Problems of Stability for Quantum Fields in External Time-Dependent Potentials*

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We discuss the algebraic aspect of the time evolution for various types of fields interacting with external potentials. This is a pure c-number problem for the classical wave equations. In the case of interaction, the equation for $s\geqslant\frac{3}{2}$ exhibits the well-known consistency troubles and the breakdown of causality studied by Velo and Zwanziger. Other equations, for example, a modified form of the Joos-Weinberg equations, are shown to lead to different troubles. As for the question of a unitary operator implementing the time-evolution automorphism in the Fock space of the in-fields, we encounter a peculiar property: The scalar coupling of $s=0$ fields (superrenormalizable) leads to the existence of a time-evolution operator, whereas electromagnetic-type couplings (renormalizable) possess at most an S matrix and no evolution operator in the interaction region. This Haag's phenomenon holds for arbitrary smooth and short-range external fields.

I. INTRODUCTION

BESIDES its usefulness as a mathematical preliminary to the much more difficult problem of a fully quantized theory, the investigation of relativistic wave equations for arbitrary mass and spin in the presence of external potentials is also important from a physical point of view: One would well expect that a system of composite particles, say atoms for instance, could be treated as elementary as long as it is subject to potentials which are slowly varying compared to its size and weak compared to its binding energy. We know, of course, that in the nonrelativistic limit we can use the Schrödinger equation for its description. However, if relativistic corrections are to be included, one would like to have something intermediate between the Schrödinger equation and the fully quantized theory (as in the case of the hydrogen atom). The relativistic wave equation emerges as a natural candidate for such a description. From this point of view, the troubles with higher-spin equations with external potentials are quite baffling and merit more investigation.

Whereas there exist many equivalent possibilities of describing noninteracting spin-s mass-m particles by covariant causal free fields fulfilling linear field equations,[1] the number of possibilities becomes rather limited if one uses these field equations for implementing interactions.[2] The discussion of field equations involving coupled fields requires considerable mathematical sophistication, and even for the simplest case of the self-coupling of a scalar neutral field, only incomplete results have been obtained.[3] On the other hand, one cannot expect every coupled-field equation obtained from coupling free fields to yield a reasonable physical theory. In many cases, the reason for this is quite elementary and does not have to be discussed on the rather sophisticated level of renormalization and space-time limiting procedures. Features of instability[2] and breakdown of causality[4] can be discussed by coupling the fields to external potentials and are typical for higher-spin fields ($s\geqq\frac{3}{2}$). The occurrence of an indefinite metric can be demonstrated for interactions with stationary external fields, and this phenomenon happens for every quantized field except the $s=\frac{1}{2}$ Dirac field.[5]

Using well-known methods[6] of functional analysis, we first discuss the algebraic aspect of time evolution. For scalar and electromagnetic couplings of fields up to spin 1, we show causality and analyticity in the coupling constant of the algebraic automorphism of time evolution. For higher spin we rederive the breakdown of causality[4] and discuss the problem of consistency. The main (and novel) part of this paper is concerned with the construction of the time-evolution operator in the Fock space of the incoming particles. Here we find that with the exception of a scalar coupling to a spin-zero field (superrenormalizable in the fully quantized theory), the unitary time-evolution operator does not exist in the interaction region. This Haag's phenomenon, which to our surprise occurs even though the external

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[1] M. Fierz, Helv. Phys. Acta 12, 3 (1939).

[2] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) A173, 211 (1939). For a recent discussion of the problem involving external interactions, see A. S. Wightman, in *Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energy, University of Miami, 1968*, edited by T. Gudehus, G. Kaiser, and A. Perlmutter (Gordon and Breach, New York, 1969).

[3] J. Glimm and A. Jaffe, Commun. Pure Appl. Math. 22, 401 (1969).

[4] G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969); 188, 2218 (1969). According to Ref. 12, these problems of causality breakdown have been discussed by J. Weinberg and S. Kusaka (unpublished); J. Weinberg, thesis, University of California, 1943 (unpublished).

[5] L. I. Schiff, H. Snyder, and J. Weinberg, Phys. Rev. 57, 315 (1940) concluded that strong external interactions lead to catastrophic instabilities for the Klein-Gordon field. We recently reinvestigated this problem and reached slightly different conclusions. See B. Schroer and J. A. Swieca, following paper, Phys. Rev. D 2, 2938 (1970).

[6] S. L. Sobolev, Transl. Amer. Math. Soc. 7, (1963); K. Friedrichs and H. Lewy, Math. Ann. 98, 192 (1928).

TABLE I. Properties of higher-spin equations. (All theories with a conserved current can be derived from a Lagrangian.)

Type	Desirable properties	Troubles
Fierz-Pauli; Rarita-Schwinger; $\psi\in(s_1,s_2)\oplus(s_2,s_1)\oplus\cdots$	Unique mass; conserved current	Noncausality; inconsistency
Joos-Weinberg; $\psi\in(s,0)\oplus(0,s)$	Conserved current; causal (c-number problem)	Several masses; lack of unitarity due to creation of unphysical mass states
Modified Joos-Weinberg; $\psi\in(s,0)\oplus(0,s)$	Unique mass	No conserved current leading to violation of unitarity
Generalized Feynman-Gell-Mann; $(\Box+m^2)\psi=V\psi$; $\psi\in(s,0)\oplus(0,s)$	Unique mass; causal	No conserved current leading to violation of unitarity ($s\geqslant 1$)
$(\Box+m^2)\psi=V\psi$; $\psi=\psi^{(s,0)}+\psi^{(0,s)}$	Unique mass; conserved current; causal	Parity doublets; lack of unitarity due to the creation of opposite parity states with indefinite metric

interaction is arbitrarily smooth and rapidly decreasing, does not impede the existence of the evolution operator outside the interaction region (i.e., the S matrix).

Finally, some of the properties of higher-spin equations are summarized in Table I.

II. REDUCTION TO c-NUMBER PROBLEM

The algebraic aspects of time evolution (automorphism of the field algebra) can be reduced to a c-number problem.[7] The precise form of this connection will be discussed after we derive the functional analysis treatment of the c-number problem. Let us first give a somewhat heuristic discussion of this point. For explanatory purposes we consider a scalar interaction of an $s=0$, massive particle:

$$(\partial_\mu\partial^\mu+m^2)A(x)=V(\mathbf{x},t)A(x), \qquad (1)$$

where $V(\mathbf{x},t)$ is a smooth function which vanishes outside a finite interval, say, outside $0<t<T$. Interpreting the free field $A_{\text{in}}(x)$ to be the field before interaction, we write the Yang-Feldman equation

$$A(x)=A_{\text{in}}(x)+\int\Delta_{\text{ret}}(x-x')V(x')A(x')dx'. \qquad (2)$$

The solution can be written in the form

$$A(x)=A_{\text{in}}(x)+\int G_R(\mathbf{x}_1t_1;\mathbf{x}_1t')V(\mathbf{x}_1't')A_{\text{in}}(x')dx' \qquad (3)$$

with, formally,

$$G_R(x;y)=\Delta_R(x-y)+\int\Delta_R(x-x_1)V(x_1)\times\Delta_R(x_1-y)dx_1+\int\int\cdots+\cdots.$$

The important fact is that the propagation function G_R is a purely classical one since (1) is linear in the field A. The problem posed by the coupled-field equation can now be subdivided into two problems:

(a) Show the existence of the classical time evolution.
(b) Demonstrate that the free field $A_{\text{out}}(x)=A(x)$, $t>T$ and $A_{\text{in}}(x)$ are unitarily related.

Before we go into the discussion of the classical problem, let us add two remarks:

(i) The essential ingredient for the reduction to a c-number problem along these lines is the linearity of the field equation in A.

