Renormalization, Diagrams, and Gauge Invariance*

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A compact formula for the renormalized transition probability in quantum electrodynamics is derived and its interpretation in terms of the diagrams is given. This formula is used next to prove the equivalence of the Coulomb gauge and the Feynman gauge. It is shown that, contrary to widespread belief, the problem of the gauge invariance becomes a very delicate one when the radiative corrections are included. The common practice of dropping k_μ and k_ν terms in the photon propagator $D_{\mu\nu}(k)$ is justified, but the justification requires a nontrivial analysis of the gauge transformations of the propagators involved.

I. INTRODUCTION

A REMARKABLE feature of the quantum theory of electromagnetic phenomena is that it can be formulated in so many different ways, all leading to the same predictions for the observable effects. The equivalence of different formulations is believed to be a result of the gauge invariance,[1] but all standard proofs of this invariance fail when applied directly to the transition amplitudes. The reason for this failure is the appearance of gauge-dependent objects (usually the electron propagators) in all explicit formulas for the transition amplitudes. So far no one has been able to express explicitly even the simplest electron transition amplitudes in terms of the gauge-invariant objects alone.[2] Until such expressions are found, the direct proof of gauge invariance of the transition amplitudes is still necessary for the logical completeness of quantum electrodynamics.

The questions of gauge invariance or gauge independence of the physical amplitudes are usually dealt with in one or two sentences, as typified by the following example taken from a recent article on the renormalization theory in quantum electrodynamics[3]: "Ultimately because of gauge invariance, the $k_\mu k_\nu$ terms will not contribute to any physical amplitude and we may drop them."

We shall prove in the present paper that such statements are justified and that we may indeed drop the gauge terms in the photon propagator.

As was pointed out by the present author,[4] the renormalization procedure, which is necessary to compute observable results, is gauge dependent, and this fact invalidates various simple proofs of gauge invariance given in the literature. The renormalization is intertwined with the gauge invariance in a very intricate way, and if a proof of gauge invariance is to have any physical significance, it should be carried out on the *renormalized* transition amplitudes or *renormalized* transition probabilities.

Out of the three renormalizations in quantum electrodynamics, the mass renormalization and the charge renormalization (or the photon-propagator renormalization) have a clear physical interpretation, but the meaning of the electron-wave-function (or the electron-propagator) renormalization is made obscure by the gauge dependence of the corresponding renormalization constant.

The purpose of this paper is to derive a new formula for the renormalized transition probability and then use it to give a simple and complete proof of its gauge invariance. Our formula does not contain explicitly the wave-function renormalization constants even when expressed in terms of the unrenormalized propagators. The elimination of all gauge-dependent renormalization constants makes the proof of the gauge invariance fairly simple.

In Sec. II we derive our formula for the renormalized transition probability. In Sec. III we describe a natural generalization of the method of Feynman diagrams to the transition probabilities. In Sec. IV we prove the independence of the transition probabilities on the gauge of the photon propagator.

II. RENORMALIZED TRANSITION PROBABILITIES

To establish our notation, we shall write the expression for the transition amplitude $T(p;q;k)$ in the form which follows directly from the reduction formula:[5]

$$T(p;q;k)=(2\pi)^4\delta_{(4)}(\sum p_i-\sum q_j-\sum k_l)\times M(p;q;k), \quad (1)$$

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[1] Various aspects of gauge invariance will be studied in detail in Sec. IV.

[2] We have no reasons to doubt that *all* observable effects in quantum electrodynamics can be expressed in terms of gauge-invariant objects (like, for example, the vacuum expectation values of the currents), but the explicit formulas may turn out to be exceedingly complicated.

[3] P. K. Kuo and D. R. Yennie, Ann. Phys. (N. Y.) 51, 496 (1969).

[4] I. Bialynicki-Birula, Phys. Rev. 155, 1414 (1967).

[5] See, for example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), p. 146; S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966), p. 102. Symbols p, q, and k will be often used here to denote sets of four-vectors $p_1\cdots p_n$, $q_1\cdots q_n$, and $k_1\cdots k_n$. We will introduce a small photon mass μ to avoid the infrared catastrophe. Throughout Secs. II and III we will use the Feynman gauge for the photon propagator.

$$M(p;q;k)=Z_2^{-n}Z_3^{-m/2}i^{2n}(-i)^m\bar{u}(p_1)\cdots\bar{u}(p_n)$$
$$\times[(m-\boldsymbol{p}_1)\cdots(m-\boldsymbol{p}_n)G^{\mu_1\cdots\mu_n}(p;q;k)$$
$$\times(m-\boldsymbol{q}_n)\cdots(m-\boldsymbol{q}_1)(\mu^2-k_1^2)\cdots(\mu^2-k_m^2)]$$
$$\times u(q_n)\cdots u(q_1)\epsilon_{\mu_1}(k_1)\cdots\epsilon_{\mu_m}(k_m)\,, \quad (2)$$

where $(2\pi)^4\delta_{(4)}G$ is the Fourier transform of the connected part[6] of the unrenormalized propagator

$$(2\pi)^4\delta_{(4)}(\sum p_i-\sum q_j-\sum k_l)G_{\mu_1\cdots\mu_m}(p;q;k)$$
$$=\int d^{4n}xd^{4n}yd^{4m}z\,\exp(i\sum p_i\cdot x_i-i\sum q_j\cdot y_j-i\sum k_l\cdot z_l)$$
$$\times(\Omega|\,T(\psi(x_1)\cdots\psi(x_n)\bar{\psi}(y_n)\cdots\bar{\psi}(y_1)$$
$$\times A_{\mu_1}(z_1)\cdots A_{\mu_m}(z_m))\Omega)_{\text{conn}}\,. \quad (3)$$

Prior to taking the mass-shell values of the initial and final momenta in the Fourier transform of the propagator, we must cancel all the singularities in G with the help of the appropriate factors $m-\boldsymbol{p}$, $m-\boldsymbol{q}$, and μ^2-k^2. The same prescription must be followed also in all similar situations encountered further in this paper. For simplicity we treated in the formula (2) all the four-momenta p_i as outgoing and all the four-momenta q_j and k_l as incoming. However, this is not a real restriction provided we allow for the negative values of the time components of these four-momenta and we interpret the resulting transition amplitudes in accordance with the requirements of crossing symmetry.

