We have been treating in this paper field-theoretical models of the bilinear type:

$$H=H_0+\int V(x)A^\dagger(x)A(x)d^3x. \qquad (50)$$

Realistic interactions are much more complicated trilinear, quadrilinear, or higher couplings. We see no reasons why the phenomenon of indefinite metric and associated eigenstates can not occur if the realistic coupling is sufficiently strong. For the same reasons as one might have overlooked this possibility in the bilinear case, one could be actually ignoring it in the realistic case.

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Finite-Dimensional Spectrum-Generating Algebras

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It is suggested that the generators of a spectrum-generating algebra are all constants of the motion, some of them having an explicit time dependence. Also suggested is a specific form of the Hamiltonian action on the spectrum-generating algebra for systems with a finite number of degrees of freedom. Well-known examples of spectrum-generating algebras are shown to fit into this framework. The stability of the suggested structure against small perturbation is discussed. The question of the generalization of the suggested structure to systems with an infinite number of degrees of freedom is briefly commented upon.

I. INTRODUCTION

SEVERAL years ago the concept of a spectrum-generating algebra (SGA) was introduced[1] as a means of algebraic description of physical systems. This was motivated by the observation that in certain problems[2] series of energy eigenstates with different energies form a basis for a single unitary irreducible representation of a Lie algebra. Thus in a mathematical sense the SGA can be thought of as a generalization of the symmetry algebra (SA). While the SA is represented irreducibly on states which are energy degenerate, a SGA may have as a basis for a single unitary irreducible representation all the energy eigenstates of a system. In fact the SGA was required to have as a subalgebra the SA of the problem.

Of these two algebras the symmetry algebra has an intuitively clear physical definition. Its generators are Hermitian operators which do not have an explicit time dependence and satisfy the following conditions.

(a) They commute with the Hamiltonian of the problem. Since they do not have an explicit time dependence they are constants of the motion.

(b) They form a Lie algebra under commutation. Namely, the commutator of two generators is a linear combination of the generators of the algebra with coefficients which are numbers.

(c) The symmetry algebra is maximal in the sense that for any energy eigenvalue the space of all degenerate states is irreducible under the algebra. This means that we do not allow "accidental" degeneracies. (Since we discuss the symmetry algebra and not the symmetry group, we have to exclude from the discussion degeneracies explainable only by discrete symmetries. However, it is easy to generalize the conditions to symmetry groups instead of symmetry algebras).

(d) The symmetry algebra is minimal in the sense that it does not have a proper subalgebra with the same properties.

On the other hand, the definition of the SGA is more mathematical. One searches for a Lie algebra of Hermitian generators which has the symmetry algebra as a subalgebra such that all the energy eigenfunctions of the physical problem which satisfy the same boundary conditions form a basis for a single unitary irreducible representation of the algebra. This may be considered a generalization of conditions (b) and (c) above. Condition (d) has an obvious generalization, but condition (a) is not generalized. Stated differently, one poses a problem of embedding all the spaces of states which are irreducible under the symmetry algebra in a space

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[1] Y. Dothan, M. Gell-Mann, and Y. Ne'eman, Phys. Rev. Letters **17**, 145 (1965); Y. Dothan and Y. Ne'eman, in *Proceedings of the Second Topical Conference on Resonant Particles*, edited by B. A. Munir (Ohio U. P., Athens, Ohio, 1965), p. 17. The same concept is also known as a noninvariance group or a dynamical group. See, e.g., N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965); A. O. Barut and A. Böhm, Phys. Rev. **139**, B1107 (1965).

[2] It is interesting that in most of the classical analogs of these problems the motion is completely degenerate in the classical sense. Namely, the motion is simply periodic instead of being multiply periodic. See, e.g., H. Goldstein, *Classical Mechanics* (Addison Wesley, New York, 1959), p. 297.

which is irreducible under a larger Lie algebra, namely, the SGA. This embedding should be such that when the representation of the SGA is considered as a representation of its symmetry subalgebra, it gives back only the embedded spaces with the correct multiplicities. Stated this way, it is not clear that a finite-dimensional SGA exists at all. More intriguing is the physical meaning of its generators, both in quantum mechanics[3] and in classical mechanics.[4]

The nature of the SGA may be abstracted as a generalization of some remarks made by Malkin and Man'ko[5] and by Lipkin.[6] Malkin and Man'ko remarked that if $\psi(q;t)$ is any solution of the time-dependent Schrödinger equation

$$\left[i\frac{\partial}{\partial t}-H(p,q)\right]\psi(q;t)=0, \tag{1}$$

then $G(p,q;t)\psi(q;t)$ is also a solution of the same equation if G satisfies the condition

$$i\frac{\partial G}{\partial t}-[H,G]=0. \tag{2}$$

In the above, q stands for the set of configuration-space coordinates and p for the set of their canonically conjugate momenta.

Condition (2) is, of course, a quantum transcription of the classical equation

$$\frac{\partial G}{\partial t}+[H,G]_{\text{P.B.}}=0, \tag{3a}$$

where P.B. indicates Poisson brackets, or

$$\frac{dG}{dt}=0. \tag{3b}$$

In other words, G is again a constant of the motion, but it is now allowed to have an explicit time dependence. It is easy to see that if G is a Hermitian operator satisfying condition (2) and if $\psi(q;t)$ is a normalized solution of Eq. (1), then $e^{iaG}\psi$ is also a normalized solution of Eq. (1) for an arbitrary real a which is independent of p, q, and t. Thus the generator G can be exponentiated to form a one-parameter group of transformations that take the set of normalized wave packet solutions of Eq. (1) into itself.

Lipkin noted that if G satisfies Eq. (2) and has a nontrivial time dependence, namely, $\partial G/\partial t\neq 0$, then $G\psi$ is a linear combination of eigenstates of H with different energies, and thereby G generates the spectrum of H. Lipkin shows explicitly how this happens in the case that G is linear in the time, namely, a boost.

In this paper we propose to adopt Eq. (2) or its classical analog Eq. (3) as a generalization of condition (a) when defining the properties of the generators of a SGA. Our generators are, therefore, always constants of the motion.

It is obvious that in order to generate a spectrum we need operators that do not commute with H. On the other hand, the totality of all operators that do not commute with H seems to be too rich a set and with no obvious algebraic properties. Restricting ourselves to the subset of operators which fulfill Eq. (2) will give us some control over the operators under consideration.

For systems with a finite number of degrees of freedom the states are labeled by a finite number of quantum numbers. Therefore, a finite-dimensional SGA should characterize a system completely.[7] As we shall see, this demand of a finite-dimensional SGA, together with Eq. (2), completely determines the type of explicit time dependence that the generators may have (at least in the classical case).

In Sec. II we propose a simple definition of a SGA and the Hamiltonian action on its generators. In Sec. III we analyze some mathematical properties of the proposed structure. We also show that the Hamiltonian action defined in Sec. II is too restrictive. In Sec. IV we propose a generalization of the Hamiltonian action. These three sections contain most of the new ideas and results of this paper. In Sec. V we discuss some well-known examples of SGA from our point of view. In Sec. VI we discuss some general problems connected with our suggested structure. The most important of these problems is the question of the stability of the structure against small perturbation. We also comment briefly on generalizing our scheme to systems with an infinite number of degrees of freedom.