(ii) In order to be able to construct a unitary operator in the Fock-space generated by the in-fields, we have to demonstrate a certain Hilbert-Schmidt property of the classical time-evolution kernel. It is essentially this property which forces us to use Hilbert-space techniques in contrast to the test-function methods employed by Capri.[7]

Unlike the one-particle nonrelativistic theory, the physical,[8] i.e., conserved, inner product for the Klein-Gordon equation is indefinite:

$$(\Phi,\chi)\equiv i\int d^3x\,\bar{\Phi}\overleftrightarrow{\partial}_0\chi=\int d^3x\,\mathbf{\Phi}^\dagger\tau_3\boldsymbol{\chi},\qquad \boldsymbol{\chi}=\frac{1}{\sqrt{2}}\begin{pmatrix}\chi+i\partial_0\chi\\ \chi-i\partial_0\chi\end{pmatrix}, \qquad (4)$$

where we introduced the sometimes-useful two-component formalism. The time-evolution operator is pseudo-unitary, i.e., isometric in the metric (4), and the time-dependent one-particle Hamiltonian (where $m=1$) is

$$H=-\tfrac{1}{2}(\tau_3+i\tau_2)\Delta+\tau_3-\tfrac{1}{2}(\tau_3+i\tau_2)V,$$

pseudo-self-adjoint. These features are important for obtaining a positive-definite Hilbert space for the quantum theory, but they are unfortunately quite useless for showing existence of the propagation kernel. A useful (but nonconserved) positive-definite metric is

[7] A. Z. Capri, J. Math. Phys. 10, 575 (1969).

[8] H. Feshbach and F. Villars, Rev. Mod. Phys. 30, 24 (1958).

the "energy" metric, or the "number" metric:

$$\|\Phi\|^2{}_E \equiv \int K\bar{\Phi}(x)K\Phi(x)d^3x + \int |\dot{\Phi}(x)|^2 d^3x$$

$$= \int |K^{1/2}\alpha(x)|^2 d^3x + \int |K^{1/2}\beta(x)|^2 d^3x\,, \quad (5)$$

with

$$K = +(-\Delta+1)^{1/2}$$

and

$$\Phi(x) = (2K)^{-1/2}\alpha(x) + (2K)^{-1/2}\bar{\beta}(x)\,,$$
$$\pi(x) = -i(\tfrac{1}{2}K)^{1/2}\alpha(x) + i(\tfrac{1}{2}K)^{1/2}\bar{\beta}(x)\,,$$

$$\|\Phi\|^2{}_N = \int |\alpha(x)|^2 d^3x + \int |\beta(x)|^2 d^3x\,. \quad (6)$$

For the existence and uniqueness of the Cauchy problem and the causality of the propagation, we note that the free energy-momentum density vector for the classical theory is

$$P^\mu(x) = (|\partial_t\Phi|^2 + |\partial_i\Phi|^2 + |\Phi|^2;\ -\partial_t\Phi\,\partial_i\Phi + \text{c.c.}) \quad (7a)$$

and fulfills the divergence equation [as a consequence of Eq. (1) for $\Phi(x)$]

$$P^\mu{}_{,\mu} = \partial_t\bar{\Phi}V(x)\phi + \text{c.c.}, \quad (7b)$$

and its zero component is just the positive-definite "energy density" as in formula (5). The Hilbert space X of functions Φ, $\dot{\Phi}$ equipped with the inner product of the energy norm is a Sobolev space, and applying Gauss's theorem to a "causal shadow" region R (stump of cylinder, Fig. 1), we obtain

$$\int_R \partial^\mu P_\mu d^4x = \|\Phi\|^2{}_{E,s(\tau)} - \|\Phi\|^2{}_{E,s(0)} + \int_M P_\mu dM^\mu\,. \quad (8)$$

From (7), it follows that the energy-momentum density is a timelike vector $P_\mu P^\mu \geqslant 0$, and hence the light-cone contribution $\int_M$ is non-negative.

From (7b), it follows that [assuming $V(x)$ is a bounded function]

$$\int_R d^4x\, P^\mu{}_{,\mu} < \int_0^\tau |\partial_t\Phi(x)\ V(x)\Phi(x) + \text{c.c.}|\, d^3x dt$$

$$= c\int_0^\tau \|\Phi\|^2{}_{E,s(t)}dt \quad (9)$$

because the Hermitian form $\partial_t\bar{\Phi}V + \text{c.c.}$ is bounded by the positive-definite energy density. This leads, with the help of (8), to the inequality

$$\|\Phi\|^2{}_{E,s(t)} - \|\Phi\|^2{}_{E,s(0)} \leq c\int_0^t \|\Phi\|^2{}_{E,s(t')}dt'\,. \quad (10a)$$

A similar inequality holds if instead of the surfaces at the ends $s(0)$ and $s(t)$ we take two arbitrary surfaces $s(t)$ and $s(t')$ with $0 \leq t' < t \leq t$. This leads to the following inequality for the logarithmic derivative:

$$\frac{d}{dt}\ln\|\Phi\|^2{}_{E,s(t)} \leq c\,, \quad (10b)$$

and via integration to

$$\|\Phi\|^2{}_{E,s(t)} \leq \|\Phi\|^2{}_{E,s(0)}e^{ct}\,. \quad (10c)$$

Immediate consequences of this inequality are the causal shadow property (Huyghen's principle): If Φ, $\pi = 0$ in $s(0)$, Φ, π vanish in the causal shadow of $s(0)$ (which immediately gives local uniqueness); and the causal propagation property: If Φ, $\pi = 0$ outside a region s, then the propagation stays in the light cone (cone of influence) spanned by s. Writing the solution of the Cauchy problem formally as

$$\Phi(x) = \int_{x_0'=0} G(x,x')\overleftrightarrow{\partial}_{x_0'}\Phi(x')d^3x'$$

is equivalent to supp $G(x,x')$ and is outside $(x-y)^2 < 0$.

The existence of a solution for the Cauchy problem is most easily shown by writing the equation of motion as

$$i\partial_t\mathbf{\Phi} = \left[i\begin{pmatrix} 0 & 1 \\ -K^2 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}\right]\mathbf{\Phi}$$

$$\text{with } \mathbf{\Phi} = \begin{pmatrix} \Phi \\ \dot{\Phi} \end{pmatrix}, \quad (11a)$$

or

$$i\partial_t\Upsilon(x) = \left[\begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix} + \begin{pmatrix} -v & -v \\ v & v \end{pmatrix}\right]\Upsilon(x) \quad (11b)$$

with

$$\Upsilon(x) = \begin{pmatrix} \alpha \\ \bar{\beta} \end{pmatrix} \text{ and } v = (2K)^{-1/2}V(x)(2K)^{-1/2}. \quad (12)$$

We note that the free part H_0 is self-adjoint in the energy or the number norm (the domain of definition Δ is again another Sobolev space $\Delta \subset X$) and that the interaction H_1 is a bounded operator in X. This boundedness implies the boundedness of $H_{\text{int}}(t)$ defined by

$$H_{\text{int}}(t) = e^{iH_0t}H_1(t)e^{-iH_0t}\,.$$

Hence the time-evolution operator

$$U(t) = e^{-iH_0t}T\exp\left[-i\int_0^t H_{\text{int}}(t')dt'\right] \quad (13)$$

exists and is bounded in energy norm (the power series converges in energy norm). With a simple additional consideration one can see that C^∞ initial data are mapped by this operator into smooth data at a later time, provided the interaction is infinitely smooth. The proof is very similar to the corresponding statement in Schrödinger theory, but since we have no use for these smoothness properties, we will refrain from proving them.

Even if we had taken the apparently more singular *electromagnetic interaction*, in which case we would have obtained (in Φ basis)

$$H_1(t)=\begin{pmatrix} +V & 0 \\ 0 & +V \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ \mathrm{div}\mathbf{A}+2A^l\partial_l-i\mathbf{A}^2 & 0 \end{pmatrix}, \quad (14)$$

we still would be dealing with an *operator which is bounded* in the energy norm. The "magnetic" part is evidently bounded in (5) and for the electric part one obtains (assuming differentiability of V and using the Schwarz inequality)

$$(V\Phi,K^2V\Phi)=(KVK^{-1}K\Phi,\, KVK^{-1}K\Phi) \leq \mathrm{const}\,(K\Phi,K\Phi).$$

The last inequality follows from Eq. (A4). Summarizing, we proved that the Cauchy initial value problem for the Klein-Gordon equation with a scalar (1) or an electromagnetic interaction (14) has an unique solution in X for data in Δ. The assumptions on the potentials are sketched just after (1). (For a precise formulation see the Appendix, Lemma 1.)