The Fourier transform of the propagator contains all the terms given by the corresponding connected Feynman diagrams, including all self-energy corrections to the external electron and photon lines. It is convenient to separate those corrections and to write G in the following factorized form:

$$G_{\mu_1\cdots\mu_m}(p;q;k)=(-i)^{2n+m}\mathcal{G}_{\mu_1\nu_1}(k_1)\cdots\mathcal{G}_{\mu_m\nu_m}(k_m)$$
$$\times G(p_1)\cdots G(p_n)F^{\nu_1\cdots\nu_m}(p;q;k)G(q_n)\cdots G(q_1)\,, \quad (4)$$

where $\mathcal{G}_{\mu\nu}(k)$ and $G(p)$ are complete (unrenormalized) one-photon and one-electron propagators. The truncated part F of the propagator G defined by this formula[7] will contain no contributions corresponding to the external lines.

The residues of the propagators $G(p)$ and of the transverse parts $\mathcal{G}_{\mu\nu}{}^{\text{tr}}(k)$ of the propagators $\mathcal{G}_{\mu\nu}(k)$ will now be expressed in terms of the wave-function renormalization constants in the standard fashion.

$$(m^2-p^2)G(p)\,|_{p^2=m^2}=(m+\boldsymbol{p})Z_2\,, \quad (5a)$$

$$(\mu^2-k^2)\mathcal{G}_{\mu\nu}{}^{\text{tr}}(k)\,|_{k^2=\mu^2}=(-g_{\mu\nu}+\mu^{-2}k_\mu k_\nu)Z_3\,, \quad (5b)$$

and the final formula for M reads

$$M(p;q;k)=Z_2^nZ_3^{m/2}\bar{u}(p_1)\cdots\bar{u}(p_n)F^{\mu_1\cdots\mu_m}(p;q;k)$$
$$\times u(q_n)\cdots u(q_1)\epsilon_{\mu_1}(k_1)\cdots\epsilon_{\mu_m}(k_m)\,, \quad (6)$$

where we have made use of the relations

$$(m+\boldsymbol{p})u(p)=2mu(p) \quad (7a)$$

and

$$k^\mu\epsilon_\mu(k)=0. \quad (7b)$$

The transition probability (or, more precisely, the density of the transition probability in momentum space) is constructed from the product of M and its complex conjugate M^*,

$$P(p;q;k)=(2\pi)^4\delta_{(4)}(\sum p_i-\sum q_j-\sum k_l)\,|M|^2\,, \quad (8)$$

$$|M|^2=Z_2^{2n}Z_3^m\rho_{\mu_1\nu_1}(k_1,\lambda_1)\cdots\rho_{\mu_m\nu_m}(k_m,\lambda_m)$$
$$\times\text{Tr}\{\Lambda(p_1,s_1)\cdots\Lambda(p_n,s_n)F^{\mu_1\cdots\mu_m}(p;q;k)$$
$$\times\Lambda(q_n,s_n')\cdots\Lambda(q_1,s_1')\bar{F}^{\nu_1\cdots\nu_m}(p;q;k)\}\,, \quad (9)$$

where $\Lambda(p,s)=u(p)\bar{u}(p)$ and $\rho_{\mu\nu}(k,\lambda)=\epsilon_\mu(k)\epsilon_\nu{}^*(k)$ are the density matrices for the initial and final electrons and photons with the specified polarizations, and the bar over F denotes the "bispinor conjugation," i.e., the Hermitian conjugation in the bispinor space with the γ^0 acting as the metric matrix.

We shall limit ourselves from now on to the study of the transition probabilities for unpolarized initial and final states of the electrons and photons. This assumption does not actually restrict the types of measurements that can be made, because all the polarization measurements can be reduced (and in practice are almost always reduced) to the measurements with unpolarized beams involving additional intermediate interactions whose role is to produce or to detect the polarization. For the unpolarized states, the density matrices are

$$\Lambda(p,\text{ unpol})=m+\boldsymbol{p}\,, \quad (10a)$$

$$\rho_{\mu\nu}(k,\text{ unpol})=-g_{\mu\nu}+\mu^{-2}k_\mu k_\nu\,, \quad (10b)$$

and the transition probability in this case is

$$P(p;q;k)=(2\pi)^4\delta_{(4)}(\sum p_i-\sum q_j-\sum k_l)$$
$$\times Z_2^{2n}Z_3^m(-g_{\mu_1\nu_1}+\mu^{-2}k_{\mu_1}k_{\nu_1})\cdots(-g_{\mu_m\nu_m}+\mu^{-2}k_{\mu_m}k_{\nu_m})$$
$$\times\text{Tr}\{(m+\boldsymbol{p}_1)\cdots(m+\boldsymbol{p}_n)F^{\mu_1\cdots\mu_m}(p;q;k)$$
$$\times(m+\boldsymbol{q}_n)\cdots(m+\boldsymbol{q}_1)\bar{F}^{\nu_1\cdots\nu_m}(p;q;k)\}\,. \quad (11)$$

Now we will again use the relations (5) and we will eliminate completely all the renormalization constants.

