi_4(x)\varphi_4(x)+\varphi_5(x)\varphi_5(x)\\
&\qquad-\varphi_6(x)\varphi_6(x)-\varphi_7(x)\varphi_7(x)]+O(\varphi^3),\\
\bar{\varphi}_4(x)&=\pm\tfrac{3}{2}\pi\varphi_5(x)\pm\frac{f}{\sqrt{2}}[(\varphi_3+\sqrt{3}\varphi_8)\varphi_5\\
&\qquad+\varphi_2\varphi_6+\varphi_1\varphi_7]+O(\varphi^3),\\
\bar{\varphi}_5(x)&=\mp\tfrac{3}{2}\pi\varphi_4(x)\mp\frac{f}{\sqrt{2}}[(\varphi_3+\sqrt{3}\varphi_8)\varphi_4\\
&\qquad+\varphi_1\varphi_6-\varphi_2\varphi_7]+O(\varphi_3),\qquad(3.24)\\
\bar{\varphi}_6(x)&=\pm\tfrac{3}{2}\pi\varphi_7(x)\pm\frac{f}{\sqrt{2}}[(-\varphi_3+\sqrt{3}\varphi_8)\varphi_7\\
&\qquad+\varphi_1\varphi_5-\varphi_2\varphi_4]+O(\varphi^3),\\
\bar{\varphi}_7(x)&=\mp\tfrac{3}{2}\pi\varphi_6(x)\mp\frac{f}{\sqrt{2}}[(-\varphi_3+\sqrt{3}\varphi_8)\varphi_6\\
&\qquad+\varphi_1\varphi_4+\varphi_2\varphi_5]+O(\varphi^3),\\
\bar{\varphi}_8(x)&=\pm(\surd\tfrac{3}{2})\frac{\pi}{f}+\varphi_8(x)\mp(\surd\tfrac{3}{2})\tfrac{3}{4}\pi\\
&\qquad\times f(\varphi_4\varphi_4+\varphi_5\varphi_5+\varphi_6\varphi_6+\varphi_7\varphi_7)+O(\varphi^3).
\end{aligned}$$

Notice that

$$\bar{\varphi}_8(x)=\pm(\surd\tfrac{3}{2})\pi/f+\varphi_8(x)+O(\varphi^2),$$

i.e., the transformation from $\varphi_8(x)$ to $\bar{\varphi}_8(x)$ contains a constant translation. This fact strongly suggests that the transformation is not an ordinary one, but that of changing a given Hilbert space into another disjoint one, as we may see from the example of the Van Hove model[15] and of others. Also its form is similar to that used by Bessler *et al.*[16] We see also that $\bar{\varphi}_\alpha(x)$ has no definite parity if we adhere to the old parity convention for $\varphi_\alpha(x)$. This is not surprising, however, since the Kuo transformation $M\to UM$ does not commute with the parity operation $M\leftrightarrow M^\dagger$. It should be emphasized that although we have the identity $\partial_\mu(\bar{M})^\dagger\partial_\mu\bar{M}=\partial_\mu M^\dagger\partial_\mu M$, we have neither

$$\textstyle\sum_\alpha\partial_\mu\bar{\varphi}_\alpha\partial_\mu\bar{\varphi}_\alpha=\sum_\alpha\partial_\mu\varphi_\alpha\partial_\mu\varphi_\alpha$$

nor $\sum_\alpha\bar{\varphi}_\alpha\bar{\varphi}_\alpha=\sum_\alpha\varphi_\alpha\varphi_\alpha$. The reason is the presence of the constant term in $\bar{\varphi}_8$ and the higher-order terms contained in the expression of $\bar{\varphi}_\alpha$ could contribute to the quadratic term of φ_α. This fact also explains the apparent disparity of the expression between φ_α and $\bar{\varphi}_\alpha$ for $\alpha=4$, 5, 6, 7. Since $U^2=1$, we could solve φ_α in terms of $\bar{\varphi}_\alpha$ exactly in the same way as in Eq. (3.24). However, if we take only a few terms in the expansion, we do not recover the correct answer for φ_α in terms of $\bar{\varphi}_\alpha$. The dilemma is again resolved if we take into account φ^3 and higher terms in the expansion of Eq. (3.24) for that purpose.

