$O(2,1)$ Analysis of Single-Particle Spectra at High Energy*

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Processes of the type proton+proton → pion+anything at high energy are discussed. The differential cross section for such reactions is expressed in terms of a three-particle → three-particle amplitude. This amplitude is then expanded into $O(2,1)$ representations. If the Pomeranchuk trajectory is the dominant $O(2,1)$ singularity at high energy, the existence of pionization and limiting fragmentation is obtained. Furthermore, the pionization products are essentially independent of the target and projectile, while the fragments of the target are independent of the projectile. Modifications in the presence of strong Regge cuts at $J=1$ are discussed.

I. INTRODUCTION

In this paper we consider processes of the type proton+proton → pion+anything; that is, processes where two hadrons initiate a reaction in which only the momentum of a definite final hadron is observed. Such processes have the simplicity of depending only on three variables, which can be taken to be the invariant mass s of the two initial hadrons and the longitudinal and transverse momenta q_L and q_T of the final observed hadron in the center-of-mass frame of the initial hadron system. Before entering into a detailed discussion of the results of this investigation, we give a brief review of several ideas on single-particle spectra.

In particular, the hypothesis of limiting fragmentation of Benecke, Chou, Yang, and Yen[1] predicts that the final pion distribution (for simplicity we shall always take the final observed particle to be a pion, although it should be emphasized that this is merely a convenience and not a necessity) described by $\omega_q d\sigma/d^3q$ approaches a constant as s increases when q, the momentum of the pion, is held fixed in the laboratory frame of either of the initial hadrons.

Feynman[2] has argued that $\omega_q d\sigma/d^3q$ approaches a constant for fixed $q_L/(s)^{1/2}$ and fixed q_T and, furthermore, that $\omega_q d\sigma/d^3q$ depends only on q_T so long as $q_L/(s)^{1/2}$ is not too large. This agrees with Benecke *et al.* in the region of fragmentation and in addition predicts pionization products whose differential cross section has a longitudinal momentum distribution proportional to dq_L/ω_q.

A model more explicit than those above is the multiperipheral model with Regge trajectory exchanges. The properties of the single-particle spectra have been discussed for this model by Pinsky and Weisberger[3] and by Silverman and Tan.[4] Their results agree with those of Benecke *et al.* and Feynman where they overlap.

In this paper we hope to present a more general analysis of inelastic reaction of the above type by means of an $O(2,1)$ analysis of the three-particle → three-particle process described by

$$A(p_2,q,p_1)=(2\pi)^3\int d^4x \times e^{-iqx}\langle p_1p_2(+)|j_\pi(x)j_\pi(0)|p_1p_2(+)\rangle. \quad (1.1)$$

In (1.1), p_1 and p_2 refer to the momenta of the two initial hadrons (protons), while q is the momentum of the final pion, $q^2=\mu^2$, and $j_\pi(x)$ is the pion current. In Sec. II it is shown that the invariant $\omega_q d\sigma/d^3q=(M^2/s)\times A(p_2,q,p_1)$ for large s, with M the mass of the proton.

In Sec. III an $O(2,1)$ analysis of A is presented. This method of analysis is inspired by the discussion of Altarelli, Brandt, and Preparata[5] concerning massive μ-pair production in inelastic proton-proton collisions. The analysis of Ref. 5 does not directly apply, however, to the purely hadronic processes considered in this paper. In Sec. IV A, a double $O(2,1)$ analysis of $A(p_2,q,p_1)$ gives $A(p_2,q,p_1)\to(p_1\cdot q)(p_2\cdot q)\beta(p_1\cdot qp_2\cdot q/s)$ as p_1 and p_2 become large for fixed q in the center-of-mass system when the Pomeranchuk trajectory is the leading $O(2,1)$ trajectory. If Regge cuts at $J=1$ are present, there will be an asymptotic term differing from the one above by factors of $\ln(p_1\cdot q)$ and $\ln(p_2\cdot q)$. In Sec. III B, a single $O(2,1)$ analysis of $A(p_2,q,p_1)$ is presented which yields $A(p_2,q,p_1)\to(p_1\cdot p_2)\beta(p_2\cdot q,\, p_1\cdot q/p_1\cdot p_2)$ in the limit of Pomeranchuk pole dominance when q remains finite in the rest system of p_2 while p_1 becomes large. Again cuts will give logarithmic factors.