II. FINITE-DIMENSIONAL SGA

Let us consider the set S of all Hermitian generators $G_a(p,q;t)$ which fulfill Eq. (2),

$$S=\left\{G_a\colon i\frac{\partial}{\partial t}G_a-[H,G_a]=0\right\} \tag{4}$$

[or their classical analogs that fulfill Eq. (3)]. In virtue of the Jacobi identity the set S has a convenient algebraic property, namely, the commutator of two members of the set calculated at equal t is again a member of the set. (In classical mechanics this is known as Poisson's theorem.[8]) In other words, S is a Lie algebra.

[3] Y. Ne'eman, Quanta and Fields 1, 55 (1970).

[4] I am indebted to R. Hermann for repeatedly calling this point to my attention.

[5] I. A. Malkin and V. I. Man'ko, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu 2, 230 (1965) [JETP Letters 2, 146 (1965)].

[6] H. J. Lipkin, in *Symmetry Principles at High Energy, Fifth Coral Gables Conference*, edited by A. Perlmutter *et al.* (Benjamin, New York, 1968), p. 266; H. J. Lipkin, in *Nuclear Physics*, edited by C. DeWitt and V. Gillet (Gordon and Breach, New York, 1969), p. 644.

[7] R. Hermann, *Lie Groups for Physicists* (Benjamin, New York, 1966), p. 98.

[8] L. D. Landau and E. M. Lipshitz, *Mechanics* (Addison Wesley, Reading, Mass., 1960), p. 137.

Since we shall treat only closed systems described in an inertial reference frame, the Hamiltonians of these systems do not depend explicitly on time and so $\partial H/\partial t=0$. Therefore, by taking a partial time derivative of Eq. (2), we see that together with any G_a the set S also contains $\partial G_a/\partial t$.

The set S possesses as a subset the set D of all Hermitian operators $L_b(p,q)$ which fulfill both Eq. (2) and $\partial L_b/\partial t\equiv 0$. They are time-independent constants of the motion,

$$D=\left\{L_b:\frac{\partial}{\partial t}L_b=0,\quad [H,L_b]=0\right\}. \tag{5}$$

Again the set D itself is closed under commutation and so forms a Lie subalgebra of the Lie algebra S. The Lie algebra D is, of course, an infinite-dimensional Lie algebra. We expect conditions (c) and (d) of the Introduction to be sufficient to ensure the existence of a finite subalgebra $D_{\rm fin}$ which is the symmetry algebra of the problem. The reason for this expectation is once again that for a system with a finite number of degrees of freedom the states are uniquely specified by a finite number of quantum numbers. This is true in particular for the energy-degenerate states, and so a finite-dimensional symmetry algebra can supply the necessary quantum numbers to distinguish energy-degenerate states.

It may be worthwhile to remark that a similar situation occurs in classical mechanics. The set D of all "reasonable" (separating[9]) real functions $L_b(p,q)$ which fulfill

$$\frac{\partial}{\partial t}L_b=0,\quad [H,L_b]_{\rm P.B.}=0 \tag{6}$$

forms an infinite-dimensional Lie algebra. However, it is possible[10] to choose a finite-dimensional Lie subalgebra $D_{\rm fin}$ such that all the generators of D are functions of the generators of $D_{\rm fin}$.

Let us note here that the choice of $D_{\rm fin}$ is not necessarily unique. For instance, in the case of a single free particle we can choose either the algebra generated by $\{\mathbf{p},\mathbf{J}\}$, namely, $E(3)$ composed of $\mathbf{p}$, the linear momentum, and $\mathbf{J}$, the angular momentum, or we may choose the algebra $\{(1/2|\mathbf{p}|)(\mathbf{J}\times\mathbf{p}-\mathbf{p}\times\mathbf{J}),\mathbf{J}\}$, namely, $SL(2,C)$.

From a mathematical point of view, the difference between these two algebras is that the first is not semisimple while the second is. This raises the question whether a demand of semisimplicity makes the choice of $D_{\rm fin}$ unique in the degenerate cases in which it is not. In this connection it is appropriate to mention that Simoni, Zaccaria, and Vitale[11] have shown in the classical case that if the symmetry algebra is semisimple, then all its Casimir operators are functionally dependent. Therefor, if we consider Hamiltonians that are functions of the Casimir operators of symmetry algebra, then for a semisimple symmetry algebra such a Hamiltonian will depend functionally on a single function of the generators. This is reminiscent of the case of a multiply periodic motion degenerating completely to a simply periodic motion. The Hamiltonian then is a function of a single integral linear combination of action variables.[2]

We shall refer to a situation where there is only one functionally independent Casimir operator of the symmetry algebra as the completely symmetric situation.

Another remark of a mathematical nature may be in order here. We know of no reason why the symmetry algebra should in general be compact. Nevertheless, from experience, we expect to find that discrete (bound) states have a finite degeneracy, and therefore a compact symmetry algebra will do for their description.

Let us turn our attention now back to the algebra S. In principle it is simple to get an idea about its structure as follows. Consider the (formal) expressions

$$p_0=e^{iHt}pe^{-iHt}, \tag{7}$$
$$q_0=e^{iHt}qe^{-iHt}.$$

The set of operators $\{p_0,q_0\}$ satisfies Eq. (2) and for a system with f degrees of freedom they constitute a complete set of $2f$ operators so that every member of S is a function $F(p_0,q_0)$.

Again we want to construct a finite-dimensional Lie algebra out of the infinite-dimensional Lie algebra of S. This seems at first sight to have already been accomplished since $\{p_0,q_0\}$ fulfill the Heisenberg commutation relations, namely,

$$[p_{0i},q_{0j}]=-i\delta_{ij}. \tag{8}$$

But this algebraic structure is useless since it does not contain any dynamical information beyond the number of degrees of freedom of our system. Stated differently, all systems with the same number of degrees of freedom and with widely different dynamics fulfill the same commutation rules and would, therefore, have led to the same algebraic structure. To bring in some information about the dynamics, let us construct all the expressions

$$(1/i)[H,p_0],\quad (1/i)[H,q_0]. \tag{9}$$

These will again be Hermitian operators and by Eq. (2) they will be equal to

$$\partial p_0/\partial t,\quad \partial q_0/\partial t. \tag{10}$$

As we mentioned, they are again members of S and therefore functionally dependent on p_0q_0, but in general are not linearly dependent on p_0q_0. Therefore, if we want to construct a Lie algebra, expressions (10) will in general yield a number of additional generators. Having chosen a linearly independent set of additional generators, we now try to close our set of generators under

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[9] A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton U. P., Princeton, N. J., 1947), p. 36.

[10] I am indebted to R. Hermann for an explanation of this point.

[11] A. Simoni, F. Zaccaria, and B. Vitale, Nuovo Cimento **51A** 448 (1967).

commutation. This will, in general, make it necessary to add even more generators to our set. In this way by calculating successively higher time derivatives and commutators, we will in the general case generate an infinite-dimensional Lie algebra. This brings us back to where we started from.

To get out of this impasse, we may get a hint from the relation between D and D_{fin}. The two sets have in common the property that a commutator of two elements in the set is again in the set. But in D_{fin} this is realized in a finite linear fashion.