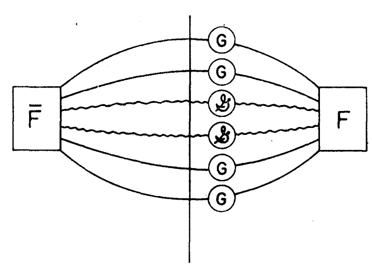

FIG. 1. Symbolic representation of the transition probability.

[6] The disconnected parts describe several transitions taking place independently.

[7] The definition of the truncated part applies only to connected propagators.

The final formula for the transition probability is

$$P(p;q;k)$$
$$=(2\pi)^4\delta_{(4)}(\sum p_i-\sum q_j-\sum k_l)\prod_i (m^2-p_i^2)$$
$$\times\prod_j (m^2-q_j^2)\prod_l (\mu^2-k_l^2)\mathcal{G}_{\mu_1\nu_1}(k_1)\cdots\mathcal{G}_{\mu_m\nu_m}(k_m)$$
$$\times\mathrm{Tr}\{G(p_1)\cdots G(p_n)F^{\mu_1\cdots\mu_m}(p;q;k)G(q_n)\cdots$$
$$G(q_1)\bar{F}^{\nu_1\cdots\nu_m}(p;q;k)\}, \quad (12)$$

where we have used the gauge invariance[8] to replace the transverse parts of the photon propagators by complete propagators. In spite of the fact that all the propagators appearing in the expression for P are the unrenormalized propagators,[9] there are no renormalization constants there. This will enable us to give in Sec. IV a rather simple proof of the gauge invariance. In the next section we will show that our formula has an attractive diagrammatic representation.

III. DIAGRAMS FOR TRANSITION PROBABILITIES

We shall now extend the method of Feynman diagrams, which was originally developed for the unrenormalized transition amplitudes, to the renormalized transition probabilities. Let us consider a pair of the Feynman diagrams, one representing a contribution to M and the other a contribution to M^*. It is natural to

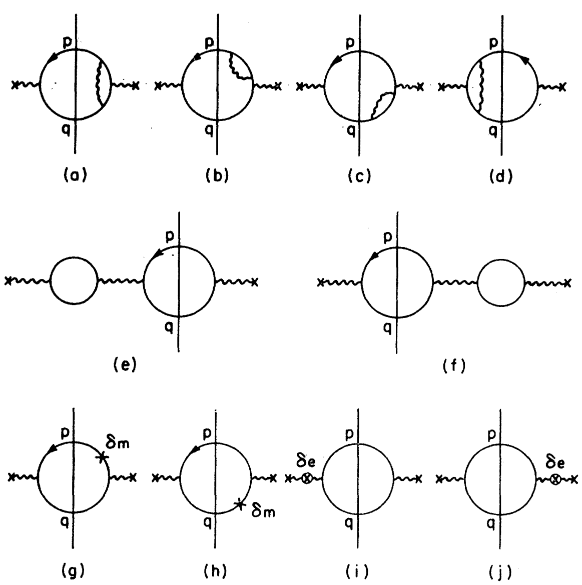

FIG. 2. Electron scattering in an external field.

[8] We shall return to this problem in Sec. IV.

[9] Actually formula (12) for P is invariant under the renormalization of the propagators. It retains its form if the normalization of the propagators is changed in the following way: $G(p)\to z_2G(p)$, $\mathcal{G}(k)\to z_3\mathcal{G}(k)$, and $F(p;q;k)\to z_2^{-n}z_3^{-m/2}F(p;q;k)$. Therefore, we can also use this formula with all propagators having their finite renormalized values.

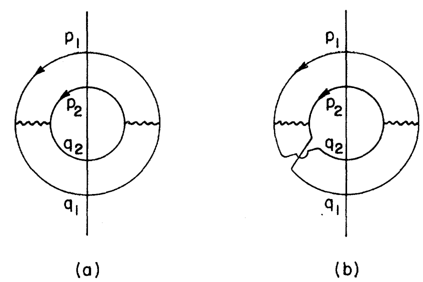

FIG. 3. Møller scattering.

draw the diagrams representing M^* as the mirror reflections of the corresponding diagrams for M with an additional reversal of all the arrows on the electron lines. With every such pair of diagrams we can associate a single *two-sided diagram*. We will draw a vertical line separating two sides of every two-sided diagram and we will adopt the convention that the part to the right (left) of the separating line corresponds to M (M^*). Our two-sided diagrams have no external lines; the old external lines have now become the *connecting lines*. Each of these lines connects two vertices lying on the opposite sides of the separating line. The connecting lines carry the initial and final particle momenta.

The general diagrammatic structure of $P(p;q;k)$ drawn in accordance with the above rules is shown in Fig. 1. The two boxes represent the truncated parts of the propagators and the circles on the solid and wavy lines represent all one-electron and one-photon propagator corrections. Thus every particular two-sided diagram has two truncated parts lying on the opposite sides of the separating line which are linked together by the connecting lines carrying possibly some self-energy corrections.

As examples of such diagrams we have shown in Figs. 2 and 3 all the fourth-order diagrams representing, respectively, the elastic electron scattering in a weak electromagnetic field and the elastic scattering of two electrons.