In Sec. IV the physical consequences of the previous formalism are described. In Sec. IV A it is shown that the double $O(2,1)$ analysis of the preceding section gives a pionization amplitude $d\sigma/d^3q=f(q_T)/\omega_q$ for large energy when the Pomeranchuk trajectory is the dominant $O(2,1)$ trajectory. This is the distribution predicted by Feynman. Furthermore, the criterion for the above formula to be valid is that both $p_1\cdot q$ and $p_2\cdot q$ be large. If the Pomeranchuk trajectory factorizes, then pioniza-

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[1] J. Benecke, T. T. Chou, C. N. Yang, and E. Yen, Phys. Rev. **188**, 2159 (1969).

[2] R. P. Feynman, Phys. Rev. Letters **23**, 1415 (1969). See also a discussion of pionization by H. Cheng and T. T. Wu, *ibid.* **23**, 1311 (1969).

[3] W. I. Weisberger (private communication).

[4] D. Silverman and Chung-I Tan, Phys. Rev. (to be published).

[5] G. Altarelli, R. A. Brandt, and G. Preparata (unpublished); Giuliano Preparata (private communication).

tion products are independent of the initial hadrons except for a normalization constant which can be determined from elastic scattering data. Modifications in the presence of Regge cuts at $J=1$ are discussed.

In Sec. IV B it is argued that the single $O(2,1)$ analysis of Sec. III along with Pomeranchuk trajectory dominance predicts limiting fragmentation of the target and projectile and also the independence of the fragments of the target on the projectile except for a known normalization constant. Regge cuts could violate limiting fragmentation logarithmically and would destroy the factorization property.

In Sec. IV C the average multiplicity $\bar{n}_\pi(s)$ for pions is derived. In the limit of Pomeranchuk trajectory dominance, a logarithmically increasing multiplicity is found and the coefficient of $\ln s$ can be determined from the analysis of Sec. IV A. If Regge cuts are present, this logarithmically increasing multiplicity could be destroyed and the determination of the coefficient of $\ln s$ from pionization products would certainly not be possible at machine energies.

In Sec. V arguments are presented connecting the $O(2,1)$ singularities of Sec. III with the usual Regge singularities in two-body reactions.

II. KINEMATICS

Consider the process shown in Fig. 1, $p_1+p_2\to q$ + anything. We shall take the particles having momenta p_1 and p_2 to be protons and the particle with momentum q to be a pion. In what follows it will become clear that neither the spin nor internal quantum numbers of the particles labeled by p_2, p_1, and q put any restriction on the formalism developed below. Call $d\sigma/d^3q=q^{-2}d^2\sigma/d\Omega dq$ the differential cross section for producing a pion of momentum q with anything else, including possibly more pions, also being produced. In the center-of-mass system of the two protons in the initial state,[6]

$$\frac{d\sigma}{d^3q}=\frac{(2\pi)^3M^2}{4pE_p\omega_q}\sum_n\prod_{i=1}^{n}\left[\frac{d^3k_i}{(2\pi)^3\rho_i}\right](2\pi)^4 \times\delta(p_1+p_2-q-\sum_{j=1}^{n}k_j)\left|(p_1p_2(+)\,|\,j_\pi(0)\,|\,n(-))\right|^2 =(M^2/4pE_p\omega_q)A(p_2,q,p_1)\,, \tag{2.1}$$

with

$$p_{1\mu}=((p^2+M^2)^2,0,0,p)\,,$$
$$p_{2\mu}=((p^2+M^2)^2,0,0,-p)\,,$$

where $j_\pi(0)$ is a pion current, $j_\pi(x)=(\Box-\mu^2)\phi_\pi(x)$, and where $q_\mu q^\mu=\mu^2$ is the physical mass of the pion. The summation over n is assumed to include an average over the initial protons' spins and a sum over all final spins. Our normalization of states is as in Ref. 6 so that $\rho_q=2\omega_q/(2\pi)^3$ for fermions. It follows easily from (2.1) that both $A(p_2,q,p_1)$ and $\omega_q d\sigma/d^3q$ are invariants which can depend only on the three invariants $p_2\cdot q$, $p_1\cdot q$, and $p_2\cdot p_1$.

Using translation invariance and completness, one can rewrite (2.1) as

$$\frac{d\sigma}{d^3q}=\frac{(2\pi)^3M^2}{4pE_p\omega_q}\sum\int d^4x \times e^{-iqx}(p_1p_2(+)\,|\,j_\pi(x)j_\pi(0)\,|\,p_1p_2(+)) =(M^2/2pE_p\omega_q)A(p_2,q,p_1)\,. \tag{2.2}$$

The $\sum$ in (2.2) refers to an average over the spins of the protons.