We now want to construct a subset of S called S_{fin} which will share the following two properties with S. (1) A commutator of two elements in the set is again in the set. (2) The time derivative of an element in the set is again in the set. But in S_{fin} this is to be realized in a finite linear fashion. The set S_{fin} will then give rise to a finite-dimensional SGA.

Consider then a finite set of Hermitian operators $G_i(p,q;t)$ having the following properties:

$$i\frac{\partial}{\partial t}G_i-[H,G_i]=0\,, \tag{11}$$

$$[G_i(p,q;t),G_j(p,q;t)]=ic_{ij}{}^kG_k(p,q;t)\,, \tag{12}$$

$$\frac{\partial}{\partial t}G_j(p,q;t)=\omega_j{}^kG_k(p,q;t)\,. \tag{13}$$

Equation (12) specifies a finite-dimensional SGA. The structure constants $c_{ij}{}^k$ are real numbers. One of the spaces on which the SGA (12) is represented irreducibly will serve as a space of states of our system. Since we did not demand that Eqs. (12) define a semisimple Lie algebra, we distinguish upper and lower indices[12] of $c_{ij}{}^k$ and $\omega_j{}^k$.

The indicated dependence of the G_i on the canonical variables p, q actually stands for their dependence on any set of dynamical variable which are not necessarily canonical. Thus we allow the G_i to involve Pauli matrices, Dirac matrices, etc.

Equation (13) contains the dynamical information about our system. The real quantities $\omega_j{}^k$ are time independent. For the moment we assume that they commute with all the generators of the SGA. Thus they may be functions of the masses and coupling constants of the problem or involve degrees of freedom that commute with all the generators of the SGA.

From Eq. (12) it follows that we can choose the G_i to be dimensionless quantities and then the $\omega_j{}^k$ have the dimensions of $(\mathrm{time})^{-1}$ or frequency. We shall refer to the matrix ω with elements $\omega_j{}^k$ as the frequency matrix. Using Eq. (11), we can rewrite Eq. (13) in the form

$$[H,G_j]=i\omega_j{}^kG_k\,. \tag{14}$$

We see that the Hamiltonian maps, by commutation, the algebra into itself.[13] Since the dynamics is specified by the $\omega_j{}^k$, we suspect that we may have different dynamical systems which have the same set of Eqs. (12) but different sets of Eqs. (14). We analyze this question in the next section. Here instead we emphasize the different content of Eqs. (12) and (14).

Equations (12), by specifying a Lie algebra, specify in effect the set of all irreducible representations of this Lie algebra. Equations (14), on the other hand, contain information about the action of the Hamiltonian on the representation spaces.

This leads us to the following program for a phenomenological analysis of a given spectrum. By looking at the states in the spectrum, their quantum numbers, and their multiplicities, we try to guess a spectrum-generating algebra to generate the given spectrum. Having done so, we have a set of structure constants $c_{ij}{}^k$ that fulfill the Jacobi identity so that Eqs. (12) indeed define a Lie algebra. We now ask what are all possible Hamiltonian actions on the algebra according to Eq. (14) and which of these reproduces the observed spacings in the given energy spectrum best. This will then specify the $\omega_j{}^k$.

III. PROPERTIES OF SGA AND HAMILTONIAN MAPPING

We now address ourselves to the question raised in the last section, namely, given a spectrum-generating algebra what are all possible modes of Hamiltonian action according to Eqs. (14). In more technical terms, suppose we are given the set of $c_{ij}{}^k$ and we ask what are all possible sets of $\omega_j{}^k$.

To investigate this point, we ask about the compatibility of the various Eqs. (12) and (13). From the general theory of Lie algebras, we know that Eqs. (12) are mutually compatible and define a Lie algebra if the $c_{ij}{}^k$ fulfill the Jacobi identity:

$$c_{ij}{}^kc_{kl}{}^m+c_{jl}{}^kc_{ki}{}^m+c_{li}{}^kc_{kj}{}^m=0\,. \tag{15}$$

Similarly Eqs. (12) and (13) are mutually compatible if the following Bargmann-type[14] identity is satisfied:

$$c_{ij}{}^k\omega_k{}^m+c_{ki}{}^m\omega_j{}^k+c_{jk}{}^m\omega_i{}^k=0\,. \tag{16}$$

Two consequences of Eqs. (15) and (16) immediately follow.

(1) If two frequency matrices $\omega'_j{}^k$ and $\omega''_j{}^k$ fulfill Eq. (16), then so does

$$\omega_j{}^k=\alpha'\omega'_j{}^k+\alpha''\omega_j{}^k\,, \tag{17}$$

where α' and α'' are time independent, real, and commute with all the G_i. Therefore if the given SGA has n

[12] G. Racah, Group Theory and Spectroscopy, lectures delivered at the Institute for Advanced Study, Princeton, p. 13, 1951 (unpublished).

[13] Equations of the form of Eq. (14) are well known in other branches of physics. See, e.g., A. M. Lane, *Nuclear Theory* (Benjamin, New York, 1964), p. 94. I am indebted to T. A. Griffy for an interesting discussion of this remark.

[14] V. Bargmann, Ann. Math. 59, 1 (1954), Eq. (4.26).

generators, the possible frequency matrices form a real vector space of dimension smaller than or equal to n^2.

(2) The following expression satisfies the Bargmann identity (16):

$$\omega_k{}^m = \alpha^l c_{kl}{}^m , \quad (18)$$

where α^l are again time-independent, real, and commute with all the G_i. We shall refer to a Hamiltonian to which a frequency matrix of the type (18) corresponds as a linear Hamiltonian since in such a case H may be represented by

$$H = -\alpha^l G_l + C , \quad (19)$$

where C commutes with all the G_i. Thus the $\omega_k{}^m$ corresponding to a linear Hamiltonian form a real vector space of dimension smaller than or equal to n.

The generators of our SGA that commute with the Hamiltonian form a Lie subalgebra. We shall refer to it as the augmented symmetry algebra and demand that it contains the symmetry algebra defined in the Introduction.

By definition, a linear Hamiltonian is a linear Casimir operator of the augmented symmetry algebra. But a linear Casimir operator is possible only for an algebra which is not semisimple. We therefore conclude that a linear Hamiltonian possesses a nonsemisimple augmented symmetry algebra.

Let us now restrict ourselves to the case where the given set of $c_{ij}{}^k$ defines by Eq. (12) a semisimple Lie algebra. In this case the Killing-Cartan metric

$$g_{il} = c_{ij}{}^k c_{lk}{}^j \quad (20)$$

is nonsingular and can be used to raise and lower indices.[12] Using Eqs. (15) and (16), it follows that for a semisimple SGA,

$$\omega_{km} = -\omega_{mk} . \quad (21)$$

Also, Eq. (16) can be rewritten as follows:

$$c_{ij}{}^k \omega_{km} + c_{jm}{}^k \omega_{ki} + c_{mi}{}^k \omega_{kj} = 0 . \quad (22)$$

From Eq. (21) we would conclude that the dimension of the linear space of possible frequency matrices ω_{km} is less than or equal to $\frac{1}{2}n(n-1)$ for a semisimple SGA. But Eq. (22) is identical with Eq. (4.26) of Bargmann's paper[14] if we identify our ω_{km} with his β_{km}. Therefore his Eq. (7.5) holds, namely, the only possible ω_{km} are the ones given by our Eq. (18):

$$\omega_{km} = \alpha^l c_{klm} . \quad (23)$$

In other words, if the SGA is semisimple, the only possible Hamiltonians in our present framework are linear Hamiltonians.