The symmetry between the two sides of the diagrams, which appears to be broken in Figs. 1–3, can be easily restored if we observe[10] that below the threshold all self-energy functions are self-adjoint, i.e.,

$$\Sigma(p)=\bar{\Sigma}(p), \quad p^2<(m+\mu)^2 \quad (13a)$$

$$\Pi_{\mu\nu}(k)=\Pi_{\mu\nu}{}^*(k), \quad k^2<(3\mu)^2. \quad (13b)$$

Therefore, the self-energy parts on any two-sided diagram can be freely shifted across the separating line without any change in the corresponding contribution to the $P(p;q;k)$. For example, instead of diagrams (b) and (c) in Fig. 2, we could have taken the two diagrams in Fig. 4.

To make this symmetry even more explicit, it is convenient to introduce the notion of the equivalence

[10] This is explained in more detail in Appendix B.

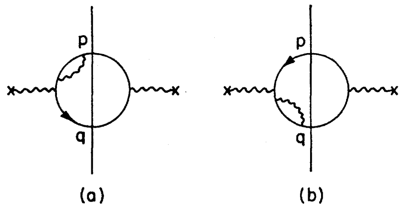

FIG. 4. Diagrams equivalent to (2b) and (2c).

of two-sided diagrams. Two diagrams are equivalent if one can be obtained from the other by shifting some self-energy corrections to the connecting lines across the separating line. On account of Eqs. (13) all equivalent diagrams give equal contributions to the transition probability. The renormalized transition probability contains, therefore, contributions from all inequivalent two-sided diagrams.

The rules for writing the transition probability directly from the two-sided diagrams can be obtained from the well-known rules for the Feynman diagrams and from formula (12) for the $P(p;q;k)$. We give these rules below for completeness.

(1) The solid and wavy lines represent the electron and the photon propagators. Depending on whether these lines lie to the right or to the left of the separating lines, we use the propagators $-iS(p)$ and $-iD_{\mu\nu}(k)$ or the adjoint propagators $i\bar{S}(p)$ and $i\bar{D}_{\mu\nu}(k)$,

$$S(p)=(m+\not{p})(m^2-p^2-i\epsilon)^{-1}, \tag{14a}$$

$$\bar{S}(p)=(m+\not{p})(m^2-p^2+i\epsilon)^{-1}, \tag{14b}$$

$$D_{\mu\nu}(k)=-g_{\mu\nu}(\mu^2-k^2-i\epsilon)^{-1}, \tag{14c}$$

$$\bar{D}_{\mu\nu}(k)=-g_{\mu\nu}(\mu^2-k^2+i\epsilon)^{-1}. \tag{14d}$$

(2) The vertices to the right (left) of the separating line represent $-ie\gamma^\mu$ $(ie\gamma^\mu)$.

(3) To every connecting electron or photon line there corresponds the factor m^2-p^2 or μ^2-k^2.

(4) There is an over-all factor $(-1)^{L+n}$, where L is the total number of closed electron loops and n is half of the total number of connecting electron lines.

Our new rules incorporate *all* the effects of the wave-function renormalization, so that the renormalization constants do not have to be inserted by hand as was the case in the usual approach. The mass renormalization and the charge renormalization must, however, still be carried out.

IV. GAUGE INVARIANCE

To define precisely the notion of the gauge invariance, we must specify the class of gauge transformations which will leave all the observable quantities invariant. There are several types of gauge transformations and they lead to different requirements for gauge invariance.

First there is the gauge transformation of the classical vector potential $\mathcal{A}_\mu$:

$$\mathcal{A}_\mu(x)\to\mathcal{A}_\mu(x)+\partial_\mu\Lambda(x), \tag{15}$$

where $\Lambda(x)$ is a given function of x. In quantum electrodynamics this classical transformation can be applied to the external potential. The requirement of the gauge invariance for any quantity C which depends on the external potential $\mathcal{A}$ can be written in the form

$$\partial_\mu\frac{\delta}{\delta\mathcal{A}_\mu}C[\mathcal{A}]=0. \tag{16}$$

Next there is the gauge transformation of the free (initial or final) photon wave functions,

$$\phi_\mu(x)\to\phi_\mu(x)+\partial_\mu\alpha(x), \tag{17}$$

which is the simplest quantum generalization of the classical potential transformation (15). Owing to the universality of the electromagnetic interactions,[11] the requirement of the gauge invariance under transformations (17) leads again to condition (16). This is best seen if we use the following relation between the photon absorption[12] amplitude T_1 and the photonless transition amplitude T_0:

$$T_1=\int d^4x\,\phi_\mu(x)\frac{\delta}{\delta\mathcal{A}_\mu(x)}T_0[\mathcal{A}]\bigg|_{\mathcal{A}=0}, \tag{18}$$

where $T_0[\mathcal{A}]$ denotes the amplitude in the presence of an external electromagnetic field.

The invariance under the gauge transformations (15) and (17) is not enough, however, to prove the equivalence of various formulations of quantum electrodynamics. We also need the invariance under the so-called operator gauge transformations, i.e., the gauge transformations which change the wave functions of virtual photons. We do not see any point in discussing such transformations in their utmost generality, especially in view of the difficulties in defining the exponential functions of the gauge operator $\Lambda(x)$. We shall restrict ourselves to only those operator gauge transformations which can be described as the gauge transformations of the free photon propagator. A transformation of this type bridges two most important formulations of quantum electrodynamics: the Coulomb gauge and the Feynman gauge formulation. The formulation in the Coulomb gauge is physically acceptable because of the use of the Hilbert space of the state vectors. The formulation in the Feynman gauge requires the introduction of the indefinite metric with its unclear physical interpretation, but it offers also significant calculational advantages. The problem of the equivalence of these two formulations is clearly one of great importance in quantum electrodynamics.

