PHYSICAL REVIEW D VOLUME 2, NUMBER 2 15 JULY 1970

$SU(3)$ Symmetry and Algebraic Realizations of Chiral Symmetry*

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(Received 30 March 1970)

The algebraic realizations of chiral symmetry obtained by Weinberg are investigated for the case of $SU(3)$ symmetry. When only p-wave interactions are taken into account, the algebraic realization of the first superconvergence condition is given by the Lie algebra of the group $SU(2)\otimes SU(6)$. The mass matrix obtained from the second superconvergence relation corresponds to degenerate masses within each of the multiplets ($V=0$, **56**) and ($V=2$, **56**). The consequences of particle classification under the group $SU(2)\otimes SU(6)$ are enumerated.

I. INTRODUCTION

By demanding the correct high-energy behavior of the forward scattering of massless pions, Weinberg[1] has recently derived the following algebraic relations:

$$[X_\alpha(\lambda),X_\beta(\lambda)]=i\epsilon_{\alpha\beta\gamma}T_\gamma \tag{1}$$

and

$$[X_\alpha(\lambda),[m^2,X_\beta(\lambda)]]\sim\delta_{\alpha\beta}. \tag{2}$$

In the above equations T_α are the generators of the isospin group, m^2 is the diagonal mass operator, and $X_\alpha(\lambda)$ are the axial-vector coupling matrices.

The commutation relations

$$[T_\alpha,T_\beta]=i\epsilon_{\alpha\beta\gamma}T_\gamma \tag{3}$$

together with Eq. (1), define the Lie algebra of the chiral symmetry group $SU(2)\otimes SU(2)$. This implies that the single-particle states that enter the tree graphs giving the forward scattering amplitudes, for each helicity, belong to unitary representations of the chiral group. Equation (1) then relates the coupling constants for the particles with different isospin, but the same helicity, that are assigned to the same representations of the chiral group.

In order to determine the helicity dependence of the operators $X_\alpha(\lambda)$, which are related to the pion-transition operators via the Goldberger-Treiman relation, further assumptions have to be made.[1] We could, for example, limit our considerations to p-wave pion transitions. This has led Cronstrom and Noga[2] to show that, for p-wave pion transitions, the commutator of Eq. (1) gets modified into the Lie algebra of the group $SU(2)\otimes SU(4)$. They also show that the double commutator of Eq. (2) is solvable for symmetric representations of $SU(4)$ and that the mass matrix obtained is

$$m^2(I,J)=m_0^2+c[J(J+1)-T(T+1)]. \tag{4}$$

The mass spectrum represented by Eq. (4) is unsatisfactory because it corresponds to hadron masses either decreasing with increasing isospin (if $c>0$) or decreasing with increasing spin (if $c<0$).

The purpose of this paper is to study what happens when the symmetry is enlarged from $SU(2)$ to $SU(3)$, and to extend the work of Cronstrom and Noga. The chiral symmetry group is now $SU(3)\otimes SU(3)$. We limit ourselves to p-wave interactions in order to obtain the helicity dependence of the operators $X_\alpha(\lambda)$, and we show in Sec. II that the first superconvergence condition now corresponds to the Lie algebra of the group $SU(2)\otimes SU(6)$ and compare the predictions of this result with experiment. In Sec. III we discuss the second superconvergence condition for p-wave interactions and solve for the mass matrix for the representations of $SU(2)\otimes SU(6)$ of low dimensionality. With the **56** representation of the $SU(6)$ subgroup we do not have the state-labeling problem; the physically available quantum numbers are sufficient to distinguish between various states within the multiplets. The mass spectrum for the baryon representations ($V=0$, **56**) and ($V=2$, **56**) shows degeneracy within each of the multiplets. In Sec. IV we compare the results of this model with the strong-coupling model and the intermediate-coupling models.