This result by itself shows that our present framework is too restrictive. To see this, we recall that the bound states of the hydrogen atom are described with a SGA which is[1] $SO(4,1)$ or[5,15] $SO(4,2)$. Both of these algebras are semisimple, but the problem does not correspond to a linear Hamiltonian.

To understand the physical implications of a semisimple SGA in the present framework, we now show that it gives rise to an energy spectrum of a set of harmonic oscillators.[13]

To see this consider the following two $n\times n$ matrices: The Killing-Cartan matrix g, Eq. (20), with elements g_{il} and the frequency matrix ω with elements ω_{km}. The matrix g is real symmetric and nonsingular; the matrix ω is real antisymmetric. We now follow the standard procedure of decoupling a set of coupled harmonic oscillators. We first transform the Killing-Cartan matrix g, Eq. (26), by a complex congruence transformation T to the unit matrix

$$TgT^\dagger = I . \quad (24)$$

Equation (14), which for a semisimple SGA may be written

$$[H,G_k] = i\omega_{km} G^m , \quad (25)$$

is transformed by T to

$$[H,G_k'] = \Omega_{km} G_m' , \quad (26)$$

where the operators $\{G_k'\}$ are the transforms of the operators $\{G_k\}$ by the complex regular matrix T. The matrix Ω is the transform of the matrix $i\omega$. All the indices are now lower indices since the right-hand side of Eq. (24) is the unit matrix. The operators G_k' are not necessarily Hermitian, but the matrix Ω is a Hermitian matrix which we can now diagonalize by a similarity transformation leading to

$$[H,A_k] = \nu_k A_k , \quad (27)$$

where there is no summation on k. The ν_k are the eigenvalues of the Hermitian matrix Ω and are all real. If the operator A_k satisfies Eq. (27), the Hermiticity of H and reality of ν_k lead to a similar equation for $A_k{}^\dagger$,

$$[H,A_k{}^\dagger] = -\nu_k A_k{}^\dagger . \quad (28)$$

Consider first the operators A_k for which $\nu_k = 0$. We form the Hermitian combinations

$$A_k + A_k{}^\dagger ; \quad (1/i)(A_k - A_k{}^\dagger) . \quad (29)$$

Some of these combinations may vanish. The nonvanishing combinations all commute with the Hamiltonian and thus generate the augmented symmetry algebra.

Now consider the operators A_k for which $\nu_k \neq 0$. Taking now matrix elements of Eq. (27) between eigenstates of H with energies E_1 and E_2, we get

$$(E_1 - E_2)\langle 1 | A_k | 2 \rangle = \nu_k \langle 1 | A_k | 2 \rangle . \quad (30)$$

Therefore, A_k can connect the states $|1\rangle$ and $|2\rangle$ only if

$$E_1 - E_2 = \nu_k . \quad (31)$$

Since ν_k is independent of the states $|1\rangle$ and $|2\rangle$, we see that the operators A_k and $A_k{}^\dagger$ act on the states as

[15] For a detailed list of references covering both possibilities see G. Györgyi, Nuovo Cimento **53A**, 717 (1968).

creation and annihilation operators of a harmonic oscillator and the spectrum obeys the characteristic equal-spacing rule.

The reader may notice that in the decoupling procedure the matrix g played the role of a kinetic-energy matrix while the matrix $i\omega$ played the role of a potential-energy matrix.[16] We may say in this sense that the matrix g carries information of a kinematical nature while the matrix $i\omega$ carries information of a dynamical nature.

In view of the result we have just proven, we are forced to relax our constraint on our Eqs. (13) or (14) since once again our present framework is at odds with the description of the bound states of hydrogen by means of $SO(4,1)$ or $SO(4,2)$. Both are semisimple, but the spectrum does not show equal spacing of the levels.

Actually the crucial point in our proof is Eq. (26). Therefore any Hamiltonian action on a SGA which can be brought to the form (26) with Ω_{km} a Hermitian matrix whose elements commute with all the generators of the SGA gives rise to a spectrum of a set of harmonic oscillators.[13]

IV. TIME DEPENDENCE OF GENERATORS AND HAMILTONIAN ACTION

Before enlarging our framework, let us still consider Eq. (14) or its equivalent, Eq. (13), in more detail. Equation (13) is a set of first-order linear differential equations with constant coefficients which determines the explicit time dependence of the G_i. Indeed the solution of this set has the form[17]

$$G_i(p,q;t)=(e^{\omega t})_i{}^j G_j(p,q;t=0)\,. \tag{32}$$

The exponential of the matrix ωt can be written as

$$e^{\omega t}=\sum_{k=1}^{s} e^{\nu_k t}\Big(\sum_{l=1}^{r_k} t^{l-1}Z_{kl}\Big)\,. \tag{33}$$

Here Z_{kl} are matrices which are polynomials in the matrix ω. The ν_k are the different eigenvalues of the matrix ω, and r_k are their multiplicities in the minimal polynomial of the matrix ω.

An alternative way of writing Eqs. (32) which does not make an explicit reference to the initial values is as follows:

$$G_i(p,q;t)=\sum_{k=1}^{l+m} e^{\nu_k t}\left[\sum_{\alpha=1}^{n_k} C_{k,\alpha}(p,q)(V_{k,\alpha})_i\frac{t^{n_k-\alpha}}{(n_k-\alpha)!}\right]. \tag{34}$$

Here the ν_k are again eigenvalues of the matrix ω, but they are now not necessarily distinct. The vectors $V_{k,\alpha}$, whose ith component is $(v_{k,\alpha})_i$, are generalized eigenvectors of the matrix ω and form a basis in which ω has the Jordan normal form. The numbers n_k are the dimensionalities of the blocks in the Jordan normal form and $n_1=\cdots=n_l=1$.

From Eqs. (33) or (34) it follows that the explicit time dependence of the generators is rather limited, namely, a polynomial in t times an exponential function in t. In particular, for a semisimple SGA we have a purely exponential time dependence. We shall refer to a constant of the motion with an explicit time dependence of a polynomial in t multiplied by an exponential function of t as having the standard time dependence. This class of functions has two obvious properties. (1) They have no singularity in the finite part of the complex t plane. They are thus a subclass of the class of entire functions. (2) The t derivative of a function in the class is again in the class. Since we want to enlarge our framework we could have tried to generate more general types of explicit time dependence. However, we shall follow a different line of reasoning.

To get a hint how we may relax our scheme, we recall that in classical systems for which the motion is multiply periodic, the frequencies are given by[18]

$$\nu_k=\frac{\partial H}{\partial J_k}, \tag{35}$$

where J_k are the action variables and the Hamiltonian is given as a function of these action variables. We recall that the action variables are single-valued real functions of p, q which have no explicit time dependence. They have vanishing Poisson brackets between themselves and with the Hamiltonian. Therefore, the frequencies ν being functions of the J's have vanishing Poisson brackets with the Hamiltonian and do not have an explicit dependence on time. They also have vanishing Poisson brackets with all the J_k's. In the present formulation the role of the J_k is played by the set of Casimir operators of the augmented symmetry algebra, supplemented by generators of the augmented symmetry algebra itself, making up a complete set of commuting operators.

