Deep-Inelastic Electroproduction and Conformal Symmetry*

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Motivated by the observed scale invariance of high-energy inelastic electron-proton scattering, we study the constraints on a Compton amplitude that follow if it is invariant under the full group of conformal coordinate transformations. Although the conformal group contains operations that interchange spacelike and timelike coordinate intervals, a large class of manifestly causal amplitudes can be constructed. We find that the strict application of conformal symmetry reduces the number of independent covariants from four to two in the case of a spin-zero target. The two covariants of the conformal Compton amplitude remain independent when it is restricted to forward scattering. Hence, invariance under the full conformal group yields no constraints on the structure functions of inelastic electron scattering other than that already provided by simple dilation invariance.

THE manner in which deep-inelastic electron-proton scattering probes the constitution of a nucleon is very different from that of an ordinary scattering experiment. The form factors of an isolated resonant state fall rapidly at large momentum transfer,[1] and the deep-inelastic region is probably characterized by the production of final hadronic states in the continuum where there is no single state to set a mass scale. Indeed, Bjorken has proposed,[2] and experiment has confirmed,[3] that the inelastic structure functions become scale invariant at high energy (ν/M_p) and large momentum transfer (k^2); they become functions of the single dimensionless parameter ν/k^2. If the structure functions are expressed in terms of a Jost-Lehmann representation, which embodies the constraints imposed by casuality, then one finds[4] that the scaling limit is directly related to a Fourier transform on the surface of the light cone of the space-time separation of two current operators. This suggests that the conformal group could have some role to play here, for it is that subgroup of general coordinate transformations which leaves the Minkowski metric invariant save for an over-all coordinate-dependent scale factor, and, in particular, it also leaves the light cone invariant. This group contains, in addition to the Poincaré group and an over-all dilation of space-time associated with scale invariance, a four-parameter set of "special" conformal transformations.

It is clear that strict conformal symmetry does not apply to ordinary high-energy scattering processes. These processes appear to be well described by dimensional quantities such as diffraction-peak widths, cutoffs in transverse momenta, and constant total cross sections; none of these features is consistent with scale invariance.[5] In the inelastic electron-proton process, on the other hand, the cross section involves a sum over all accessible hadronic final states; no quantities which exhibit an intrinsic length need appear, and conformal symmetry could apply at high energy.

With these considerations in mind, we examine here the constraints imposed by conformal symmetry on the Compton amplitude for a virtual photon scattering on a spin-zero target. We shall require strict conformal symmetry; that is, we shall require invariance with a nondegenerate vacuum. The absorptive part of this Compton amplitude for forward scattering gives the scalar-target analog of the electroproduction structure functions. Although this strict application of conformal invariance requires that the target particle be massless, such a conformal model of the Compton amplitude could still describe the high-energy behavior of the scattering of virtual photons on a massive target.

Dilation invariance alone requires that the Compton amplitude be scale invariant. The additional symmetry under special conformal transformations provides a further restriction on the tensor covariant decomposition of the amplitude: It reduces the number of inde-

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[1] Above threshold, the differential cross sections for several resonance production processes not only fall rapidly with increasing momentum transfer, but are remarkably close to the elastic differential cross sections. This is exhibited in Figs. 16–18 of W. K. H. Panofsky's review in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968), p. 35.

[2] J. D. Bjorken, Phys. Rev. **179**, 1547 (1969).

[3] E. D. Bloom, D. H. Coward, H. DeStaebler, J. Drees, G. Miller, L. W. Mo, R. E. Taylor, M. Breidenbach, J. I. Friedman, G. C. Hartmann, and H. W. Kendall, Phys. Rev. Letters **23**, 930 (1969); **23**, 935 (1969).

[4] D. G. Boulware and L. S. Brown (unpublished); L. S. Brown, Lectures given at The Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1969, to be published in *Lectures in Theoretical Physics*; see also B. L. Ioffe, Phys. Letters **30B**, 123 (1969).