III. $O(2,1)$ EXPANSION OF $A(p_2,q,p_1)$

It will prove convenient to expand $A(p_2,q,p_1)$ into $O(2,1)$ harmonics in order to connect properties of pion production such as pionization, limiting fragmentation, and average multiplicity with Regge-type singularities familiar from two-body reactions. Although the kinematic configuration in (2.2) has sufficient symmetry to render an $O(3,1)$ expansion natural, we shall, nevertheless, work only with the smaller $O(2,1)$ group for the sake of simplicity.

A. Double $O(2,1)$ Expansion

Since A is an invariant function of $p_2\cdot q$, $p_1\cdot q$, and $p_2\cdot p_1$, we may choose any coordinate system which is convenient. To begin, let us choose a system where

$$q=\mu(1,0,0,0)=(q_t,q_x,q_y,q_z)\,,$$
$$p_1=M(\cosh\zeta_1,\ \sinh\zeta_1\cos\varphi,\ \sinh\zeta_1\sin\varphi,\ 0)\,, \tag{3.1}$$
$$p_2=M(\cosh\zeta_2,\ -\sinh\zeta_2,\ 0,\ 0)\,.$$

The three invariants become

$$s_1=p_1\cdot q=M\mu\cosh\zeta_1\,,$$
$$s_2=p_2\cdot q=M\mu\cosh\zeta_2\,, \tag{3.2}$$
$$s=p_2\cdot p_1=M^2(\cosh\zeta_1\cosh\zeta_2+\sinh\zeta_1\sinh\zeta_2\cos\varphi)\,,$$

so that A can be considered as a function of the independent variables ζ_2, φ, and ζ_1. Now expand A in $O(2,1)$ harmonics according to[7]

$$A(\zeta_2,\varphi,\zeta_1)=\sum_m\int_{-1/2-i\infty}^{-1/2+i\infty}d\Lambda_2\int_{-1/2-i\infty}^{-1/2+i\infty}d\Lambda_1 \times A_m{}^{\Lambda_2\Lambda_1}d_{0m}{}^{\Lambda_2}(\zeta_2)e^{im\varphi}d_{m0}{}^{\Lambda_1}(\zeta_1)\,. \tag{3.3}$$

For simplicity we neglect the discrete series and assume that terms in ζ_1 and ζ_2 which decrease less rapidly than $(\cosh\zeta_1)^{-1/2}$ and $(\cosh\zeta_2)^{-1/2}$ are handled explcitly.

[6] S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966).

[7] M. Toller, Nuovo Cimento **37**, 631 (1965).

The inversion of (3.3) is

$$A_m^{\Lambda_2\Lambda_1}=\frac{-\cot\pi\Lambda_2\cot\pi(\Lambda_1-m)(2\Lambda_2+1)(2\Lambda_1+1)}{32\pi}\times\int_0^{2\pi}d\varphi\int_1^\infty d\cosh\zeta_2\int_1^\infty d\cosh\zeta_1 d_{0m}{}^{\Lambda_2}(\zeta_2)\times e^{-im\varphi}d_{m0}{}^{\Lambda_1}(\zeta_1)A(\zeta_2,\varphi,\zeta_1)\,. \quad (3.4)$$

When ζ_1 (ζ_2) becomes large, the behavior of (3.3) is governed by the leading singularities in Λ_1 (Λ_2). When both ζ_1 and ζ_2 become large, the asymptotic behavior in ζ_1 and ζ_2 is governed by the leading singularities α_1 and α_2 in Λ_1 and Λ_2. Keeping only the leading singularities, (3.3) becomes

$$A(\zeta_2,\varphi,\zeta_1)\underset{\zeta_1,\zeta_2\to\infty}{\longrightarrow}(\cosh\zeta_2)^{\alpha_2}(\cosh\zeta_1)^{\alpha_1}\beta(\varphi) \quad (3.5)$$

if the leading singularities are poles. If cuts in Λ_1 and Λ_2 are present, (3.5) will have additional logarithmic terms in $\cosh\zeta_1$ and $\cosh\zeta_2$. We shall not write these logarithmic terms explicitly, but neither shall we assume that they are not present. The question of cuts versus poles will be discussed more fully somewhat later.

