# Symmetries Imposed on Two-Particle Systems. III. Energy Dependence of Partial-Wave Amplitudes

WILLIAM H. KLINK AND GRANVILLE J. SMITH II*
*Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52240*
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From the assumption that the $S$ operator transforms as an irreducible tensor operator under an approximate symmetry group, called a two-particle symmetry group, it is shown that partial-wave scattering amplitudes can be calculated. Two models, based on different two-particle symmetry groups, are presented. Investigation of the first model makes it clear that an energy-dependent amplitude can be calculated, but the amplitude cannot be identified as a unique partial-wave amplitude. This problem suggests the investigation of the second model, which is then used to calculate $S$-wave proton-proton elastic scattering phase shifts. Further research along these lines is suggested.

## I. INTRODUCTION

A BASIC problem in high-energy physics is the formulation of models capable of predicting the structure of scattering amplitudes that describe strong-interaction scattering processes. Since no fundamental theory of strong interactions exists, many approaches towards the solution of this problem are being investigated.

It is natural that any attempt to construct a model of a scattering amplitude begin with some consideration of the model-independent features of the scattering amplitude. It is generally believed that scattering amplitudes must satisfy crossing, analyticity, and unitarity.[1] These model-independent properties are derivable from relativistic quantum field theory for certain special scattering processes (e.g., pion-nucleon scattering)[2]; however, it is not generally considered essential that these properties be derivable from relativistic quantum field theory, as many physicists assume these properties as fundamental.[3]

A great deal of work has been done in an attempt to find a model that satisfies these three conditions. The recently developed Veneziano model[4] satisfies analyticity and crossing, but it does not satisfy unitarity.[5] Dispersion models[6] satisfy analyticity and unitarity, but they do not satisfy crossing; a similar statement can be made about the Regge-pole model.[7]

In addition to analyticity, crossing, and unitarity, further restrictions on the structure of scattering amplitudes come from conservation laws and symmetry arguments. It is well known that Lorentz and translational invariance imply that scattering amplitudes are Poincaré scalars; also, the conservation of internal quantum numbers suggests that they are at least "approximate" $SU(3)$ scalars. These conditions have been used to analyze scattering amplitudes in several ways. For example, from the assumption that the scattering amplitude is a Poincaré scalar it is possible to derive the Jacob-Wick partial-wave expansion[8] and Toller's expansions.[9] Also it has been postulated that scattering amplitudes might approximately satisfy a "higher symmetry." This idea was used in the formulation of Barut's $O(4,2)$ model,[10] and in the two-particle symmetry model proposed by Klink.[11]

Now the point is that though much is known about the properties of scattering amplitudes, yet the problem of mathematically determining the structure of scattering amplitudes is not solved. This means, for example, that the energy dependence of scattering amplitudes has not been theoretically predicted. It is partially the purpose of this paper to discuss the two-particle symmetry model and to demonstrate how this model might be used to calculate the energy dependence of partial-wave scattering amplitudes.

The two-particle symmetry model is developed from two assumptions. First, it is assumed that the two-particle space of the direct product group $\mathcal{P}\otimes K$ ($\mathcal{P}$ denotes the Poincaré group and $K$ denotes an internal symmetry group) is spanned by a representation space associated with an irreducible unitary representation of a group $G$. $G$, which will be called the two-particle symmetry group, must contain $\mathcal{P}\otimes K$ as a subgroup. The second assumption is that the $T$ operator transforms as a tensor under $G$ and as a scalar under $\mathcal{P}\otimes K$. These two transformation requirements do not completely specify the transformation properties of the $T$ operator. The remaining transformation labels are fixed by choosing one component of the $T$ operator to correspond to the physical two-particle $T$ matrix. In Sec. II these two assumptions are discussed in detail and it

* Present address: Department of Physics, Grinnell College, Grinnell, Iowa 50112.

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is shown how one might use them to find the energy dependence of partial-wave amplitudes.

Unitarity, analyticity, and crossing[12] will not be discussed in this paper; instead, our main purpose is to consider several specific two-particle symmetry groups and to calculate the energy dependence of partial-wave scattering amplitudes using these groups. This is done in Sec. III. It is found that for these two-particle symmetry groups, partial-wave amplitudes cannot be uniquely identified unless they are $S$-wave amplitudes. Thus proton-proton $S$-wave phase shifts are calculated and compared with the experimental low-energy elastic scattering phase shifts.

Several mathematical problems are discussed in the appendices. In particular, Appendix A deals with the question: How do unitary irreducible representation spaces of $G$ decompose into unitary irreducible representation spaces of $\mathcal{P}\otimes K$ when $G$ is restricted to $\mathcal{P}\otimes K$? A strong condition is found that allows us to say immediately that certain groups are not two-particle symmetry groups. In Appendix D it is shown that $InO(1,n)$ satisfies this condition. The unitary irreducible representations of $InO(1,n)$ are discussed in Appendix B and in Appendix C the Clebsch-Gordan coefficients that must be known in order to find the partial-wave amplitudes are calculated.

## II. REVIEW OF TWO-PARTICLE SYMMETRIES

### A. Two-Particle Symmetry Groups

In this section a method for calculating the energy dependence of partial-wave scattering amplitudes describing the relativistic reaction

$$1+2\rightarrow 3+4 \tag{1}$$

will be discussed. This method is based on a two-particle symmetry group which by definition satisfies two conditions. First, it contains the Poincaré group and some internal-symmetry group as subgroups, that is,

$$G\supset\mathcal{P}\otimes K\,, \tag{2}$$

with $G$, $\mathcal{P}$, and $K$ denoting the two-particle symmetry group, the Poincaré group, and the internal-symmetry group, respectively. The product symbol $\otimes$ means direct product. The second condition is that $G$ have a representation space associated with a unitary irreducible representation that covers the two-particle scattering space of $\mathcal{P}\otimes K$. The implications of these conditions will be made clear after the notion of two-particle spaces has been defined.

Consider single-particle wave functions in a representation space of $\mathcal{P}\otimes K$ labeled by mass, spin, the eigenvalues of the Casimir operators of $K$ (denoted by $m$, $J$, and $I$, respectively), and a complete set of diagonal quantum numbers, the three-momenta, spin component, and diagonal quantum numbers of $K$ (denoted by $\mathbf{p}$, $\sigma$, and $i$, respectively). The norm of the wave function $\psi_{[m_l,J_l,I_l]}(\mathbf{p}_l,\sigma_l,i_l)$ describing the $l$th particle is

$$\|\psi_{[m_l,J_l,I_l]}\|^2=\sum_{\sigma_l=-J_l}^{+J_l}\sum_{i_l}\int\frac{d\mathbf{p}_l}{E_l}\times|\psi_{[m_l,J_l,I_l]}(\mathbf{p}_l,\sigma_l,i_l)|^2<\infty\,, \tag{3}$$

so that $\psi$ is an element in the Hilbert space $\mathcal{H}(m^2,J,I)$ of unitary irreducible representations of $\mathcal{P}\otimes K$.

Now the wave function formed by coupling two noninteracting-particle wave functions [say particles 1 and 2 in Eq. (1)] has the norm

$$\|\psi_{[m_1,J_1,\sigma_1,I_1;m_2,J_2,\sigma_2,I_2]}\|^2=\sum_{J=J_0}^{\infty}\sum_{I=I_0}^{\infty}\int_{(m_1+m_2)^2}^{\infty}dm^2\sum_{\sigma=-J}^{J}\sum_{i}\int\frac{d\mathbf{p}}{E}\times|\psi(m,J,I,\mathbf{p},\sigma,i)|^2<\infty\,, \tag{4}$$

with $E^2=\mathbf{p}^2+m^2$ and $m^2=(p_1+p_2)^2$. $J_0$ and $I_0$ are fixed by the single-particle labels $J_1$, $J_2$ and $I_1$, $I_2$, respectively. The Hilbert space of unitary representations spanned by two-particle states of $\mathcal{P}\otimes K$ is

$$\mathcal{H}(m_1,J_1,I_1,\sigma_1;m_2,J_2,I_2,\sigma_2)=\int_{(m_1+m_2)^2}^{\infty}dm^2\sum_{J=J_0}^{\infty}\sum_{I=I_0}^{\infty}\mathcal{H}(m,J,I)\,. \tag{5}$$

Now denoting the Hilbert space of unitary irreducible representations of $G$ by $\mathcal{H}(\chi)$, it is required that $\chi$, labeling the unitary irreducible representations of $G$, be chosen such that

$$\mathcal{H}(\chi)=\mathcal{H}(m_1,J_1,I_1,\sigma_1;m_2,J_2,I_2,\sigma_2)\,. \tag{6}$$

Note that Eq. (6) implies that $\chi$ is fixed by the single-particle labels of the particles in the two-particle system. Any group that satisfies Eqs. (2) and (6) is by definition a two-particle symmetry group.

Before discussing how two-particle symmetry groups might determine the structure of partial-wave amplitudes, consider the problem of finding a group $G$ satisfying Eqs. (2) and (6). It is not difficult to find many groups that satisfy Eq. (2), but it is not obvious which groups have representations that satisfy Eq. (6). It is, in fact, worthwhile to consider how $\mathcal{H}(\chi)$ reduces into constituents $\mathcal{H}(m,J,I)$ when $G$ is restricted to $\mathcal{P}\otimes K$, since this reduction process will indicate restrictions on the structure of $G$ and hence will simplify the task of choosing $G$.

In Appendix A it is shown that a group[13] $G$ can be decomposed into double cosets with respect to its

[12] Crossing in the two-particle symmetry model has already been discussed. See W. H. Klink, Phys. Rev. **181**, 2027 (1969).

[13] The restriction to groups with induced representations simplifies the mathematical problem and yet allows a wide choice of $G$. Induced representations and the subgroup theorem are discussed by Mackey. See G. W. Mackey, University of Chicago Mathematics Department report, 1955 (unpublished).

inducing subgroup[14] $H$ and its subgroup $\mathcal{P}\otimes K$,

$$G=\bigcup_D Hg_D(\mathcal{P}\otimes K). \tag{7}$$

The double-coset representatives $g_D$ are a function of continuous parameters, one of which can be related to the mass of the two-particle system. If Eq. (6) is satisfied, other continuous double-coset parameters can have no physical interpretation. Thus two-particle symmetry groups have double-coset representatives that are a function of one continuous parameter. This restriction allows one to say (with little work) that certain groups do not qualify as two-particle symmetry groups.

### B. Dynamics from $G$

Two-particle symmetry groups have been introduced in order to make statements about the partial-wave scattering amplitude for the reaction of Eq. (1). This can be done if it is assumed that the $T$ operator transforms as an irreducible tensor operator under $G$ and if it is assumed that one component of the irreducible tensor operator corresponds to the physical two-particle $T$ matrix. There is no apparent criterion for choosing the transformation properties of the $T$ operator except, of course, it must transform as a scalar under $\mathcal{P}\otimes K$ (excluding any symmetry breaking in $K$). Since the choice of transformation properties of the $T$ operator (equivalently, the choice of the component of the $T$ operator that corresponds to the two-particle $T$ matrix) is arbitrary, it is necessary to treat the choice as a parameter in the model.

We will denote the reduced amplitude resulting from the partial-wave expansion as[11]

$$\mathcal{A}_{J,\sigma_1,\sigma_2,\sigma_3,\sigma_4}(\sqrt{s})=\langle[\chi_f]x,y\,|\,T_1{}^{[\chi_s]}\,|\,[\chi_i]x,y\rangle, \tag{8}$$

with the subscript on $T$ indicating that it is a $\mathcal{P}\otimes K$ scalar and the superscript indicating that it transforms as an irreducible tensor under $G$. $|[\chi]x,y\rangle$ is a basis state in the Hilbert space $\mathcal{H}(\chi)$ with $[\chi]$ the representation labels of $G$, $x$ the set of quantum numbers arising from the subgroups $\mathcal{P}\otimes K$, and $y$ the additional labels necessary to uniquely specify the basis state. Note that if $\mathcal{H}(\chi)$ spans the two-particle space of $\mathcal{P}\otimes K$, $x$ completely labels the basis state.