[5] It is also clear from purely theoretical considerations that one cannot have a complete, conformally invariant theory, for strict conformal symmetry demands that the two-point function be that of a massless, free field. If the two-point function is free, all the Green's functions are those of a free field [P. G. Federbush and K. A. Johnson, Phys. Rev. **120**, 1926 (1960); R. Jost in *Lectures on Field Theory and the Many-Body Problem*, edited by E. Caianiello (Academic, New York, 1961); R. F. Streater and A. S. Wightman, *PCT, Spin, Statistics and All That* (Benjamin, New York, 1964); K. Pohlmeyer, Commun. Math. **12**, 204 (1969)]. It is not clear that this is a real difficulty, however, for we are not interested in a complete conformally invariant field theory but rather only in the high-energy behavior of a very restricted class of amplitudes such as those which are associated with inelastic electroproduction. We require only that the high-energy limits of *these* amplitudes be conformally invariant.

pendent, gauge-invariant covariants from four to two.[6] This symmetry also strongly restricts the functional form of the scalar amplitudes associated with these two covariants. They become functions of only two parameters. Although a special conformal transformation can map a spacelike coordinate interval into a timelike interval, we can express these scalar amplitudes in terms of a conformally invariant integral representation that is manifestly causal. We find that the two conformal covariants remain independent when the Compton amplitude is evaluated for forward scattering. Thus full conformal symmetry yields no further information on the two inelastic structure functions other than that already provided by simple dilation invariance.

We shall now review the nature of the conformal group and construct the two conformal covariants of the Compton amplitude. We exhibit the causal, conformal integral representation for the scalar amplitudes in Appendix A, and we prove the completeness of the covariants in Appendix B.

The conformal group is that subgroup of general space-time coordinate transformations $x^\mu \to \bar{x}^\mu(x)$ which leaves the Minkowski metric [which we take to have signature $(-,+,+,+)$] invariant save for an over-all scale factor,

$$\bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^\lambda}{\partial \bar{x}^\mu}\frac{\partial x^\kappa}{\partial \bar{x}^\nu}\eta_{\lambda\kappa} = \lambda(x)\eta_{\mu\nu}. \quad (1)$$

Its name is appropriate in that it leaves the cosine of infinitesimal "angles" $dx^\mu dy_\mu (dx^2 dy^2)^{-1/2}$ invariant. The connected subgroup that we shall use is generated by the infinitesimal transformations

$$\bar{x}^\mu = x^\mu + \delta x^\mu(x), \quad (2)$$

which, in view of the definition (1), obey

$$\partial_\mu \delta x_\lambda + \partial_\lambda \delta x_\mu = \delta\lambda \eta_{\mu\lambda}. \quad (3)$$

The trace of this constraint identifies $\delta\lambda$ as

$$\delta\lambda = \tfrac{1}{2}\partial_\mu \delta x^\mu. \quad (4)$$

If we operate on Eq. (3) with ∂_ν and antisymmetrize in the indices μ and ν, we obtain

$$\partial_\lambda(\partial_\mu \delta x_\nu - \partial_\nu \delta x_\mu) = (\eta_{\nu\lambda}\partial_\mu - \eta_{\mu\lambda}\partial_\nu)\delta\lambda. \quad (5)$$

Similarly, we may operate on this result with ∂_κ and antisymmetrize in the indices λ and κ to arrive at

$$(\eta_{\nu\lambda}\partial_\kappa\partial_\mu - \eta_{\nu\kappa}\partial_\lambda\partial_\mu - \eta_{\mu\lambda}\partial_\kappa\partial_\nu + \eta_{\mu\kappa}\partial_\lambda\partial_\nu)\delta\lambda = 0. \quad (6)$$

It is easily verified, by identifying various indices,[7] that this requires that

$$\partial_\mu\partial_\nu\delta\lambda = 0. \quad (7)$$

Thus $\delta\lambda$ is a linear function of x, and Eq. (3) can be integrated immediately to give the remarkable result that the full connected conformal group is a 15-parameter Lie group generated by the infinitesimal transformations[8]

$$\delta x^\mu = \delta a^\mu + \delta\omega^\mu{}_\nu x^\nu + \delta\rho\, x^\mu + (\delta c^\mu x^2 - 2x^\mu \delta c^\nu x_\nu). \quad (8)$$