Thus, we are led to consider the class of gauge transformations under which the free photon propagator

[11] Charges are coupled to the photons in the same manner as they couple to the external electromagnetic field.

[12] The same argument can be applied to the photon emission amplitude.

undergoes the following change:

$$D_{\mu\nu}(x-y) \to D_{\mu\nu}(x-y)+i\partial_\mu g_\nu(x-y) + i\partial_\nu g_\mu(x-y). \quad (19)$$

Four functions $g_\mu(x-y)$ will be restricted by the requirement that the photon propagator after the transformation will have again the basic properties of the Feynman propagator, i.e.,

$$D_{\mu\nu}(x)=\begin{cases} D_{\mu\nu}{}^{(+)}(x), & x^0>0 \\ D_{\mu\nu}{}^{(-)}(x), & x^0<0 \end{cases} \quad (20)$$

where $D_{\mu\nu}{}^{(\pm)}$ contain, respectively, only positive or negative frequencies and

$$D_{\mu\nu}{}^{(+)}(x)=-[D_{\mu\nu}{}^{(+)}(-x)]^*, \quad (21a)$$

$$D_{\mu\nu}{}^{(+)}(x)=-[D_{\mu\nu}{}^{(-)}(x)]^*. \quad (21b)$$

It follows from properties (20) and (21) that the Fourier transform $D_{\mu\nu}(k)$ of the photon propagator is an even function of k,

$$D_{\mu\nu}(k)=D_{\mu\nu}(-k), \quad (22)$$

and has its poles in the k_0 variable displaced off the real axis by the Feynman $i\epsilon$ prescription. The gauge transformation leading from the Feynman gauge to the Coulomb gauge clearly belongs to this class. The free photon propagators in these two gauges have the form

$$D_{\mu\nu}{}^F(k)=(k^2+i\epsilon)^{-1}g_{\mu\nu}, \quad (23a)$$

$$D_{\mu\nu}{}^C(k)=(k^2+i\epsilon)^{-1}\{g_{\mu\nu}-[k^2-(n\cdot k)^2]^{-1} \times[k_\mu k_\nu-(k\cdot n)(k_\mu n_\nu+n_\mu k_\nu)]\}, \quad (23b)$$

where n_μ is the unit vector in the time direction. The Fourier transforms of the corresponding g_μ functions have the form

$$g_\mu(k)=(k^2+i\epsilon)^{-1}[k^2-(n\cdot k)^2]^{-1}[n_\mu(n\cdot k)-\tfrac{1}{2}k_\mu]. \quad (24)$$

The requirement of the gauge invariance under the general gauge transformation (19) of the photon propagator will be stated in the following form: The renormalized transition amplitudes are invariant under the gauge transformations of the photon propagator in every order of perturbation theory.

The key word in this statement is *renormalized*, because the unrenormalized transition amplitudes, which are calculated directly from Feynman diagrams, are *not* gauge invariant.

The first attempt to prove the invariance of the transition amplitudes under the gauge transformation (19) was made by Feynman in his classic paper[13] on quantum electrodynamics. However, as was pointed out in Ref. 3, his proof is incorrect, because he (i) did not include the effects of the renormalization and (ii) disregarded the singularities of the propagators on the mass shell.[14]

(i) The wave-function renormalization was not included in Feynman's discussion, though it contributes gauge-dependent terms to the observed transition amplitudes.

(ii) The radiative corrections to the external electron lines bring in the factors $(m-\not{p})^{-1}$, which become singular in the limit $p^2\to m^2$. Gauge-dependent contributions resulting from such terms were also not included in Feynman's proof.

The combined gauge-dependent terms arising from these two sources fortunately cancel out in every order of perturbation theory, so that the final conclusion reached by Feynman, that the Coulomb gauge and the Feynman gauge are equivalent, is still valid. This equivalence has been studied in detail in a recent paper by Tatur.[15]

In the present paper we bypass many complications in the proof of the gauge invariance of transition amplitudes by working directly with renormalized transition probabilities. Formula (12) for $P(p;q;k)$ does not contain explicitly the electron wave-function renormalization constants which were the source of most of the complications encountered by Tatur in his proof of gauge invariance.

The proof of the gauge invariance given below consists of showing the invariance of $P(p;q;k)$ under the infinitesimal change of the photon propagator,

$$D_{\mu\nu}(k)\to D_{\mu\nu}(k)+k_\mu\delta g_\nu(k)+\delta g_\mu(k)k_\nu. \quad (25)$$

The invariance under (25) will guarantee the invariance under the finite gauge transformations, since these transformations clearly form a group.

The gauge transformation (25) causes the following change (to the lowest order in δg) in any propagator G:

$$\delta G=2\int(dk)k_\mu\delta g_\nu(k)\frac{\delta}{\delta D_{\mu\nu}(k)}G, \quad (26)$$

where $(dk)=(2\pi)^{-4}\,d^4k$.

The functional differentiation with respect to $D_{\mu\nu}(k)$ has a simple description in terms of the diagrams. When acting on any diagram which represents a contribution to G, $\delta/\delta D_{\mu\nu}$ produces a set of diagrams with two additional external photon lines.[16] Every diagram of this set is obtained from the original diagram for G by opening one internal photon line and symmetrizing the two new external photon lines.