We now explain in precise terms what we mean by the "reduction to the c-number problem." The classical evolution $U(t)$ evidently gives the automorphism of the quantized fields by

$$\begin{pmatrix} A(\mathbf{x},t) \\ \Pi(\mathbf{x},t) \end{pmatrix}=\int d\mathbf{x}'\begin{pmatrix} U_{11}(\mathbf{x},\mathbf{x}';t) & U_{12}(\mathbf{x},\mathbf{x}';t) \\ U_{21}(\mathbf{x},\mathbf{x}';t) & U_{22}(\mathbf{x},\mathbf{x}';t) \end{pmatrix} \times\begin{pmatrix} A_{\mathrm{in}}(\mathbf{x}') \\ \Pi_{\mathrm{in}}(\mathbf{x}') \end{pmatrix}, \quad (15\mathrm{a})$$

where the U's are the matrix elements of the operator U written as x-space kernels. This formula makes mathematical sense because the smeared-out fields at time t, for example,

$$\int A(\mathbf{x},t)f(\mathbf{x})=\int\int f(\mathbf{x})U_{11}(\mathbf{x},\mathbf{x}';t)A_{\mathrm{in}}(\mathbf{x}')d^3xd^3x' +\int\int f(\mathbf{x})U_{12}(\mathbf{x},\mathbf{x}';t)\Pi_{\mathrm{in}}(\mathbf{x}')d^3xd^3x', \quad (15\mathrm{b})$$

exist, since the free fields can be smeared with any L^2-integrable functions, and by our Hilbert-space construction we know that the operator U (and its adjoint) maps L^2-integrable functions into such functions. If we would not work within the Fock representation but in an arbitrary representation of the canonical commutation relation, we would have to know much more detailed properties of U as a test-function mapping. The pseudo-unitarity of U in the indefinite metric is responsible for the fact that the new field operators are canonical. By "algebraic automorphism" we mean precisely a canonical mapping of this type. The fact that this automorphism preserves causality becomes evident if we use the "physicists' notation"

$$A(\mathbf{x},t)=\int_{x_0'=0} G(x,x')\overleftrightarrow{\partial}_{x_0}A_{\mathrm{in}}(x')d^3x'. \quad (15\mathrm{c})$$

Here G is the solution with δ-function initial value (for the time derivative) which is, of course, related to the kernels of the U matrix:

$$[A(\mathbf{x},x_0),A(\mathbf{y},\mathbf{y}_0)] =\int\int G(x,x')\overleftrightarrow{\partial}_{x_0'}G(yy')\overleftrightarrow{\partial}_{y_0'}[A_{\mathrm{in}}(x'),A_{\mathrm{in}}(y')] =\int_{x_0'=0} G(x,x')\overleftrightarrow{\partial}_{x_0'}G(y,y')d^3x'.$$

The result is independent of the hypersurface over which we integrate since the inner product (4) is conserved (same as pseudo-unitarity of U). Choosing the surface to be $x_0'=y_0'$, we obtain

$$[A(x)A(y)]=G(x,y) \quad (15\mathrm{d})$$

and hence the local commutability is reduced to the causal Cauchy propagation for the c-number solution.

The treatments of this "classical" propagation for an $s=\frac{1}{2}$ field and an $s=1$ field are analogous. For $s=\frac{1}{2}$, one uses instead of the energy norm the positive-definite conserved "physical" norm (ϕ=4-component spinor function)

$$(\Phi,\Phi)=\int\bar{\Phi}^{\mathrm{tr}}\Phi d^3x,$$

with the conserved 4-vector

$$j^\mu=(\bar{\Phi}^{\mathrm{tr}}\Phi,\bar{\Phi}^{\mathrm{tr}}\gamma^0\gamma^i\Phi), \quad (16)$$

with

$$j^\mu(x)j_\mu(x)\geqslant 0,$$

which follows by pure algebraic manipulations using the properties of the γ matrices. For this case, one obtains for the causal shadow region

$$\|\Phi\|^2{}_{s(t)}\leqslant\|\Phi\|^2{}_{s(0)}. \quad (17)$$

The existence of time evolution for couplings without derivatives is as easily demonstrated as in Schrödinger theory.

For the case of $s=1$, one uses the symmetric energy-momentum tensor T and constructs the energy-momentum density vector:

$$P^\mu(x)=(\tfrac{1}{2}(\boldsymbol{\pi}^2+v_0{}^2+\mathbf{v}^2+(\mathrm{rot}\mathbf{v})^2,\, \pi^l f_l{}^i+v^iv_0), \quad i=1,2,3 \quad (18)$$

$$\pi^k=(\partial^0v^k-\partial^kv^0)\,;\quad f_{\mu\nu}=(\partial_\mu v_\nu-\partial_\nu v_\mu).$$

The energy density is positive and $P^\mu P_\mu\geqslant 0$. The latter is a consequence of the property of $P^\mu(x)$ that $P^0(x)$ can only vanish together with $\mathbf{P}(x)$. The field equation incorporating scalar $V(x)$ and electromagnetic A_μ interaction is

$$(D_nD^n+m^2)V^\mu-D_nD^\mu V^n=V(x)V^\mu \tag{19}$$

when

$$D_n=\partial_n+ieA_n.$$

Again the divergence is bounded by the positive-definite Hermitian energy density, which is a positive-definite form in the variables $\boldsymbol{\pi}$, $\mathbf{V}$, rot$\mathbf{V}$, and V^0:

$$\partial_\nu T^{0\nu}=-\bar{\pi}^k(IV)_k+V^0\partial^\nu V_\nu+\text{c.c.}, \tag{20}$$

with

$$(IV)_\nu=2ieA_\mu(\partial^\mu V_\nu-\partial_\nu V^\mu)-2ieA_\nu\partial^\mu V_\mu -A_\nu A^\mu V_\mu+V(x)V_\nu,$$

$$[-V(x)+1]\partial^\nu V_\nu=ie\partial_\mu F^{\mu\nu}V_\nu+ieA^\nu V_\nu +\partial^\nu V(x)V_\nu-ieA^\nu V(x)V_\nu,$$

and

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu.$$

Again the boundedness follows for bounded interaction functions with bounded derivatives (at least if $V\neq 1$):

$$\|V\|^2_{E,s(t)}\leqslant\|V\|^2_{E,s(0)}e^{c|t|}.$$

From this we obtain the causality.

The boundedness of the interaction operator in the energy norm follows in the same way as in the scalar case.

With

$$\psi=\begin{pmatrix}\boldsymbol{\Phi}\\ \boldsymbol{\pi}\end{pmatrix},$$

we obtain

$$i\frac{\partial}{\partial t}\psi=(H_0+H_1)\psi, \tag{21}$$

with

$$H_0=i\begin{pmatrix}0 & 1-\text{grad div}\\ -1+\Delta-\text{grad div} & 0\end{pmatrix}$$

and, for example,

$$H_1=\begin{pmatrix}0 & 0\\ V & 0\end{pmatrix}$$

for the scalar interaction. H_0 is self-adjoint and H_1 is bounded in the energy metric. Hence

$$U(t)=e^{-iH_0t}T\exp\left[-i\int_0^t H_{\text{int}}(t')dt'\right] \tag{22}$$

exists and is a bounded operator. Similarly one shows the existence of the propagation operator for the electromagnetic interaction.

The c-number time evolution leads to an algebraic automorphism and the commutator function can be related to the propagation kernel of the Cauchy problem. The proofs are analogous to the ones given for the case $s=0$.

III. TIME EVOLUTION AND S MATRIX

The classical equations that we have studied all have a current which is conserved even in the presence of the interaction. The space integral over the conserved current defines, therefore, a conserved inner product. This inner product for the $s=0$ case and also the $s=1$ case [see Eq. (4)] is indefinite, whereas for the $s=\frac{1}{2}$ case we are dealing with a positive-definite inner product. The classical time-evolution operator whose existence we have shown is accordingly pseudo-unitary in this metric (for $s=0$ or 1) or unitary (for $s=\frac{1}{2}$). For interactions which are finitely extended in time, one has no problem with asymptotic limits and the (pseudo-) unitarity property also holds for the time evolution which maps the function before the interaction into the function after the interaction.