Now, let us consider the second example,

$$M(\varphi)=\frac{1+if\varphi(x)}{1-if\varphi(x)}.$$

In this case, the situation is really peculiar. One can exactly solve the equation $UM(\varphi)=M(\bar{\varphi})$ without any approximation by

$$\begin{aligned}if\bar{\varphi}(x)=[(U-1)+if(U+1)\varphi(x)]\\ \times[(U+1)+if(U-1)\varphi(x)]^{-1}.\qquad(3.25)\end{aligned}$$

The problem is that both matrices $(1+U)$ and $(1-U)$ have no inverses and hence we cannot expand the denominator in a power series of $\varphi(x)$. But one can evaluate the denominator in the standard way to get

---

[14] The author owes this remark to a discussion with Professor H. Umezawa and would like to express his gratitude. A similar criterion is also used in the superconductivity theory to select the superconducting solution against the normal one.

[15] L. Van Hove, Physica **18**, 145 (1952); R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **29**, No. 12 (1955); K. Friedricks, *Mathematical Aspects of Quantum Theory of Fields* (Interscience, New York, 1953).

[16] L. Bessler, T. Muta, H. Umezawa, and D. Welling, Phys. Rev. D **2**, 349 (1970).

finally

$$\bar{\varphi}_\alpha(x) = -\frac{1}{f^2}\frac{1}{\Delta}\varphi_\alpha(x) \quad (\alpha=1,2,3),$$

$$\bar{\varphi}_8(x) = \frac{1}{3f^2}\frac{1}{\Delta}[\varphi_8(x)+\sqrt{2}\varphi_0(x)+(\sqrt{6})f^2\det\varphi(x)],$$

$$\bar{\varphi}_0(x) = \frac{1}{3f^2}\frac{1}{\Delta}[\sqrt{2}\varphi_8(x)+2\varphi_0(x)-\sqrt{3}f^2\det\varphi(x)],$$

$$\bar{\varphi}_4(x) = -\frac{1}{\sqrt{2}f}\frac{1}{\Delta}\{\varphi_2\varphi_6+\varphi_1\varphi_7-\varphi_5 \times[-\varphi_3+(\sqrt{\tfrac{1}{3}})\varphi_8+(\sqrt{\tfrac{2}{3}})\varphi_0]\},$$

$$\bar{\varphi}_5(x) = \frac{1}{\sqrt{2}f}\frac{1}{\Delta}\{\varphi_1\varphi_6+\varphi_2\varphi_7 -\varphi_4[-\varphi_3+(\sqrt{\tfrac{1}{3}})\varphi_8+(\sqrt{\tfrac{2}{3}})\varphi_0]\}, \tag{3.26}$$

$$\bar{\varphi}_6(x) = \frac{1}{\sqrt{2}f}\frac{1}{\Delta}\{\varphi_4\varphi_2-\varphi_5\varphi_1 +\varphi_7[\varphi_3+(\sqrt{\tfrac{1}{3}})\varphi_8+(\sqrt{\tfrac{2}{3}})\varphi_0]\},$$

$$\bar{\varphi}_7(x) = -\frac{1}{\sqrt{2}f}\frac{1}{\Delta}\{-\varphi_4\varphi_1-\varphi_5\varphi_2 +\varphi_6[\varphi_3+(\sqrt{\tfrac{1}{3}})\varphi_8+(\sqrt{\tfrac{2}{3}})\varphi_0]\},$$

where Δ and $\det\varphi$ are given by

$$\Delta = \tfrac{1}{2}\sum_{\alpha=1}^{3}\varphi_\alpha(x)\varphi_\alpha(x)-\tfrac{1}{6}[\varphi_8(x)+\sqrt{2}\varphi_0(x)]^2,$$

$$\det\varphi(x) = \frac{1}{3\sqrt{2}}\sum_{\alpha,\beta,\gamma=0}^{8} d_{\alpha\beta\gamma}\varphi_\alpha(x)\varphi_\beta(x)\varphi_\gamma(x) -\tfrac{1}{2}\sqrt{3}\varphi_0(x)\sum_{\alpha=1}^{8}\varphi_\alpha(x)\varphi_\alpha(x). \tag{3.27}$$

Since Δ^{-1} is a singular operator, we really do not know how to interpret it mathematically. This example definitely illustrates the fact that the Kuo transformation is singular.[17]

So far we have not discussed the form of generators of the $SW(3)$ group in the nonlinear theory. This is in general very complicated. However, it is well known[12] that one can give their exact form if $M(\varphi)$ is given by $M(\varphi)=(1+if\varphi)(1-if\varphi)^{-1}$ and hence we shall restrict ourselves only to this case.