The action of $\delta/\delta D_{\mu\nu}$ on any complete propagator containing all insertions can be expressed also in a different form. Every such propagator contains, in addition to the contribution from any internal photon line, all the self-energy corrections to this line. In other

[13] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

[14] Many others have fallen into the same trap (see Ref. 4). In several monographs on quantum electrodynamics and in many original papers where the problem of equivalence of various gauges was discussed, we find references to Feynman's paper or direct "proofs" of the gauge invariance of the *unrenormalized* transition amplitudes.

[15] S. Tatur, Acta Phys. Polon. **A37**, 71 (1970).

[16] For simplicity we shall assume that the external photon lines are *not* represented by the free photon propagators in G but that these lines are attached directly to the corresponding vertices.

words, $D_{\mu\nu}$ never appears alone in the propagator but always in the combination

$$D_{\mu\nu}+D_{\mu\lambda}\Pi^{\lambda\rho}D_{\rho\nu}+D_{\mu\lambda}\Pi^{\lambda\sigma}D_{\sigma\tau}\Pi^{\tau\rho}D_{\rho\nu}+\cdots=\mathcal{G}_{\mu\nu}. \quad (27)$$

Thus $\delta/\delta D_{\mu\nu}$, when acting on any complete propagator truncated in all external photon lines, can be expressed in terms of $\delta/\delta\mathcal{G}_{\mu\nu}$,

$$\frac{\delta}{\delta D_{\mu\nu}(k)}=\int(dk')\frac{\delta\mathcal{G}_{\lambda\rho}(k')}{\delta D_{\mu\nu}(k)}\frac{\delta}{\delta\mathcal{G}_{\lambda\rho}(k')}. \quad (28)$$

The differentiation with respect to $\mathcal{G}_{\mu\nu}$ removes an internal photon line together with all the self-energy corrections to this line. Using formula (27), we obtain

$$\frac{\delta\mathcal{G}_{\lambda\rho}(k')}{\delta D_{\mu\nu}(k)}=(2\pi)^4\delta_{(4)}(k-k')\tfrac{1}{2}[P^{\mu}{}_{\lambda}(k)P^{\nu}{}_{\rho}(k)+P^{\mu}{}_{\rho}(k)P^{\nu}{}_{\lambda}(k)]+P^{\mu}{}_{\alpha}(k)P^{\nu}{}_{\beta}(k)\mathcal{G}_{\lambda\sigma}(k')\Pi^{\alpha\beta\sigma\tau}(k,-k,k',-k')\mathcal{G}_{\tau\sigma}(k')+(\text{higher terms}), \quad (29)$$

where

$$P^{\mu}{}_{\lambda}(k)\equiv D^{-1\,\mu\rho}(k)\mathcal{G}_{\rho\lambda}(k)=\delta_{\lambda}{}^{\mu}+\Pi^{\mu\rho}(k)D_{\rho\lambda}(k)+\cdots, \quad (30)$$

$$\Pi^{\alpha\beta\sigma\tau}(k,-k,k',-k')=\frac{\delta\Pi^{\sigma\tau}(k')}{\delta\mathcal{G}_{\alpha\beta}(k)}, \quad (31)$$

and the higher terms involve products of 2, 3, ..., functions $\Pi^{\mu\nu\lambda\rho}$.

We shall use now the generalized Ward identity[17] (GWI) to learn more about the structure of δG. For the propagator $H^{\mu_1\cdots\mu_m}(p;q;k)$, which is obtained from $G_{\mu_1\cdots\mu_m}(p;q;k)$ by truncating only the external photon lines,

$$G_{\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m)=(-i)^m\mathcal{G}_{\mu_1\nu_1}(k_1)\cdots\mathcal{G}_{\mu_m\nu_m}(k_m)\times H^{\nu_1\cdots\nu_m}(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m), \quad (32)$$

this identity reads

$$k_\mu H^{\mu\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k,k_1\cdots k_m)=e\sum_{i=1}^{n}[H^{\mu_1\cdots\mu_m}(p_1\cdots,p_i-k,\cdots p_n;q_n\cdots q_1;k_1\cdots k_m)-H^{\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots,q_i+k,\cdots q_1;k_1\cdots k_m)]. \quad (33)$$

We give a new detailed proof of this identity in Appendix A. In the simplest case of no external electron lines, we have

$$k_\mu H^{\mu\mu_1\cdots\mu_m}(-;-;kk_1\cdots k_m)=0. \quad (34)$$

The reasoning of Appendix A can also be applied to $\Pi^{\mu\nu}$, so that we also have

$$k_\mu\Pi^{\mu\nu}(k)=0. \quad (35)$$

This last equation when used together with

$$k_\mu\Pi^{\mu\nu\lambda\rho}(k,-k,k',-k')=0 \quad (36)$$

[which is a special case of (34)] gives

$$k_\mu\delta g_\nu(k)\frac{\delta}{\delta D_{\mu\nu}(k)}=k_\mu\delta\tilde{g}_\nu\frac{\delta}{\delta\mathcal{G}_{\mu\nu}(k)}, \quad (37)$$

where

$$\delta\tilde{g}_\nu(k)\equiv\delta g_\lambda(k)P^{\lambda}{}_{\nu}(k). \quad (38)$$