Here the constant parameters δa^μ and $\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}$ describe the translations and homogeneous Lorentz transformations of the Poincaré group, $\delta\rho$ gives a simple scale change (dilation), and δc^μ are the four parameters of the special conformal transformations. The action of such a special conformal transformation on a tensor is described by a local Lorentz transformation and a dilation that depend linearly upon the coordinate:

$$\partial\bar{x}^\mu/\partial x^\nu = \delta_\nu{}^\mu + \partial_\nu \delta x^\mu, \quad (9)$$

where

$$\partial_\mu \delta x_\nu = -\delta\omega_{\mu\nu}(x) + \eta_{\mu\nu}\delta\rho(x), \quad (10)$$

with

$$\delta\omega_{\mu\nu} = 2(\delta c_\mu x_\nu - x_\mu \delta c_\nu) \quad (11a)$$

and

$$\delta\rho(x) = -2\delta c_\lambda x^\lambda. \quad (11b)$$

There are various discrete operations that leave the Minkowski metric invariant save for a scale factor. In addition to the usual reflections of the extended Poincaré group, the inverse radius transformation

$$x^\mu \to \bar{x}^\mu = \kappa x^\mu (x^2)^{-1} \quad (12)$$

is a discrete conformal transformation. We shall not require invariance under this transformation, but simply note that it may be used to produce connected conformal transformations. In particular, since the double application of the inverse radius transformation gives the identity, the sequence of an inverse radius transformation followed by a space-time translation followed in turn by another inverse radius transformation yields an over-all operation which is continuously connected to the identity. It is straightforward to check that the infinitesimal version of this four-parameter set of transformations is a special conformal transformation. Thus, finite special conformal transformations can be produced by such a sequence with a finite transla-

[6] This statement is not precisely correct since a third conformal covariant that is also gauge invariant can be constructed. However, as we show in Appendix B, the amplitude associated with this covariant is necessarily noncausal and hence can be discarded.

[7] This result depends upon the dimensionality of space-time. In two dimensions $\delta\lambda$ is subject only to the condition that it satisfy Laplace's equation, and we encounter the familiar statement that all analytic functions provide a conformal map.

[8] We note, incidentally, the connection of this result with the class of coordinate transformations that can be generated by a symmetrical, *traceless*, conserved stress-energy tensor $T^{\mu\nu}$. This tensor will generate a coordinate symmetry if the current $J^\mu = T^{\mu\nu}\delta x_\nu$ is conserved or, since the stress tensor is conserved, if $T^{\mu\nu}\partial_\mu \delta x_\nu = 0$. Because the stress tensor can assume arbitrary values constrained only by the condition that it be symmetrical and traceless, this condition is tantamount to Eq. (3), and we conclude that the most general δx^μ is simply one of the general conformal transformations given by Eq. (8). Alternatively, if a current is formed from various coordinate moments of such a stress tensor (e.g., $T^{\mu\nu}x_\nu$, $T^{\mu\nu}x^2$, etc.), then this current will be conserved only for those moments associated with the generators of the conformal group.

tion; in this manner one obtains

$$\bar{x}^\mu = (x^\mu + c^\mu x^2)\sigma(x)^{-1}, \quad (13)$$

where

$$\sigma(x) = 1 + 2cx + c^2x^2. \quad (14)$$

Under this transformation a finite space-time interval $(x-y)^2$ becomes

$$(\bar{x}-\bar{y})^2 = (x-y)^2\sigma(x)^{-1}\sigma(y)^{-1}. \quad (15)$$

It is clear that the remainder of the conformal group, the Poincaré group and dilations, also leave the finite interval $(x-y)^2$ invariant except for a constant scale factor in the case of a dilation. We thus learn that the full conformal group not only leaves the infinitesimal interval dx^2 invariant up to a scale factor, but that it also leaves a finite interval invariant in the same sense. In particular, the light cone $(x-y)^2=0$ is preserved by all conformal transformations. On the other hand, the denominator $\sigma(x)$ can vanish for some finite points x^μ, and the special conformal transformation maps such points to infinity; moreover, the denominator $\sigma(x)$ can become negative for certain regions of x^μ, and thus a special conformal transformation can take points that were originally in a spacelike relation into points that are in a timelike relation. Nevertheless, we construct in Appendix A an integral representation for a wide class of amplitudes that is manifestly causal and that is invariant under infinitesimal conformal transformations.