Motivated by this remark, we now allow the $\omega_i{}^j$ to be Hermitian functions of the generators L_k that commute with the Hamiltonian, restricting them by the demand that they commute with all the L_k. Thus the $\omega_i{}^j$ are now time-independent functions of the masses and coupling constants of the problem, the Casimir operators of the algebra generated by the L_k and the degrees of freedom that commute with all the G_i. Since the $\omega_i{}^j$ are still not allowed to have an explicit time dependence, the solution to Eq. (13) as an equation between classical quantities is still Eq. (33). Indeed, as we shall see explicitly in the next section, the classical description of the Kepler problem fits into the present framework.

On the other hand, as equations between operators Eqs. (13) are not necessarily valid since if the G_i is Hermitian so is $\partial G_i/\partial t$ but $\omega_i{}^j(L_k)G_j$ is not necessarily

[16] H. Goldstein, Ref. 2, p. 326.

[17] See, e.g., F. R. Gantmacher, *Applications of the Theory of Matrices* (Interscience, New York, 1959), p. 137.

[18] H. Goldstein, Ref. 2, p. 292.

Hermitian. We now replace the right-hand side of Eqs. (13) by its symmetrized form although we do not have any compelling argument in favor of this ansatz.

We now write the set of generators of the SGA as

$$\{G_i\}=\{L_j;K_\alpha\},\quad i=1,\ldots,n,\quad j=1,\ldots,p;\quad \alpha=p+1,\ldots,n. \tag{36}$$

Since the $\{L_j\}$ form a subalgebra, we now have the following detailed form of Eqs. (12) and (14):

$$[L_i,L_j]=ic_{ij}{}^kL_k,\quad i,\,j,\,k=1,\ldots,p \tag{37a}$$

$$[L_i,K_\alpha]=ic_{i\alpha}{}^kL_k+ic_{i\alpha}{}^\beta K_\beta,\quad i,\,k=1,\ldots,p;\,\alpha,\,\beta=p+1,\ldots,n \tag{37b}$$

$$[K_\alpha,K_\beta]=ic_{\alpha\beta}{}^kL_k+ic_{\alpha\beta}{}^\gamma K_\gamma,\quad k=1,\ldots,p;\,\alpha,\,\beta,\,\gamma=p+1,\ldots,n \tag{37c}$$

$$[H,L_i]=0, \tag{38a}$$

$$[H,K_\alpha]=i\omega_\alpha{}^k(L_i)L_k+i\tfrac{1}{2}\{\omega_\alpha{}^\beta(L_i)K_\beta+K_\beta\omega_\alpha{}^\beta(L_i)\}, \tag{38b}$$

$$[\omega_\alpha{}^k(L_i),L_j]=0,\quad [\omega_\alpha{}^\beta(L_i),L_j]=0. \tag{38c}$$

Calculating the commutator of H with Eqs. (37), we get compatibility conditions on the ω. Unfortunately these compatibility conditions are not very transparent, and we have been unable to draw conclusions on their basis.

Equations (38b) allow us now to derive a result analogous to Eq. (30). To this end we choose a representation in which the Hamiltonian and all the Casimir operators of the augmented symmetry algebra are simultaneously diagonal. The basis states of this representation are then eigenstates of all the $\omega_\alpha{}^k$ and $\omega_\alpha{}^\beta$. Also the operators L_k have matrix elements only between energy-degenerate states. Therefore, taking matrix elements of Eqs. (38b) between states $|1\rangle$ and $|2\rangle$ with energies E_1 and E_2 which are different, we get

$$(E_1-E_2)\langle 1|K_\alpha|2\rangle=\tfrac{1}{2}i[\omega_\alpha{}^\beta(1)+\omega_\alpha{}^\beta(2)]\langle 1|K_\beta|2\rangle. \tag{39}$$

Therefore, the generator K_α may connect the states $|1\rangle$ and $|2\rangle$ only when E_1-E_2 is a nonzero eigenvalue of the matrix $\frac{1}{2}i[\omega_\alpha{}^\beta(1)+\omega_\alpha{}^\beta(2)]$. Thus, if we know the $\omega_\alpha{}^\beta$, we can calculate the energy spacings. In a phenomenological analysis we have to find the $\omega_\alpha{}^\beta$ from the knowledge of the energy spacings.

V. EXAMPLES

In this section we discuss a few examples in order to illustrate explicitly the points made in the preceding general discussion.

A. Free Spinless Particle

Here the Galilei group serves as the spectrum-generating algebra.[6,19] To see this, consider first the generators of space translations $\mathbf{P}$ and velocity transformations $\mathbf{K}$. In terms of the Cartesian canonical momenta and coordinates, they are given as follows:

$$\mathbf{P}=\mathbf{p},\quad \mathbf{K}=m\mathbf{q}-\mathbf{p}t. \tag{40}$$

We see that in terms of the operators $\mathbf{p}_0$ and $\mathbf{q}_0$ they are just

$$\mathbf{P}=\mathbf{p}_0,\quad \mathbf{K}=m\mathbf{q}_0, \tag{41}$$

and have essentially the canonical commutation relations

$$[P_i,P_j]=0,\quad [K_i,K_j]=0,\quad [P_i,K_j]=-im\delta_{ij}I, \tag{42}$$

where I is the identity operator.

These operators already have the standard explicit time dependence. However, since t appears in $\mathbf{K}$ linearly, the operator $\mathbf{K}\times\mathbf{P}$ is independent of t and will furnish additional generators for the symmetry algebra. These are, up to a factor m, the generators of rotations $\mathbf{J}$, where

$$\mathbf{J}=\mathbf{q}\times\mathbf{p}. \tag{43}$$

They have the following commutators among themselves and with the operators $\mathbf{P}$ and $\mathbf{K}$:

$$[J_i,J_j]=i\epsilon_{ijk}J_k,\quad [J_i,P_j]=i\epsilon_{ijk}P_k,\quad [J_i,K_j]=i\epsilon_{ijk}K_k. \tag{44}$$

The operators $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, and the identity now close on an algebra. This algebra is the SGA. The Hamiltonian H maps this algebra into itself by commutation,

$$[H,P_i]=0,\quad [H,J_i]=0,\quad [H,K_i]=-iP_i. \tag{45}$$

(We may of course consider H as a generator of the SGA.) In particular, the generators $\{\mathbf{P},\mathbf{J}\}$ form the symmetry subalgebra. The Casimir operators of this SA are not functionally independent since for a single spinless particle $\mathbf{J}\cdot\mathbf{P}=0$.