First, defining the canonical conjugate matrix $\Pi(x)$ by

$$\Pi(x) \equiv \frac{\delta\mathcal{L}}{\delta\dot{\varphi}(x)} = \frac{1}{1+f^2\varphi(x)\varphi(x)}\partial_0\varphi(x)\frac{1}{1+f^2\varphi(x)\varphi(x)}, \tag{3.28}$$

we impose the ordinary canonical commutation relation

$$[\Pi_a{}^b(x),\varphi_d{}^c(y)] = -i\delta_a{}^c\delta_d{}^b\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{3.29}$$

with the rest of commutators being zero, where a, b, c, and d are the matrix indices and assumes values 1, 2, 3. This matrix $\Pi(x)$ should not be confused with that defined in Sec. II. Defining[18] the two matrices $A_a{}^b(x)$ and $V_a{}^b(x)$ by

$$A_a{}^b(x) = \frac{1}{f}\Pi_a{}^b(x)+f[\varphi(x)\Pi(x)\varphi(x)]_a{}^b,$$
$$V_a{}^b(x) = i\{[\Pi(x)\varphi(x)]_a{}^b-[\varphi(x)\Pi(x)]_a{}^b\}, \tag{3.30}$$

one can easily compute the usual equal-time commutation relations

$$[V_b{}^a(x),V_d{}^c(y)] = [A_b{}^a(x),A_d{}^c(y)] = (\delta_b{}^cV_d{}^a(x)-\delta_d{}^aV_b{}^c(x))\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$[A_b{}^a(x),V_d{}^c(y)] = (\delta_b{}^cA_d{}^a(x)-\delta_d{}^aA_b{}^c(x))\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{3.31}$$

Also, one can check

$$[V_b{}^a(x),\varphi_d{}^c(y)] = (\delta_b{}^c\varphi_d{}^a(x)-\delta_d{}^a\varphi_b{}^c(x))\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$[V_b{}^a(x),\Pi_d{}^c(y)] = (\delta_b{}^c\Pi_d{}^a(x)-\delta_d\Pi_b{}^c(x))\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$[A_b{}^a(x),\varphi_d{}^c(y)] = \frac{-i}{f}(\delta_d{}^a\delta_b{}^c+f^2\varphi_d{}^a(x)\varphi_b{}^c(x))\delta^{(3)}(\mathbf{x}-\mathbf{y}), \tag{3.32}$$
$$[A_b{}^a(x),\Pi_d{}^c(y)] = if\{\delta_d{}^a(\varphi(x)\Pi(x))_b{}^c +\delta_b{}^c(\Pi(x)\varphi(x))_d{}^a\}\delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

Setting now

$$V_b{}^a(t) = \int_{x_0=t} d^3x\, V_b{}^a(x),$$
$$A_b{}^a(t) = \int_{x_0=t} d^3x\, A_b{}^a(x), \tag{3.33}$$

we find that $A_b{}^a(t)$ and $V_b{}^a(t)$ form the Lie algebra of the $SW(3)$ group. One also obtains

$$\exp[\epsilon_b{}^aA_a{}^b(t)]M_d{}^c(x)\exp[-\epsilon_b{}^aA_a{}^b(t)] = [e^{+\epsilon}M(x)e^{+\epsilon}]_d{}^c, \tag{3.34}$$

$$\exp[\epsilon_b{}^aV_a{}^b(t)]M_d{}^c(x)\exp[-\epsilon_b{}^aV_a{}^b(t)] = [e^{-\epsilon}M(x)e^{+\epsilon}]_d{}^c \tag{3.35}$$

if $M(\varphi)$ is expressed by $M(\varphi)=(1+if\varphi)(1-if\varphi)^{-1}$ and

[17] A similar singular character of a finite $SW(2)$ transformation in the nonlinear realization can be found in A. Salam and J. Strathdee, Phys. Rev. **184**, 1750 (1969), when we set $\omega=\frac{1}{2}\pi$ in Eq. (1.7) of their paper.

[18] See Ref. 12. This form has been originally suggested by T. K. Kuo and M. Sugawara, Phys. Rev. **151**, 1181 (1966).

if we neglect the question of the convergence with respect to power-series expansion of $\varphi(x)$. Therefore, setting $V=\exp[\frac{3}{2}\pi i(Y-Y_5)]$, we find

$$VM_d{}^c(x)V^{-1}=[U\cdot M(x)]_d{}^c\,,$$

where $U=\exp(\sqrt{3}\pi i\lambda_8)$, as is required in the Kuo transformation. Thus at the formal level, everything proceeds well. However, as we noted, this transformation which changes $\varphi(x)$ into $\bar{\varphi}(x)$ is highly singular in reality and somewhere the formal method must be breaking down. The likely explanation is that $Y_5=-A_3{}^3(t)$ is not a mathematically well-defined operator. It can be probably defined only as a functional in a certain domain of direct product space $\mathcal{H}\otimes\mathcal{H}$ of the underlying Hilbert space $\mathcal{H}$ in such a way that it is linear in the first $\mathcal{H}$ but antilinear in the second one. In other words, the matrix elements of Y_5 can be defined in a certain domain of $\mathcal{H}$, but Y_5 itself will not even be an unbounded self-adjoint operator in $\mathcal{H}$. Thus, $\exp[\frac{3}{2}\pi i(Y-Y_5)]$ is not well defined mathematically and it represents at best a formal operator changing one Hilbert space into another disjoint one as in the model of Nambu and Jona-Lasinio.