However, the existence of a unitary operator in the field-theoretical state space of the corresponding theory which transforms the field at one time to the field at another time (in particular, the existence of a unitary S matrix) requires further restrictions on the interactions. Let us first discuss this in the case of a scalar field with scalar interaction. As an analog to Eq. (5), we write for the field operator

$$\begin{aligned}A(x)&=(2K)^{-1/2}a(x)+(2K)^{-1/2}b^\dagger(x),\\ \pi(x)&=-i(\tfrac{1}{2}K)^{1/2}a(x)+i(\tfrac{1}{2}K)^{1/2}b^\dagger(x),\end{aligned} \tag{23}$$

where (in the absence of interactions) $a^\dagger(x)$ and $a(x)$ are the usual creation and annihilation operators of particles. The existence of a unitary operator $U(t)$ which implements the canonical transformation from, say, A and π at time zero to time t is equivalent to the problem whether the $a(\mathbf{x},t)$ and $b^\dagger(\mathbf{x},t)$ belong to the "no-particle" representation if this was the case at time zero. Since the time evolution of the particle operators is given by the classical differential equation (11b), we write

$$\begin{pmatrix}a\\ b^\dagger\end{pmatrix}_t=e^{-iH_0t}T\exp\left[-i\int_0^t H_{\text{int}}(t')dt'\right]\begin{pmatrix}a\\ b^\dagger\end{pmatrix}_{t=0} \equiv\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}a\\ b^\dagger\end{pmatrix}_{t=0}, \tag{24}$$

where the M matrix is the pseudo-unitary classical evolution matrix. The pseudo-unitarity is necessary and sufficient for the canonical structure of the $a(t)$ and $b(t)$, i.e., the validity of the canonical formalism at a later time t.[9] The existence of a unitary operator in the Fock space of the a and b (operators before interaction) which implements this canonical transformation leads to conditions for the M matrix, and therefore restricts the interaction H_1. The problem is equivalent to the existence of a vacuum for the $a(t)$ and $b(t)$ in the Fock space of the in-field. For the case of Bose statistics, this

[9] The field is assumed to be complex, i.e., $a^\dagger=(a)^\dagger$, $b=(b^\dagger)^\dagger$.

again is equivalent to $L=M_{11}^{-1}M_{12}$ being a Hilbert-Schmidt (H.S.) operator in $L^2(d^3p)$. (This has already been shown by Shale.[10]) A similar statement holds for Fermi statistics, where one has to pay attention to the possible eigenvalue zero of $M_{11}M_{11}^\dagger$. We show the sufficiency of the above condition for the existence of a vacuum $\Omega(t)$ in Fock space. If L is H.S., there exist two orthonormal bases $\{f_i\}$ and $\{g_k\}$ and positive numbers λ_i such that $L=\sum\lambda_i f_i\otimes g_i$. Consider the vector $\chi=\exp(-\sum a_i^\dagger b_i^\dagger)\Omega$, where

$$a_i^\dagger=\int d^3p\; f_i(p)a^\dagger(p)\,,\quad b_i^\dagger=\int d^3p\; g_i(b)b^\dagger(p)\,.$$

The norm N of χ turns out to be given by

$$N^2=\begin{cases}\prod_i(1-\lambda_i^2)^{-1} & \text{for Bose statistics} \quad (25)\\ \prod_i(1+\lambda_i^2) & \text{for Fermi statistics}, \quad (26)\end{cases}$$

which is finite if and only if L is H.S. In the case of Bose statistics, we used the fact that the λ_i's have to be smaller than 1 as a consequence of the already demonstrated canonical structure. For the same reason, L is H.S. if and only if M_{12} is H.S. It can easily be checked that $\Omega(t)=N^{-1}\chi(t)$ is annihilated by $a(t)$ and $b(t)$.

Let us analyze the above condition for the unitarity of the S matrix in terms of the specific models. We start with a spin-zero field coupled to an external scalar field (11b). For the M matrix, we get the expression (in a formal way of writing)

$$M_{ik}=e^{\pm iKt}\biggl(1+\sum_{n,m}(-i)^n\,\mathrm{sgn}\int_{-T}^{t}dt_n e^{\pm iKt_n}\times v(t_n)e^{\pm iKt_n}\int_{-T}^{t_{n-1}}dt_{n-1}\cdots\biggr). \quad (27)$$

The sum over m involves terms with the same number of time integrations. It is a crucial fact that in the off-diagonal part of M, all terms contain a part $e^{\pm iKt}ve^{\pm iKt}$ with the *same sign* in the exponent. This enables us to apply the before-mentioned criterion for the unitarity of the S matrix. For arbitrary t, M_{12} can be split into a bounded part and a part which is H.S. This is the case because

$$\int_{t_1}^{t_2}dt\; e^{iKt}ve^{iKt} \quad (28)$$

has a H.S. norm which is the square root of

$$\int d^3xd^3x'\int_{t_1}^{t_2}dtdt'[\Delta^+(x-x')]^2V(x)V(x')\,, \quad (29)$$

which is smaller than a t-independent constant. $M_{12}(t)$ is a H.S. operator and there exist therefore for each t a vacuum $\Omega(t)$ which is annihilated by $a(t)$ and $b(t)$. The situation changes in the case of minimal electromagnetic interaction. The interaction part of the Hamiltonian in the form analogous to (11b) is given by

$$H_1=\begin{pmatrix}v_+ & v_-\\ v_- & v_+\end{pmatrix}+\begin{pmatrix}z & z\\ -z & -z\end{pmatrix},$$
$$v_\pm=(\tfrac{1}{2}K)^{1/2}A^0(1/2K)^{1/2}\pm(1/2K)^{1/2}A^0(\tfrac{1}{2}K)^{1/2}, \quad (30)$$
$$z=(1/2K)^{1/2}(\mathbf{A}^2-i\{\partial_e,A^e\})(1/2K)^{1/2}.$$

Since the free part of the Hamiltonian is the same as before, all the off-diagonal terms of M contain again a term of the form (28). Concentrating on the Born term of M_{12}, we find the following expression for the square of the H.S. norm:

$$\int d^3xd^3x'\int_{-T}^{t}dtdt'A_\mu(x)A_\nu(x')(g^{\mu\nu}-\partial^\mu\partial^\nu)\times[\Delta^+(x-x')]^2. \quad (31)$$

From (31) one can read off that only the Born term of the S matrix, where the time integration runs over the whole support of $A_\mu(\cdot,t)$ is a H.S. operator. The time-evolution operator for a time t in the interaction region involves a Born term which is not H.S. It is interesting to note that only the *magnetic part* $\mathbf{A}$ has the effect of making the expression (31) infinite. The pure electric case has the same virtues as the coupling to a scalar field, as was discussed previously. But since the high-energy behavior of higher-order terms is steadily improving (for $s=0$), we expect for the case of electromagnetic interaction the following qualitative picture to be true. The time-evolution operator does not exist for t in the interaction region but for $t>T$. This is due to the breakdown of the H.S. property for the Born term, whereas all higher terms are H.S. even in the interaction region. Unfortunately, we were not able to find a useful estimate for the H.S. norm of the higher terms.

The case of $s=\frac{1}{2}$ can be discussed with the same methods. Instead of working with the spinor amplitudes (15) Φ and $\Phi^{-\mathrm{tr}}$, one goes over by a Foldy-Wouthuysen transformation to amplitudes $\alpha_i(x)$ and $\bar{\beta}_i(x)$ [corresponding to (12)]. The "number" norm and the "physical" norm are the same in this case, and the interaction operator in the number norm is bounded (if the interaction is given by bounded functions without derivative coupling) and self-adjoint. The propagation matrix

$$M=e^{-iH_0t}\sum_{n=0}^{\infty}\int_0^t e^{iH_0t_n}Ve^{-iH_0t_n}dt_n\times\int_0^{t_n}\cdots\int_0^{t_2}e^{iH_0t_1}Ve^{-iH_0t_1}dt_1 \quad (32)$$

[10] D. Shale, Transl. Amer. Math. Soc. 103, 149 (1961).

is of the form

$$M=\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{33}$$

As mentioned before, it is sufficient (Fermi statistics) for the existence of a unitary S matrix to show the H.S. property of $M_{11}^{-1}M_{12}$. The scalar and the electromagnetic interaction look very much the same, but the propagater has a worse high-energy behavior than in the case of spin zero. One expects, therefore, a similar statement about the existence of the S matrix to be true as in the case of spin zero and electromagnetic interaction.