Now the Wigner-Eckart theorem can be used to reduce the partial-wave amplitude [Eq. (8)] into a Clebsch-Gordan coefficient of $G$ times an unknown coefficient which is a constant for a given reaction:

$$\begin{aligned}\langle[\chi_f]x\,|\,T_1{}^{[\chi_s]}\,|\,[\chi_i]x\rangle=&\langle[\chi_f]x\,|\,[\chi_s]1;[\chi_i]x\rangle\\ &\times\langle[\chi_f]m_3,m_4,J_3,J_4,I_3,I_4,\sigma_3,\sigma_4\|\,T^{[\chi_s]}\\ &\times\|[\chi_i]m_1,m_2,J_1,J_2,I_1,I_2,\sigma_1,\sigma_2\rangle.\end{aligned} \tag{9}$$

$\langle[\chi_f]\|T^{[\chi_s]}\|[\chi_i]\rangle$ is a function of the representation labels of $G$, but $[\chi_i]$ and $[\chi_f]$ are partially fixed by single-particle quantum numbers of the particles in Eq. (1) and $[\chi_s]$ is assumed fixed[15]; therefore $\langle[\chi_f]\|\times T^{[\chi_s]}\|[\chi_i]\rangle$ is a constant for a given reaction. This implies that all of the dynamics in Eq. (8) is contained in the Clebsch-Gordan coefficient $\langle[\chi_f]x|[\chi_s]1;[\chi_i]x\rangle$ of $G$.

Note that the matrix element [Eq. (8)] can be uniquely associated with a partial-wave scattering amplitude that describes a *specific* reaction, since $[\chi_i]$ and $[\chi_f]$ are fixed by single-particle quantum numbers [see Eq. (6)], and $x$ is the set of all two-particle quantum labels. Thus, the usual notation for a partial-wave amplitude $\mathcal{A}_{J,\eta}(\sqrt{s})$ can be used for Eq. (8); the two-particle energy, $\sqrt{s}=[(p_1+p_2)^2]^{1/2}$, and the total angular momentum $J$ are in the set $x$, whereas the degeneracy labels $\eta$ (i.e., single-particle helicity or spin-component labels) come from $[\chi_i]$ and $[\chi_f]$.

In this work we will restrict our attention to low-energy elastic scattering, below any inelastic thresholds so that, from unitarity, it is possible to write $\mathcal{A}_{J,\eta}(\sqrt{s})$ as

$$\mathcal{A}_{J,\eta}(\sqrt{s})=e^{i\delta_J(\sqrt{s})}\sin\delta_J(\sqrt{s}). \tag{10}$$

Using Eqs. (9) and (10), the phase shifts $\delta_J(\sqrt{s})$ can be calculated and compared with the experimental phase shifts. Note, however, that Eq. (9) actually holds for any sort of two-body reaction, elastic or not.

It should be emphasized that any statement about the dynamics of a scattering system depends upon $G$ and the assumed transformation properties $[\chi_s]$ of the $T$ operator. $[\chi_s]$ can be considered a parameter in the model at this point, but when explicit two-particle symmetry groups are considered we will find that $[\chi_s]$ is not completely arbitrary. Also, no statement can be made about the static properties of the particles in the scattering system, since this information is used as input into the model.

## III. SPECIFIC TWO-PARTICLE SYMMETRY GROUPS

### A. $InSL(n,c)$

The Poincaré group can be written in matrix form[16] as

$$\mathcal{P}=\left\{\begin{pmatrix}\Lambda & H(a)\Lambda^{-1+}\\ 0 & \Lambda^{-1+}\end{pmatrix}\right\}, \tag{11}$$

with $\Lambda\in SL(2,c)$ and $H(a)$ a $2\times2$ Hermitian matrix. A group with similar structure is the semidirect product group $InSL(n,c)$:

$$InSL(n,c)=\left\{\begin{pmatrix}\Gamma & H(A)\Gamma^{-1+}\\ 0 & \Gamma^{-1+}\end{pmatrix}\right\}, \tag{12}$$

[14] How to get the inducing subgroup $H$ is given by W. H. Klink, in *Lectures in Theoretical Physics XID*, edited by K. T. Mahanthappa and W. E. Brittin (Gordon and Breach, New York, 1969).

[15] Ideally $[\chi_s]$ would remain the same for all reactions.

[16] See, for example, W. H. Klink, J. Math. Phys. **10**, 1477 (1969).

TABLE I. Little groups of certain representations of $InO(1,3)$, $InO(1,4)$, and $InO(1,5)$ are given along with the invariants that label the representations (representation labels) and the eigenvalues of complete sets of diagonal operators (diagonal labels) in the representation spaces.

| Group | $InO(1,3)$ = Poincaré group | | | | $InO(1,4)$ | | | $InO(1,5)$ | |
|---|---|---|---|---|---|---|---|---|---|
| Representation | Timelike | Spacelike | Lightlike | Null-like | Timelike | Spacelike | Null-like | Timelike | Spacelike |
| Standard vector | $\hat{p}^T=(m,0,0,0)$ | $\hat{p}^T=(0,0,0,\rho)$ | $\hat{p}^T=(\omega,0,0,\omega)$ | $\hat{p}\equiv 0$ | $\hat{p}^T=(M,0,0,0,0)$ | $\hat{p}^T=(0,0,0,0,\rho)$ | $\hat{p}\equiv 0$ | $\hat{p}^T=(M,0,0,0,0,0)$ | $\hat{p}^T=(0,0,0,0,0,\rho)$, |
| Little group | $O(3)$ | $O(1,2)$ | $E_2$ | $O(1,3)$ | $O(4)$ | $O(1,3)$ | $O(1,4)$ | $O(5)$ | $O(1,4)$ |
| Representation labels | $m^2, J=0,\frac{1}{2},1,\ldots$ | $\rho^2, J$; $\mathrm{Re}J=\frac{1}{2}$; $-\infty\leq \mathrm{Im}J\leq\infty$ for principal series | $\eta$; $0\leq\eta\leq\infty$; $\mathrm{Im}\eta=0$ for principal series | $\beta$; $\mathrm{Re}\beta=0$; $-\infty\leq\mathrm{Im}\beta\leq\infty$ for principal series; $j_0=0,\frac{1}{2},1,\ldots$ | $M^2, n$; $j_0=0,\frac{1}{2},1,\ldots$; $n\geq j_0$ | $\rho^2$; $j_0=0,\frac{1}{2},1,\ldots,\beta$; $\mathrm{Re}\beta=0$; $-\infty\leq\mathrm{Im}\beta\leq\infty$ for principal series | $k_0, c$; $k_0$ discrete and $c$ continuous, complex | $M^2,\mu,\lambda$ | $k_0, c$; $k_0$ discrete and $c$ continuous, complex |
| Diagonal labels | $\mathbf{p}$, $-J\leq\sigma\leq+J$ | $\mathbf{p}$, $\sigma=0,\pm1,\pm2,\ldots$ or $\sigma=\pm\frac{1}{2},\pm\frac{3}{2},\ldots$ | $\sigma=0,\pm1,\pm2,\ldots$ or $\sigma=\pm\frac{1}{2},\pm\frac{3}{2},\ldots$ | $J=j_0,j_0+1,j_0+2,\ldots$; $-J\leq\sigma\leq J$ | $\mathbf{p}$; $j_0\leq J\leq n$; $-J\leq\sigma\leq+J$ | $\mathbf{p}$; $J=j_0,j_0+1,j_0+2,\ldots$; $-J\leq\sigma\leq+J$ | $j_0=0,\frac{1}{2},1,\ldots$; $n\geq j_0$; $n\geq J\geq j_0$; $-J\leq\sigma\leq+J$ | $j_0=0,\frac{1}{2},1,\ldots$; $n\geq j_0$; $n\geq J\geq j_0$; $-J\geq\sigma\geq+J$ | $j_0=0,\frac{1}{2},1,\ldots$; $n\geq j_0$; $n\geq J\geq j_0$; $-J\geq\sigma\geq+J$ |

with $\Gamma\in SL(n,c)$ and $H(A)$ an $n\times n$ Hermitian matrix. $InSL(n,c)$ contains $\mathcal{P}$ and the internal-symmetry group $SU(n-2)$ as subgroups, but its double cosets contain many continuous parameters[17] and therefore, recalling the argument after Eq. (6), $InSL(n,c)$ is not a useful two-particle symmetry group.

### B. $InO(1,n)$

Consider the semidirect product group $InO(1,n)$:

$$InO(1,n)=\left\{\begin{pmatrix}\Gamma & A\\ 0 & 1\end{pmatrix}\right\},\qquad(13)$$

with $\Gamma\in O(1,n)$, the group of $(n+1)\times(n+1)$ matrices satisfying

$$\Gamma\gamma\Gamma^{+}=\gamma,\quad \gamma=\begin{pmatrix}1 & 0\\ 0 & -I_n\end{pmatrix},$$

$I_n$ the $n$-dimensional identity, and $A$ an $(n+1)$-dimensional vector. $InO(1,n)$ contains $\mathcal{P}$ and $O(n-3)$ as a direct-sum subgroup, $\mathcal{P}\oplus O(n-3)$, and it is not difficult to show that its double cosets defined by Eq. (7) contain one continuous parameter (Appendix D). Thus, $InO(1,n)$ seems a reasonable candidate for a two-particle symmetry group.

### C. $InO(1,4)$

The simplest example arises when $n=4$. The internal-symmetry group, in this case, is the identity. It is obvious that $G\supset\mathcal{P}\oplus 1$, since $\mathcal{P}$ can be written as

$$\mathcal{P}=\left\{\begin{bmatrix}\Lambda & 0 & a\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\},\qquad(14)$$

with $\Lambda\in O(1,3)$ and $a$ a four-vector.

In order to find if a representation space of $G$ satisfies Eq. (6), it is necessary to calculate the representations of $G$. All representations of $G$ can be found using little-group techniques in exactly the same way the representations of $\mathcal{P}$ are found. Since this analysis is well understood for the Poincaré group it is easy to obtain the representations of $G$. The four classes of representations for both groups are listed in Table I [18] (see also Appendix B).

In Appendix D it is shown that the Hilbert space of "timelike" unitary representations of $InO(1,4)$ can be written as

$$\mathcal{H}(U^{[M,n,j_0]})\approx\int_{M^2}^{\infty}dm^2\sum_{J=j_0}^{n}\mathcal{H}(U^{[m,J]}),\qquad(15)$$

[17] There are eight parameters for $n=4$ and more than eight for $n>4$.

[18] For a discussion of the Poincaré group, see J. F. Boyce, R. Delbourgo, A. Salam, and J. Strathdee, International Centre for Theoretical Physics, Trieste, Report No. IC/67/9, 1967 (unpublished). For a discussion of $O(4)$, see B. Kurşunoğlu, *Modern Quantum Theory* (Freeman, San Francisco, 1962).

with $\mathcal{H}(U^{[m,J]})$ the Hilbert space of timelike unitary representations of $\mathcal{P}$. We see that in the representation $U^{[M,n,j_0]}$ the physical mass is continuous from $M^2$ to $\infty$ and the spin occurs in finite towers. In principle, a two-particle symmetry group contains an infinite tower of spins, since one wants to calculate all partial-wave amplitudes in a coherent manner. However, infinite numbers of partial-wave amplitudes (or equivalently, phase shifts) are never used to fit partial and total cross sections so that, in practice, having finite towers of spins imposes no restriction on our calculation.