II. FIRST SUPERCONVERGENCE CONDITION

Let us consider the forward scattering of a massless *octet* of pseudoscalar mesons, the target being any hadron state. We can express the scattering amplitude in terms of a part which is symmetric under the interchange of the $SU(3)$ indices of the mesons and a part which is antisymmetric. We impose the condition that these amplitudes, as evaluated from tree graphs, have the high-energy behavior allowed by Regge theory. Considerations similar to those in Ref. 1, for the odd amplitude, lead to the commutator

$$[X_\alpha(\lambda),X_\beta(\lambda)]=if_{\alpha\beta\gamma}F_\gamma, \tag{5}$$

where F_γ are now the generators of the unitary symmetry group $SU(3)$, and $X_\alpha(\lambda)$ are the meson coupling matrices. The matrices $X_\alpha(\lambda)$ are related to the Feynman amplitude for the process $B(n_\alpha;T,T_z,Y;\lambda)\rightarrow$

* Work supported by the U. S. Atomic Energy Commission, Contract No. AT(11-1)-1428.
[1] S. Weinberg, Phys. Rev. **177**, 2604 (1969); Phys. Rev. Letters **22**, 1023 (1969).
[2] C. Cronstrom and M. Noga, Nucl. Phys. **B15**, 61 (1970); M. Noga and C. Cronstrom, Phys. Rev. D **1**, 2414 (1970).

$B(n_\beta; t,t_z,y; \lambda')+P_\gamma$ by the relation

$$M(B_\alpha \to B_\beta+P_\gamma)=2F_\pi^{-1}(m_\alpha^2-m_\beta^2) \times\langle n_\beta; t,t_z,y; \lambda' | X_\gamma | n_\alpha; T,T_z,Y; \lambda\rangle . \quad (6)$$

In Eq. (6), meson coupling matrices are diagonal in helicity, and F_π is the pion decay constant. The $SU(3)$ representations of B_α, B_β are denoted by n_α, n_β.

The second superconvergence condition, Eq. (2), was obtained by Weinberg[1] by demanding that, for the amplitudes which are even under the exchange of the meson indices, no "exotic" mesons (with $T=2$) be exchanged in the t channel. In the case of $SU(3)$ symmetry, a direct generalization is possible, and we demand that the t-channel amplitudes transforming under $SU(3)$ as the **27**, **10**, or **10*** representations be zero. We then obtain

$$[X_\alpha(\lambda),[m^2,X_\beta(\lambda)]]\sim\delta_{\alpha\beta}+d_{\alpha\beta\gamma}m_\gamma^2, \quad (7)$$

where m_γ^2 is defined by Eq. (7) to transform as an octet. The mass matrix m^2 commutes with F_α and with the generators of the spin group $SU(2)_J$.

Let us limit ourselves to p-wave interactions. The invariance group is $K=SU(3)\otimes SU(2)_J$, defined by the Lie algebra

$$[F_\alpha,F_\beta]=if_{\alpha\beta\gamma}F_\gamma, \quad (8)$$
$$[J_a,J_b]=i\epsilon_{abc}J_c,$$

and

$$[F_\alpha,J_a]=0.$$

We denote the meson operators by $D_{\alpha a}$, which transforms as a octet under $SU(3)$, and as a vector under ordinary rotations. The components of $D_{\alpha a}$ which are diagonal in helicity are the operators $X_\alpha(\lambda)$.

The complete helicity dependence of $D_{\alpha a}$ will be given by the commutator $[D_{\alpha a},D_{\beta b}]$. This quantity is odd under the interchange of the indices (αa) and (βb) and can be written

$$[D_{\alpha a},D_{\beta b}]=if_{\alpha\beta\gamma}A_\gamma(ab)+i\epsilon_{abc}B_c(\alpha\beta). \quad (9)$$

The right-hand side of Eq. (9) can be considered to be the t-channel contribution from tree graphs. We exclude the exchange of exotic mesons; this condition is satisfied if

$$B_c(\alpha\beta)=\delta_{\alpha\beta}\tfrac{2}{3}B_c+d_{\alpha\beta\gamma}D_{\gamma c}, \quad (10)$$

where

$$[J_a,B_b]=i\epsilon_{abc}B_c \quad\text{and}\quad [F_\alpha,B_c]=0.$$

The numerical coefficient in front of B_c in Eq. (10) is introduced for later convenience.

The first superconvergence condition, Eq. (5), for p waves is

$$[D_{\lambda 3},D_{\mu 3}]=if_{\lambda\mu\nu}F_\nu. \quad (11)$$

Then Eqs. (11) and (8) completely determine the right-hand side of Eq. (9) as having the form

$$[D_{\lambda a},D_{\mu b}]=i\delta_{ab}f_{\lambda\mu\nu}F_\nu+i\epsilon_{abc}(\tfrac{2}{3}\delta_{\lambda\mu}B_c+d_{\lambda\mu\nu}D_{\nu c}). \quad (12)$$