The conformal transformation laws of tensor fields are readily inferred from their behavior under general coordinate transformations; we need only restrict the general coordinate change to a conformal transformation. For example, under a general coordinate change, a scalar field $\chi(x)$ of weight ω obeys

$$\bar{\chi}(\bar{x}) = \left(\det\frac{\partial\bar{x}}{\partial x}\right)^{\omega/4}\chi(x). \quad (16)$$

In particular, a dilation $\bar{x}=\rho x$ is represented by

$$\bar{\chi}(\bar{x}) = \rho^\omega\chi(x), \quad (17)$$

which shows that the weight ω corresponds to the dimension of the field. If this field χ is the source for a massless spin-zero particle $(-\partial^2\phi=\chi)$, then it must have a weight $\omega=-3$. Another example is provided by a vector current. The conservation of this current,

$$\partial_\mu j^\mu(x) = 0, \quad (18)$$

is maintained by a general coordinate transformation only if it is a vector density which has a weight $\omega=-3$ and transforms according to

$$\bar{j}^\mu(\bar{x}) = \frac{\partial\bar{x}^\mu}{\partial x^\nu}j^\nu(x)\left(\det\frac{\partial\bar{x}}{\partial x}\right)^{-1}. \quad (19)$$

Again, the restriction of the general coordinate change to a conformal transformation gives the correct conformal transformation law for a conserved vector current.

We are now in a position to construct the tensor structure of a conformally invariant Compton amplitude that describes the scattering of a virtual photon on a spin-zero target. It is convenient to consider first a generating functional in which the currents are contracted with an external vector potential $A_\mu(x)$:

$$j^\mu(x) \to \int (dx)A_\mu(x)j^\mu(x). \quad (20)$$

Here we take A_μ to transform as a vector of weight -1, so that the integral is a conformal scalar. Now gauge invariance requires that the vector potential occur only through the field-strength tensor

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \quad (21)$$

The curl of a vector of weight -1 behaves as a tensor of weight -2 under general coordinate changes, and hence $F_{\mu\nu}$ transforms as such a tensor under the operations of the conformal group. Accordingly, all we need do to construct the tensor structure of the gauge-invariant,[9] conformally symmetric Compton amplitude is to couple two field-strength tensors $F_{\mu\nu}(x)$ and $F_{\mu'\nu'}(x')$ together in a conformally symmetric manner. We cannot simply contract the indices of the two field-strength tensors using the Minkowski metric $\eta^{\mu\mu'}$ for, according to Eq. (10), conformal tensors at different space-time points undergo different, coordinate-dependent Lorentz rotations. Hence we must construct a conformal metric tensor $h^{\mu\mu'}(x,x')$ such that the index μ transforms like a vector field at the point x while the index μ' transforms like a vector field at the point x'.

The construction of $h^{\mu\mu'}(x,x')$ is facilitated by the remark that $\ln(x-x')^2$ transforms in an additive manner under the conformal operation (15):

$$\ln(\bar{x}-\bar{x}')^2 = \ln(x-x')^2 - \ln\sigma(x) - \ln\sigma(x'). \quad (22)$$

The additional terms are annihilated if we differentiate with respect to both coordinates, and the logarithm behaves effectively as if it were a scalar. Since the derivative of a scalar is a vector, we conclude that

$$h^{\mu\mu'}(x,x') = (x-x')^2\partial^\mu\partial'^{\mu'}\ln(x-x')^2 \quad (23)$$

[9] Here we have concealed a small technical problem. In order to obtain conformal scalar amplitudes we must, as we shall discuss in a subsequent paper, first work with the target particles off mass shell and then pass to the mass-shell limit. The off-shell amplitude is not gauge invariant but rather obeys a Ward identity that relates its divergence to a vertex function. However, it is not difficult to prove that a conformally invariant vertex function is uniquely determined to be that of a free field. Thus, if we subtract the free-field Born approximation from the Compton amplitude, the remainder is gauge invariant, and we need consider only the structure of this remainder.