The physically interesting object is the matrix element [Eq. (8)] which can be written as

$$\mathcal{A}_{J,\sigma_1,\sigma_2,\sigma_3,\sigma_4}(\sqrt{s})=\langle[M_f^2,n_f,j_{0f}]\sqrt{s},\mathbf{p},J,\sigma;\sigma_3,\sigma_4| \times T_1^{[\chi_s]}|[M_i^2,n_i,j_{0i}]\sqrt{s},\mathbf{p},J,\sigma;\sigma_1,\sigma_2\rangle. \quad (16)$$

The representation labels $[n,j_0,M]$ are fixed by the single-particle quantum numbers labeling the particles in the reaction of Eq. (1) in the following way. The c.m. energy squared of the initial system has the spectrum $(m_1+m_2)^2\leq s\leq\infty$ and it is known that $M_i^2\leq s\leq\infty$ [Eq. (D14)]; therefore, $M_i^2$ is chosen to be $M_i^2=(m_1+m_2)^2$. Similarly, $M_f^2$ is fixed at $(m_3+m_4)^2$. The two-particle spin angular momentum equals the minimum value of the total angular momentum but $j_0\leq J$ [Eq. (D14)]; therefore, $j_{0i}$ and $j_{0f}$ are fixed by $|s_1-s_2|\leq j_{0i}\leq|s_1+s_2|$ and $|s_3-s_4|\leq j_{0f}\leq|s_3+s_4|$, with $s_i$, the spin angular momentum of the $i$th particle. Finally, the "principal quantum numbers" $n_i$ and $n_f$ are fixed by the number of partial waves one wants to consider. This number is, in principle, fixed by experiment.

There are four possible ways one can choose $T_1^{[\chi_s]}$ to transform, but interestingly only one yields "reasonable" nontrivial results. This can be seen in the following way. Consider a basis state in a representation space of $G$ with the same transformation properties as $T$ (i.e., $|[\chi]1\rangle$). The tensor product decomposition

$$|[\chi]1\rangle|[M_i^2,n_i,j_{0i}]pJ\sigma\rangle =\sum\int\langle[M_f^2,n_f,j_{0f}]pJ\sigma|[\chi_s]1;[M_i^2,n_i,j_{0i}]pJ\sigma\rangle \times|[M_f^2,n_f,j_{0f}]pJ\sigma\rangle \quad (17)$$

always results in a condition

$$p_i+p_s=p_f, \quad (18)$$

where $p_i$ and $p_f$ [$p$ denotes a five-vector; see Eq. (C3)] are related to the four-momenta in the c.m. frame by

$$p_i=\begin{pmatrix}\sqrt{s}\\0\\0\\0\\(s-M_i^2)^{1/2}\end{pmatrix},\quad p_f=\begin{pmatrix}\sqrt{s}\\0\\0\\0\\(s-M_f^2)^{1/2}\end{pmatrix}.$$

Obviously, energy and momenta are not conserved if the four-momentum content of $p_s$ is nonzero so that it is necessary to restrict $p_s$ to

$$p_s=\begin{pmatrix}0\\0\\0\\0\\x\end{pmatrix}.$$

Referring to Table I it is obvious that $p_s$ must be "spacelike" or "null." If $p_s$ is spacelike, then in the c.m. frame Eq. (18) becomes

$$\begin{pmatrix}\sqrt{s}\\0\\0\\0\\(s-M_i^2)^{1/2}\end{pmatrix}+\begin{pmatrix}0\\0\\0\\0\\\rho\end{pmatrix}=\begin{pmatrix}\sqrt{s}\\0\\0\\0\\(s-M_f^2)^{1/2}\end{pmatrix}, \quad (19)$$

so that energy and momenta are conserved. However, equating the fifth components in Eq. (19) fixes $s=(M_i^2-\rho^2-M_f^2)^2/4\rho^2+M_i^2$. Thus all transformation properties of $T_1^{[\chi_s]}$ are unreasonable (i.e., nonconservation of energy) or trivial (i.e., $s$ fixed) unless $[\chi_s]$ is a "null-like" representation.

Thus, we want to calculate the Clebsch-Gordan coefficient that occurs in the reduction of the matrix element [Eq. (16)] when $[\chi_s]$ is null-like. This has been done in Appendix C, where it is shown [Eq. (C16)] that

$$\langle[M_f^2,n_f,j_{0f}],\sqrt{s},\mathbf{p},J,\sigma,n_s',j_{0s}',j_s'|[M_i^2,n_i,j_{0i}], \sqrt{s},\mathbf{p},J,\sigma;[k_0,c],n_s,1\rangle =D_{n_s',j_{0s}',j_s',0;n_s,0,0,0}{}^{[k_0,c]}(\Gamma_c)\delta(\hat{p}-\Gamma_c p), \quad (20)$$

with $\Gamma_c$ a representative of $O(1,4)/O(4)$ defined by $\Gamma_c p=\hat{p}$, and $D^{[k_0,c]}(\Gamma_c)$ an $O(1,4)$ representation matrix defined by

$$D_{n_s',j_{0s}',j_s',0;n_s,0,0,0}{}^{[k_0,c]}(\Gamma_c)=\langle[k_0,c]n_s',j_{0s}',j_s',\sigma_s'=0| \times U(\Gamma_c)|[k_0,c]n_s,j_{0s}=j_s=\sigma_s=0\rangle. \quad (21)$$

$U(\Gamma_c)$ is a unitary operator associated with $\Gamma_c$ and $|[k_0,c]n_s,j_{0s},j_s,\sigma_s\rangle$ is a basis state in the representation space of $O(1,4)$ (see Table I).

Equation (20) is a disappointing result since all quantum numbers that label the reaction (except $M^2$) have "dropped out" of the Clebsch-Gordan coefficient.[19] On the other hand, the Clebsch-Gordan coefficient does have a nontrivial energy dependence, since the parameters in $\Gamma_c$ are functions of the four-momenta, $\hat{p}=\Gamma_c p$. An explicit functional form for the $D$ functions can be obtained by analytic continuation of the Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$; however, since it is not possible to make any comparisons with experiment, these functions will not be discussed.[20]

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[19] If $j_s=0$, then $J=0$. If $j_s\neq 0$, then $J$ is not uniquely determined.

[20] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 2. See also D. A. Akyeampong, J. F. Boyce, and M. A. Rashid, International Centre for Theoretical Physics, Trieste, Report No. IC/67/74, 1967 (unpublished).

It is worth mentioning several conclusions one can make about $InO(1,n)$ from studying the group $InO(1,4)$. First, the timelike representations always contain the timelike representations of $\mathcal{P}$, and therefore it is these representations that must be tested to see if Eq. (6) is valid. Second, choosing the $T$ operator to transform as a timelike or "lightlike" representation will always violate conservation of energy. Also, if $T$ transforms as a null representation, then the angular momentum label will never appear in the Clebsch-Gordan coefficient [Eq. (9)] so that all partial waves that occur in the representation are predicted to be the same. Finally, the matrix elements [Eq. (9)] are nontrivial functions of the energy if $T$ transforms as a spacelike representation and $n>4$. Each of these statements can be proved by simple generalizations of the $InO(1,4)$ analysis; they indicate new prospects for physically interesting two-particle symmetry groups, the simplest being $InO(1,5)$.

### D. $InO(1,5)$

The two-particle symmetry group $InO(1,5)$ contains both $\mathcal{P}$ and $O(2)$ subgroups. $O(2)$ will be considered the internal-symmetry group and its irreducible representations will be denoted by $Q$ (charge or baryon number). The irreducible representations of $InO(1,5)$ are divided into the same four classes as $\mathcal{P}$; the timelike and spacelike irreducible representation labels and a complete set of commuting observables are given in Table I. We have used the fact that an $O(n)$ eigenstate is uniquely labeled by the invariants in the chain $O(n)\supset O(n-1)\supset\cdots\supset O(2)$. The six-momentum $P$ ($|P|^2=M^2$) is related to the two-particle c.m. four-momenta

$$p=\begin{bmatrix}\sqrt{s}\\0\\0\\0\end{bmatrix}$$

by

$$P=\begin{bmatrix}\sqrt{s}\\0\\0\\0\\(s-M^2)^{1/2}\sin\theta\\(s-M^2)^{1/2}\cos\theta\end{bmatrix},\tag{22}$$

with $-\pi\leq\theta\leq\pi$ and $M^2\leq s\leq\infty$.

The timelike representations of $InO(1,5)$ contain the timelike representations of $\mathcal{P}$, and in Appendix D, Eq. (D18), the Hilbert-space decomposition with respect to representations of $\mathcal{P}\oplus O(2)$ is given as

$$\mathcal{H}(U^{[M,\mu,\lambda]})\approx\int_{M^2}^{\infty}dm^2\sum_{n,j_0}\sum_{J=j_0}^{n}\sum_{Q=-\infty}^{\infty}\mathcal{H}(U^{[m,J,Q]}).\tag{23}$$

The sums on $n$, $j_0$ and $J$ are fixed by $(\mu,\lambda)$, the irreducible representation labels of $O(5)$, and can easily be determined by investigating $O(5)$ weight diagrams.

The $T$ operator must transform as a "spacelike" representation under $G$ and as a scalar under $\mathcal{P}\oplus O(2)$. These transformation requirements do not completely specify the transformation properties of $T$ and it is necessary, in addition, to assume some value for $n_s$. It is convenient to pick $n_s=0$.

Now the matrix element [Eq. (8)] can be written using the Wigner-Eckart theorem and the Clebsch-Gordan coefficients derived in Appendix C, Eq. (C25):

$$\begin{aligned}&\langle[M_f^2,\mu_f,\lambda_f]\mathbf{P}_f,n,j_0,J,\sigma|T_1^{[k_0,c,\rho^2]}\\&\quad\times|[M_i^2,\mu_i,\lambda_i]\mathbf{P}_i,n,j_0,J,\sigma\rangle\\&=\delta(P_f-P_i-P_s)\delta(\hat{P}_i-\tilde{\Gamma}P_i)\delta(\hat{P}_s-\Gamma_D\tilde{\Gamma}P_s)\\&\quad\times D_{0,0,0,0;n,j_0,J,\sigma}{}^{[\mu_f,\lambda_f]}(\mathbf{P}_f,\tilde{\Gamma})D_{0,0,0,0;n_s,0,0,0}{}^{[k_0,c]}\\&\quad\times(\mathbf{P}_s,\Gamma_D\tilde{\Gamma})D_{0,0,0,0;n,j_0,J,\sigma}{}^{[\mu_f,\lambda_i]}(\mathbf{P}_i,\tilde{\Gamma})\\&\quad\times\langle[M_f^2,\mu_f,\lambda_f]\|T^{[k_0,c,\rho^2]}\|[M_i^2,\mu_i,\lambda_i]\rangle,\end{aligned}\tag{24}$$

with $\tilde{\Gamma}$ a representative of $O(1,5)/O(4)$,

$$\Gamma_D=\begin{bmatrix}\cosh D&0&\sinh D\\0&I_4&0\\\sinh D&0&\cosh D\end{bmatrix},$$

and $\sinh D=(M_f^2-M_i^2+\rho^2)/2M_i\rho$. The $O(5)/O(4)$ "rotations" $(\mathbf{P}_i,\tilde{\Gamma})$ and $(\mathbf{P}_f,\tilde{\Gamma})$ and the $O(1,4)/O(4)$ rotations $(\mathbf{P}_s,\Gamma_D\tilde{\Gamma})$ are defined by

$$(\mathbf{P},\Gamma)=\Gamma_c(\Gamma P)\Gamma\Gamma_c^{-1}(P),\tag{25}$$

with $\Gamma_c(P)$ a representative of $O(1,5)/O(5)$ defined by $\Gamma_c(P)\,P=\hat{P}$. The first and last $D$ functions are matrix elements of an $O(5)$ representation and the second $D$ function is an $O(1,4)$ representation matrix element.

All degeneracy parameters have been set equal to zero. As was pointed out in Sec. II B there is no physical motivation for setting these "extra" labels to zero. However, there is a very good practical reason, namely, it is then possible to evaluate explicitly the $D$ functions in Eq. (24). This evaluation gives the energy dependence of the $D$ functions for the special case when the degeneracy parameters are zero.