It is now easy to show that

$$[B_a,D_{\lambda b}]=i\epsilon_{abc}D_{\lambda c}, \quad (13a)$$
$$[B_a,B_b]=i\epsilon_{abc}B_c, \quad (13b)$$

and that the vector operator $V_a=J_a-B_a$ commutes with F_α, B_b, and $D_{\lambda b}$. The operator V_a satisfies the commutation relations of the group $SU(2)$ and

$$[V_a,V_b]=i\epsilon_{abc}V_c. \quad (14)$$

Equations (8)–(14) show that F_α, B_a, and $D_{\alpha a}$ lead to a closed algebra and are the generators of $SU(6)$, and V_b are the generators of $SU(2)$. Thus the first superconvergence condition leads to the Lie group of $SU(2)\otimes SU(6)$, and the hadron states belong to unitary representations (reducible, or irreducible) of this group.

It should be noted that Eq. (10) is not the only possible choice for the form of $B_c(\alpha\beta)$ which leads to a closed algebra. We could equally well have used

$$B_c(\alpha\beta)=\xi\delta_{\alpha\beta}\tfrac{2}{3}B_c+\eta d_{\alpha\beta\gamma}D_{\gamma c},$$

where ξ, $\eta=\pm1$. However, only the choice $\xi=\eta=1$ leads to a compact Lie algebra. We limit ourselves to the compact algebraic realization of the superconvergence conditions.

A. Particle Spectrum

The physical states have well-defined transformation properties under the invariance group $K=SU(3)\otimes SU(2)_J$. We can expand these states in terms of the representations of the group $SU(2)_V\otimes SU(6)$. The coupling constants for p-wave pseudoscalar meson transitions are all determined in this scheme because the meson coupling matrices are just some of the generators of the subgroup $SU(6)$.

For the physical states we write

$$|n;(T,T_z,Y);J,J_z\rangle=\sum_{B_z,V_z}\begin{pmatrix} B & V & J \\ B_z & V_z & J_z\end{pmatrix} \times|N;n,(T,T_z,Y);B,B_z\rangle|V,V_z\rangle, \quad (15)$$

where N, n are the dimensionalities of the $SU(6)$ and $SU(3)$ representations, and the B spin and V spin are coupled to give the physical spin J, using Clebsch-Gordan coefficients.

The transformation properties of the meson-transition operator under the invariance group $SU(3)\otimes SU(2)_J$ and the Wigner-Eckhart theorem allow us to write the matrix elements of the operator $D_{\lambda a}$ in the form

$$\langle n_\alpha,(T,T_z,Y);J,J_z|D_{\lambda a}|n_\beta,(T',T'_z,Y');J',J'_z\rangle =\sum_\gamma G(\gamma;n_\beta J';n_\alpha J)\begin{pmatrix} n_\beta & \mathbf{8} & n_\alpha(\gamma) \\ \beta & \lambda & \alpha\end{pmatrix} \times\begin{pmatrix} J' & 1 & J \\ J'_z & a & J_z\end{pmatrix}, \quad (16)$$

where $SU(3)$ and $SU(2)$ Clebsch-Gordan coefficients have been used.

We also have

$$\langle N_\alpha; n_\alpha(T,T_z,Y); B,B_z; V,V_z| \times D_{\lambda a}|N_\beta; n_\beta(T',T'_z,Y); B',B'_z; V',V'_z\rangle = \delta_{VV'}\delta_{V_zV'_z}\begin{pmatrix} N_\beta & \mathbf{35} & N_\alpha \\ (n_\beta B') & (\mathbf{8},\mathbf{3}) & (n_\alpha(\gamma),B)\end{pmatrix} \times\begin{pmatrix} n_\beta & \mathbf{8} & n_\alpha(\gamma) \\ \beta & \lambda & \alpha\end{pmatrix}\begin{pmatrix} B' & 1 & B \\ B'_z & a & B_z\end{pmatrix}, \quad (17)$$

where the right-hand side of Eq. (17) has a $SU(6)$ reduced matrix element and $SU(3)$ and $SU(2)$ Clebsch-Gordan coefficients.