is a conformal tensor[10] of weight 0. Therefore,

$$F_{\mu\nu}(x)h^{\mu\mu'}(x,x')h^{\nu\nu'}(x,x')F_{\mu'\nu'}(x') \quad (24a)$$

is a conformal scalar. The divergence of an antisymmetrical tensor is a vector. Hence

$$[\partial^\nu F_{\mu\nu}(x)]h^{\mu\mu'}(x,x')[\partial'^{\nu'}F_{\mu'\nu'}(x')] \quad (24b)$$

is also a conformal scalar. We establish in Appendix B that these are a complete set of gauge-invariant covariants for a causal amplitude. The Compton amplitude is identified by the variation of the external vector potential, and we obtain

$$\begin{aligned} C^{\mu\mu'}(x,x';p,p') &= \partial_\nu\partial'_{\nu'}[h^{\mu\mu'}(x,x')h^{\nu\nu'}(x,x')-h^{\nu\mu'}(x,x')h^{\mu\nu'}(x,x')] \\ &\quad\times[(x-x')^2]^{-2}C_1(x,x';p,p') \\ &\quad+(\delta_\nu{}^\mu\partial^2-\partial^\mu\partial_\nu)(\delta_{\nu'}{}^{\mu'}\partial'^2-\partial'^{\mu'}\partial'_{\nu'})h^{\nu\nu'}(x,x') \\ &\quad\times[(x-x')^2]^{-1}C_2(x,x';p,p'). \quad (25) \end{aligned}$$

Here the scalar amplitudes C_1 and C_2 are conformally invariant. A wide class of such functions is exhibited by the integral representation of Appendix A. This integral representation is manifestly causal and involves a weight function depending upon only two parameters.

To relate our scalar amplitudes to the structure functions of electroproduction, we must first Fourier transform Eq. (25) and then go to the forward scattering limit. This process produces the replacements

$$\partial_\nu\to ik_\nu, \quad \partial'_\mu\to -ik_\mu, \quad (x-x')^\mu\to -i\partial/\partial k_\mu \quad (26)$$

in the tensor structures. Since the Fourier transforms of the scalar amplitudes scale, they become essentially functions only of the single variable

$$\rho=\nu/k^2=-pk/k^2. \quad (27)$$

Using the definition

$$\begin{aligned} W^{\mu\nu} &= \mathrm{Im}C^{\mu\nu} \\ &= [p^\mu-k^\mu(pk/k^2)][p^\nu-k^\nu(pk/k^2)]W_2 \\ &\quad+(g^{\mu\nu}-k^\mu k^\nu/k^2)W_1, \quad (28) \end{aligned}$$

we find that

$$W_1=2\,\mathrm{Im}[2\rho C_1'(\rho)-\rho^2C_1''(\rho)+2C_2(\rho)+2\rho C_2'(\rho)] \quad (29)$$

and

$$\nu W_2=-2\,\mathrm{Im}\rho[C_1''(\rho)+C_2''(\rho)], \quad (30)$$

where the prime denotes a derivative with respect to ρ.

We see from the above equations that, apart from scaling, conformal invariance puts no restrictions on W_1 and W_2. It should be emphasized that this result is not due to our having considered the simpler problem of spin-zero targets. For the spin-$\frac{1}{2}$ case one would still have at least these two invariants, and hence no relation between the structure functions.

[10] This structure, save for an over-all difference in weight, is that of the vacuum polarization $\langle(j^\mu(x)j^{\mu'}(x'))_+\rangle$ of a free, massless, spin-$\frac{1}{2}$ field which is conformally covariant.

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APPENDIX A

Here we shall construct a wide class of conformally symmetric scalar amplitudes that are manifestly causal. We begin by considering the amplitude associated with the graph in Fig. 1. This is the Green's function for source fields χ of the form

$$\chi_1(x_1)=\phi_{12}(x_1)\phi_{13}(x_1)\phi_{14}(x_1), \quad \text{etc.}, \quad (A1)$$

in which ϕ_{ab} is a free field of mass m_{ab},

$$\langle T(\chi_1(x_1)\chi_2(x_2)\chi_3(x_3)\chi_4(x_4))\rangle = \prod_{a\neq b}\Delta_+(x_a-x_b;\,m_{ab}). \quad (A2)$$