With the help of Eqs. (27)–(31), (35), and (36) the change in the complete photon propagator under the gauge transformation (25) can be shown to have the form

$$\delta\mathcal{G}_{\mu\nu}(k)=k_\mu\delta\tilde{g}_\nu+\delta\tilde{g}_\mu(k)k_\nu. \quad (39)$$

Thus the change in $\mathcal{G}_{\mu\nu}$ is essentially of the same type as the initial change in $D_{\mu\nu}$, only the infinitesimal gauge function δg_μ has gotten renormalized. With the help of the last formula, Eq. (37) can be written in the more transparent form

$$\delta D_{\mu\nu}(k)\frac{\delta}{\delta D_{\mu\nu}(k)}=\delta\mathcal{G}_{\mu\nu}(k)\frac{\delta}{\delta\mathcal{G}_{\mu\nu}(k)}. \quad (40)$$

The change in the complete one-electron propagator $G(p)$ can also be explicitly evaluated:

$$\delta G(p)=\int(dk)\delta\mathcal{G}_{\mu\nu}(k)\frac{\delta G(p)}{\delta\mathcal{G}_{\mu\nu}(k)}=\int(dk)k_\mu\delta\tilde{g}_\nu(k)H^{\mu\nu}(p;p;k,-k). \quad (41)$$

With the help of the GWI this can be expressed in the form

$$\delta G(p)=\delta N(p)G(p)+G(p)\delta\bar{N}(p), \quad (42)$$

where

$$\delta N(p)\equiv ie^2\int(dk)\delta\tilde{g}_\nu(k)G(p-k)\Gamma^\nu(p-k;p;-k), \quad (43a)$$

$$\delta\bar{N}(p)\equiv -ie^2\int(dk)\delta\tilde{g}_\nu(k)\times\Gamma^\nu(p;p+k;-k)G(p+k). \quad (43b)$$

In our proof of the gauge invariance we shall need the property of $\delta\bar{N}(p)$ that it is equal below the thresh-

[17] This identity is often referred to as the Ward-Takahashi identity, but the name Ward-Green-Fradkin-Takahashi identity would be more appropriate {see H. S. Green, Proc. Phys. Soc. (London) **A66**, 873 (1953); E. S. Fradkin, Zh. Eksperim. i Teor. Fiz. **29**, 258 (1955) [Soviet Phys. JETP **2**, 361 (1956)]; Y. Takahashi, Nuovo Cimento **6**, 371 (1957)}.

old to the bispinor conjugate of $\delta N(p)$,

$$\delta\bar{N}(p)=\overline{\delta N(p)}. \tag{44}$$

The proof of Eq. (44) is given in Appendix B. Here we would like to point out that, if this relation holds, then the property of the electron propagator of being self-adjoint under the bispinor conjugation,

$$G(p)=\bar{G}(p) \quad \text{below threshold}, \tag{45}$$

is not changed by the gauge transformation.

It is only in the proof of Eq. (44) that we need to impose some restrictions on the gauge functions $\delta g_\mu(k)$. We could relax the restrictions imposed on these functions at the beginning of this section and still have Eq. (44) satisfied, but such generalizations do not seem to be very interesting.

Unlike Σ and $\Pi^{\mu\nu}$, the propagators H and F are in general only weakly connected[18] and this causes a minor complication in the calculation of δH or δF. It is easier to evaluate δH and then to obtain δF from the formula

$$\begin{aligned}F^{\mu_1\cdots\mu_m}&(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m)\\&=i^{2n}G^{-1}(p_1)\cdots G^{-1}(p_n)\\&\quad\times H^{\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m)\\&\quad\times G^{-1}(q_1)\cdots G^{-1}(q_n).\end{aligned} \tag{46}$$

Since H contains the contributions from all the diagrams, with the only exception of the self-energy corrections to the external photon lines, the derivative of H with respect to $\mathcal{G}_{\mu\nu}$ is given by the formula

$$\begin{aligned}2i\frac{\delta}{\delta\mathcal{G}_{\mu\nu}(k)}&H^{\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m)\\&=H^{\mu\nu\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k,-k,k_1\cdots k_m)\\&\quad+\sum_D H^{\mu\cdots}(\cdots;\cdots;k\cdots)H^{\nu\cdots}(\cdots;\cdots;-k\cdots).\end{aligned} \tag{47}$$

In the last sum containing the contributions from these diagrams, which have become disconnected when the photon line was removed, we have shown only the dependence on the relevant variables. Using the GWI, we obtain

$$\begin{aligned}\delta H^{\mu_1\cdots\mu_m}&(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m)\\&=-ie\int(dk)\delta\tilde{g}_\nu(k)\sum_{i=1}^{n}[H^{\nu\mu_1\cdots\mu_m}(p_1\cdots,p_i-k,\cdots p_n;q_n\cdots q_1;-k,k_1\cdots k_m)\\&\quad-H^{\nu\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots,q_i+k,\cdots q_1;-k,k_1\cdots k_m)]-ie\int(dk)\delta\tilde{g}_\nu(k)\sum_D\sum_{i=1}^{l}[H^{\cdots}(\cdots,p_i-k,\cdots;\cdots;\cdots)\\&\quad-H^{\cdots}(\cdots;\cdots,q_i+k,\cdots;\cdots)]H^{\nu\cdots}(\cdots;\cdots;-k\cdots).\end{aligned} \tag{48}$$