For the $SU(6)$ representations of low dimensionality it is easy to obtain these reduced matrix elements as solutions to the nonlinear relations between them given by the commutator relation of Eq. (12). This method was first suggested by Lee.[3] For the representations **35**, **56**, and **70** these reduced matrix elements have been tabulated by Cook and Murtaza[4] and by Carter and Coyne.[5]

The coupling constant can now be written in terms of a (6-J) symbol and the reduced matrix element for the group $SU(6)$, and we have

$$G(\gamma; n_\beta J'; n_\alpha J) = \begin{pmatrix} N_\beta & \mathbf{35} & N_\alpha \\ (n_\beta B') & (\mathbf{8},\mathbf{3}) & (n_\alpha(\gamma)B)\end{pmatrix}[(2B+1)(2J'+1)]^{1/2} \times(-1)^{1+V+J+B'}\begin{Bmatrix} V & J & B \\ 1 & B' & J'\end{Bmatrix}. \quad (18)$$

Hermiticity of the generators leads to the following relation between G and its complex conjugate $\bar{G}$:

$$G(\gamma; n_\beta J'; n_\alpha J) = \bar{G}(\gamma; n_\alpha J; n_\beta J') \times\left(\frac{n_\beta}{n_\alpha}\right)^{1/2}\left(\frac{2J'+1}{2J+1}\right)^{1/2}(-1)^{J'-J}\xi_1(n_\alpha,\mathbf{8},n_\beta) \times\xi_2(\mathbf{8},n_\alpha,n_\beta)\xi_3(\mathbf{8},\bar{n}_\beta,\bar{n}_\alpha)\xi_1(n_\beta,\mathbf{8},n_\alpha), \quad (19)$$

where the phase factors ξ_1, ξ_2, and ξ_3 are defined by deSwart.[6]

The decay rates for the processes $B(n_\alpha,J_\alpha)\to B(n_\beta,J_\beta)+P_\lambda(k)$ are evaluated using Eqs. (6) and (16), and we have[1,2]

$$\Gamma(B_\alpha\to B_\beta+P_\lambda) = \frac{2\mathbf{k}^3}{\pi F_\pi^2(2J_\alpha+1)} \times\left|\sum_\gamma G(\gamma; n_\alpha J_\alpha; n_\beta J_\beta)\begin{pmatrix} n_\alpha & \mathbf{8} & n_\beta(\gamma) \\ \alpha & \lambda & \beta\end{pmatrix}\right|^2, \quad (20)$$

where $\mathbf{k}$ is the momentum of the emitted meson.

B. Applications

There is strong evidence that the baryons can be classified under $SU(3)$ into singlets, octets, and decuplets and the mesons into singlets and octets.[7,8] These $SU(3)$ multiplets can be grouped into supermultiplets belonging to representations of $SU(2)_V\otimes SU(6)$, and the simplest possibilities are the ones that have been considered earlier in connection with the quark model.[9] The representations of physical interest are ($V=0$, **35**), ($V=1$, **35**) for mesons and ($V=0$, **56**), ($V=2$, **56**), and ($V=1$, **70**) for baryons. The ($V=0$, **56**) multiplet includes the usual nucleon octet ($J^P=\frac{1}{2}^+$), and the p-wave decuplet resonances ($J^P=\frac{3}{2}^+$). The ($V=2$, **56**) multiplet has octets with $J^P=(\frac{3}{2})^+$, $(\frac{5}{2})^+$ and decuplets with $J^P=(\frac{1}{2})^+$, $(\frac{3}{2})^+$, $(\frac{5}{2})^+$, and $(\frac{7}{2})^+$. The ($V=1$, **70**) multiplet classifies all the known negative-parity baryons.[8]

Equations (18) and (20) now allow an evaluation of all p-wave interaction coupling constants and decay rates. The results for the ($V=0$, **56**) representation are well known. The D/F ratio for the axial-vector coupling of the $(\frac{1}{2})^+$ baryon octet is predicted[10] to be $\frac{3}{2}$. The octet-octet, octet-decuplet, and the decuplet-decuplet couplings are all determined in terms of one parameter which is determined by comparison with the experimentally determined g_A/g_V ratio for the nucleons. We thus have[11]

$$\left|\frac{\langle\Delta^+|\pi^0|p\rangle}{\langle p|\pi^0|p\rangle}\right|^2 = 1.28 \quad\text{and}\quad \frac{\langle\Delta^+|\pi^0|\Delta^+\rangle}{\langle p|\pi^0|p\rangle} = \frac{1}{5}. \quad (21)$$

The decay widths for $\Delta(1236)\to N\pi$, $\Sigma^*(1385)\to\Sigma\pi$, $\Lambda\pi$, and $\Xi^*(1530)\to\Xi\pi$ are predicted to be the same as in $SU(6)$ theory[2], viz.,

$$\Gamma(\Delta)=76\text{ MeV},\quad \Gamma(\Sigma^*\to\Sigma\pi)=3.3\text{ MeV},\quad \Gamma(\Sigma^*\to\Lambda\pi)=24.0\text{ MeV},$$

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and

$$\Gamma(\Xi^* \to \Xi\pi) = 8.9 \text{ MeV}.$$

Similar predictions can be made for the $(V=2, \mathbf{56})$ multiplet for p-wave interactions. However, these resonances are produced and are seen to decay mainly via other partial waves and a comparison with experiment cannot be made at the present time. One prediction of interest is a common D/F ratio of $\frac{3}{2}$ for axial-vector coupling between baryons of the same spin belonging to the representations $(V, \mathbf{56})$.