Since the source fields χ are composed of local free fields, this amplitude is manifestly causal. We obtain a causal integral representation for the four-point function if we integrate this amplitude over all the various masses m_{ab} with an arbitrary weight function $f(m_{ab}^2)$. It is convenient, however, to integrate over the derivative of the two-point propagator with respect to its mass rather than over the original function. Such a derivative clearly does not spoil the causality. We may make use of the explicit construction

$$\frac{\partial}{\partial m^2}\Delta_+(x;\,m)=\frac{-i}{(4\pi)^2}\int_0^\infty\frac{d\lambda}{\lambda}e^{i\lambda x^2-im^2/4\lambda} \quad (A3)$$

to write this integral representation in the form

$$\begin{aligned} M(x_1,x_2,x_3,x_4) &= \int_0^\infty\prod dm_{ab}^2 f(m_{ab}^2)\int_0^\infty\prod\left(\frac{d\lambda_{ab}}{\lambda_{ab}}\right) \\ &\quad\times\exp\left[i\sum_{a\neq b}\left(\lambda_{ab}(x_a-x_b)^2-\frac{m_{ab}^2}{4\lambda_{ab}}\right)\right]. \quad (A4) \end{aligned}$$

Invariance under a special conformal transformation requires that

$$M(\bar{x})=\prod\sigma(x_a)^3M(x). \quad (A5)$$

This invariance guarantees dilation invariance as well since, according to the structure of the conformal group, the composition of a special conformal transformation with a translation produces a dilation, and the amplitude is manifestly translationally invariant. Now [cf. Eq. (15)]

$$(\bar{x}_a-\bar{x}_b)^2=(x_a-x_b)^2\sigma(x_a)^{-1}\sigma(x_b)^{-1}, \quad (A6)$$

so that if we scale the integration parameters appropriately,

$$\lambda_{ab}\to\lambda_{ab}\sigma(x_a)\sigma(x_b) \quad (A7)$$

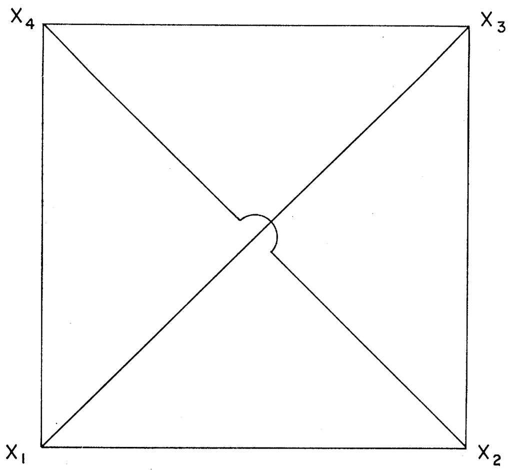

FIG. 1. A causal graph.

and

$$m_{ab}^2 \to m_{ab}^2\sigma(x_a)\sigma(x_b)\,, \tag{A8}$$

we obtain a conformal symmetric representation if the weight $f(m_{ab}^2)$ obeys

$$f(m_{ab}^2)=f(m_{ab}^2\sigma(x_a)\sigma(x_b))\,. \tag{A9}$$

This homogeneity condition requires that the weight be a function of only two parameters[11] which we may choose to be

$$u=\frac{m_{13}^2m_{24}^2}{m_{12}^2m_{34}^2},\quad v=\frac{m_{14}^2m_{23}^2}{m_{12}^2m_{34}^2}. \tag{A10}$$

It should be emphasized that the integration parameters λ_{ab} and m_{ab}^2 range over only positive values. Hence we have manifest invariance only in those circumstances where all the $\sigma(x_a)$ have the same sign. This, while it is not sufficient for invariance under the full conformal group, guarantees invariance under the infinitesimal subgroup. The full conformal invariance can be exhibited, however, by an analytic continuation to Euclidean space-time where the $\sigma(x_a)$ always remain positive. Accordingly, we have a causal integral representation invariant under infinitesimal conformal transformations with a weight function of the form

$$f(m_{ab}^2)=g(u,v)\,. \tag{A11}$$

This integral representation may be extended to fields of different weight by inserting appropriate powers of λ_{ab} in the integrand.