For $s=1$, one can again find a representation of H such that H_0 looks the same as in the case of spin zero. The same criterion for the existence of the time-evolution operator can be applied. Since the propagater has in this case such a bad high-energy behavior, there are difficulties even in proving the existence of the S matrix. We note, however, that the algebraic aspect of the time-evolution is perfectly all right; it is just on the level of states where things become awkward. It is also interesting to note in this connection that the $s=1$ electromagnetic interaction belongs to the class of nonrenormalizable couplings in the fully quantized theory.

IV. HIGHER-SPIN EQUATIONS, CANONICAL STRUCTURE

In generalizing our discussion to higher spins, we should be aware of several restrictions which have to be imposed on the free-field equation which is used to implement interaction by adding a coupling term. Writing the field equation as a first-order differential system,

$$(\Gamma^\mu\partial_\mu+m\Gamma)\psi=J\psi, \tag{34}$$

the free equation ($J=0$) should satisfy the following requirements.

(a) Lorentz invariance: $S(\Lambda)\Gamma^\mu S^{-1}(\Lambda)=\Lambda^\mu{}_\nu\Gamma^\nu$, $S\Gamma S^{-1}=\Gamma$.

(b) Unique mass: $(\partial_\mu\partial^\mu+m^2)\psi=0$ as a consequence of the free-field equation.

(c) Spins: There should be exactly $2s+1$ linearly independent solutions of the free-field equation for each $\mathbf{p}$ and sign of the energy.

(d) There should be a matrix β such that

$$\beta\Gamma^\mu=\Gamma^{\mu\dagger}\beta,\quad \beta\Gamma=\Gamma^\dagger\beta,\quad \beta=\beta^\dagger,\quad \beta^2=1.$$

Requirements (b) and (c) result from the well-known fact[2] that the free equation ($J=0$) must yield only physically acceptable solutions and that a description of just one particle should lead exactly to $2s+1$ classical solutions. Otherwise, the action of the external field will result in the production of particles of unacceptable characteristics. An example of pathological equations are the (unmodified) Joos-Weinberg equations.[11] Writing them in the Yang-Feldman form

$$\psi(x)=\psi_{\text{in}}(x)+\int G_{R1}{}^{(0)}(x-x')J(x')\psi(x')dx',$$

with $G_{R1}{}^{(0)}$ being the retarded Green's function to the differential operator

$$\begin{pmatrix} -1 & D(im^{-1}(\partial^0\sigma_0-\partial^i\sigma_i)) \\ D(im^{-1}\partial^\mu\sigma_\mu) & -1 \end{pmatrix}, \tag{35}$$

one sees immediately that $G_{R1}{}^{(0)}$ has, in addition to the physical particle pole, also unphysical poles. As was shown by Wightman,[2] this leads to the unacceptable consequences mentioned before. We would like to point out, however, that Weinberg's S-matrix rules do not come from these equations. They are rather related to the "modified" Joos-Weinberg equations whose criticism is more subtle (see Sec. V). The last requirement, (d) (which is equivalent to the existence of a free Lagrangian), is sufficient for the existence of a conserved current:

$$j^\mu=\bar\psi\Gamma^\mu\psi,\quad \bar\psi=\psi^\dagger\beta.$$

The interactions must be restricted to $\beta J=J^\dagger\beta$. The scalar product

$$(\psi,\Phi)=\int d^3x\,\bar\psi\Gamma^0\Phi \tag{36}$$

is then time independent. The time independence of such a scalar product leads to the pseudo-unitarity of the c-number problem and is necessary for establishing the unitarity of the quantized theory, as we have already seen for the case of spin $s\leqq 1$ (Sec. II). For the case of arbitrary spin, we consider a potential which acts in the interval $-T<t<+T$ and suppose that the c-number problem has a solution, i.e., for given Cauchy data a unique solution of (42) exists for all t. We introduce the free quantized fields

$$\psi^{\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[a_\lambda{}^{\text{in}}(\mathbf{p})u_\lambda(\mathbf{p})e^{+ipx}+b_\lambda{}^{\text{in}\dagger}(\mathbf{p})v_\lambda(\mathbf{p})e^{-ipx}],$$

$$\psi^{\text{out}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[a_\lambda{}^{\text{out}}(\mathbf{p})u_\tau(\mathbf{p})e^{+ipx}+b_\lambda{}^{\dagger\text{out}}(\mathbf{p})v_\lambda(\mathbf{p})e^{-ipx}], \tag{37}$$

with $u_\lambda(p)$ ($v_\lambda(p)$) being plane-wave solutions of the free equation corresponding to positive (negative) energies and given helicities. The interacting field is given by

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\int d^3p[a_\lambda{}^{\text{in}}(\mathbf{p})u_\lambda(\mathbf{p},x)+b_\lambda{}^{\text{in}\dagger}(\mathbf{p})v_\lambda(\mathbf{p},x)], \tag{38}$$

with $u_\lambda(p,x)$ and $v_\lambda(p,x)$ being c-number solutions of

[11] H. Joos, Fortschr. Phys. 10, 65 (1962); S. Weinberg, Phys. Rev. 133, B1318 (1964).

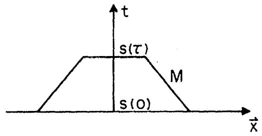

FIG. 1. "Causal shadow" region for which energy inequality is derived.

Eq. (34) satisfying

$$\left.\begin{aligned} u_\lambda(\mathbf{p},x) &= u_\lambda(\mathbf{p})e^{ipx} \quad (39)\\ v_\lambda(\mathbf{p},x) &= v_\lambda(\mathbf{p})e^{-ipx} \quad (40)\end{aligned}\right\} \text{ for } t \leqq -T.$$

By construction, $\psi(x)=\psi^{\text{in}}(x)$ for $t<T$. Since we want $\psi(x)=\psi^{\text{out}}(x)$ for $t>T$, the following relations hold for the particle operators:

$$a_\lambda{}^{\text{out}}(p)=\int M_{\lambda\lambda'}{}^{11}(p,p')a_{\lambda'}{}^{\text{in}}(p')d^3p' + \int M_{\lambda\lambda'}{}^{12}(p,p')b_{\lambda'}{}^{\text{in}\dagger}(p')d^3p', \quad (41)$$

$$b_\lambda{}^{\text{out}\dagger}(p)=\int M_{\lambda\lambda'}{}^{21}(p,p')a_{\lambda'}{}^{\text{in}}(p')d^3p' + \int M_{\lambda\lambda'}{}^{22}(p,p')b_{\lambda'}{}^{\text{in}\dagger}(p')d^3p', \quad (42)$$

with (physical scalar product)

$$\begin{aligned} M_{\lambda\lambda'}{}^{11}(p,p') &= (u_\lambda(p)e^{-ipx},u_{\lambda'}(p',x)),\\ M_{\lambda\lambda'}{}^{12}(p,p') &= (u_\lambda(p)e^{-ipx},v_{\lambda'}(p,x)),\\ M_{\lambda\lambda'}{}^{21}(p,p') &= (-1)^{2s+1}(v_\lambda(p)e^{-ipx},u_{\lambda'}(p,x)),\\ M_{\lambda\lambda'}{}^{22}(p,p') &= (-1)^{2s+1}(v_\lambda(p)e^{-ipx},v_{\lambda'}(p,x)),\end{aligned} \quad (43)$$

where $t>T$. A prerequisite for the unitarity of the S matrix is that a^{out} and b^{out} given by (44) satisfy the usual commutation (or anticommutation) relations, which means that M should satisfy (in matrix notation)

$$MSM^\dagger=S, \quad (44)$$

where

$$M=\begin{pmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix},\quad S=\begin{pmatrix}1 & 0\\ 0 & (-1)^{2s+1}\end{pmatrix}. \quad (45)$$