The superconvergence condition for odd amplitudes [Eq. (12)] leads to simple relations between scattering amplitudes.[11] The matrix elements of the left-hand side of Eq. (12) are the scattering amplitudes, and they are given in terms of the spin-nonflip (δ_{ab} term) and the spin-flip (ϵ_{abc} term) amplitudes by the right-hand side. The spin-nonflip amplitudes alone survive in the forward direction and this leads to predictions for relations among total cross sections. Again, the results for the $(V=0, \mathbf{56})$ representation are well known. For the nucleon octet the scattering amplitudes may be written as

$$T(P_\alpha B_a, P_\beta B_b) = f + \boldsymbol{\sigma}\cdot\mathbf{n}g,$$

where the f amplitudes are the spin-nonflip amplitudes. For the f amplitudes Kuriyan and Sudarshan[11] obtain

$$\begin{aligned} f(p\pi^+,p\pi^+) &= f(nK^+,nK^+) = \tfrac{1}{2}f(pK^+,pK^+) \\ &= (1/\sqrt{2})f(p\pi^-,n\pi^0) = f(p\pi^+,\Sigma^+K^+) \\ &= -\sqrt{2}f(p\pi^-,\Sigma^0K^0) = -(\sqrt{\tfrac{2}{3}})f(p\pi^-,\Lambda K^0). \end{aligned} \quad (22)$$

These relations are valid for all angles. In the forward direction only the f amplitudes survive, and the optical theorem allows us to relate these to the total cross section. The first three equalities are the well-known Johnson-Treiman relations[12]

$$\sigma(p\pi^+) = \sigma(nK^+) = \tfrac{1}{2}\sigma(pK^+). \quad (23a)$$

We can derive similar results for decuplet particles with spin $\frac{1}{2}$ belonging to any $SU(2)_V\otimes SU(6)$ representation. Let us limit our considerations to the $\Delta(1930; J^P=\frac{1}{2}^+)$ isomultiplet belonging to the $(V=2, \mathbf{56})$ representation, or the $\Delta(1640, J^P=\frac{1}{2}^-)$ isomultiplet belonging to the $(V=1, \mathbf{70})$ representation. We can derive the following relations for the spin-nonflip amplitudes:

$$\begin{aligned} \tfrac{1}{3}f(\pi^+\Delta^{++}) &= f(\pi^+\Delta^+) = -f(\pi^+\Delta^0) = -\tfrac{1}{3}f(\pi^+\Delta^-) \\ &= \tfrac{1}{3}f(K^+\Delta^{++}) = \tfrac{1}{2}f(K^+\Delta^+) = f(K^+\Delta^0), \end{aligned} \quad (23b)$$

and

$$f(K^+\Delta^-) = 0. \quad (23c)$$

These relations can again be translated into relations among the corresponding total cross sections using the optical theorem:

$$\begin{aligned} \tfrac{1}{3}\sigma(\pi^+\Delta^{++}) &= \sigma(\pi^+\Delta^+) = \sigma(\pi^+\Delta^0) = \tfrac{1}{3}\sigma(\pi^+\Delta^-) \\ &= \tfrac{1}{3}\sigma(K^+\Delta^{++}) = \tfrac{1}{2}\sigma(K^+\Delta^+) = \sigma(K^+\Delta^0). \end{aligned} \quad (23d)$$

In this section we have derived the Lie algebra of $SU(2)\otimes SU(6)$ as being the algebraic realization of Weinberg's superconvergence condition for amplitudes which are odd under the exchange of the meson indices, and we have shown that the predictions are similar to those made using noninvariance groups.[11]