The coordinate system described by (3.1) is not a very transparent one in terms of physical quantities. In particular, the meaning of φ is not immediately clear. In order to relate φ to a more natural experimental variable, let us consider the center-of-mass system of p_2 and p_1 in which

$$\begin{aligned}p_2&=((p^2+M^2)^{1/2},0,0,-p)\,,\\ p_1&=((p^2+M^2)^{1/2},0,0,p)\,,\\ q&=(q_0,\mathbf{q},q_3)\,,\end{aligned} \quad (3.6)$$

where $\mathbf{q}$ is a two-component vector. In terms of these variables the invariants s_2, s_1, and s become

$$\begin{aligned}s_2&=p_2\cdot q=p(q_0+q_3)\,,\\ s_1&=p_1\cdot q=p(q_0-q_3)\,,\end{aligned} \quad (3.7)$$

in the limit that s, s_1, and s_2 are large. Thus

$$s_1s_2=p^2(q_0{}^2-q_3{}^2)=\tfrac{1}{2}s(\mathbf{q}^2+\mu^2)\,.$$

But, from (3.2),

$$s_1s_2=M^2\mu^2\cosh\zeta_1\cosh\zeta_2$$

and

$$s=M^2\cosh\zeta_1\cosh\zeta_2(1+\cos\varphi)$$

for large s_1 and s_2. Comparing the two above expressions for s_1s_2, one obtains

$$1+\cos\varphi=2\mu^2/(\mu^2+\mathbf{q}^2)\,. \quad (3.8)$$

Equation (3.8) shows that the $\cos\varphi$ dependence in (3.3) and (3.5) is equivalent to the $\mathbf{q}^2$ dependence which is the transverse momentum dependence of the pion in the center-of-mass system. This relation remains valid so long as s, s_1, and s_2 are large.

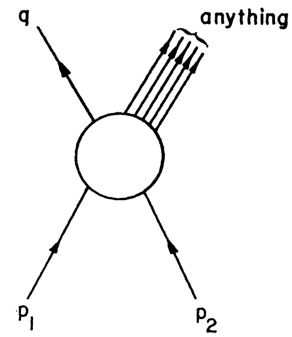

FIG. 1. Diagram for the process $p_1+p_2\to q+$anything.

B. Single $O(2,1)$ Expansion

If either ζ_1 or ζ_2 is not large, then (3.5) should not be a good approximation to (3.3). In fact, if only ζ_1 (ζ_2) becomes large there is no advantage to expanding in Λ_2 (Λ_1). For later discussions it is convenient to choose a new coordinate system in which

$$\begin{aligned}p_2&=M(1,0,0,0)\,,\\ q&=\mu(\cosh\xi,-\sinh\xi\cos\psi,\sinh\xi\sin\psi,0)\,,\\ p_1&=M(\cosh\eta,\sinh\eta,0,0)\,,\end{aligned} \quad (3.9)$$

with the corresponding relations for the invariants

$$\begin{aligned}s_2&=M\mu\cosh\xi\,,\\ s&=M^2\cosh\eta\,,\\ s_1&=M\mu(\cosh\xi\cosh\eta+\sinh\xi\sinh\eta\cos\psi)\,.\end{aligned} \quad (3.10)$$

Now $\cosh\xi=\cosh\zeta_2$, as is clear from a comparison of the two expressions for s_2 in (3.2) and (3.10). Also, when $\cosh\zeta_1$ becomes large, $\cosh\eta$ becomes large so long as s_2 remains finite.

Thus, the natural expansion for large s_1 and finite s_2 is

$$A(\zeta_2,\varphi,\zeta_1)=A(\xi,\psi,\eta)=\sum_m\int_{-1/2-i\infty}^{-1/2+i\infty}d\Lambda\,A_m{}^\Lambda(\xi,\psi)d_{m0}{}^\Lambda(\eta)\,, \quad (3.11)$$

which for large $\cosh\eta$ becomes equal to

$$A(\xi,\psi,\eta)\underset{\eta\to\infty}{\longrightarrow}(\cosh\eta)^\alpha\beta(\psi,\xi)\,,$$

where α is the leading singularity in Λ of $A_m{}^\Lambda$ and where we have assumed that the leading singularity is a simple pole. Again, ln $\cosh\eta$ terms will be discussed later.

IV. PHYSICAL CONSEQUENCES

In this section an analysis of $d\sigma/d^3q$ is given in terms of the Regge singularities which occur in two-body interactions. At this point we do not provide an argument that the Λ_2, Λ_1 singularities of $A_m^{\Lambda_2\Lambda_1}$ in (3.3) and the singularities of A_m^{Λ} in (3.11) are at the same positions as $O(2,1)$ singularities in two-body reactions. We shall, however, assume that such is the case and postpone arguments directed at this result until Sec. V.