We are primarily interested in the energy dependence of the $D$ functions, and in order to find this dependence we make use of the fact that these functions are eigenfunctions of the Laplace-Beltrami operator, the differential operator associated with the quadratic Casimir operator

$$\sum_{i=1}^{5}X_i^2=1$$

of $O(5)/O(4)$. In a spherical basis its form is[21]

$$\begin{aligned}&\left\{\frac{1}{\sin^3\alpha}\frac{\partial}{\partial\alpha}\sin^3\alpha\frac{\partial}{\partial\alpha}+\frac{1}{\sin^2\alpha}\right.\\&\left.\quad\times\left[\frac{1}{\sin^2\psi}\frac{\partial}{\partial\psi}\sin^2\psi\frac{\partial}{\partial\psi}+\frac{1}{\sin^2\psi}\nabla^2\right]\right\}F(\alpha,\psi,\theta,\phi)\\&\qquad=\epsilon(\epsilon+3)F(\alpha,\psi,\theta,\phi),\end{aligned}\tag{26}$$

[21] Reference 20, p. 235.

with $\epsilon$ a constant. This is a separable differential equation and has the solution

$$F_{n,J,\sigma}{}^{\epsilon}(\alpha,\psi,\theta,\varphi)=\frac{A_\epsilon(\sin\alpha)^n}{T_\epsilon}\frac{d^{n+1}}{d\cos\theta^{n+1}}[P_{\epsilon+1}(\cos\alpha)]\times\frac{i^J(\sin\psi)^J}{T_n}\frac{d^{J+1}(\cos n\psi)}{d\cos\psi^{J+1}}Y_\sigma{}^J(\theta,\varphi)\,,\quad(27)$$

with $A_\epsilon$ a constant,

$$T_\epsilon=[\epsilon^2(\epsilon^2-1^2)(\epsilon^2-2^2)\cdots(\epsilon^2-n^2)]^{1/2},$$

and $T_n=[n^2(n^2-1^2)\cdots(n^2-J^2)]^{1/2}$. Now these functions are equal to the special $D$ functions

$$F_{nJ\sigma}{}^{\epsilon}(\alpha,\psi,\theta,\varphi)=D_{0,0,0,0;n,0,J,\sigma}{}^{[\mu,\lambda]}(\mathbf{P},\tilde{\Gamma})\qquad(28)$$

with $\epsilon$ directly related to $[\mu,\lambda]$.[22] The energy dependence is known if a connection between the four-momenta and the variables $\alpha$, $\psi$, $\theta$, and $\varphi$ can be made.

In order to see this connection, consider the action of $R$, a representative of $O(5)/O(4)$, on the vector $P_i{}^T\equiv(\sqrt{s},0,0,0,0,(s-M_i{}^2)^{1/2})$ ($T$ denotes transpose)[23]:

$$R\tilde{P}_i=\begin{bmatrix}\sqrt{s}\\(s-M_i{}^2)^{1/2}\sin\alpha\sin\psi\sin\theta\cos\varphi\\(s-M_i{}^2)^{1/2}\sin\alpha\sin\psi\sin\theta\sin\varphi\\(s-M_i{}^2)^{1/2}\sin\alpha\sin\psi\cos\theta\\(s-M_i{}^2)^{1/2}\sin\alpha\cos\psi\\(s-M_i{}^2)^{1/2}\cos\alpha\end{bmatrix}.\qquad(29)$$

This is the most general form for the six-momenta $P$ in a spherical basis, the same basis used to write Eq. (26). Next consider the action of $(\mathbf{P}_i,\tilde{\Gamma})$, a representative of $O(5)/O(4)$, on $\tilde{P}_i$:

$$(\mathbf{P}_i,\tilde{\Gamma})\tilde{P}_i=\begin{bmatrix}\sqrt{s}\\0\\0\\-M_i\sinh\gamma\cosh\beta\\-M_i\sinh\beta\end{bmatrix},\qquad(30)$$

with $\sqrt{s}=M_i\cosh\gamma\cosh\beta$ and $\coth\beta=-\coth D$.[24] $(\mathbf{P}_i,\tilde{\Gamma})\tilde{P}_i$ is obviously not a general six-momentum vector because $\tilde{\Gamma}$ has been constructed such that the three-momentum content of $P_i$ is zero and the four-momentum of $P_s$ (i.e., the first four components of $P_s$) is zero. Comparing Eqs. (29) and (30), we see that $\theta=\varphi=\psi=0$ in the c.m. frame and

$$\cos\alpha=\tfrac{1}{2}\rho/(s-M_i{}^2)^{1/2}.\qquad(31)$$

The above argument for calculating the energy dependence has been carried through for the rotation $(\mathbf{P}_i,\tilde{\Gamma})$, but it also holds for $(\mathbf{P}_f,\tilde{\Gamma})$ and $(\mathbf{P}_s,\Gamma_D\tilde{\Gamma})$. However, the $O(1,4)$ matrix elements $D^{(k_0,c)}(\mathbf{P}_s,\Gamma_D\tilde{\Gamma})$ do not satisfy Eq. (26) but instead are eigenfunction solutions of

$$\left(\frac{1}{\sinh^3\alpha}\frac{\partial}{\partial\alpha}\sinh^3\alpha\frac{\partial}{\partial\alpha}+\frac{1}{\sinh^2\alpha}\nabla_4{}^2\right)G(\alpha,\psi,\theta,\varphi)=IG(\alpha,\psi,\theta,\varphi)\,,\quad(32)$$

with $I$ a constant and $\nabla_4{}^2$ the four-dimensional Laplace-Beltrami operator. The solution to this equation is a hypergeometric function with a hyperbolic argument. It is not necessary to evaluate these functions, since the $O(1,4)$ matrix element

$$D_{0,0,0,0;0,0,0,0}{}^{[k_0,c]}(\mathbf{P}_s,\Gamma_D\tilde{\Gamma})$$

is a constant. This can be seen by writing the solution of Eq. (32) as a Jacobi polynomial and then setting all indices to zero.[25] Thus, the final form for the matrix element [Eq. (24)] in the c.m. frame is

$$\langle[M_f{}^2,\mu,\lambda]\sqrt{s},n,j_0=0,J,\sigma|\times T_1{}^{[k_0,c,\rho]}|[M_i{}^2,\mu,\lambda]\sqrt{s},n,j_0=0,J,\sigma\rangle=\frac{A(\sin\alpha_f)^n(\sin\alpha_i)^n}{(T_\epsilon)^2(T_n)^2}\frac{d^{n+1}}{d\cos\theta^{n+1}}P_{\epsilon+1}(\cos\alpha_i)\times\frac{d^{n+1}}{d\cos\theta^{n+1}}P_{\epsilon+1}(\cos\alpha_f)\,,\quad(33)$$

with $A$ a constant that contains the $O(1,4)$ matrix element, the reduced matrix element, and the normalization constant $A_\epsilon$.

If a specific reaction is considered, then several statements can be made about the matrix element, Eq. (33). Thus, consider proton-proton elastic scattering; this fixes $M_i{}^2=M_f{}^2=(2m_p)^2$. The spin angular momentum is either 0 or 1, so representations of $O(5)$ must be picked that contain the minimum angular momentum representations $j_0=0$, 1. We identify $j_0$ as the spin angular momentum, since $j_0\leq J$. The angular momentum con-

[22] $\epsilon$ is uniquely related to a specific subset of all $O(5)$ representations. See Table II.

[23] We make use of the theory of homogeneous spaces (see Appendix B) in this calculation. Obviously $O(5)$ decomposes into right cosets with respect to

$$\begin{pmatrix}O(4)&0\\0&1\end{pmatrix}$$

as

$$O(5)=\bigcup_c\begin{pmatrix}O(4)&0\\0&1\end{pmatrix}R_c,$$

with $R_c$ a representative of $O(5)/O(4)$. We assume that $R$ and $(P,\tilde{\Gamma})$ are associated with the *same* arbitrary point $c$ in the homogeneous space $O(5)/O(4)$ so that $x$ is defined by $x=x_0R=x_0(P,\tilde{\Gamma})$. $x_0=(0,0,0,0,1)$ is the stabilizer with respect to

$$\begin{pmatrix}O(4)&0\\0&1\end{pmatrix}.$$

Now $R$ and $(P,\tilde{\Gamma})$ are parametrized differently, $R$ in terms of $(\lambda,\alpha,\theta,\varphi)$ and $(P,\tilde{\Gamma})$ in terms of the components of the four-momenta, and it is the relation between the parameters that we wish to find. This is readily done by considering $x_0R=x_0(P,\tilde{\Gamma})$.

[24] These relations come from the $\delta$-function conditions $P_i=\tilde{\Gamma}\tilde{P}_i$ and $P_s=\Gamma_D\tilde{\Gamma}P_s$. See Appendix C, after Eq. (C25). Note that $\alpha=\beta_4$, $\beta=\alpha_5$, and $\gamma=\alpha_4$.

[25] Reference 20, pp. 169–175.

TABLE II. Lower-dimensional representations of $O(5)$ are analyzed with respect to their $O(4)$ and angular momentum content. The label $\epsilon$ picks out those representations of $O(5)$ which contain the $O(4)$ representations $\{(n,0), (n-1, 0), \ldots, (0,0)\}$.

| $O(5)$ representation dimension | $O(4)$ representation dimension | $\epsilon$ | $(n,j_0)$ | $J$ |
|---|---|---|---|---|
| 1 | 1 | 0 | (0,0) | 0 |
| 4 | 4 | ··· | (1,0) | 0, 1 |
| 5 | 1, 4 | 1 | (1,0), (0,0) | 0, 0, 1 |
| 10 | 9, 1 | ··· | (2,0), (0,0) | 0, 0, 1, 2 |
| 14 | 9, 4, 1 | 2 | (2,0), (1,0), (0,0) | 0, 0, 0, 1, 1, 2 |
| 16 | 8, 8 | ··· | (2,1), (2,1) | 1, 1, 2, 2 |
| 20 | 16, 4 | ··· | (3,0), (1,0) | 0, 0, 1, 1, 2, 3 |
| 30 | 16, 9, 4, 1 | 3 | (3,0), (2,0), (1,0), (0,0) | 0, 0, 0, 0, 1, 1, 1, 2, 2, 3 |

tent of the lower-dimensional representations of $O(5)$ is given in Table II.

Now Eq. (33) is valid for $j_0=0$; therefore, it is possible to calculate the amplitudes $\mathcal{A}_{\epsilon n j_0}{}^d=\mathcal{A}_{\epsilon n 0}{}^d$, where the superscript indicates the dimension of the $O(5)$ representation. All partial waves that occur in the representation are predicted to be the same unless $n=0$ (i.e., $J=0$), since the angular momentum dependence has dropped out of the amplitude. Thus, the amplitudes $\mathcal{A}_{\epsilon 00}{}^d$ are unique $S$-wave amplitudes whereas $\mathcal{A}_{\epsilon 10}{}^d$, for example, is not unique ($J=0, 1$). The phase shifts associated with $\mathcal{A}_{100}{}^5$, $\mathcal{A}_{200}{}^{14}$, and $\mathcal{A}_{110}{}^5$ are plotted as functions of the energy in Figs. 1–3.[26] We have also looked at phase shifts associated with $\mathcal{A}_{\epsilon n 0}{}^d$ when $n$ is large ($n\geq 5$) and have found that they are nearly constant at all energies.

Ideally, the spacelike mass parameter can be adjusted such that the experimental phase shifts are predicted. We have calculated phase shifts for several values of $\rho$ between 1 and 600 MeV. The maximum value of $\rho$ is restricted by the minimum value of the laboratory energy according to Eq. (34), which comes directly from Eq. (31):

$$\rho_{\max}<2(2m_pE_{\text{lab}})^{1/2}. \tag{34}$$

The phase shifts plotted in Figs. 1–3 have been calculated for $50 \leq E_{\text{lab}} \leq 400$ MeV, except in the case $\rho=1$ MeV. In that case the energy dependence has been checked down to 1 MeV.