With regard to the physical scalar product (43), we remark that in accordance with the theorem on spin and statistics, it is positive definite for half-integer spin and indefinite for integer spin, for solutions of the free equation. As a form, however, the scalar product is only positive definite for $s=\frac{1}{2}$. This immediately suggests that in all other cases troubles with indefinite metric and complex energies will occur for a stationary external field. This point has been discussed in the following paper.[5]

Whatever additional difficulties we encounter for spin equations with *external time-dependent interactions*, there can be *no problem with indefinite metric*.[7] This is evident from the fact that one state space is generated by the physical field $\psi_{\text{in}}(x)$ before the interaction takes place and that the Yang-Feldman equation expresses explicitly the Heisenberg field ψ and the out-field ψ_{out} as operators in $\mathcal{H}_{\text{in}}$. The consideration of Johnson and Sudarshan[12] applies *only* to the time-independent case. The incorrect application of Schwinger's variational principle in the case of a time-dependent external potential was already pointed out by Velo and Zwanziger.[4]

The Fierz-Pauli[2] field equations are examples of wave equations fulfilling the requirements, but they are not genuine hyperbolic equations in the usual mathematical terminology because the time derivatives have a non-invertible coefficient matrix:

$$(\Gamma^\mu\partial_\mu+Bm)\psi=0,\quad s=\text{odd} \quad (46)$$

$$(\Gamma^{\mu\nu}\partial_\mu\partial_\nu+Bm^2)\psi=0,\quad s=\text{even} \quad (47)$$

with $\det\Gamma^0=0$ for odd s, and $\det\Gamma^{00}=0$ for even s.

As an example, let us mention the Rarita-Schwinger form of the Fierz-Pauli equation for $s=\frac{3}{2}$:

$$\Gamma^\mu{}_{(n\lambda)}=g_{n\lambda}\gamma^\mu-g_\lambda{}^\mu\gamma_n-g^\mu{}_n\gamma_\lambda-\gamma_n\gamma^\mu\gamma_\lambda, \quad (48)$$

$$B_{(n\lambda)}=g_{n\lambda}+\gamma_n\gamma_\lambda. \quad (49)$$

This is the canonical form[13] of the most general first-order equation for a vector-spinor $\psi_\mu(x)$ with (a) the irreducibility condition $\partial^\nu\psi_\mu=0=\gamma^\nu\psi_\nu$ following from the field equation and (b) the conservation law for a bilinear Hermitian current following algebraically from the field equation.

V. INCONSISTENCIES AND NONCAUSALITIES FOR HIGHER SPIN

The method of energy inequalities for deriving causality of the c-number solution breaks down for fields with spins $s\geqslant\frac{3}{2}$. The divergence of the free energy-momentum density ("charge-current" density in the case of half-integer spin) cannot be bounded by the energy density (charge density). A change in the definition of the energy-momentum density by taking parts of the interaction into the definition, hence working with a modified (external field-dependent) density, which saves the inequality, can only be done at the expense of losing the property $P_\mu(x)P^\mu(x)\geqslant 0$. This forces one to construct a propagation cone which is different from the one in Fig. 1. Indeed, looking directly at the local propagation cone by employing the methods of characteristics for symmetric hyperbolic system of equations, Velo and Zwanziger were able to show that the local cone will depend on the external field, and hence one loses causality in the sense of the Minkowski cone. Even worse, many interactions for higher-spin

[12] K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N. Y.) **13**, 126 (1961).

[13] E. E. Fradkin, Zh. Eksperim. i Teor. Fiz. **32**, 1479 (1957) [Soviet Phys. JETP **5**, 1203 (1957)].

fields lead to mathematical inconsistencies.[2,14,15] For explanatory purposes, let us look at the following two scalar interactions of a spin-$\frac{3}{2}$ Rarita-Schwinger field:

$$(\Gamma^{\mu}{}_{(\nu\chi)}\partial_\mu + mB_{(\nu\chi)})\psi^\chi = \begin{cases} \Phi\psi_\nu & \text{(50a)} \\ \Phi\psi_\nu + \Phi\gamma_\nu\gamma^\mu\psi_\mu . & \text{(50b)} \end{cases}$$

The second equation results from replacing m in the free equation by $m-\Phi$.

Writing down the zero component multiplied with γ^0, we obtain one subsidiary condition:

$$\chi \equiv \partial^i\psi_i + (\gamma^i\partial_i - m)\gamma^k\psi_k + \begin{cases} -\Phi\gamma_0\psi_0 \text{ for (50a)} & \text{(51a)} \\ \Phi\gamma^k\psi_k \text{ for (50b)}. & \text{(51b)} \end{cases}$$

In order to obtain another subsidiary condition necessary to arrive at eight components, we contract the field operator once with γ^μ and once with ∂^μ:

$$(2\gamma\cdot\partial\gamma_\nu + 2\partial_\nu)\psi^\nu - 3m\gamma_\nu\psi^\nu = \begin{cases} \Phi\gamma_\nu\psi^\nu \\ -3\Phi\gamma_\nu\psi^\nu , \end{cases} \tag{52a}$$

$$m(\partial_\nu + \gamma\cdot\partial\gamma_\nu)\psi^\nu = \begin{cases} \partial_\nu\Phi\psi^\nu \\ (\partial_\nu + \gamma\cdot\partial\gamma_\nu)\Phi\psi^\nu . \end{cases} \tag{52b}$$

For the coupling (50b), the two equations lead to the subsidiary condition

$$\tfrac{3}{2}(m-\Phi)^2\gamma_\nu\psi^\nu = [(\partial_\nu + \gamma\cdot\partial\gamma_\nu)\Phi]\psi^\nu \tag{53a}$$

or

$$\Lambda \equiv \gamma_\nu\psi^\nu - \tfrac{2}{3}(m-\Phi)^{-1}[(\partial_\nu + \gamma\cdot\partial\gamma_\nu)\Phi]\psi^\nu = 0 , \tag{53b}$$

whereas for (50a), an equation which has no differentiation acting on ψ^ν, no subsidiary condition (relation between Cauchy data for a system of first-order equations) can be obtained. Therefore in the case of (50a) there are more independent components in the interaction region than in the free region. This leads to a disaster in the propagation from the interacting into the free region (i.e., inconsistency with "switching off" of ϕ) because one has to match a lesser number of Cauchy data to a prescribed larger number of components.[2,4,13] This is a heuristic discussion of what we mean by "inconsistency trouble." The equation (50b), which was obtained by replacing $m \to m-\Phi$, does not lead to such a trouble because the number of subsidary conditions in the interaction region is the same as that in the free-field region.[15]

Now to the causality breakdown. Following Velo and Zwanziger we illustrate this first in the simple model of a spin-1 field interacting with an external symmetric tensor field (an antisymmetric $T^{\mu\nu}$ would lead to causality):

$$(\partial_\nu\partial^\nu + m^2)V_\mu - \partial_\mu\partial^\nu V_\nu - T_\mu{}^\nu V_\nu = 0 . \tag{54}$$

The zero component as well as the 4-divergence leads to constraints involving only time derivatives of first order:

$$\chi \equiv \partial_i[\partial^i V_0 - \partial_0 V^i] + m^2 V_0 - T_0{}^\nu V_\nu = 0 , \tag{55a}$$

$$\Phi \equiv m^2\partial^\mu V_\mu - \partial_\mu T^{\mu\nu} V_\nu = 0 . \tag{55b}$$

Since the standard techniques for hyperbolic equations are only applicable to systems in which the matrices of the highest derivatives are symmetric (Hermitian), we rewrite our system by using (55b) twice:

$$(\partial^\mu\partial_\mu + m^2)V_\nu - m^{-2}\partial_\nu\partial_\mu T^{\mu\chi}V_\chi - m^{-2}\partial_\mu T_\nu{}^\mu\partial^\chi V_\chi + m^{-2}\partial_\mu T_\nu{}^\mu m^{-2}\partial_\lambda T^{\lambda\chi}V_\chi - T_\nu{}^\mu V_\mu = 0 . \tag{56}$$