A. Pionization

Pionization products are those pions which maintain a finite momentum in the center-of-mass system of the initial protons as the energy of the protons becomes very large. For such particles, the analysis performed in Sec. III A is relevant since both s_1 and s_2, and hence both $\cosh\zeta_1$ and $\cosh\zeta_2$, become large. Thus,

$$A(\zeta_2,\varphi,\zeta_1) \to (\cosh\zeta_2)^{\alpha_2}(\cosh\zeta_1)^{\alpha_1}\beta(\varphi) \qquad (3.5)$$

as $s\to\infty$ for fixed q in the center-of-mass system. For the moment take only the contribution of the Pomeranchuk pole to (3.5). Then

$$A(\zeta_2,\varphi,\zeta_1) \to (\cosh\zeta_2)(\cosh\zeta_1)\beta(\varphi)\,,$$

which can be written as

$$A\to\frac{s}{M^2}\frac{\beta(\varphi)}{1+\cos\varphi}=\frac{s}{M^2}f(\mathbf{q}^2)\,. \qquad (4.1)$$

Equation (4.1) shows that the pionization products approach a limiting distribution[2] since

$$\frac{d\sigma(q)}{d^3q}\to\frac{f(\mathbf{q}^2)}{\omega_q} \qquad (4.2)$$

for large s, as can be seen by a comparison of (2.1) and (4.1). Furthermore, for this limiting distribution $\omega_q d\sigma/d^3q$ is independent of the longitudinal momentum q_3 of the pion. Also, in this case of Pomeranchuk pole dominance the pionization products will obey a factorization property similar to that in elastic scattering. This factorization property can be written

$$\frac{d\sigma_{AB}/d^3q}{d\sigma_{CD}/d^3q}=\frac{\beta_{AAP}\beta_{BBP}}{\beta_{CCP}\beta_{DDP}}\,, \qquad (4.3)$$

where $d\sigma_{AB}/d^3q$ is the differential cross section for $A+B\to\pi(q)+$anything and similarly for $d\sigma_{CD}/d^3q$. β_{AAP} is the coupling of the A particle to the Pomeranchuk trajectory at zero momentum transfer, the same coupling which occurs in forward elastic scattering and total cross-section formulas, and similarly for β_{BBP}, β_{CCP}, and β_{DDP}. The transverse momentum distribution $f(\mathbf{q}^2)$ in (4.2) does not seem to be determined in any way from the assumption of Pomeranchuk pole dominance. Indeed, $f(\mathbf{q}^2)$ appears to be a very model-dependent function.

If one gives up the assumption of Pomeranchuk pole dominance and allows cuts in the Λ_1 and Λ_2 planes with branch points at $\Lambda_1=1$, $\Lambda_2=1$, then the asymptotic form of $A(\zeta_2,\varphi,\zeta_1)$ will not be so simple as in (3.5). With cuts one must allow terms such as $(\cosh\zeta_1)(\cosh\zeta_2)\times[\ln\cosh\zeta_1]^{c_1}[\ln\cosh\zeta_2]^{c_2}$ in the asymptotic expansion of A. With such terms as these, a limiting distribution would not occur if $c_1+c_2>0$ and the limiting distribution would be approached only logarithmically if $c_1+c_2<0$ for all terms involving logarithms. The factorization of the pionization products as given in (4.3) would be violated if $c_1+c_2>0$ and would be approached logarithmically if $c_1+c_2<0$ for all logarithmic factors. Also, the observed pion can now depend on q_3, the logitudinal component of q in the center-of-mass system, in a logarithmic way, although at sufficiently large values of s the q_3 dependence goes away. That is, terms like $\ln p(q_0-q_3)\to\ln p$ for $\ln p\gg\ln(q_0-q_3)$. At machine energies where logarithmic effects may not be detectable in production processes, such as the ones considered here, the only real distinction between Pomeranchuk pole dominance and cut effects is probably the factorization property (4.3).

B. Limiting Fragmentation

Suppose now that s_2 remains finite as s becomes large. Then the single $O(2,1)$ expansion in Sec. III B is appropriate. In the coordinate frame defined by (3.9), the pion is seen to be a fragment of the target particle. Thus, when s_2 remains finite as $s\to\infty$, we are in the region where the asymptotic properties of the fragments of the particle labeled by p_2 can be investigated.