III. MASS MATRIX

In the forward direction, for given helicity states, the second superconvergence condition takes the form given in Eq. (7). Let us generalize this to all helicities by considering p-wave interactions. We then have to evaluate the double commutator

$$[D_{\alpha a},[m^2,D_{\beta b}]]. \quad (24)$$

The Jacobi identity gives the relation

$$\begin{aligned} [D_{\alpha a},[m^2,D_{\beta b}]] &- [D_{\beta b},[m^2,D_{\alpha a}]] \\ &= i\epsilon_{abc}\delta_{\alpha\beta}[m^2,B_c] + i\epsilon_{abc}d_{\alpha\beta\gamma}[m^2,D_{\gamma c}]. \end{aligned} \quad (25)$$

The commutator of Eq. (24) has the transformation properties under the invariance group $K=SU(3)\otimes SU(2)_J$, given by the direct product $(\mathbf{8},\mathbf{3})\otimes(\mathbf{8},\mathbf{3})$, because $D_{\alpha a}$ belongs to the $(\mathbf{8},\mathbf{3})$ representation of this group. It is assumed that the mass matrix is invariant under the invariance group K. The Jacobi identity [Eq. (25)] ensures that the double commutator $[D_{\alpha a},[m^2,D_{\beta b}]]$ will not contain terms with the tensorial characters $(\mathbf{8}_a,\mathbf{1})$, $(\mathbf{8}_a,\mathbf{5})$; $(\mathbf{10},\mathbf{1})$, $(\mathbf{10},\mathbf{5})$, $(\mathbf{10}^*,\mathbf{1})$, $(\mathbf{10}^*,\mathbf{5})$, and $(\mathbf{27},\mathbf{3})$. The absence of exotic mesons is now expressed by demanding that the t-channel amplitudes with the tensorial characters $(\mathbf{27},\mathbf{1})$, $(\mathbf{27},\mathbf{5})$, $(\mathbf{10},\mathbf{3})$, and $(\mathbf{10}^*,\mathbf{3})$ be zero. We are then left with amplitudes having transformation properties which can be associated with the transformation properties of observable mesons. We interpret the amplitudes as being the amplitudes given by the t-channel exchange of these observable mesons. We have the vector meson octet $(\mathbf{8}_a,\mathbf{3})$ associated with the exchange of the $\rho(765)$, $\omega(783)$, and $K^*(890)$; and we have the tensor meson nonet $(\mathbf{1}\oplus\mathbf{8}_s, \mathbf{5})$ to which belong the $A_2(1310)$, $K^*(1420)$, and $f^0(1260)$ and the $f^*(1515)$. We also have the axial-vector meson and the scalar meson nonets associated with the t-channel amplitudes with the tensorial characters $(\mathbf{1}\oplus\mathbf{8}_s, \mathbf{3})$ and $(\mathbf{1}\oplus\mathbf{8}_s, \mathbf{1})$, respectively. Not all the members of the axial-vector nonet have been identified experimentally.[7,8] The mesons $A_1(1070)$, $K^*(1230)$, and $D(1280)$ are well established. Of the members of the scalar meson nonet, less is known. The usual identification[8,9] is to include the $\delta(960)$, $K\pi(1100)$, $\sigma(750)$, and $S^*(1070)$ in the scalar meson nonet.

We evaluate the double commutator of Eq. (24) by putting in a complete set of intermediate states and

[12] K. Johnson and S. B. Treiman, Phys. Rev. Letters 14, 189 (1965).

solve for the mass matrix from the conditions imposed by the absence of exotic mesons, as discussed above. These conditions may be expressed in a compact form by using crossing matrices.[13] Let the initial and final states be denoted by $|n_\beta J_\beta\rangle$ and $|n_\alpha J_\alpha\rangle$, respectively, and the intermediate states by $|n_s J_s\rangle$. Then we have

$$\sum_{(\gamma,\gamma';n_s,J_s)} G(\gamma;nJ;n_sJ_s)\bar{G}(\gamma';nJ;n_sJ_s) \times(n_t|X_{ts}|n_s,\gamma\gamma')(J_t|X'_{ts}|J_s) \times[2m^2(n_s,J_s)-m^2(n_\alpha,J_\alpha)-m^2(n_\beta,J_\beta)]=0 \quad (26a)$$

for

$$n_t=\mathbf{27}\,,\quad J_t=0 \text{ or } 2 \quad (26b)$$

and

$$n_t=\mathbf{10} \text{ or } \mathbf{10}^*,\quad \text{with}\quad J_t=1. \quad (26c)$$

The conditions of Eq. (26) are expressed as difference equations for $m^2(n,J)$, using the known crossing matrix elements[13] for $SU(3)$ and $SU(2)$ and the definition of the coupling constants G given by Eq. (16). By choosing $|n_\alpha J_\alpha\rangle$ and $|n_\beta J_\beta\rangle$ appropriately, it is straightforward to show that the superconvergence conditions expressed by Eq. (26) lead to

$$m^2(\mathbf{8},\tfrac{1}{2})=m^2(\mathbf{10},\tfrac{3}{2}) \quad (27)$$

for the members of the $(V=0,\mathbf{56})$ representation. This is just the usual $SU(6)$ result of mass degeneracy between the nucleon octet and the $\Delta(1236)$ decuplet.