This equation leads to the following equation for vectors n_μ orthogonal on the local characteristic cone:

$$\det\{n^2 g_\mu{}^\nu - m^{-2}T^{\chi\lambda}n_\mu n_\chi - (\mu \leftrightarrow \nu) - m^{-4}T_\mu{}^\chi T^{\lambda\nu}n_\chi n_\lambda\} = (n^2)^2(n^2 - m^{-2}n\cdot T\cdot n)^2 = 0 . \tag{57}$$

Up to this point we followed exactly the discussion of Velo and Zwanziger. Now we ask the question about the field equation for the subsidary combination χ and Φ which follow from the symmetric system (56). A straightforward (but lengthy) computation yields

$$\begin{aligned} \partial_0\Phi &= -m^2\chi + m^{-2}\partial_\mu T_0{}^\mu\Phi , \\ \partial_0\chi &= m^{-2}\partial^i\partial_i\Phi + \Phi - m^{-4}\partial^i\partial_\mu T_i{}^\mu\Phi . \end{aligned} \tag{58a}$$

This evidently implies the second-order equation

$$(\partial_\mu\partial^\mu + m^2)\Phi = m^{-2}\partial_\nu\partial_\mu T^{\mu\nu}\Phi . \tag{58b}$$

Hence the subsidiary combination propagates also noncausally as a consequence of (56). Since the characteristic equation for the propagation cone is

$$(n^2 - m^{-2}n\cdot T\cdot n) = 0 ,$$

the subsidary condition $\Phi = 0 = \chi$ (which is equivalent to $\Phi = 0 = \partial_0\Phi$), which is to be interpreted as an initial condition, *only gets rid of half of the noncausal factors* in (57).

As has been stated in Sec. II already, the causality or lack of causality of the c-number problem reflects itself in the validity or breakdown of local commutativity for the quantized fields. In fact, the classical propagation kernel for the solution of the Cauchy problem is identical to the field (anti-) commutator. Whereas in the spin-one case we could have avoided this noncausal behavior by restricting the couplings to interactions with scalar or electromagnetic external fields (as we did in Sec. I), this would have not been possible for $s \geqq \frac{3}{2}$. The discovery of this peculiar behavior of higher-spin equations has been attributed to Weinberg and Kusaka. The present authors have become aware of the causality breakdown as a result of the work of Velo and Zwanziger.

Using, again, methods similar to those used by Velo and Zwanziger, we briefly demonstrate this in the case of an $s = \frac{3}{2}$ Rarita-Schwinger field. The consistent scalar

[14] This type of inconsistency has been pointed out for the case of spin 2 and minimal electromagnetic interaction by P. Federbush, Nuovo Cimento **19**, 572 (1961). It was also discussed by Velo and Zwanziger, Ref. 4.

[15] It can be shown that for arbitrary equations of the Rarita-Schwinger or Fierz-Pauli type the replacement $m \to m-\Phi$, $m^2 \to m^2 - m\Phi$ does not lead to inconsistency trouble.

coupling [50(b)] in a more explicit form is

$$[\gamma^\mu\partial_\mu-(m-\Phi)]\psi_\nu-\partial_\nu C[(\partial_\kappa+\gamma\cdot\partial\gamma_\kappa)\Phi]\psi^\kappa$$
$$-[(\partial_\nu+\gamma\cdot\partial\gamma_\nu)\Phi]C\partial_\kappa\psi^\kappa$$
$$-[(\partial_\nu+\gamma\cdot\partial\gamma_\nu)\Phi]C\gamma\cdot\partial C[(\partial_\kappa+\gamma\cdot\partial\gamma_\kappa)\Phi]\psi^\kappa$$
$$+[(\partial_\nu+\gamma\cdot\partial\gamma_\nu)\Phi]C\tfrac{3}{2}(m-\Phi)\gamma_\kappa\psi^\kappa+\gamma_\nu(m-\Phi)\gamma^\kappa\psi_\kappa=0; \quad (59)$$

here $C=\frac{2}{3}/(m-\Phi)^2$.

The characteristic determinant for the normals n_μ on the local propagation cone is ($\Phi^\nu=\partial^\nu\Phi$)

$$\det[\delta_\nu{}^\kappa\gamma^\mu n_\mu-n_\nu C(\Phi^\kappa+\gamma_\mu\Phi^\mu\gamma^\kappa)-(\nu\leftrightarrow\kappa)-(\Phi_\nu+\gamma_\mu\Phi^\mu\cdot\gamma_\nu)$$
$$\times C^2 n_\lambda\gamma^\lambda(\Phi^\kappa+\gamma_\mu\Phi^\mu\gamma^\kappa)]$$
$$=(n^2)^4[n^2-C^2\epsilon_{\mu\nu\kappa\lambda}n^\kappa\Phi^\lambda\epsilon^{\mu\nu\sigma\alpha}\Phi_\sigma n_\alpha]^4. \quad (60)$$

In order to show that in the transition from (50b) to (59) nothing was lost, we have to demonstrate that (59) leads to an equation of motion for the subsidiary components χ, Λ. Using only (59), we obtain a system of first-order equations in time for χ and Λ which is equivalent to the second-order equation for Λ:

$$(\partial_\kappa\partial^\kappa+m^2)\Lambda-\gamma\cdot\partial C[(\partial^\kappa+\gamma\cdot\partial\gamma^\kappa)\Phi]\partial_\kappa\Lambda$$
$$+(\text{terms with lower derivation})=0. \quad (61)$$

The characteristic determinant is

$$(n^2)^2[n^2-C^2\epsilon_{\mu\nu\kappa\lambda}n^\kappa\phi^\lambda\epsilon^{\mu\nu\sigma\alpha}\Phi_\sigma n_\alpha]^2; \quad (62)$$

hence the subsidiary condition $\Lambda=0$, $\partial_0\Lambda=0$ (equivalent to $\chi=0$) at a hypersurface eliminates only half of the noncausal factors.

The causality situation does certainly not improve with more complicated interactions. The electromagnetic (minimal or not) is noncausal and the characteristic determinant is (for the minimal case)

$$(n^2)^4\left(n^2-\frac{4}{9}\frac{e^2}{m^4}\tilde{F}_\nu\cdot n\tilde{F}^\nu\cdot n\right)^4,$$

where $\tilde{F}_\nu\cdot n\equiv\tilde{F}_{\nu\alpha}n^\alpha$ and $\tilde{F}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$ (in the case of minimal coupling). The equation of motion corresponding to (61) is

$$(D_\mu D^\mu+m^2)\Lambda-\tfrac{2}{3}iem^{-2}D^\mu\tilde{F}_\mu\cdot\gamma(-\gamma^\mu D_\mu+2m)\Lambda$$
$$-ieF^{\mu\nu}\sigma_{\mu\nu}\Lambda=0 \quad (63)$$

and leads to the determinant

$$(n^2)^2\left(n^2-\frac{4}{9}\frac{e^2}{m^4}\tilde{F}_\nu\cdot n\tilde{F}^\nu\cdot n\right)^2, \quad (64)$$

and hence the subsidiary condition can only eliminate half of the noncausally propagating components. A tensor interaction leads to the same difficulty as the example (50a), and hence is *inconsistent* in the sense explained there. From the examples studied, one might be inclined to suspect the validity of the following statement:

Any external interaction involving a field of spin with $s>1$ *is always noncausal.*

For higher-spin Fierz-Pauli and Rarita-Schwinger equations one can demonstrate this statement, but we have not been able to give a general proof which is independent of the coupling scheme on which the field equation is built. Looking at spin 2, one can see that even the minimal electromagnetic interaction becomes inconsistent. The loss of consistency for the minimal electromagnetic interaction going from $s=\frac{3}{2}$ to $s=2$ could create the suspicion that except for couplings to scalar fields, if one replaces m by $m-\phi$ in the free-field equation, all interactions for $s\geqslant 2$ are inconsistent. However, as was pointed out by Federbush,[13] for $s=2$ and electromagnetic coupling one is able to add to the minimal interaction a "consistency completion."

For the quantized version the difference of the propagation case versus the Minkowski case implies that the commutators for boson fields (anticommutators for fermion fields) do not vanish outside the light cone or vanish outside a cone *contained in the light cone.*[16] Both cases ("noncausal" and "supercausal") are in contradiction with the conventional concept of Einstein causality for spacelike separated observables.