Assuming that only the Pomeranchuk pole dominates as $\cosh\eta\to\infty$ in (3.11), one obtains

$$A(p_2,q,p_1)\to(1/M^2)s\gamma(\mathbf{q},q_3) \qquad (4.4)$$

or

$$\omega_q\frac{d\sigma}{d^3q}\to\gamma(\mathbf{q},q_3)\,. \qquad (4.5)$$

Equation (4.5) indicates that the hypothesis of limiting fragmentation[1] is correct when the Pomeranchuk trajectory is the dominant singularity in (3.10). Further, there is again a factorization property which says that

$$\frac{d\sigma_{BC}/d^3q}{d\sigma_{DC}/d^3q}=\frac{\beta_{BPP}}{\beta_{DPP}}\,, \qquad (4.6)$$

where $d\sigma_{BC}/d^3q$ is the cross section for $B+C\to\pi(q)+$anything when $\pi(q)$ is a fragment of the particle C, and similarly for $d\sigma_{DC}/d^3q$. β_{BPP} and β_{DPP} are as in (4.3). Equation (4.6) implies that the limiting distribution for a fragment of the target is independent of the projectile at high energy, except for a constant factor which can be determined from the coupling of the projectile to the Pomeranchuk trajectory at zero momentum transfer.

In case the Pomeranchuk trajectory does not dominate cuts at $\Lambda=1$, one would expect limiting fragmentation to be approached or even violated logarithmically.

Further, cuts would violate the factorization property (4.6).

C. Multiplicities

The average multiplicity $\bar{n}_\pi(s)$ of pions is given by

$$\bar{n}_\pi(s)=\frac{1}{\sigma(s)}\int d^3q\,\frac{d\sigma}{d^3q}(p_2,q,p_1)\,,\qquad (4.7)$$

where $\sigma(s)$ is the total spin-averaged proton-proton cross section at a given s and where the integral in (4.7) extends over the allowed phase space. In our notation $\bar{n}_\pi(s)$ means the average multiplicity for the specific pion (π^+, π^-, or π^0) to which $d\sigma/d^3q$ refers. To get the average multiplicity for all kinds of pions produced in p-p collisions, we have to add $\bar{n}_{\pi^+}(s)+\bar{n}_{\pi^-}(s)+\bar{n}_{\pi^0}(s)$.

If one has no information about the transverse momentum distributions $f(\mathbf{q}^2)$ in (4.2) and the $\mathbf{q}^2$ dependence of $\gamma(\mathbf{q},q_3)$ in (4.5), it is difficult to estimate the multiplicity of pions. However, if the $\mathbf{q}^2$ dependence of $f(\mathbf{q}^2)$ and $\gamma(\mathbf{q},q_2)$ decreases faster than $(\mathbf{q}^2)^{-1}$, then the leading behavior of $\bar{n}_\pi(s)$ can be calculated when the Pomeranchuk pole is the leading singularity. In this case, (4.7) can be written as

$$\sigma(s)\bar{n}_\pi(s)=\int d^2q\int_{(\mu^2+\mathbf{q}^2)/2M-M/2}^{(\mu^2+\mathbf{q}^2)/2\epsilon M-M\epsilon/2}\frac{dq_3\gamma_2(\mathbf{q},q_3)}{(\mathbf{q}^2+q_3^2+\mu^2)^{1/2}}$$
$$+\int_{-\epsilon p}^{\epsilon p}dq_3\int\frac{d^2qf(\mathbf{q}^2)}{(\mathbf{q}^2+q_3^2+\mu^2)^{1/2}}$$
$$+\int d^2q\int_{-(\mu^2+\mathbf{q}^2)/2\epsilon M+M\epsilon/2}^{M/2-(\mu^2+\mathbf{q}^2)/2}\frac{(dq_3\gamma_1\mathbf{q},q_3)}{(\mathbf{q}^2+q_3^2+\mu^2)^{1/2}}\,.\qquad (4.8)$$

Equation (4.8) is derived in the Appendix. The first term in (4.8) is an integration over the fragments of the particle labeled by p_2 and the integration is done in the system where $p_2=M(1,0,0,0)$. The third term is a similar expression for the fragments of p_1 with the integration being done in the system where $p_1=M(1,0,0,0)$. The second term in (4.8) is an integration over pionization products including some very fast pions q_3 of order ϵp, which, indeed, one may not wish to call pionization. One easily obtains

$$\bar{n}_\pi(s)\underset{s\to\infty}{\longrightarrow}\frac{\ln s}{\sigma(s)}\int d^2q\,f(\mathbf{q}^2)\,,\qquad (4.9)$$

with the logarithmic term coming from the second term on the right-hand side of (4.8). It should be emphasized, however, that whether one wishes to associate this logarithmic increase with fast pionization products or fast fragments of the target and projectile is completely a matter of taste.