It is interesting to note that the dilatation operator defined as

$$D=(1/2m)(\mathbf{P}\cdot\mathbf{K}+\mathbf{K}\cdot\mathbf{P})=\tfrac{1}{2}(\mathbf{p}\cdot\mathbf{q}+\mathbf{q}\cdot\mathbf{p})-(\mathbf{p}^2/m)t \tag{46}$$

is a constant of the motion with an explicit time dependence. It is instructive to rewrite D in the following more suggestive form:

$$D=\tfrac{1}{2}(\mathbf{p}\cdot\mathbf{q}+\mathbf{q}\cdot\mathbf{p})-2Ht, \tag{47}$$

where we have substituted for the free single-particle Hamiltonian

$$H=\mathbf{p}^2/2m. \tag{48}$$

The dilatation operator D has the following commutation relations with the generators of the SGA:

$$[D,P_i]=iP_i,\quad [D,J_i]=0,\quad [D,K_i]=-iK_i, \tag{49}$$

and the commutator of D and H is given by

$$[H,D]=-i(\mathbf{P}^2/m)I. \tag{50}$$

Since $\mathbf{P}^2/m$ commutes with all the generators of the

[19] For a construction of the representations of the Galilei group, see Jean-Marc Levy-Leblond, J. Math. Phys. 4, 776 (1963).

symmetry algebra it can be thought of as $\omega_D{}^I$ in the sense of Eq. (38b). Therefore, we could consider as SGA the algebra generated by **P**, **J**, **K**, D, and the identity operator. The Hamiltonian does map this algebra into itself by commutation.

B. Three-Dimensional Isotropic Harmonic Oscillator

It is customary to regard $SU(3,1)$ as the SGA of this problem.[20] We prefer to consider a contracted version of $SU(3,1)$ as the SGA. Consider the operators

$$\begin{aligned}\mathbf{P}&=\mathbf{p}\cos\omega t+m\omega\mathbf{q}\sin\omega t,\\ \mathbf{K}&=m\mathbf{q}\cos\omega t-(\mathbf{P}/\omega)\sin\omega t.\end{aligned}\tag{51}$$

They satisfy Eq. (2) and are normalized in such a way that when the frequency ω goes to zero at finite t they go over to the operators **P** and **K** of the former problem. Again we have

$$\mathbf{P}=\mathbf{p}_0,\quad \mathbf{K}=m\mathbf{q}_0,\tag{52}$$

and the commutation relations

$$[P_i,P_j]=0,\quad [K_i,K_j]=0,\quad [P_i,K_j]=-im\delta_{ij}I,\tag{53}$$

and again the operators **P** and **K** have the standard time behavior. We can now use the fact that $\cos^2\omega t+\sin^2\omega t=1$ to eliminate t from the following second-order polynomials in **P** and **K**:

$$\mathbf{K}\times\mathbf{P},\quad P_iP_j+\omega^2K_iK_j.\tag{54}$$

These are, up to factors of m and 2ω, the generators of rotation **J** and the quadropole generators Q_{ij},

$$\begin{aligned}\mathbf{J}&=\mathbf{q}\times\mathbf{p},\\ Q_{ij}&=(1/2m\omega)(p_ip_j+m^2\omega^2q_iq_j),\end{aligned}\tag{55}$$

with the following commutators among them:

$$\begin{aligned}&[J_i,J_j]=i\epsilon_{ijk}J_k,\\ &[J_i,Q_{jk}]=i(\epsilon_{ijl}Q_{lk}+\epsilon_{ikl}Q_{jl}),\\ &[Q_{ij},Q_{kl}]=i\tfrac{1}{4}(\delta_{ik}\epsilon_{jlm}+\delta_{il}\epsilon_{jkm}+\delta_{jk}\epsilon_{ilm}+\delta_{jl}\epsilon_{ikm})J_m.\end{aligned}\tag{56}$$

These are commutation relations of $U(3)$. The algebra of $SU(3)$ is generated by the three components of **J** and the traceless part of the tensor Q. The trace of the tensor Q is just the Hamiltonian divided by ω. We, therefore, have here an example of the linear case. The operators **J**, Q_{ij}, $[1/(m\omega)^{1/2}]\mathbf{P}$, $(\omega/m)^{1/2}\mathbf{K}$, and the identity are now dimensionless operators closing on an algebra since

$$\begin{aligned}&[J_i,(1/m\omega)^{1/2}P_j]=i\epsilon_{ijk}(1/m\omega)^{1/2}P_k,\\ &[J_i,(\omega/m)^{1/2}K_j]=i\epsilon_{ijk}(\omega/m)^{1/2}K_k,\\ &[Q_{ij},(1/m\omega)^{1/2}P_k]\\ &\qquad=i\tfrac{1}{2}\{(\omega/m)^{1/2}K_i\delta_{jk}+(\omega/m)^{1/2}K_j\delta_{ik}\},\\ &[Q_{ij},(\omega/m)^{1/2}K_k]\\ &\qquad=-i\tfrac{1}{2}\{(1/m\omega)^{1/2}P_i\delta_{jk}+(1/m\omega)^{1/2}P_j\delta_{ik}\}.\end{aligned}\tag{57}$$

These equations together with Eq. (37) are the commutation relations of a contracted version of $SU(3,1)$ and exhibit explicitly the SGA. The Hamiltonian maps this algebra into itself as follows:

$$\begin{aligned}&[H,J_i]=0,\\ &[H,Q_{ij}]=0,\\ &[H,(1/m\omega)^{1/2}P_i]=i\omega[(\omega/m)^{1/2}K_i],\\ &[H,(\omega/m)^{1/2}K_i]=-i\omega[(1/m\omega)^{1/2}P_i],\end{aligned}\tag{58}$$

exhibiting explicitly the augmented symmetry algebra $U(3)$ generated by J_i and Q_{ij}. Again the Casimir operators of this $U(3)$ algebra are functionally dependent since $Q_{ij}J_j=0$, and[21] $H^2=\frac{3}{4}\omega^2(C_2+3)$, where C_2 is the second-order Casimir operator of $SU(3)$.

To have a noncontracted version of $SU(3,1)$, one has to change the commutators in Eq. (53). This can be fairly easily accomplished in a classical treatment by multiplying P_i and K_i by functions of the Hamiltonian. The functions thus calculated contain one free parameter[22] which allows one to make the SGA either a compact $SU(4)$ or a noncompact $SU(3,1)$. It is not clear what one gains this way from our present point of view.

C. Free Spinless Field

We did not deal with field theories at all but we mention this example because of the formal similarity with the former two examples. We shall use this example later when we make some speculations on systems with an infinite number of degrees of freedom.

We are looking for conserved currents v_α which depend on space-time both explicitly and implicitly through the field operators and their partial derivatives. For the massive spinless field $\phi(x)$, the following are examples of such currents:

$$\begin{aligned}a_\alpha(k,x)&=\partial_\alpha\phi\cos kx+k_\alpha\phi\sin kx,\\ b_\alpha(k,x)&=-\partial_\alpha\phi\sin kx+k_\alpha\phi\cos kx,\end{aligned}\tag{59}$$

where the constant four-vector k_α is a possible four-momentum of a single-meson state

$$k^2=m^2.\tag{60}$$

We have thus exhibited 2∞ conserved vectors. We define their charges by

$$\begin{aligned}A(k)&=\int[\partial_0\phi\cos(k_0t-\mathbf{k}\cdot\mathbf{x})\\ &\qquad+k_0\phi\sin(k_0t-\mathbf{k}\cdot\mathbf{x})]d^3x,\\ B(k)&=\int[-\partial_0\phi\sin(k_0t-\mathbf{k}\cdot\mathbf{x})\\ &\qquad+k_0\phi\cos(k_0t-\mathbf{k}\cdot\mathbf{x})]d^3x.\end{aligned}\tag{61}$$

[20] R. C. Hwa and J. Nuyts, Phys. Rev. **145**, 1188 (1966).