The natural question to ask therefore is whether there are other types of field equations (i.e., without subsidiary equations) which are free of causality problems. Consider, for example, the already mentioned Joos-Weinberg equation. We modify the corresponding Yang-Feldman equation by writing

$$\psi(x)=\psi_{\text{in}}(x)+\int\hat{G}_R{}^0(x-x')J(x')\psi(x'),$$

where $G_R{}^0=L_R{}^{-1}$ with the poles in addition to $p^2=m^2$ being projected out, i.e.,

$$G_R{}^0(x)=C\int\frac{L^*(p)}{P_\epsilon{}^2-m^2}e^{ipx}d^4p,$$

with

$$L=\begin{pmatrix}-1 & D(im^{-1}(\partial^0\sigma_0-\partial^i\sigma_i))\\ D(im^{-1}\partial^\mu\sigma_\mu) & -1\end{pmatrix},$$

$$L^*=\begin{pmatrix}1 & D(im^{-1}(\partial^0\sigma_0-\partial^i\sigma_i))\\ D(im^{-1}\partial^\mu\sigma_\mu) & 1\end{pmatrix}.$$

One can show that this new Yang-Feldman equation (which does not bring in unphysical solution for external interactions) formally leads to Weinberg's S-matrix rules:

$$S=T\exp\left[-i\int\psi^*(x')J(x')\psi(x')d^4x'\right],$$

$$\psi^*=\psi^\dagger\gamma_0,\quad \gamma_0=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

In this case there are no *obvious* difficulties with causality because the field equation belonging to the modified Yang-Feldman equation is $(\partial_\mu\partial^\mu+m^2)\psi=cL^*J(x)\psi$ and is therefore causal outside the region of interaction. However, it does *not* belong to the class of *strictly causal* equations because in the interaction region the hyperbolic operator L^* is dominating over the Klein-Gordon operator. An investigation of such non-strictly causal equations has not been carried out. Another difficulty is the requirement of a conserved inner product. In the interaction-free case, we have

$$(\psi,\psi)=\int\tilde{\psi}(p)^\dagger\gamma_0\tilde{\psi}(p)\frac{d^3p}{2p_0}=i\int\psi^*(\mathbf{x})\overleftrightarrow{\partial_0}\psi(x)d^3x\,.$$

We have not been able to find an interaction J and an expression for (ψ,ψ) which agree with the above for $J=0$, such that (ψ,ψ) is conserved.

The original (unprojected) Joos-Weinberg equations have a conserved current at the expense of the occurrence of unphysical particles. The modified Joos-Weinberg equations have physical particles but do not seem to lead to a conserved current. Closely related to this is the trouble with generalizing the Feynman–Gell-Mann equations for the electromagnetic interaction of an $s=\frac{1}{2}$ particle to higher spins. Using the current of the original Joos-Weinberg equation and rewriting it in terms of $(2s+1)$ components Φ (i.e., following the same procedure which leads from the $s=\frac{1}{2}$ Dirac formulation to the Feynman–Gell-Mann formalism), one easily checks that there is no interaction of the form $(\Box+m^2)\Phi=J\cdot\Phi$ which keeps the current conserved. Of course, one can rescue the conservation by using the current

$$j_\mu(x)=\bar{\Phi}^{\dot\alpha}(x)\overleftrightarrow{\partial_\mu}\Phi_\alpha(x)+\text{H.c.}$$

if one does not try to relate the $\Phi^{\dot\alpha}$ to Φ_α. This means that one introduces two independent spinor fields. This causes the well known "parity doubling" of states. In the case of $s=\frac{1}{2}$ this is equivalent to having two $s=\frac{1}{2}$ particles, one with positive and the other with negative metric. Because of the metric trouble (the interaction mixes the different states), this idea of parity doubling must also be rejected.

Note added in proof. After the completion of this work we learned that Dr. Bongaarts has obtained analogous results as to the unitarity of the S-matrix for the case of spin $\frac{1}{2}$ [P. J. M. Bongaarts, Ann. Phys. (N. Y.) **56**, 108 (1970)].

APPENDIX

The following lemma is important for the proof of existence for the solutions of the partial differential equations (A1), (11a), and (14) for a particle of spin zero in an external electromagnetic field $(A^0=V,\mathbf{A})$. The precise assumptions are stated below. The equation of motion is given in the two-component formalism:

$$i\partial_t\Phi=(H_0+H_1)\Phi\,,\quad \Phi=\begin{pmatrix}\Phi_0\\ \Phi_1\end{pmatrix},$$
$$H_0=i\begin{pmatrix}0 & 1\\ -K^2 & 0\end{pmatrix},\quad K=+(1-\Delta)^{1/2}, \tag{A1}$$
$$H_1=\begin{pmatrix}V & 0\\ -i\mathbf{A}^2+\{\partial_l,A^l\} & V\end{pmatrix},\quad A^0=V\,.$$

Φ is an element of the Hilbert space X with the energy norm $(5)^{1/2}$ derived from the scalar product

$$(\Phi,\Phi)=(K\Phi_0,K\Phi_0)+(\Phi_1,\Phi_1)\,. \tag{A2}$$

The domain of H_0 is the set

$$\Delta(H_0)=\{\Phi\in X\,|\,H_0\Phi\in X\}\,.$$

H_0 is self-adjoint. We state now a lemma which was used previously.

Lemma. For fixed t let A^μ and $\partial_\nu A^\mu$ be elements of $L_\infty(d^3x)$. Furthermore, let the Fourier transforms of A^0 and $\partial_\nu A^0$ be in $L_1(d^3x)$. Then $H_1(t)$ is a bounded operator in X.

Proof. It is sufficient to estimate $(\phi,H_1\psi)$ in terms of $\|\Phi\|\,\|\psi\|$:

$$(\Phi,H_1\psi)=(K\Phi_0,KV\psi_0)+(\Phi_1,V\psi_1)-i(\Phi_1,\mathbf{A}^2\psi_0)+(\Phi_1,(\partial_lA^l)\psi_0)\,. \tag{A3}$$

The first term can be estimated if KVK^{-1} is a bounded operator in $L^2(d^3x)$. But this follows from the estimate

$$\left|\int d^3p\tilde{\Phi}(\mathbf{p})\left(\frac{1+\mathbf{p}^2}{1+\mathbf{q}^2}\right)^{1/2}\tilde{V}(\mathbf{p}-\mathbf{q})\tilde{\psi}(q)\right|$$
$$\leqq\int d^3s\,|\tilde{V}(\mathbf{s})|\,F(\mathbf{s})\|\Phi\|_2\|\psi\|_2\,, \tag{A4}$$
$$F(\mathbf{s})=\sup_{\mathbf{p}}\left|\frac{1+\mathbf{p}^2}{1+(\mathbf{p}-\mathbf{s})^2}\right|^{1/2},$$
$$F(\mathbf{s})\leq(4+\mathbf{s}^2)^{1/2}\,.$$

The existence of the right-hand side of (A4) follows from the assumption

$$\tilde{V}\in L_1\quad\text{and}\quad p_\mu\tilde{V}\in L_1\,. \tag{A5}$$

The second term is bounded again by (A5):

$$|(\Phi_1,V\psi_1)|\leqq\|V\|_\infty\|\Phi_1\|_2\|\psi_1\|_2\,. \tag{A6}$$

The third term can be treated by the following rough estimate:

$$|(\Phi_1,\mathbf{A}^2\psi_0)|\leqq\|\mathbf{A}^2\|_\infty\|\Phi_1\|_2\|K\psi_0\|_2\,. \tag{A7}$$

Finally, the last term is

$$|(\Phi_1,\partial_lA^l\psi_0)|\leqq(\|\partial_lA^l\|_\infty+3(\sum_l\|A^l\|_\infty))\times\|\Phi_1\|_2\|K\psi_{0-2}\,. \tag{A8}$$

From Eqs. (A4)–(A8) one obtains the inequality

$$|(\Phi,H_1\psi)\leq\text{const}\,\|\Phi\|\,\|\psi\|\,.$$

[16] Although we only explicitly established the connection between the commutator and the c-number Cauchy propagator in the case of a scalar equation, such a connection can be shown to exist in general.