IV. DISCUSSION

As we noted in the previous sections, mass formulas for mesons can be invariant under the Kuo transformation only for linear realizations of the $SW(3)$ group, but not for nonlinear ones. This conclusion agrees with that reached in (I). A similar consideration demonstrates the rather surprising fact that the boson-boson scattering amplitudes are not also invariant in general under the Kuo transformation, in the case of the nonlinear theory at least, if we use the so-called tree-diagram calculation. This fact is rather curious since the Kuo transformation can be alternatively understood to be a substitution of the field $\varphi(x)$ by another field $\bar{\varphi}(x)$ which is a function of $\varphi(x)$ itself. Salam *et al.* showed[19] some years ago that the scattering amplitude should remain invariant under various choices of canonical field variables. However, our example appears to refute this theorem. A likely explanation is again that our transformation from $\varphi(x)$ to $\bar{\varphi}(x)$ is so singular that the theorem of Salam *et al.* would not be applicable. As we emphasized in (I) and also in the previous sections, the choice between $\varphi(x)$ and $\bar{\varphi}(x)$ as the canonical field implies a choice of two different worlds. Both choices, however, give the same mass formulas and the same scattering amplitudes if we interchange the roles of $\bar{\epsilon}_0$ and $\bar{\epsilon}_8$ (or $\bar{a}$) and those of ϵ_0 and ϵ_8 (or a). As we have stressed, the choice is a matter of convention in the sense that either ordinary $SU(3)$ at $a=0$ or the chimeral $SU(3)$ at $a=2$ could be accepted as the good symmetry. This fact is related with the arbitrary relative parity convention for particles with zero and nonzero hypercharge. Since this relative parity cannot be determined experimentally[20] and is a matter of convention, it implies that we have no method to decide which of the ordinary and chimeral $SU(3)$ should become the good symmetry in the exact $SW(3)$ limit if the Goldstone boson appear in that limit. When both symmetries are simultaneously realized, it automatically implies the parity-doublet structure for all particle multiplets but the parity assignment still remains a matter of convention.

We have seen that the mass formulas for $m_\pi{}^2$ and $m_K{}^2$ in the nonlinear case agree with those obtained in (II) and (III). But this agreement will be destroyed if we add a derivative-type $(3,3^*)\oplus(3^*,3)$ interaction of the form

$$\mathcal{L}_1{}''=-\epsilon_0\tilde{u}_0(x)-\epsilon_8\tilde{u}_8(x)\,,\tag{4.1}$$

where $\tilde{u}_\alpha(x)$ is given by

$$\tilde{u}_\alpha(x)=\tfrac{1}{2}\,\mathrm{Tr}[(T(x)+T^\dagger(x))\lambda_\alpha]\tag{4.2}$$

with

$$\begin{aligned}(T)_{b'}{}^a&=\epsilon^{acd}\epsilon_{b'e'f'}\partial_\mu M_c{}^{e'}\partial_\mu M_d{}^{f'}\,,\\(T)_b{}^{a'}&=\epsilon^{a'c'd'}\epsilon_{bef}\partial_\mu M_{c'}{}^e\partial_\mu M_{d'}{}^f=[(T)_{a'}{}^b]^\dagger\,.\end{aligned}\tag{4.3}$$

Then, the addition of Eq. (4.1) gives[21] mass formulas which have a more complicated dependence upon ϵ_0 and a as has been noted by some authors.[21] However, the trouble with this argument lies in the derivative nature of Eq. (4.3). Although it belongs formally to the $(3,3^*)\oplus(3^*,3)$ representation, it has in reality a more complicated $SW(3)$ transformation property. If we add a derivative-type interaction to a Lagrangian, the canonical formalism tells us that the definition of canonical conjugate $\Pi(x)$ of $\varphi(x)$ will be accordingly modified. Hence forms of infinitesimal generators $A_b{}^a$ and $V_b{}^a$ will be altered also in a rather complicated way. With respect to these new generators, the derivative interaction $\mathcal{L}_1{}''$ given by Eqs. (4.1) and (4.3) does not behave as a simple $(3,3^*)\oplus(3^*,3)$-type interaction any longer. Therefore, we should not, strictly speaking, include such a derivative term at all within the framework of Gell-Mann, Oakes, and Renner theory. This also implies that the definition of the so-called covariant derivatives in the $SW(3)$ theory must be modified if the $SW(3)$ is not exact. When there is an $SW(3)$-violating interaction, the generators $V_b{}^a$ and $A_b{}^a$ are now time-dependent, and the ordinary proof of defining the covariant derivatives will not necessarily go through. However, if the $SW(3)$-violating interaction does not contain any derivatives, then we will not have these troubles in the definition of the covariant derivativse.