For the $(V=2,\mathbf{56})$ representation, the mass matrix conditions are obtained as follows. We first choose $|n_\alpha J_\alpha\rangle$ and $|n_\beta J_\beta\rangle$ to be $|\mathbf{10},J+1\rangle$ and $|\mathbf{10},J-1\rangle$. This limits the sum over J_s in Eq. (26) to one term. The superconvergence condition giving $n_t=\mathbf{27}$ and $J_t=2$ for the above choice of the initial and final states leads to the difference equation

$$5[2m^2(\mathbf{8},J)-m^2(\mathbf{10},J+1)-m^2(\mathbf{10},J-1)] =8[2m^2(\mathbf{10},J)-m^2(\mathbf{10},J+1)-m^2(\mathbf{10},J-1)]. \quad (28)$$

Next we apply the same constraint for the initial state $|\mathbf{8},J-1\rangle$ and the final state $|\mathbf{10},J+1\rangle$ and obtain the difference equation

$$5[2m^2(\mathbf{8},J)-m^2(\mathbf{8},J-1)-m^2(\mathbf{10},J+1)] =-2[2m^2(\mathbf{10},J)-m^2(\mathbf{8},J-1)-m^2(\mathbf{10},J+1)]. \quad (29)$$

The superconvergence condition $n_t=27$, $J_t=0$ leads to the following constraint on the mass matrix for the choice of $|\mathbf{10},J\rangle$, and $|\mathbf{10},J\rangle$ as the initial and final states:

$$\sum_{J_s}(2J_s+1)\left[[m^2(\mathbf{8},J_s)-m^2(\mathbf{10},J)]\begin{Bmatrix}V & J_s & \tfrac{1}{2}\\ 1 & \tfrac{3}{2} & J\end{Bmatrix}^2 +2[m^2(\mathbf{10},J_s)-m^2(\mathbf{10},J)]\begin{Bmatrix}V & J_s & \tfrac{3}{2}\\ 1 & \tfrac{3}{2} & J\end{Bmatrix}^2\right]=0. \quad (30)$$

[13] P. Carruthers, *Introduction to Unitary Symmetry* (Interscience, New York, 1966), Chap. 7, and references therein.

Equations (28)–(30) lead to a degenerate mass spectrum for the $(V=2,\mathbf{56})$ representation. This is just the trivial solution for difference equations and will also be a solution of any other difference equation obtained from the other superconvergence conditions. We thus find that in going to a larger invariance group [viz., $SU(3)\otimes SU(2)$], the absence of exotic meson exchanges in the t channel constitute stronger constraints on the mass matrix, and the superconvergence conditions yield degenerate mass spectra for the symmetric representations of interest, and a mass spectrum corresponding to Eq. (4) is not obtained.

We have not studied the $(V=1,\mathbf{70})$ representation for the following reasons. As mentioned in Sec. III, these negative-parity baryons decay strongly into the positive-parity baryons belonging to the $(V=0,\mathbf{56})$ representation. The p-wave interaction of these negative-parity baryons has not been studied experimentally, and the coupling constants are not known. Secondly, the reduction of the direct product $\mathbf{70}\otimes\mathbf{35}$ contains two $\mathbf{70}$ representations, and this would introduce an unknown parameter. Finally we would have the problem of mixing: The representation $(V=1,\mathbf{70})$ contains two octets with $J=\frac{3}{2}$ and two octets with $J=\frac{1}{2}$. The assignment of the $\frac{3}{2}^-$ and $\frac{1}{2}^-$ octets of baryons would in general require two more parameters in the form of mixing angles.