[21] L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw Hill, New York, 1968), p. 241.

[22] N. Mukunda, L. O'Raifeartaigh, and E. C. G. Sudarshan, Phys. Rev. Letters **15**, 1041 (1965).

They fulfill the following commutation relations:

$$[A(k),A(k')]=0,\quad [B(k),B(k')]=0,\quad [A(k),B(k')]=-ik_0\delta^{(3)}(\mathbf{k}-\mathbf{k}')(2\pi)^3. \tag{62}$$

The four-momentum operator P_α maps this infinite Heisenberg algebra into itself:

$$[P_\alpha,A(k)]=ik_\alpha B(k),\quad [P_\alpha B(k)]=-ik_\alpha A(k). \tag{63}$$

Our discussion up to this point is completely equivalent to the standard introduction of creation and annihilation operators.

For the massless spinless field, the same expressions (59) with $k^2=0$ give conserved currents but there are other conserved currents that are peculiar to this case. They are

$$c_\alpha(x)=\partial_\alpha\phi,\quad d_\alpha(k,x)=k_\alpha\phi-\partial_\alpha\phi k\cdot x, \tag{64}$$

where now k_α is an arbitrary four-vector. We do not go beyond this formal analogy but one should bear in mind that in a theory with an infinite number of degrees of freedom there may exist conserved four-vectors whose charges diverge (do not annihilate the vacuum).

D. Hydrogen Atom

We write the Hamiltonian of the system as follows:

$$H=p^2/2m-ze^2/q \tag{65}$$

(for hydrogen itself, $z=1$). We confine our detailed discussion to the bound states but similar considerations are applicable to the scattering states. The structure of the symmetry algebra was pointed out explicitly by Fock[23] and Bargmann,[24] but it is implicit in Pauli's work.[25] The symmetry algebra is $SU(2)\otimes SU(2)$, generated by the angular momentum vector

$$\mathbf{J}=\mathbf{q}\times\mathbf{p} \tag{66}$$

and the Pauli-Lenz vector (properly normalized)

$$\mathbf{A}=\frac{1}{(-2mH)^{1/2}}\left[\tfrac{1}{2}(\mathbf{J}\times\mathbf{p}-\mathbf{p}\times\mathbf{J})+mze^2\frac{\mathbf{q}}{q}\right]. \tag{67}$$

Bacry[26] calculated the classical explicit form of the generators that enlarge the symmetry algebra to an $SO(4,1)$ SGA. His expressions contain an arbitrary function $\theta(H)$, and a constant related to the normalization of the generators so that they yield the semisimple algebra of $SO(4,1)$. In our approach, $\theta(H)$ is a definite function, namely, $[-8H^3/m(ze^2)^2]^{1/2}t$. On the other hand, we actually construct the algebra of $SO(4,2)$. We comment later on the difference between these two algebras. In the following, we write classical expressions for the generators and calculate Poisson brackets in order to exhibit the algebraic structure without getting into the questions connected with noncommutativity of p's and q's.

Define then

$$\begin{aligned}
U_4&=-\mathbf{p}\cdot\mathbf{q}\sin\varphi-N\left(1+\frac{2Hq}{ze^2}\right)\cos\varphi,\\
\mathbf{U}&=\mathbf{p}\left[q\cos\varphi-\frac{1}{mze^2}N(\mathbf{p}\cdot\mathbf{q})\sin\varphi\right]+\mathbf{q}N\frac{1}{q}\sin\varphi,\\
V_4&=-\mathbf{p}\cdot\mathbf{q}\cos\varphi+N\left(1+\frac{2Hq}{ze^2}\right)\sin\varphi,\\
\mathbf{V}&=-\mathbf{p}\left[q\sin\varphi+\frac{1}{mze^2}N(\mathbf{p}\cdot\mathbf{q})\cos\varphi\right]+\mathbf{q}N\frac{1}{q}\sin\varphi.
\end{aligned} \tag{68}$$

In these expressions,

$$N=\left[\frac{m(ze^2)^2}{-2H}\right]^{1/2} \tag{69}$$

and the function φ is given by

$$\varphi=\left[\frac{-2H}{m(ze^2)^2}\right]^{1/2}\mathbf{p}\cdot\mathbf{q}-\left[\frac{-8H^3}{m(ze^2)^2}\right]^{1/2}t. \tag{70}$$

The reader will recognize the coefficient of the t in φ as the classical frequency.[27] A slightly different form for φ is the following:

$$\varphi=\left[\frac{-2H}{m(ze^2)^2}\right]^{1/2}(\mathbf{p}\cdot\mathbf{q}-2Ht). \tag{71}$$

The expression $\mathbf{p}\cdot\mathbf{q}-2Ht$ resembles Eq. (47), but in the present problem it is not conserved and also fulfills only the last two Eqs. (49). The U_α and V_α ($\alpha=1,\ldots,4$) all fulfill Eq. (2). To write down the Poisson brackets between them, which define the SGA, let us define

$$\begin{aligned}
M_{4i}&=A_i,\\
M_{ij}&=\tfrac{1}{2}\epsilon_{ijk}J_k,\\
M_{5\alpha}&=U_\alpha,\\
M_{6\alpha}&=V_\alpha,\\
M_{56}&=\left[\frac{m(ze^2)^2}{-2H}\right]^{1/2},\\
M_{AB}&=-M_{BA},\quad A,B=1,\ldots,6.
\end{aligned} \tag{72}$$

We then have

$$[M_{AB},M_{CD}]_{\text{P.B.}}=-(g_{BC}M_{AD}-g_{BD}M_{AC}+g_{AD}M_{BC}-g_{AC}M_{BD}), \tag{73}$$

[23] V. Fock, Z. Physik **98**, 145 (1935).
[24] V. Bargmann, Z. Physik **99**, 576 (1936).
[25] W. Pauli, Z. Physik **36**, 336 (1926).
[26] H. Bacry, Nuovo Cimento **41A**, 222 (1966).

[27] H. Goldstein, Ref. 2, p. 304.

where g_{AB} is diagonal with diagonal elements $\{1, 1, 1, 1, -1, -1\}$. The way the Hamiltonian maps this algebra into itself is as follows:

$$[H,M_{\alpha\beta}]_{\text{P.B.}}=0\,,\quad \alpha,\beta=1,\ldots,4$$

$$[H,M_{56}]_{\text{P.B.}}=0\,,$$

$$[H,U_\alpha]_{\text{P.B.}}=-\left[\frac{-8H^3}{m(ze^2)^2}\right]^{1/2}V_\alpha\,, \tag{74}$$

$$[H,V_\alpha]_{\text{P.B.}}=\left[\frac{-8H^3}{m(ze^2)^2}\right]^{1/2}U_\alpha\,.$$

In this example the augmented symmetry algebra is $SO(4)\otimes O(2)$ and the ω are functions of the Casimir operators of $SO(4)$. To explain what happens if we construct $SO(4,1)$ as a SGA, we note that

$$V_\alpha=\left[\frac{-2H}{m(ze^2)^2}\right]^{1/2}M_{\alpha\beta}U_\beta\,. \tag{75}$$

The U_α are mapped into themselves as follows:

$$[H,U_\alpha]_{\text{P.B.}}=\frac{4H^2}{m(ze^2)^2}M_{\alpha\beta}U_\beta\,, \tag{76}$$

but now the $\omega_{\alpha\beta}$ do not have vanishing Poisson brackets with all the $M_{\gamma\delta}$.