### E. $InO(1,n)$, $n>5$

It might be hoped that the angular momentum dependence problem is solved by considering larger groups ($n>5$). This is not the case, and in fact the matrix elements become more ambiguous for larger groups because new eigenvalues arise which have no physical interpretation. Also, the transformation properties of the $T$ operator are less well defined by demanding that it transform as a scalar under $\mathcal{P}\oplus K$ so that a large number of additional assumptions must be made with no physical motivation. Finally, when $\mathcal{P}$ and $K$ are contained as a direct-sum subgroup in $G$, spin-statistics violations arise. That is, one representation of $G$ can contain both half-integer and integer representation of the angular momentum.

[26] Experimental and one-pion-exchange model $S$-wave phase shifts have been taken from M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. 169, 1128 (1968).

## IV. CONCLUSION

It has been shown how a two-particle symmetry model can be used to derive the energy dependence of partial-wave amplitudes. The only parameters in the model come from the quantity $\langle[\chi_f]\|T^{[\chi_s]}\|[\chi_i x\rangle$ [Eq. (9)] and the transformation properties of the $T$ operator under the two-particle symmetry group. However, as was shown in Sec. III, the $T$-operator transformation properties are not completely arbitrary, being restricted by the demand that the $T$ operator transform as a scalar under $\mathcal{P}\otimes K$. It is possible that demanding that $T$ be unitary further restricts its transformation properties. These restrictions would arise if the unitarity condition, which implies that the tensor product space $\mathcal{H}([\chi]\otimes[\chi])$ uniquely contains the identity, is a "stronger" restriction than the $\mathcal{P}\otimes K$ scalar condition.

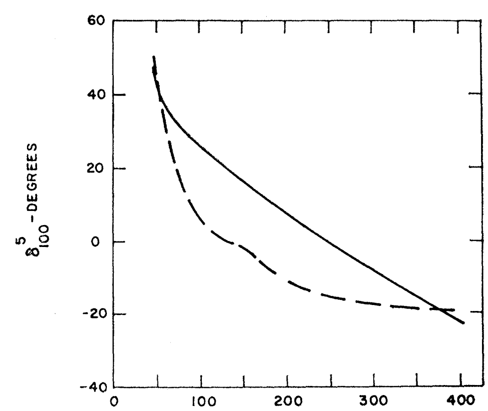


FIG. 1. $S$-wave proton-proton elastic scattering phase shift $\delta_{100}{}^5$ is plotted as a function of beam energy for $\rho=550$ MeV. The point at which $\delta_{100}{}^5$ crosses zero degrees is nearly independent of $\rho$. The experimental phase shift (solid line) is plotted for comparison.

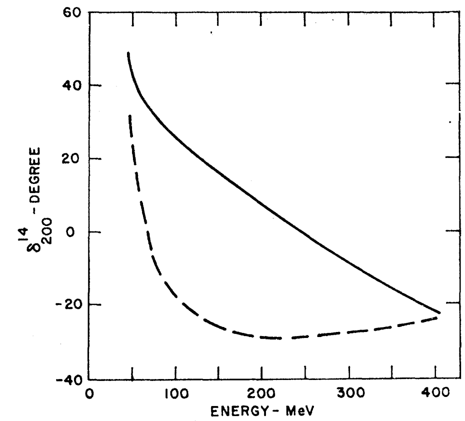


FIG. 2. $S$-wave proton-proton elastic scattering phase shift $\delta_{200}{}^{14}$ is plotted as a function of beam energy for $\rho=550$ MeV. The point at which $\delta_{200}$ crosses zero degrees is nearly independent of $\rho$. The experimental phase shift (solid line) is plotted for comparison.

It should be pointed out that since we have taken a group-theoretical approach towards calculating partial-wave cross sections using unitary representations, such cross sections will always be square-integrable functions of the energy; yet experimentally it seems that partial-wave cross sections are non-square-integrable functions of the energy.[27] This would imply that nonunitary representations be considered somewhat in the spirit in which Toller has used nonunitary representations to get Regge behavior from group theory.[9]

For the $InO(1,n)$ two-particle symmetry group, it has been shown that no unique association between the matrix element Eq. (16) [or Eq. (24)] and a partial-wave amplitude is possible unless the total angular momentum is zero. (See Figs. 1–3 for the prediction of the model for $J=0$.) The angular momentum dependence is lost because the Poincaré group and internal-symmetry group are considered as a direct sum in $G$. It is possible that the angular momentum dependence will not be lost if $\mathcal{P}$ and $K$ are considered as direct-product subgroups of $G$. In addition, it can be shown that spin-statistics violations arise if $\mathcal{P}\oplus K$ is used instead of $\mathcal{P}\otimes K$. Thus, whether the direct-product or direct-sum subgroup is considered appears to be an important consideration and work is being started on the problem of calculating matrix elements for $\mathcal{P}\otimes K$ as a subgroup of $G$.

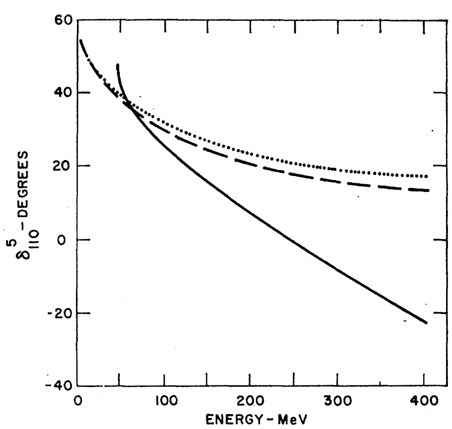


FIG. 3. Proton-proton elastic scattering phase shift $\delta_{110}{}^{5}$ (dashed line) is plotted as a function of beam energy for $\rho=140$ MeV. The one-pion-exchange phase shift (dotted line) and experimental phase shift (solid line) are plotted for comparison.

[27] For some comments on this problem see M. Toller, Nuovo Cimento **37**, 631 (1967).

Finally, it is interesting to note that the threshold behavior of the amplitude Eq. (33) is dominated by the factor $(\sin\alpha)^{2n}$ ($\alpha_i\simeq\alpha_f$ for $\rho\gg1$). $\sin\alpha$ can be written as a function of $s$ and has the singularity $(s-4m^2)^{-n}$ for equal-mass scattering processes. It is known that the kinematic singularity in an equal-mass partial-wave amplitude is proportional to $(s-4m^2)^{-J}$.[28] Thus the kinematic singularity naturally arises if $n=J$ and after the kinematic singularity is removed the amplitude is well behaved near threshold. Since this threshold behavior was not built into the model in any way, we are encouraged to investigate two-particle symmetry models further.

## APPENDIX A: REDUCTION OF $\mathcal{H}(\chi)$

For two-particle symmetry groups that have unitary irreducible representations which can be written as induced representations, Mackey's subgroup theorem can be used to reduce $\mathcal{H}(\chi)$ into constituents $\mathcal{H}(m,J,I)$.[13] Let $H$ be a subgroup of $G$ with representation $L$ defined on a Hilbert space $\mathcal{H}(L)$.[14] The representations of $G$ induced by $L$ are defined on a Hilbert space $\mathcal{H}(U^L)$ of square-integrable functions that map $g\in G$ into $\mathcal{H}(L)$ and have the property $f(hg)=L(h)f(g)$, $h\in H$. The induced representations are defined by

$$U^L(g_0)f(g)=f(gg_0)\,, \tag{A1}$$

where $g_0$ is a fixed element of $G$ and $U^L$ denotes the unitary representations of $G$ induced by the representation $L$ of $H$.

Now $G$ can be decomposed into a union over double cosets with respect to $H$ and $\mathcal{P}\otimes K$:

$$G=\bigcup_D Hg_D(\mathcal{P}\otimes K)\,, \tag{A2}$$

with $g_D$ a double-coset representative. Mackey has shown that the unitary representation $U^L(h')$, $h'$ an element of $\mathcal{P}\otimes K$, decomposes into unitary representation $U^{L_D}(h')$ as a direct integral:

$$U^L(h')\approx\int dD\; U^{L_D}(h')\,, \tag{A3}$$

where $L_D\equiv L(g_Dh_D{}^{-1})$ is a representation of the sub-

[28] J. D. Jackson and G. E. Hite, Phys. Rev. **169**, 1248 (1968).

group $H_D = g_D^{-1} H g_D \cap \mathcal{P} \otimes K$ defined on the Hilbert space $\mathcal{H}(L_D)$. The representation $U^{L_D}(h')$ is defined on the subspace $\mathcal{H}_D(U^L)$ of $\mathcal{H}(U^L)$ labeled by the double-coset parameters $D$. The representation of $\mathcal{P}\otimes K$ induced by $L_D$ is, in general, reducible so that it is necessary to decompose $U^{L_D}$ further into a direct sum of irreducible representations of $\mathcal{P}\otimes K$; then from this decomposition it is obvious whether or not $U^L$ can be chosen such that Eq. (6) is satisfied.

## APPENDIX B: ANALYSIS OF GROUP $InO(1,n)$

The group $InO(1,n)$ can be written in matrix form as

$$InO(1,n) = \left\{ \begin{pmatrix} \Gamma & A \\ 0 & 1 \end{pmatrix} \right\}, \tag{B1}$$

with $\Gamma$ an arbitrary $(n+1)$-dimensional orthogonal matrix that leaves the metric

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}$$

invariant,[29]

$$\Gamma \gamma \Gamma^T = \gamma, \tag{B2}$$

and $A$ an arbitrary $(n+1)$-dimensional vector. The group law of multiplication is matrix multiplication:

$$\begin{pmatrix} \Gamma & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma' & A' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Gamma\Gamma' & \Gamma A' + A \\ 0 & 1 \end{pmatrix}. \tag{B3}$$

Since $InO(1,n)$ is a semidirect product group, all of its representations can be written as induced representations, with the inducing subgroups defined by

$$H = \{ g \mid g \in G,\ e^{iK\cdot gAg^{-1}} = e^{iK\cdot A} \}, \tag{B4}$$

where $e^{iK\cdot A}$ is a representation of

$$\begin{pmatrix} I_{n+1} & A \\ 0 & 1 \end{pmatrix}$$

with $K$ an $(n+1)$-dimensional vector that labels the irreducible representation of the translations $A$.[30]

Equation (B4) implies that

$$\Gamma K = K. \tag{B5}$$

There are four classes of vectors satisfying Eq. (B5), and associated with each class is a class of equivalent irreducible representations of $InO(1,n)$. It is sufficient to choose a "standard vector" in each class of vectors to characterize all irreducible representations of $InO(1,n)$.

[29] $I_n$ denotes the $n$-dimensional identity matrix.

[30] The scalar product is defined with respect to the metric

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}$$

so that $K \cdot A = K^T \gamma A$.

### 1. $K\cdot K > 0$, Timelike Representations

Consider first the class of vectors defined by $K\cdot K > 0$ with standard vector $\hat{K}^T = (M,0,0,\ldots,0)$, $M$ real and positive.[31] Using Eq. (B5),

$$H = \begin{pmatrix} 1 & 0 & A \\ 0 & O(n) & \\ 0 & 0 & 1 \end{pmatrix}, \tag{B6}$$

which has representations

$$L = \exp(i\hat{K}\cdot A) D_{\eta\eta'}^{[\chi']}(O(n)), \tag{B7}$$

where $D_{\eta\eta'}^{[\chi]}(O(n))$ are $O(n)$ representation matrices, $[\chi]$ denotes the set of irreducible representation labels, and $\eta$ denotes a complete set of eigenvalues of diagonal operators in the representation space $\mathcal{H}(\chi)$ of $O(n)$.

The irreducible representation labels of $InO(1,n)$ are $[M,\chi]$ with $M^2 = K\cdot K$ while a complete set of diagonal quantum numbers can be chosen to be $\mathbf{K}$ and $\eta$.