Another result follows rather trivially from what we have said. That is, the multiplicity of π^+, π^-, and π^0 should all become the same at large s. If the Pomeranchuk pole dominates, the coefficient of the $\ln s$ term should be the same for π^+, π^-, and π^0. But, even if the Pomeranchuk pole does not dominate, the multiplicities should become the same because of $I=0$ exchange dominance.

V. CONNECTION WITH SINGULARITIES IN TWO-BODY REACTIONS

In this section we attempt to relate the singularity structure in Λ_2 and Λ_1 of the $A_m{}^{\Lambda_2\Lambda_1}$ appearing in (3.3) and the singularities in Λ of the $A_m{}^{\Lambda}$ appearing in (3.11) to the complex angular momentum singularities which occur in two-body reactions. First, we shall argue that singularities which appear in two-body processes will also appear in $A_m{}^{\Lambda_2\Lambda_1}$ and $A_m{}^{\Lambda}$; then it will be claimed that these are the only singularities present. For the discussion in this section only scalar particles are considered.

Suppose now that we are given a forward-scattering off-shell two-body amplitude

$$f(p\cdot k,k^2)=-i\int d^4x\,e^{ikx}(p_2|\,T(j(x)j(0))\,|\,p_2)$$

and its absorptive part

$$F(p\cdot k,k^2)=\int d^4x\,e^{ikx}(p_2|\,j(x)j(0)\,|\,p_2)\,.\qquad (5.1)$$

We can write

$$A(p_2,q,p_1)=\int d^4k_1d^4k_2F(p_2\cdot k_2,k_2^2)B(k_2,q,k_1)$$
$$\times F(p_1\cdot k_1,k_1^2)+\tilde{A}(p_2,q,p_1)\,.\qquad (5.2)$$

The integral in (5.2) includes only the region $\bar{k}_i^2=k_{i0}^2-k_{i1}^2-k_{i2}^2<0$. We include other regions of integration in $\tilde{A}$. What we shall attempt to argue is that the $A_m{}^{\Lambda_2\Lambda_1}$ defined by (3.4) picks up the Λ singularities in F. To see this write

$$F(p\cdot k,k^2)=\sum_{\rho=1}^{2}\int d\Lambda\,D_{0,\rho0}{}^{\Lambda}(p,k)F_\rho{}^{\Lambda}(|\bar{k}|,k_3)\,,\qquad (5.3)$$

where the $D_{0,\rho0}{}^{\Lambda}$ are $O(2,1)$ representation functions in a mixed basis whose explicit properties will not be needed for this discussion. Substituting (5.3) into (5.2), one obtains

$$A_m{}^{\Lambda_2\Lambda_1}\alpha\sum_{\rho_1\rho_2}\int d^4k_1d^4k_2D_{m,\rho_20}{}^{\Lambda_2}(k_2)B(k_2,q,k_1)D_{\rho_10,0}{}^{\Lambda_1}(k_1)$$
$$\times F_{\rho_2}{}^{\Lambda_2}(|\bar{k}_2|,k_{23})F_{-\rho_1}{}^{-\Lambda_1-1}(|\bar{k}_1|,k_{13})+\tilde{A}_m{}^{\Lambda_2\Lambda_1}\,.\qquad (5.4)$$

Now $F_\rho{}^{\Lambda}$ appears explicitly in (5.4) and the singularities in Λ_2 (Λ_1) of $F_{\rho_2}{}^{\Lambda_2}$ ($F_{-\rho_1}{}^{-\Lambda_1-1}$) will appear in $A_m{}^{\Lambda_2\Lambda_1}$ unless there is a delicate cancellation in the integral appearing in (5.4). We can find no reason for such a cancellation to occur and so assume that singularities in Λ of $F_p{}^\Lambda$ also appear in $A_m{}^{\Lambda_2\Lambda_1}$. A completely analogous argument shows that the singularities of $F_p{}^\Lambda$ also appear in $A_m{}^\Lambda$.