APPENDIX B

In order to verify that the Compton covariants exhibited in the text form a complete set, we shall exploit the isomorphism between the 15-parameter conformal group and the group of pseudo-orthogonal transformations in six dimensions,[12,13] $SO(4,2)$. The advantage of this isomorphism is that it provides us with *linear* representations of the conformal group which involves nonlinear transformations of space-time. Under an infinitesimal $SO(4,2)$ transformation, a six-vector ξ^A $(A=0,1,2,3,5,6)$ undergoes the variation

$$\delta\xi^A=\delta\Omega^A{}_B\xi^B\,, \tag{B1}$$

in which the infinitesimal rotation parameters $\delta\Omega_{AB}$ form an antisymmetrical set,

$$\delta\Omega_{AB}=-\delta\Omega_{BA}\,. \tag{B2}$$

The isomorphism is easily verified if one endows the six-space with a metric g_{AB} of signature $(-++++-)$ and makes the correspondence $(\mu,\nu=0,1,2,3)$

$$\begin{aligned}&\delta\omega_{\mu\nu}=\delta\Omega_{\mu\nu}\,,\quad \delta a_\mu=\tfrac{1}{2}(\delta\Omega_{\mu5}-\delta\Omega_{\mu6})\,,\\ &\delta c_\mu=-\tfrac{1}{2}(\delta\Omega_{\mu5}+\delta\Omega_{\mu6})\,,\quad \delta\rho=\delta\Omega_{56}\,.\end{aligned} \tag{B3}$$

In particular, if we set

$$\xi_+=\xi_5+\xi_6\,, \tag{B4}$$

then

$$\begin{aligned}\delta(\xi^\mu/\xi_+)=&\,\delta a^\mu+\delta\omega^\mu{}_\nu(\xi^\nu/\xi_+)+\delta\rho(\xi^\mu/\xi_+)\\ &+\delta c_\nu[\eta^{\mu\nu}\xi^\lambda\xi_\lambda/\xi_+^2-2\xi^\mu\xi^\nu/\xi_+^2]-\delta c^\mu\xi^A\xi_A/\xi_+^2.\end{aligned} \tag{B5}$$

Clearly if $\xi^A\xi_A=0$, the quantity ξ^μ/ξ_+ transforms like a four-vector. Hence, if we restrict ξ^A to the six-space light cone (an invariant restriction), we can project into space-time by setting

$$x^\mu=\xi^\mu/\xi_+. \tag{B6}$$

We can now project from fields in space-time to fields in the six-space. In the case of a scalar field, the mapping

$$\phi(x)=(\xi_+)^{-\omega}\Phi(\xi) \tag{B7}$$

makes $\phi(x)$ transform as a scalar field of weight ω if $\Phi(\xi)$ is a scalar field in the six-space. In order that ϕ involve only the four space-time coordinates x^μ, this relationship must be homogeneous in ξ of degree zero, or

$$\xi^A\frac{\partial}{\partial\xi^A}\Phi(\xi)=\omega\Phi(\xi)\,. \tag{B8}$$

We may map between a vector current $j^\mu(x)$ and a six-space field $J^A(\xi)$ that transforms as a six-vector in a similar manner. In this case a factor of $(\xi_+)^3$ must appear to make $j^\mu(x)$ have weight -3. The six-vector field is rotated by a space-time translation while the four-vector current is not altered by this operation. This behavior requires that the projection have the form

$$j^\mu(x)=(\xi_+)^3\{J^\mu(\xi)-x^\mu[J_5(\xi)+J_6(\xi)]\}\,. \tag{B9}$$

Here the six-vector field $J^A(\xi)$ must be homogeneous of

[11] Our integral representation has some correspondence to the work of N. F. Bali, D. D. Coon, and A. Katz, J. Math. Phys. 10, 1939 (1969).

[12] P. A. M. Dirac, Ann. Math. 37, 429 (1936).

[13] G. Mack and A. Salam, Ann. Phys. (N.Y.) 53, 174 (1969).

degree -3 to keep $j^\mu(x)$ a function of x^μ alone,

$$\xi^B \frac{\partial}{\partial \xi^B} J^A(\xi) = -3J^A(\xi). \tag{B10}$$

Our construction does not guarantee that the vector current j^μ will transform correctly under special conformal transformations. This is assured if we adopt the constraint

$$\xi_A J^A(\xi) = 0, \tag{B11}$$

a constraint that is akin to the light-cone restriction needed for ξ^A. We note that the map is invariant under a type of gauge transformation; it is invariant under the replacement

$$J^A(\xi) \to J^A(\xi) + \xi^A \Omega(\xi). \tag{B12}$$

Thus the six-vector field $J^A(\xi)$ has only four effective independent components, and $J^A(\xi)$ can be determined from $j^\mu(x)$; the map is reversible. Finally, we note that the conservation of the four-vector current j^μ requires that the six-vector current J^A obey

$$L^A{}_B J^B(\xi) = J^A(\xi), \tag{B13}$$

in which

$$L_{AB} = \xi_A \partial/\partial\xi^B - \xi_B \partial/\partial\xi^A \tag{B14}$$

is an operator that preserves the light cone.