In exactly the same way as is done for the Poincaré group,[32] it is not difficult to calculate the action of an arbitrary group element $(\Gamma,A) \in InO(1,n)$ on the basis state $|[M,\chi]\mathbf{K}\eta\rangle$:

$$U(\Gamma,A)|[M,\chi]\mathbf{K},\eta\rangle = e^{i\Gamma K\cdot A} \sum_{\eta'} D_{\eta'\eta}^{[\chi]}(\mathbf{K},\Gamma) \times |[M,\chi]\Gamma\mathbf{K},\eta'\rangle. \tag{B8}$$

$(K,\Gamma)$ is an arbitrary element of $O(n)$ defined by

$$(K,\Gamma) = \Gamma_c(\Gamma K)\Gamma\Gamma_c^{-1}(K), \tag{B9}$$

with $\Gamma_c(K)$ a coset representative of $O(1,n)/O(n)$ defined by $\hat{K} = \Gamma_c(K)K$ [i.e., $\Gamma_c(K)$ is the analog of a boost in the Poincaré group]; $\Gamma_c(\Gamma K)$ is similarly defined.

### 2. $K\cdot K < 0$, Spacelike Representations

A second class of irreducible representations is given by choosing $\hat{K}^T = (0,0,\ldots,0,\rho)$, $\rho$ real and positive. Using Eq. (B5), the inducing subgroup is

$$H = \begin{pmatrix} O(1,n-1) & 0 & A \\ 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix}, \tag{B10}$$

which has representations

$$L = \exp(i\hat{K}\cdot A) D_{\eta\eta'}^{[\chi]}(O(1,n-1)), \tag{B11}$$

with $D_{\eta\eta'}^{[\chi]}(O(1,n-1))$ representation matrices of $O(1,n-1)$. Now $[\chi]$ labels representations of $O(1,n-1)$ and $\eta$ is the set of eigenvalues of operators diagonal in the representation space $\mathcal{H}(\chi)$ of $O(1,n-1)$.

The irreducible representations of $InO(1,n)$ are labeled by $[\rho,\chi]$; a basis state in the representation space $\mathcal{H}(\rho,\chi)$ is labeled by $[\rho,\chi]$, $\mathbf{K}$, and $\eta$.

[31] The "caret" notation, i.e., $\hat{K}$, will be used to denote standard vectors.

[32] P. Moussa and R. Stora, in *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, 1964), Vol. VIIa.

### 3. $K\cdot K=0$, Null-Like and Lightlike Representations

The class defined by $K\cdot K=0$ has two standard vectors, $\hat{K}=0$ and $\hat{K}=(\omega,0,0,\ldots,0,\omega)$. The null-like representations are derived if $\hat{K}=0$, and they are just the representations of $O(1,n)$. The lightlike representations arise if $\hat{K}=(\omega,0,\ldots,0,\omega)$. The inducing subgroup is the Euclidean group in $n-1$ dimensions. In contrast to the spacelike and timelike cases, the standard vector label $\omega$ does not partially label the irreducible representations.

Using the analysis outlined above, it is easy to find all representations of $InO(1,n)$ provided the representations of the inducing subgroups are known. In Table I the important results of this analysis are listed for $InO(1,3)$, $InO(1,4)$, and $InO(1,5)$.

## APPENDIX C: $InO(1,n)$ CLEBSCH-GORDAN COEFFICIENTS, SPECIAL CASES

Clebsch-Gordan coefficients resulting from the decomposition of tensor-product representations of $InO(1,n)$ can be obtained using techniques discussed in Ref. 14. In this appendix the derivation of the Clebsch-Gordan coefficients resulting from the tensor-product decomposition of null-like and timelike representations of $InO(1,4)$ and of spacelike and timelike representations of $InO(1,5)$ will be given.

### 1. Twofold Tensor-Product Decomposition of Null-Like and Timelike Representations of $InO(1,4)$

The right-coset decompositions of $InO(1,4)$ with respect to the timelike and null-like inducing subgroups are

$$G=\bigcup_c Hg_c \tag{C1}$$

and

$$G=Ge\,, \tag{C2}$$

with

$$G=InO(1,4)\,,\quad H=\begin{bmatrix}1 & 0 & \\ 0 & O(4) & A\\ 0 & 0 & 1\end{bmatrix},\quad g_c=\begin{pmatrix}\Gamma_c & 0\\ 0 & 1\end{pmatrix},$$

and $\Gamma_c$ an arbitrary right-coset representative of $O(1,4)/O(4)$. The null-like representations have as their inducing subgroup the whole group so that the identity element $e$ is the only coset representative.

Induced unitary irreducible timelike representations are defined by the action of the unitary operator $U(\Gamma,A)$ $(\Gamma,A)\in InO(1,4)$, on square-integrable functions over right cosets, $f_{J\sigma}(\Gamma_c)$, as

$$U(\Gamma,A)f_{J\sigma}(\Gamma_c)=\exp(i\hat{p}\cdot\Gamma_cA)\times\sum_{J'=j_0}^{n}\sum_{\sigma'=-J}^{+J}D_{J\sigma,J'\sigma'}{}^{[n,j_0]}(k)f_{J'\sigma'}(\Gamma_{c'})\,, \tag{C3}$$

with $k$ an element of $O(4)$, $\Gamma_{c'}$ defined by $\Gamma_c\Gamma=k\Gamma_{c'}$, and $D_{J\sigma,J'\sigma'}{}^{[n,j_0]}(O(4))$ representation matrices of $O(4)$.[33]

Unitarity irreducible null-like representations are just the representations of $O(1,4)$:

$$D_{n,j_0,J,\sigma;n',j_0',J',\sigma'}{}^{[k_0,c]}(\Gamma)\equiv\langle[k_0,c]n,j_0,J,\sigma|U(\Gamma)\times|[k_0,c]n',j_0',J',\sigma'\rangle\,, \tag{C4}$$

with $U(\Gamma)$ the unitary irreducible operator associated with the element $\Gamma$ of $O(1,4)$ and $|[k_0,c]n,j_0,J,\sigma\rangle$ a basis element in the representation space $\mathcal{H}(k_0,c)$ of $O(1,4)$.

The first step in the decomposition of the twofold tensor-product space is the decomposition of the outer product group $(G_1,G_2)$ into double cosets $(g_{D1},g_{D2})$ with respect to $(H_1,H_2)$ and the diagonal subgroup $(G,G)$ of $(G_1,G_2)$:

$$(G_1,G_2)=\bigcup_{D1,D2}(H_1,H_2)(g_{D1},g_{D2})(G,G)\,. \tag{C5}$$

The method for calculating double cosets outlined in Appendix D can be used to show $(g_{D1},g_{D2})=(e,e)$, so that the inducing subgroup in the subspace of the tensor-product space labeled by double cosets is

$$H_D\equiv(g_{D1},g_{D2})^{-1}(H_1,H_2)(g_{D1},g_{D2})\cap(G,G)=\begin{bmatrix}1 & 0 & A\\ 0 & O(4) & \\ 0 & 0 & 1\end{bmatrix}. \tag{C6}$$

The right-coset decomposition of $G$ with respect to $H_D$ is given by

$$G=\bigcup_c\begin{bmatrix}1 & 0 & A\\ 0 & O(4) & \\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}\Gamma_c & 0\\ 0 & 1\end{pmatrix}, \tag{C7}$$

with $\Gamma_c$ a representative of $O(1,4)/O(4)$.

Now the Clebsch-Gordan coefficients can be written as

$$\langle[M^2,n,j_0]\mathbf{p},J,\sigma;\eta|[M_1^2,n_1,j_{01}]\mathbf{p},J_1,\sigma_1;[k_0,c]n_2,j_{02},J_2,\sigma_2\rangle=\int d\Gamma_cD_{\hat{p},J',\sigma';\mathbf{p},J,\sigma}{}^{[M^2,n,j_0]}(\Gamma_c)\times D_{\hat{p}_1,J_1',\sigma_1';\mathbf{p},J_1,\sigma_1}{}^{[M_1^2,n_1,j_{01}]}(\Gamma_c)\times D_{n_2',j_{02}',J_2',\sigma_2';n_2,j_{02},J_2,\sigma_2}{}^{[k_0,c]}(\Gamma_c)\,, \tag{C8}$$

with the first two "$D$ functions" defined by

$$D_{\hat{p},J',\sigma';p,J,\sigma}{}^{[M^2,n,j_0]}(\Gamma_c)=\langle[M^2,n,j_0]\hat{p},J',\sigma'|U(\Gamma_c)\times|[M^2,n,j_0]\mathbf{p},J,\sigma\rangle\,. \tag{C9}$$

$|[M^2,n,j_0]\mathbf{p},J,\sigma\rangle$ is a basis element in the representation space $\mathcal{H}(M^2,n,j_0)$ of $InO(1,4)$. The last $D$ function is

[33] Subscript and superscript labels are given in Table I.

defined by Eq. (C4). $\eta$ is the set of degeneracy parameters $\{J_1',\sigma_1',n_2',j_{02}',J_2',\sigma_2'\}$ and the labels $J'$ and $\sigma'$ are given by $|J_1'-J_2'|\leq J'\leq |J_1'+J_2'|$ and $\sigma'=\sigma_1'+\sigma_2'$.[34]

The $D$ functions can be evaluated since it is not difficult to calculate the action of the irreducible unitary operator $U(\Gamma)$ on a basis element [see Eq. (B8)]:

$$U(\Gamma)|[M^2,n,j_0]\mathbf{p},J,\sigma\rangle=\sum_{J',\sigma'} D_{J',\sigma';J,\sigma}{}^{[n,j_0]}(\mathbf{p},\Gamma)\times|[M^2,n,j_0]\Gamma\mathbf{p},J',\sigma'\rangle \quad (C10)$$

and

$$U(\Gamma)|[k_0,c]n,j_0,J,\sigma\rangle=\sum_{n',j_0',J',\sigma'} D_{n',j_0',J',\sigma';n,j_0,J,\sigma}{}^{[k_0,c]}(\Gamma)\times|[k_0,c]n',j_0',J',\sigma'\rangle, \quad (C11)$$

with $(\mathbf{p},\Gamma)$ an element of $O(4)$ defined by

$$(\mathbf{p},\Gamma)=\Gamma_c(\Gamma p)\Gamma\Gamma_c^{-1}(p). \quad (C12)$$

$\Gamma_c^{-1}(p)$ is a representative of $O(1,4)/O(4)$ that "boosts" $\hat{p}$ to $p$, $p=\Gamma_c^{-1}(p)\hat{p}$. For the special case $\Gamma=\Gamma_c$, $(\mathbf{p},\Gamma_c)$ equals the identity. This is proved by deriving the most general form for $\Gamma_c$ consistent with

$$\begin{bmatrix} M\\0\\0\\0\\0\end{bmatrix}=\Gamma_c^{-1}\begin{bmatrix}\sqrt{s}\\0\\0\\0\\(s-M^2)^{1/2}\end{bmatrix}.$$

Recalling that $\Gamma_c$ is a representative of $O(1,4)/O(4)$ [Eq. ( C7)], it is easy to see that

$$\Gamma_c=\begin{bmatrix}(\sqrt{s})/M & 0 & (s-M^2)^{1/2}/M\\ 0 & I_3 & 0\\ (s-M^2)^{1/2}/M & 0 & (\sqrt{s})/M\end{bmatrix}, \quad (C13)$$

so that $\Gamma_c=\Gamma_c(p)$. Then, calculating $(\mathbf{p},\Gamma_c)$, we find that

$$\begin{aligned}(\mathbf{p},\Gamma_c)&=\Gamma_c(\Gamma_c p)\Gamma_c\Gamma_c^{-1}(p)\\&=\Gamma_c\Gamma_c^{-1}\\&=e.\end{aligned} \quad (C14)$$

Now using Eq. (C14), the Clebsch-Gordan coefficient, Eq. (C8), reduces to

$$\begin{aligned}\langle[M^2,n,j_0]\mathbf{p},J,\sigma;\eta|[M_1^2,n_1,j_{01}]\mathbf{p},J_1,\sigma_1;\\ [k_0,c]n_2,j_{02},J_2,\sigma_2\rangle\\ =\int d\Gamma_c\delta(\hat{p}-\Gamma_c p)\delta_{\sigma,\sigma'}\delta_{J,J'}\delta_{\sigma_1,\sigma_1'}\\ \times D_{n_2',j_{02}',J_2',\sigma_2';n_2,j_{02},J_2,\sigma_2}{}^{[k_0,c]}(\Gamma_c)\\ =D_{n_2',j_{02}',J_2,\sigma_2';n_2,j_{02},J_2,\sigma_2}{}^{[k_0,c]}(\Gamma_c),\end{aligned} \quad (C15)$$

[34] Primed subscripts are fixed in such a manner that the $D$ functions transform as their associated square-integrable functions. See Ref. 37.

with $\Gamma_c$ fixed by $\hat{p}=\Gamma p$, $\sigma_2'=\sigma-\sigma_1'$, and $|J_1-J_2'|\leq J\leq|J_1+J_2'|$.