Now suppose that $A_m{}^{\Lambda_2\Lambda_1}$ has a particular singularity in Λ_2 at a value α. Then $A_m{}^\Lambda$ will also have this singularity at $\Lambda=\alpha$ since

$$A_m{}^\Lambda=\int d\Lambda_1 A_m{}^{\Lambda\Lambda_1}e^{im\varphi}d_{m0}{}^\Lambda(\zeta_1)\,.$$

Now if $A_m{}^\Lambda$ has a singularity in Λ at α,

$$A(\xi,\psi,\eta)\underset{\eta\to\infty}{\longrightarrow}(\cosh\eta)^\alpha\beta(\psi,\xi) \tag{5.5}$$

from (3.11). In the limit $\eta\to\infty$ in (5.5) s_2 remains finite [see (3.9)]. Suppose for the moment that $q_0<0$. Then the difference between

$$\int d^4x\, e^{-iqx}(p_1p_2(+)|\,j(x)j(0)\,|p_1p_2(+)) \tag{5.6}$$

and the imaginary part of

$$\int d^4x\, e^{-iqx}(p_1p_2(-)|\,T(j(x)j(0))\,|p_1p_2(+)) \tag{5.7}$$

would be the semidisconnected parts of the three-body unitarity relation.[8] For small s_2 there should be no delicate balance between the completely connected and the semidisconnected parts of the unitarity relation for large $\cosh\eta$. Thus, if (5.6) has a $(\cosh\eta)^\alpha$ behavior, so will (5.7). However, if the completely connected part (5.5) has a $(\cosh\eta)^\alpha$ behavior for $q_0>0$, it will also have such a behavior for $q_0<0$. Thus the singularities in the angular momentum plane of (5.6) will be the same as those of (5.7) and hence the same as in two-body reactions.

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APPENDIX

To obtain Eq. (4.8), and in particular to obtain the limits on the integrals appearing in that equation, we begin with (4.7), which for large s becomes

$$\sigma(s)\bar{n}_\pi(s)=\int d^2q\,dq_3\frac{d\sigma}{d^3q}(p_2,q,p_1)$$

$$=\int d^2q\int_{-p}^{-\epsilon p}dq_3\frac{d\sigma}{d^3q}+\int d^2q\int_{-\epsilon p}^{\epsilon p}dq_3\frac{d\sigma}{d^3q}+\int d^2q\int_{\epsilon p}^{p}dq_3\frac{d\sigma}{d^3q} \tag{A1}$$

written in the center-of-mass system,

$$p_2\approx(p+M^2/2p,\,0,\,0,\,-p)\,,$$
$$p_1=(p+M^2/2p,\,0,\,0,\,p)\,,$$

and where ϵ is a fixed number not necessarily small. In order to evaluate the first term on the right-hand side of (A1), we transform to the coordinate system where $p_2=(M,0,0,0)$. If q_μ' labels the components of q in this system, then

$$q_3'=q_3\frac{(p^2+M^2)^{1/2}}{M}+(q_3^2+\mathbf{q}^2+\mu^2)^{1/2}\frac{p}{M}$$

or

$$q_3'\approx\frac{q_3M}{2p}+\frac{p}{M}\frac{\mu^2+\mathbf{q}^2}{2|q_3|}\,, \tag{A2}$$

when q_3 is large and negative. Using (A2) and the fact that $\omega_q d\sigma/d^3q$ is an invariant, the first term on the right-hand side of (A1) becomes

$$\int d^2q\int_{(\mu^2+\mathbf{q}^2)/2M-M/2}^{(\mu^2+\mathbf{q}^2)/2\epsilon M-M\epsilon/2}\frac{dq_3}{(\mathbf{q}^2+q_3^2+\mu^2)^{1/2}}\omega_q\frac{d\sigma}{d^3q}\,,$$

which is equal to the first term on the right-hand side of (4.8) by use of (4.5). The second term of (A1) becomes equal to the second term of (4.8) upon use of (4.2). To evaluate the third term of (A1), one transforms to the system where $p_1=(M,0,0,0)$. Now calling q_μ' the components of q in this system, one obtains

$$q_3'=q_3\frac{(p^2+M^2)^{1/2}}{M}-(\mathbf{q}^2+q_3^2+\mu^2)^{1/2}\frac{p}{M}$$

or

$$q_3'\approx\frac{q_3M}{2p}-\frac{\mu^2+\mathbf{q}^2}{2q_3}\frac{p}{M} \tag{A3}$$

for positive q_3. Using (A3) and (4.5), one immediately derives that the third term on the right-hand side of (A1) is equal to the third term on the right-hand side of (4.8).

[8] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge U. P., Cambridge, England, 1966).