Finally, we notice that in our theory our underlying fundamental group was the $SW(3)$ rather than the

[19] S. Kamefuchi, S. L. O'Raifeartaigh, and S. Salam, Nucl. Phys. **28**, 529 (1961). In the axiomatic field theory, this fact is known as the Borchers theorem. H. J. Borchers, Nuovo Cimento **15**, 784 (1960).

[20] P. T. Mathews, Nuovo Cimento **6**, 642 (1957).

[21] E.g., Y. M. P. Lam and Y. Y. Lee, Phys. Rev. D (to be published); Phys. Rev. Letters **23**, 734 (1969).

$W(3)$ group. As we emphasized, the $W(3)$ symmetry emerges when we let $A\to\infty$, $\epsilon_0\to 0$ in such a way that the product $\epsilon_0 A$ and a remain finite. In such a case, the $SW(3)$-violating term $\epsilon_0 u^{(0)}(x)+\epsilon_8 u^{(8)}(x)$ will vanish identically, leaving terms proportional to $\sigma^{(\alpha)}(x)$ ($\alpha=0$, 8), which belongs to the $(3,3^*)\oplus(3^*,3)$ representation of the larger $W(3)$ group. Hence, it will be interesting to investigate this case in some detail. From the result of Sec. III, one finds in this limit,

$$\begin{aligned} m_\pi^2&=2f^2\gamma(1+a)\,,\\ m_K^2&=2f^2\gamma(1-\tfrac{1}{2}a)\,,\\ m_\eta^2&=2f^2\gamma(1+a)=m_\pi^2\,,\\ m_X^2&=2f^2\gamma(1-2a)\,, \end{aligned} \tag{4.4}$$

with $\tan(2\theta)=2\sqrt{2}$ (or $\tan\theta=-\sqrt{2}$). From these, we see $m_\pi=m_\eta=0$ at $a=-1$, i.e., both π and η become soft at $a=-1$. Also, at $a=\frac{1}{2}$, we get $m_X=0$, implying that the X meson becomes a Goldstone boson there. Such behavior is very much different from what we assumed in (II)—that the symmetry associated with the conservation law $\partial_\mu A_\mu^{(-2)}(x)=0$ at $a=\frac{1}{2}$ is a good symmetry without any Goldstone boson. However, this is expected because we have no scalar mesons in the present model discussed in Sec. III and the existence of scalar mesons is presumably necessary to assure a good symmetry at $a=\frac{1}{2}$ as we emphasized in (II).

Lastly, the requirement $\gamma>0$ which results from the positivity condition of the Hilbert space implies $\epsilon_0(2+A)<0$. Hence for the case $2+A>0$ we must have $\epsilon_0<0$ and any theory with $\epsilon_0>0$ will give rise to ghost mesons. The same situation arises for $a<-1$ or $a>2$. Therefore, these cases do not make sense in the present context. Since the positive definiteness of the theory was so heavily utilized in (II) and (III), and since this property will not be enjoyed if a ghost appears, this suggests that points $a=-1$ and $a=2$ may not necessarily be essential singular points of theory as we suggested in (II) and (III). It may be that for $a<-1$ and $a>2$, we would have ghost pions and kaons if we blindly continued analytically the value of the parameter a beyond $-1<a<2$. However, the existence of the Kuo transformation which brings the domain $-1<a<2$ outside itself implies an existence of another reasonable solution for intervals $a<-1$ and $a>2$. If this reasoning is correct, the picture which emerges is the one that we have at least two distinct domains given by $-1<a<2$ and $a<-1$ or $a>2$ where all physical quantities are expressed by at least two distinct analytic functions of a in the respective domains. If this idea should turn out to be the right one, then the point $a=-1$ may not necessarily be a singular point and it may be possible to use a perturbation method around this point. A preliminary investigation for the $\pi N\to\pi N$ scattering amplitude shows that one can compute the first-order perturbation correction with respect to $1+\alpha$ and the result exactly agrees with the ordinary computation of the so-called σ term. This is an encouraging fact, but a more extensive analysis is needed to confirm this idea.