IV. DISCUSSION

We have shown that the superconvergence condition on the odd amplitudes for p-wave interactions leads to the Lie algebra of the group $SU(2)\otimes SU(6)$. Dynamical groups with the same Lie algebra have been investigated earlier by Mahantappa and Sudarshan[14] while seeking an invariance group which would allow πNN coupling in the static limit. However, the interpretation of the generators of the $SU(6)$ subgroup is different. We relate the p-wave pseudoscalar-meson decay amplitudes to the generators $D_{\alpha a}$. This possibility also has been investigated earlier by Kuriyan and Sudarshan[11] by *postulating* that the dynamical "noninvariance" group is the group $SU(6)$ in which the above interpretation of the generators $D_{\alpha a}$ is true. We have therefore shown that the assumptions that are made in obtaining the first superconvergence condition on the amplitudes derived from tree graphs lead naturally to the intermediate-coupling model of Kuriyan and Sudarshan.[11] The group $SU(2)\otimes SU(6)$ gives rise to a particle spectrum which agrees with experiment.

The decay rates and the coupling constants for the $(V=0,\mathbf{56})$ representation are the same as the ones predicted by the usual $SU(6)$ theory. We also predict relations similar to the Johnson-Treiman relations for the scattering of p-wave pseudoscalar mesons and decuplet baryons with spin $\frac{1}{2}$ and either parity.

[14] K. T. Mahantappa and E. C. G. Sudarshan, Phys. Rev. Letters 14, 165 (1965).

The intermediate-coupling models do not have analogs of the second superconvergence conditions [Eqs. (26)]. The only solution for the mass matrix obtained from these equations for the representations ($V=0$, **56**) and ($V=2$, **56**) is the trivial one of mass degeneracy within each of these multiplets. In the strong-coupling model,[15] such a result would have implied the vanishing of the scattering amplitude itself. Here the mass degeneracy is an acceptable solution, while not a satisfactory one.

[15] T. Cook, C. J. Goebel, and B. Sakita, Phys. Rev. Letters **15**, 35 (1965).

In conclusion, we believe that this work gives a better perspective to the results of Cronstrom and Noga.[2]

ACKNOWLEDGMENTS

The author wishes to thank Professor S. P. Rosen for his guidance and encouragement and Dr. M. Noga for numerous discussions regarding his work.

PHYSICAL REVIEW D VOLUME 2, NUMBER 2 15 JULY 1970

Critique of a Proposed Dynamical Group for Relativistic Quantum Mechanics*

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(Received 21 January 1970)

The dynamical group $\tilde{G}_5$ for relativistic quantum mechanics phenomenologically suggested by Aghassi, Roman, and Santilli is derived from the analysis of symmetry properties of Lagrangians and corresponding equations of motion for a free relativistic particle. All physical observables such as position, momentum, angular momentum, and mass squared are represented by well-defined operators which close the algebra of the dynamical group $\tilde{G}_5$. The unitary irreducible representations of this group, which are possible states of the physical system, are found. The particles accommodated in the single unitary irreducible representations have various spins starting from the lowest spin value and going up to infinity in integral steps. The mass-squared operator $P_\mu P^\mu$ lies in the enveloping algebra of $\tilde{G}_5$, and its eigenvalues are not necessarily quantized and can have any positive or negative values. It is pointed out that this group has several failures and thus it cannot be accepted as the reliable dynamical group for particles within relativistic quantum mechanics.

I. INTRODUCTION

THE hypothesis that the dynamics of the quantal interacting system can be completely described by some dynamical group has been verified for almost all interesting quantum-mechanical problems.[1] In the approach using dynamical groups, instead of postulating the Hamiltonian for the quantum-mechanical system we postulate a dynamical group. Then the quantum-mechanical wave functions are supposed to form the basis for the unitary irreducible representation of the group in question which is generated by the operators of the physical observables. The same idea was consequently used in strong-interaction physics with the great hope of predicting hadron states with their masses and mutual coupling constants.[2,3] This approach to particle physics became rather popular recently because various bootstrap schemes[4] and superconvergence relations following from the proper Regge behavior of the scattering amplitude are formulated in the group-theoretical language.[5]

One essential shortcoming of the models mentioned above is connected with the mass spectrum. The relation for the mass spectrum is solvable only in models which are not fully relativistic invariant, such as those in strong coupling[6] and in bootstrap theory,[7] while in the models with relativistic invariance the condition imposed on the hadron masses[5] can be solved only if rough approximations are made.[8]

Other attempts which were made to obtain the mass spectrum in relativistic-invariant theories tried to combine the Poincaré group with some semisimple

* Supported in part by the U. S. Atomic Energy Commission.
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