We write the six-space analog of the completely off-mass-shell Compton amplitude as $M^{AA'}(\xi,\xi';\eta,\eta')$, where η and η' are the coordinates of the scalar source fields with weight -3. This amplitude is homogeneous of degree -3 in each of the coordinates, obeys the constraints

$$\xi_A M^{AA'} = 0 = M^{AA'} \xi'_{A'}, \tag{B15}$$

and has the crossing properties

$$M^{AA'}(\xi,\xi';\eta,\eta') = M^{AA'}(\xi,\xi';\eta',\eta) = M^{A'A}(\xi',\xi;\eta,\eta'). \tag{B16}$$

It is a simple matter to show that, save for irrelevant gauge terms proportional to ξ^A and $\xi'^{A'}$, this amplitude can be expressed in terms of four covariants constructed from g_{AB} and the four available six-vectors. Each of these covariants multiplies a homogeneous scalar that is completely crossing symmetric.

We can write two of these structures as

$$M_1{}^{AA'} = \epsilon^{ABCDEF} \epsilon^{A'B'C'}{}_{DEF} L_{BC} L'_{B'C'} \times (\xi\xi')^{-3} (\eta\eta')^{-3} F_1(\alpha,\beta) \tag{B17}$$

and

$$M_2{}^{AA'} = \epsilon^{ABCD'E'F} \epsilon^{A'B'C'DE}{}_F L_{BC} L'_{B'C'} L_{DE} L'_{D'E'} \times (\xi\xi')^{-3} (\eta\eta')^{-3} F_2(\alpha,\beta), \tag{B18}$$

where ϵ^{ABCDEF} is the completely antisymmetric invariant in six dimensions with $\epsilon^{012356} = +1$, and

$$\alpha = \frac{(\xi\eta)(\xi'\eta')}{(\xi\xi')(\eta\eta')}, \quad \beta = \frac{(\xi\eta')(\xi'\eta)}{(\xi\xi')(\eta\eta')}. \tag{B19}$$

They are automatically conserved, and they project into linear combinations of the space-time covariants employed in the text. We can write the remaining two covariants in the form

$$M_3{}^{AA'} = \frac{1}{3!} \epsilon^{ABCDEF} \xi_B \frac{\partial(\alpha+\beta)}{\partial \xi^C} \epsilon^{A'B'C'}{}_{DEF} \xi'_{B'} \frac{\partial(\alpha+\beta)}{\partial \xi'^{C'}} \times (\xi\xi')^{-3} (\eta\eta')^{-3} G(\alpha,\beta) - (1/2!) \epsilon^{ABCDEF} \xi_B \xi'_C \eta_D \eta'_E \epsilon^{A'B'C'D'E'}{}_F \xi_{B'} \xi'_{C'} \xi_{D'} \eta'_{E'} \times (\xi\xi')^{-3} (\eta\eta')^{-3} H(\alpha,\beta). \tag{B20}$$

Current conservation turns out to require that

$$G(\alpha,\beta) = [2(\alpha+\beta) - 1 - (\alpha-\beta)^2]^{-3/2} f(\alpha+\beta) \tag{B21}$$

and

$$H(\alpha,\beta) = \left(\frac{4(\alpha+\beta)^3 - 3(\alpha+\beta)^2 + (\alpha+\beta) - 2(\alpha-\beta)^2}{2(\alpha+\beta) - 1 - (\alpha-\beta)^2} \right) \times G(\alpha,\beta), \tag{B22}$$

where $f(\alpha+\beta)$ is arbitrary. (This establishes that these two covariants are indeed independent of our previous pair, for otherwise the conservation condition would allow at least one arbitrary function of α and β.) The invariant functions G and H have singularities in space-like regions. This gives a noncausal behavior which we must exclude. Accordingly, there are only two conserved, causal amplitudes that are conformally symmetric, the structures that we exhibited in the text.