VI. DISCUSSION

Summing up what we said up to this point, we have the following algebraic structure defining a SGA.

(a′) The generators of a SGA are Hermitian operators which are all constants of the motion. Some of them may have an explicit time dependence.

(b′) They form a Lie algebra under commutation with structure constants that are numbers.

The Hamiltonian maps this algebra into its generalized enveloping algebra by commutation such that (1) the subalgebra of operators that commute with the Hamiltonian contains the symmetry algebra (we referred to it as the augmented symmetry algebra) and (2) the image of the operators that do not commute with the Hamiltonian is a linear combination of the generators of the algebra with coefficients that commute with all the generators of the augmented symmetry algebra.

(c′) Given a set of eigenstates of the Hamiltonian, a spectrum-generating algebra is maximal relative to this set in the sense that the linear space generated by these states is irreducible under the action of the algebra. (In order not to consider artificial situations which are correct mathematically but are nevertheless uninteresting from a physical point of view, the given set of energy states should in some sense form a band.)

(d′) A spectrum-generating algebra is minimal in the sense that it does not have a proper subalgebra with the same properties.

Having given a definition to a SGA, we want to introduce some notion of stability of the structure against small perturbations. This is a necessity because one uses approximate Hamiltonians to describe physical systems. Let us suppose then that we have a Hamiltonian $H(\lambda)$ dependent on a parameter λ such that, for small enough λ,

$$H(\lambda)=H_0+\lambda H_I\,. \tag{77}$$

Let us suppose that for $\lambda=0$ we have our algebraic structure

$$[G_i,G_j]=ic_{ij}{}^kG_k\,, \tag{78a}$$

$$[H_0,G_j]=i\tfrac{1}{2}(G_k\omega_j{}^k+\omega_j{}^kG_k)\,, \tag{78b}$$

and that for $\lambda\neq 0$ we also have an algebraic structure

$$[F_i(\lambda),F_j(\lambda)]=ic_{ij}{}'F_k(\lambda)\,, \tag{79a}$$

$$[H(\lambda),F_j(\lambda)]=i\tfrac{1}{2}(F_k(\lambda)\omega_j{}^k(\lambda)+\omega_j{}^k(\lambda)F_k(\lambda))\,, \tag{79b}$$

where the operators $F_i(\lambda)$ may depend on λ and their number may be different from the number of the G_i. The notion of stability we want to introduce is that in some sense as λ tends to zero Eqs. (79) reduce to Eqs. (78). Let us first discuss the relations between Eqs. (79a) and (78a). Letting λ tend to zero in (79a), we get a contracted version of the SGA (79a), which we write

$$[\tilde{F}_i,\tilde{F}_j]=i\tilde{c}_{ij}{}^k\tilde{F}_k\,. \tag{80}$$

We then demand that the SGA (78a) is a subalgebra of (80). Therefore, by appropriate choice of basis we may identify the G_i with a subset of the $\tilde{F}_i$. We also demand that as λ tends to zero Eqs. (79b) are such that they map this subalgebra of the $\tilde{F}_i$ into itself. Therefore, some of the $\omega_j{}^k(\lambda)$ should tend to the appropriate $\omega_j{}^k$ as λ tends to zero.

We may encounter a situation where the following two conditions are fulfilled.

(1) The number of G_k's is equal to the number of $F_k(\lambda)$'s so that Eqs. (80) and (78a) are identical.

(2) The matrix elements of the $F_i(\lambda)$ in the representation relevant for the set of Eqs. (79) get contracted to the matrix elements of the G_k in the representation relevant for the set of Eqs. (78) as λ tends to zero.

In such a case we shall say that the perturbation Hamiltonian H_I is preserving the Hilbert space of the Hamiltonian H_0.

If, in addition, the Lie algebras defined by Eqs. (78a) and (79a) are identical, we shall say that the perturbation Hamiltonian is structure preserving. The usual assumptions one makes about the electromagnetic and weak interactions amount to the assumption that they are structure preserving, at least as far as the symmetry algebra of $SU(3)$ is concerned.

Actually our last remark is quite an extrapolation outside of the realm of systems with a finite number of degrees of freedom with which we dealt up to now. We therefore make now some elementary remarks about generalizing our scheme to systems with an infinite number of degrees of freedom.

For systems with a finite number of degrees of freedom our scheme involved two elements: the introduction of constants of the motion which have an explicit time dependence and the action of the Hamiltonian on these constants of the motion.

The generalization of the first element to systems with an infinite number of degrees of freedom was already mentioned in discussing the free spinless field. We consider conserved tensors $t_{\alpha,\alpha_1,\ldots,\alpha_n}$,

$$\partial^\alpha t_{\alpha,\alpha_1,\ldots,\alpha_n}=0\,, \tag{81}$$

which may depend explicitly on the space-time point x. We then form the space integrals

$$T_{\alpha_1,\ldots,\alpha_n}=\int t_{0,\alpha_1,\ldots,\alpha_n}d^3x\,. \tag{82}$$

If these space integrals do not vanish identically, they form constants of the motion which may depend explicitly on time. (We do not consider in this elementary discussion symmetries of the Nambu-Goldstone type.)

As is well known, the generators of the Poincaré algebra can be built in this way out of the conserved symmetric energy-momentum tensor. The following interesting problem arises here: Can one build conserved tensors out of the components of the energy-momentum tensor, the components of conserved $SU(3)\otimes SU(3)$ currents, and the space-time point x other than the $SU(3)\otimes SU(3)$ charges and the Poincaré generators?

For a system with an infinite number of degrees of freedom, we expect an infinite-dimensional SGA. The question now is what is the analog of the action of the Hamiltonian on this algebra. One possibility is to consider the action of the momentum four-vector, namely, to demand that

$$[P_\alpha,T_{\alpha_1,\ldots,\alpha_n}] \tag{83}$$

is linear in the T's with some condition on the coefficients of the linear combinations.

Alternatively, one may argue that the interesting object is the mass spectrum and not the energy spectrum and, therefore, one should consider

$$[P^\alpha,[P_\alpha,T_{\alpha_2,\ldots,\alpha_n}]]\,. \tag{84}$$

Since we lack a solvable nontrivial relativistic invariant field theory, it is hard to put forward even heuristic arguments at this point.

Let us close with a general remark concerning particle physics. Our last remarks would tend to suggest a somewhat conservative if not reactionary point of view. According to this point of view, the spectrum of masses, spins, and internal quantum numbers would result from consideration of integrated quantities like $T_{\alpha_1,\ldots,\alpha_n}$ and not of local current components.

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