Finally, as a special example of Eq. (C15), consider the case when $|[k_0,c]n_2,j_{02},J_2,\sigma_2\rangle$ transforms as a scalar under the Poincaré group. This situation arises only if $j_{02}=J_2=\sigma_2=0$:

$$\begin{aligned}\langle[M^2,n,j_0]\mathbf{p},J,\sigma;j_{02}',n_2',J_2',\sigma_2'|[M_1^2,n_1,j_{01}]\mathbf{p},J_1,\sigma_1;\\ [k_0,c]n_2,j_{02}=J_2=\sigma_2=0\rangle\\ =D_{n_2',j_{02}',J_2',\sigma_2';n_2,0,0,0}{}^{[k_0,c]}(\Gamma_c).\end{aligned} \quad (C16)$$

### 2. Twofold Tensor-Product Decomposition of Spacelike and Timelike Representations of *InO*(1,5)

The subgroups $H_1$ and $H_2$ that induce unitary irreducible timelike and unitary irreducible spacelike representations of $InO(1,5)$ are given in Appendix B, Eqs. (B6) and (B10), respectively. The double-coset decomposition of the outer product group $(G_1,G_2)$ with respect to its diagonal subgroup $(G,G)$ and $(H_1,H_2)$ is

$$(G_1,G_2)=\bigcup_{D1,D2}(H_1,H_2)(g_{D1},g_{D2})(G,G), \quad (C17)$$

with

$$(g_{D1},g_{D2})=\left[\begin{pmatrix}\Gamma_D & 0\\ 0 & 1\end{pmatrix},e\right],$$

$$\Gamma_D=\begin{bmatrix}\cosh D & 0 & \sinh D\\ 0 & I_4 & 0\\ \sinh D & 0 & \cosh D\end{bmatrix},$$

obtained by using the technique in Appendix D. Using the definition of $H_D$ and $(g_{D1},g_{D2})$, it is easy to see that

$$H_D=\begin{bmatrix}1 & 0 & A\\ 0 & O(4) & \\ 0 & 0 & 1\end{bmatrix}, \quad (C18)$$

so that the right-coset decomposition of $G$ with respect to $H_D$ is

$$G=\bigcup_c\begin{bmatrix}1 & 0 & A\\ 0 & O(4) & \\ 0 & 0 & 1\end{bmatrix}\begin{pmatrix}\tilde{\Gamma}_c & 0\\ 0 & 1\end{pmatrix},$$

with $\tilde{\Gamma}_c$ a representative of $O(1,5)/O(4)$.

Now the Clebsch-Gordan coefficient can be written as

$$\begin{aligned}\langle[M^2,\mu,\lambda]\mathbf{P},n,j_0,J,\sigma;\eta||[M_1^2,\mu_1,\lambda_1]\mathbf{P}_1,n_1,j_{01},J_1,\sigma_1;\\ [\rho^2,k_0,c]\mathbf{P}_2,n_2,j_{02},J_2,\sigma_2\rangle\\ =\int d\tilde{\Gamma}\, D_{P_D,n',j_0',J',\sigma';\mathbf{P},n,j_0,J,\sigma}{}^{[M^2,\mu,\lambda]}(\tilde{\Gamma})\\ \times D_{\hat{P}_1,n_1',j_{01}',J_1',\sigma_1';\mathbf{P}_1,n_1,j_{01},J_1,\sigma_1}{}^{[M_1^2,\mu_1,\lambda_1]}(\tilde{\Gamma})\\ \times D_{\hat{P}_{2'},n_2',j_{02}',J_2',\sigma_2';\mathbf{P}_2,n_2,j_{02},J_2,\sigma_2}{}^{[\rho^2,k_0,c]}(\Gamma_D\tilde{\Gamma}).\end{aligned} \quad (C19)$$

The first $D$ function, defined by

$$D_{P_D,n',j_0',J',\sigma';\mathbf{P},n,j_0,J,\sigma}{}^{[M^2,\mu,\lambda]}(\tilde{\Gamma}) = \langle[M^2,\mu,\lambda]P_D,n',j_0'J',\sigma'| \times U(\tilde{\Gamma})|[M^2,\mu,\lambda]\mathbf{P},n,j_0,J,\sigma\rangle\,, \quad \text{(C20)}$$

must transform the same as a square-integrable function in the tensor-product representation space so that $n'$, $j_0'$, $J'$, and $\sigma'$ become fixed by the degeneracy parameters $\eta=\{n_1',j_{01}',J_1',\sigma_1',n_2',j_{02}',J_2',\sigma_2'\}$ and $P_D=\Gamma_D{}^{-1}\hat{P}_2+\hat{P}_1$. The second and third $D$ functions, defined by

$$D_{\hat{P}_1,n_1',j_{01}',J_1',\sigma_1';\mathbf{P}_1,n_1,j_{01},J_1,\sigma_1}{}^{[M_1{}^2,\mu_1,\lambda_1]}(\tilde{\Gamma}) = \langle[M_1{}^2,\mu_1,\lambda_1]\hat{P}_1,n_1',j_{01}',J_1',\sigma_1'| \times U(\tilde{\Gamma})|[M_1{}^2,\mu_1,\lambda_1]\mathbf{P}_1,n_1,j_{01},J_1,\sigma_1\rangle \quad \text{(C21)}$$

and

$$D_{\hat{P}_2,n_2',j_{02}',J_2',\sigma_2';\mathbf{P}_2,n_2,j_{02},J_2,\sigma_2}{}^{[\rho^2,k_0,c]}(\Gamma_D\tilde{\Gamma}) = \langle[\rho^2,k_0,c]\hat{P}_2,n_2',j_{02}',J_2',\sigma_2'|\,U(\Gamma_D\tilde{\Gamma}) \times|[\rho^2,k_0,c]\mathbf{P}_2,n_2,j_{02},J_2,\sigma_2\rangle\,, \quad \text{(C22)}$$

have the same transformation properties as the square-integrable functions upon which the timelike and spacelike representations are defined. Each $D$ function can be simplified, since it is known how $U(\tilde{\Gamma})$ [$U(\Gamma_D\tilde{\Gamma})$] acts on a basis element. That is,

$$U(\tilde{\Gamma})|[M^2,\mu,\lambda]\mathbf{P},n,j_0,J,\sigma\rangle = \sum_{n',j_0',J',\sigma'} D_{n',j_0',j',\sigma';n,j_0,J,\sigma}{}^{[\mu,\lambda]}(\mathbf{P},\tilde{\Gamma}) \times|[M^2,\mu,\lambda]\Gamma\mathbf{P},n',j_0',J',\sigma\rangle\,, \quad \text{(C23)}$$

with $(\mathbf{P},\tilde{\Gamma})$ a representative of $O(5)/O(4)$ and $D^{[\mu,\lambda]}\times(\mathbf{P},\tilde{\Gamma})$ and $O(5)$ representation matrix; and

$$U(\Gamma_D\tilde{\Gamma})|[\rho^2,k_0,c]\mathbf{P}_2,n_2,j_{02},J_2,\sigma_2\rangle = \sum_{n_2',j_{02}',J_2',\sigma_2'} D_{n_2',j_{02}',J_2',\sigma_2';n_2,j_{02},J_2,\sigma_2}{}^{[k_0,c]} \times(\mathbf{P}_2,\Gamma_D\tilde{\Gamma})\,, \quad \text{(C24)}$$

with $(\mathbf{P}_2,\Gamma_D\tilde{\Gamma})$ a representative of $O(1,4)/O(4)$ and $D^{[k_0,c]}(\mathbf{P}_2,\Gamma_D\tilde{\Gamma})$ an $O(1,4)$ representation matrix.

Thus, the Clebsch-Gordan coefficients become

$$\begin{aligned}\langle[M^2,\mu,\lambda]\mathbf{P},n,j_0,J,\sigma;\,\eta\,||[M_1{}^2,\mu_1,\lambda_1]\mathbf{P}_1,n_1,j_{01},J_1,\sigma_1;&\\ [\rho^2,k_0,c]\mathbf{P}_2,n_2,j_{02},J_2,\sigma_2\rangle&\\ =(P_D-\tilde{\Gamma}P)(\hat{P}_1-\tilde{\Gamma}P_1)(\hat{P}_2-\Gamma_D\tilde{\Gamma}P_2)&\\ \times D_{n',j_0',J',\sigma';n,j_0,J,\sigma}{}^{[\mu,\lambda]}(\mathbf{P},\tilde{\Gamma})&\\ \times D_{n_1',j_{01}',J_1',\sigma_1';n_1,j_{01},J_1,\sigma_1}{}^{[\mu_1,\lambda_1]}(\mathbf{P}_1,\tilde{\Gamma})&\\ \times D_{n_2',j_{02}',J_2',\sigma_2';n_2,j_{02},J_2,\sigma_2}{}^{[k_0,c]}(\mathbf{P}_2,\Gamma_D\tilde{\Gamma})\,,&\quad \text{(C25)}\end{aligned}$$

with $\tilde{\Gamma}$ and $\Gamma_D$ fixed by the $\delta$ functions.

Now consider the special case when the $D$ function [Eq. (C22)] transforms as a scalar under Poincaré transformation and charge transformation; then $j_{02}=J_2=\sigma_2=0$ and

$$P_2=\begin{pmatrix}0\\0\\0\\0\\\rho\sin\theta\\\rho\cos\theta\end{pmatrix}.$$

The $\delta$-function conditions can be used to show that $P=P_1+P_2$ and in the c.m. frame (i.e., the physical three-momentum content of $P$ and $P_1$ is zero) this condition can be written generally as

$$\begin{pmatrix}\sqrt{s}\\0\\0\\0\\(s-M^2)^{1/2}\sin\theta\\(s-M^2)^{1/2}\cos\theta\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\\rho\sin\theta_2\\\rho\cos\theta_2\end{pmatrix}+\begin{pmatrix}\sqrt{s}\\0\\0\\0\\(s-M_{12})^{1/2}\sin\theta_1\\(s-M_{12})^{1/2}\cos\theta_1\end{pmatrix}. \quad \text{(C26)}$$

Now $\tilde{\Gamma}$ must be the most general element of $O(1,5)/O(4)$ consistent with Eq. (C26), the $\delta$-function conditions $\hat{P}_1=\tilde{\Gamma}P_1$ and $\hat{P}_2=\Gamma_D\Gamma P_2$. A general element $\tilde{\Gamma}$ of $O(1,5)/O(4)$ (no conditions) can be written as

$$\tilde{\Gamma}^{-1}=\begin{pmatrix}\cosh\alpha_1&\sinh\alpha_1&0&0&0&0\\\sinh\alpha_1&\cosh\alpha_1&0&0&0&0\\0&0&I_4&&&\\0&0&&&&\\0&0&&&&\\0&0&&&&\end{pmatrix}\begin{pmatrix}\cosh\alpha_2&0&\sinh\alpha_2&0&0&0\\0&1&0&0&0&0\\\sinh\alpha_2&0&\cosh\alpha_2&0&0&0\\0&0&0&I_3&&\\0&0&0&&&\\0&0&0&&&\end{pmatrix}$$
$$\times\cdots\times\begin{pmatrix}\cosh\alpha_5&0&0&0&0&\sinh\alpha_5\\0&I_4&&&&0\\0&&&&&0\\0&&&&&0\\0&&&&&0\\\sinh\alpha_5&0&0&0&0&\cosh\alpha_5\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0\\0&\cos\beta_1&0&0&0&\sin\beta_1\\0&0&I_3&&&0\\0&0&&&&0\\0&0&&&&0\\0&-\sin\beta_1&0&0&0&\cos\beta_1\end{pmatrix}\begin{pmatrix}I_2&&0&0&0&0\\&0&0&0&0&0\\0&0&\cos\beta_2&0&0&\sin\beta_2\\0&0&0&I_2&&0\\0&0&0&&&0\\0&0&-\sin\beta_2&0&0&\cos\beta_2\end{pmatrix}$$
$$\times\begin{pmatrix}I_3&&&0&0&0\\&&&0&0&0\\&&&0&0&0\\0&0&0&\cos\beta_3&0&\sin\beta_3\\0&0&0&0&1&0\\0&0&0&-\sin\beta_3&0&\cos\beta_3\end{pmatrix}\begin{pmatrix}I_4&&&&0&0\\&&&&0&0\\&&&&0&0\\&&&&0&0\\0&0&0&0&\cos\beta_4&\sin\beta_4\\0&0&0&0&-\sin\beta_4&\cos\beta_4\end{pmatrix}. \quad \text{(C27)}$$

The condition $\hat{P}_2=\Gamma_D\hat{\Gamma}P_2$ implies that $\cosh\alpha_5 \sinh D = -\cosh D \sinh\alpha_5$, and $\hat{P}_1=\hat{\Gamma}P_1$, with $P_1{}^T=(\sqrt{s},0,0,0,(s-M_1{}^2)^{1/2}\sin\theta_1,(s-M_1{}^2)^{1/2}\cos\theta_1)$ implies $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=0$, $\beta_4=\theta_1$, and $\sqrt{s}=M\cosh\alpha_4\cosh\alpha_5$. These conditions can be used to write the rotations $(\mathbf{P}_1,\hat{\Gamma})$, $(\mathbf{P}_2,\Gamma_D\hat{\Gamma})$, and $(\mathbf{P},\hat{\Gamma})$ explicitly in terms of the components of $P_1$, $P_2$, and $P$.

## APPENDIX D: REDUCTION OF $\mathcal{H}(M,\chi)$ TO $\mathcal{H}(m,J,I)$

In this appendix the Hilbert space $\mathcal{H}(M,\chi)$ of unitary irreducible timelike representations $U^{[M,\chi]}$ of $G=InO(1,n)$ will be decomposed into irreducible constituents when $G$ is restricted to $H'=\mathcal{P}\oplus O(n-3)$. It is convenient to use Mackey's subgroup theorem, which can be implemented if the double-coset decomposition of $G$ with respect to its subgroups $H=InO(n)$ and $H'$ is known. That is, double-coset representatives $g_D$ defined by the double-coset decomposition

$$G=\bigcup_D Hg_DH' \tag{D1}$$

or, what is equivalent,

$$O(1,n)=\bigcup_D O(n)g_D[O(1,3)\oplus O(n-3)] \tag{D1$'$}$$

must be calculated. Given $g_D$, $H_D\equiv g_D{}^{-1}Hg_D\cap H'$ can be calculated and $U^L(H')$ decomposed into a direct integral (sum) over $U^{L_D}(H')$.

In order to find double-coset representatives, it proves convenient to introduce homogeneous spaces. The following definitions and notation are taken from Gel'fand *et al.*[35] Let the elements in $G$ transform some space $X$ into itself. If for every $x,y\in X$ there exists $g\in G$ such that $y=xg$, then $G$ is said to be transitive on $X$ and $X$ is called homogeneous with respect to $G$. In the homogeneous space $X$ there exists some point $x_0$ such that $x_0=x_0h$, $h\in H$; $x_0$ is called the stabilizer of $G$ with respect to $H$. If $x=x_0g$, then it is also true that $x=x_0hg$, so that for each $x\in X$ there exists a corresponding right-coset representative defined by $g=hg_c$, $g$ and $g_c\in G$.

Now consider the decomposition of $G$ with respect to $H$:

$$G=\bigcup_c Hg_c \tag{D2}$$

or

$$\begin{pmatrix} O(1,n) & \{A\} \\ 0 & 1 \end{pmatrix}=\bigcup_c \begin{bmatrix} 1 & 0 & \{A\} \\ 0 & O(n) & \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} g_c & 0 \\ 0 & 1 \end{pmatrix}. \tag{D3}$$

The representative $g_c\in G$ is equivalent to a vector $x$ in $X=O(1,n)/O(n)$. The point of stability with respect to $O(n)$ is the vector $x_0{}^T=(1,0,0,\ldots,0)$ since

$$x_0=x_0\begin{pmatrix} 1 & 0 \\ 0 & O(n) \end{pmatrix}. \tag{D4}$$

It is easy to see that an arbitrary point $x$ in the homogeneous space can be reached through the action of a right-coset representative on the stabilizer point $x_0$.

The double-coset representative $g_D$ defines a new point $x_D=x_0g_D$, and an arbitrary point in $X$ can be written as

$$x=x_Dh', \tag{D5}$$

with $h'\in H'$. Note that $H'$ contains a subgroup $H_D$ defined by

$$H_D=\{h'\,|\,h'\in H',\ x_D=x_Dh'\}. \tag{D6}$$

$H_D$ is just that subgroup of $H'$ that induces representations in the subspaces $\mathcal{H}_D(U^L)$ of $\mathcal{H}(U^L)$ labeled by double-coset parameters $D$.[36]

Now if one chooses

$$g_D=\begin{bmatrix} \cosh D & 0 & \sinh D \\ 0 & I_{n-1} & 0 \\ \sinh D & 0 & \cosh D \end{bmatrix}, \tag{D7}$$

with $0\leq D\leq\infty$, then Eq. (D5) is satisfied. The stability point of $H_D$ is $x_D=(\cosh D,0,\ldots,0,\sinh D)$ and the restriction $x_D=x_Dh'$ [Eq. (D6)] implies

$$H_D=\begin{bmatrix} 1 & 0 & 0 & 0 & A \\ 0 & O(3) & 0 & 0 & \\ 0 & 0 & O(n-4) & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{D8}$$

The representations $L_D$ of $H_D$ are

$$L_D=L(g_Dh'g_D{}^{-1}) = e^{iK_D\cdot A}D_{\sigma\sigma'}{}^{[j]}(O(3))D_{\beta\beta'}{}^{[\alpha]}(O(n-4)), \tag{D9}$$

with $K_D\equiv g_D\hat{K}$, $D_{\sigma\sigma'}{}^{[j]}(O(3))$ reducible representation matrices of $O(3)$ and $D_{\beta\beta'}{}^{[\alpha]}(O(n-4))$ reducible representation matrices of $O(n-4)$.

The mass is directly related to the double-coset label by

$$m^2=K_D\cdot K_D=M^2(1+2\sinh^2 D) \tag{D10}$$

so that Eq. (D8) can be written as

$$U^{[M^2,\chi]}(\mathcal{P}\oplus O(n-3)) \approx\int_{M^2}^{\infty}dm^2U^{[m^2,j,\alpha]}(\mathcal{P}\oplus O(n-3)). \tag{D11}$$

Since $[j]$ and $[\alpha]$ label reducible representations, it is necessary to decompose them into a sum of irreducible representations, $[J,I]$; then using the relation

$$U^{\Sigma[m^2,J,I]}(\mathcal{P}\oplus O(n-3))=\sum U^{[m^2,J,I]}(\mathcal{P}\oplus O(n-3)),$$

[35] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. 5.

[36] W. H. Klink, J. Math. Phys. **10**, 606 (1969).

[37] The Frobenius reciprocity theorem is proved by Mackey, Ref. 13.

Eq. (D11) can be further decomposed. Now $U^{[m^2,J,I]} \times(\mathcal{P}\oplus O(n-3))$ is, in general, a reducible representation of $\mathcal{P}\oplus O(n-3)$ and must be further reduced into irreducible constituents. The multiplicity in each reduction can be found using the Frobenius reciprocity theorem.[37] We will illustrate the problem of the decomposition of $\mathcal{H}(U^{[m^2,x]}(\mathcal{P}\oplus O(n-3)))$ by considering two examples.

First, consider the timelike representations $U^{[M^2,n,j_0]}$ of $InO(1,4)$. The decomposition of $U^{[M^2,n,j_0]}(H')$, $H'=\mathcal{P}\oplus 1$, can be written [using Eq. (D11)] as

$$U^{[M^2,n,j_0]}(\mathcal{P})\approx\int_{M^2}^{\infty} dm^2 U^{[m^2,j]}(\mathcal{P})\,. \qquad \text{(D12)}$$

The reducible representations $U(j)$ decompose into an irreducible representation $U(J)$ with multiplicity 1,

$$U(j)=\sum_{J=j_0}^{n} U(J)\,,$$

so that Eq. (D12) can be rewritten as

$$U^{[M^2,j_0,n]}(\mathcal{P})\approx \int_{M^2}^{\infty} dm^2 \sum_{J=j_0}^{n} U^{[m^2,J]}(\mathcal{P})\,, \qquad \text{(D13)}$$

with $U(m^2,J)$ an irreducible representation of $H_D$. The spectrum of $J$ is obtained by investigating the weight diagrams of $O(4)$ representations. The representations $U^{[m^2,J]}(\mathcal{P})$ are obviously irreducible representations of $\mathcal{P}$.

The decomposition of the Hilbert space $\mathcal{H}(U^{[M^2,j_0,n]})$ can be taken directly from Eq. (D13):

$$\mathcal{H}(U^{[M^2,n,j_0]})\approx\int_{M^2}^{\infty} dm^2 \sum_{J=j_0}^{n} \mathcal{H}(U^{[M^2,J]}(\mathcal{P}))\,. \qquad \text{(D14)}$$

As another example, consider the timelike representations of $InO(1,5)$. The internal-symmetry group $O(2)$ has representations labeled by an integer $Q$; Eq. (D11) is written as

$$U^{[M^2,\mu,\lambda]}(\mathcal{P}\oplus O(2)) \approx\int_{M^2}^{\infty} dm^2 U^{[m^2,J]}(\mathcal{P}\oplus O(2))\,. \qquad \text{(D15)}$$

Note that no labels $[\alpha]$ appear because the representations of $H_D$ are completely labeled by $[m^2,j]$. Again $U(m^2,j)$ is reducible and $U^{[m^2,j]}$ must be written as a direct sum over representations induced by irreducible representations of $H_D$:

$$U^{[M^2,\mu,\lambda]}(\mathcal{P}\oplus O(2)) \approx\int_{M^2}^{\infty} dm^2 \sum_J U^{[m^2,J]}(\mathcal{P}\oplus O(2))\,. \qquad \text{(D16)}$$

The spectrum of $J$ can be found by investigating $O(5)$ weight diagrams.

The representations $U^{[m^2,J]}(\mathcal{P}\oplus O(2))$ are not irreducible representations of $\mathcal{P}\oplus O(2)$; however, it is easy to decompose the identity into irreducible representations of $O(2)$ so that finally

$$U^{[M^2,\mu,\lambda]}(\mathcal{P}\oplus O(2)) \approx\int_{M^2}^{\infty} dm^2 \sum_J \sum_{Q=-\infty}^{+\infty} U^{[m^2,J,Q]}(\mathcal{P}\oplus O(2))\,. \qquad \text{(D17)}$$

The Hilbert-space decomposition is

$$\mathcal{H}(U^{[M^2,\mu,\lambda]}) \approx\int_{M^2}^{\infty} dm^2 \sum_J \sum_{Q=-\infty}^{+\infty} \mathcal{H}(U^{[m^2,J,Q]})\,. \qquad \text{(D18)}$$