It may appear at a first glance that the gauge change of H is less singular on the mass shell of the electron momenta than H and that it will not contribute, therefore, to the transition probability. A closer examination of the first integral in Eq. (48) reveals, however, a loophole in this argument. The propagator

$$H^{\nu\mu_1\cdots\mu_m}(p_1\cdots,p_i-k,\cdots p_n;q_n\cdots q_1;-k,k_1\cdots k_m)$$

contains always the (weakly connected) terms which have a pole in p_i^2 on the mass shell,

$$\begin{aligned}H^{\nu\mu_1\cdots\mu_m}&(p_1\cdots,p_i-k,\cdots p_n;q_n\cdots q_1;-k,k_1\cdots k_m)\\&=-eG(p_i-k)\Gamma^\nu(p_i-k;p;-k)\\&\quad\times H^{\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m)\\&\quad+(\text{terms regular in }p_i^2).\end{aligned} \tag{49}$$

An analogous contribution singular in q_i^2 is also found in the propagator

$$H^{\nu\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots,q_i+k,\cdots q_1;-k,k_1\cdots k_m).$$

No such terms are found, however, in the second integral in Eq. (48). Therefore, the gauge change in H can be written in the form

$$\begin{aligned}\delta H^{\mu_1\cdots\mu_m}&(p;q;k)\\&=\sum_{i=1}^{n}[\delta N(p_i)H^{\mu_1\cdots\mu_m}(p;q;k)\\&\quad+H^{\mu_1\cdots\mu_m}(p;q;k)\delta\bar{N}(q_i)]\\&\quad+(\text{terms less singular on mass shell}).\end{aligned} \tag{50}$$

We can now use Eqs. (46), (42), and (50) to obtain the formulas for the gauge changes in the truncated propagators F and $\bar{F}$. After simple manipulations with the use of Eqs. (44) and (45) and the relation

$$\delta G^{-1}(p)=-G^{-1}(p)\delta N(p)-\delta\bar{N}(p)G^{-1}(p), \tag{51}$$

we finally obtain

$$\begin{aligned}\delta F^{\mu_1\cdots\mu_m}&(p;q;k)\\&=-\sum_{i=1}^{n}[\delta\bar{N}(p_i)F^{\mu_1\cdots\mu_m}(p;q;k)\\&\quad+F^{\mu_1\cdots\mu_m}(p;q;k)\delta N(q_i)]\\&\quad+(\text{terms vanishing on mass shell}),\end{aligned} \tag{52a}$$

[18] The removal of one internal photon (or electron) line may lead to a disconnected diagram consisting of two separate connected pieces.

$$\delta\bar{F}^{\mu_1\cdots\mu_m}(p;q;k) = -\sum_{i=1}^{n}[\delta\bar{N}(q_i)\bar{F}^{\mu_1\cdots\mu_m}(p;q;k) + \bar{F}^{\mu_1\cdots\mu_m}(p;q;k)\delta N(p_i)] + (\text{terms vanishing on mass shell}). \quad (52b)$$

To complete the proof of the gauge invariance of the transition probability $P(p;q;k)$, we must now only collect all the gauge terms resulting from the change in the photon propagators $\mathcal{G}_{\mu\nu}(k_l)$, in the electron propagators $G(p_i)$ and $G(q_j)$, and in the truncated propagators F and $\bar{F}$, and show that their sum is equal to zero.

First we observe that, owing to the GWI for F and $\bar{F}$ (cf. Appendix B),

$$k_\mu F^{\mu\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k,k_1\cdots k_m) = e\sum_{i=1}^{n}[G^{-1}(p_i)G(p_i-k)F^{\mu_1\cdots\mu_m}(p_1\cdots,p_i-k,\cdots p_n;q_n\cdots q_1;k_1\cdots k_m) - F^{\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots,q_i+k,\cdots q_1;k_1\cdots k_m)G(q_i+k)G^{-1}(q_i)], \quad (53)$$

and the conjugate formula for $\bar{F}$, the gauge terms resulting from $\delta\mathcal{G}_{\mu\nu}$'s do not contribute to P on the mass shell.

Next we collect all the contributions resulting from the gauge changes in the remaining propagators, and we find that the gauge changes in the one-electron propagators cancel exactly the gauge changes in the truncated propagators F and $\bar{F}$.

This completes the proof of the gauge invariance of the transition probabilities in quantum electrodynamics under the gauge transformations of the photon propagator.

The Coulomb and Feynman gauges are completely[19] equivalent in all calculations of observable transition probabilities in quantum electrodynamics.

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APPENDIX A

The first step in the proof of the GWI (33) for the complete truncated propagators will be the derivation of a rudimentary form of the generalized Ward identity for the Feynman propagator $K_F[x,y|\mathfrak{A}]$ in an external electromagnetic field. This propagator obeys the inhomogeneous Dirac equation

$$(m-i\gamma^\mu\partial_\mu+e\gamma^\mu\mathfrak{A}_\mu)K_F[x,y|\mathfrak{A}]=\delta_{(4)}(x-y) \quad (\text{A1})$$

and satisfies the Feynman boundary conditions. Differentiating Eq. (A1) with respect to $\mathfrak{A}_\mu(z)$ and solving the resulting equation for $[\delta/\delta\mathfrak{A}_\mu(z)]K_F$, we obtain

$$\frac{\delta}{\delta\mathfrak{A}_\mu(z)}K_F[x,y|\mathfrak{A}]=-eK_F[x,z|\mathfrak{A}]\gamma^\mu K_F[z,y|\mathfrak{A}]. \quad (\text{A2})$$

Differentiating the propagator K_F m times and using systematically Eq. (A2) after every differentiation, we obtain a sum of $m!$ terms corresponding to all the diagrams having one open electron line and m photon vertices distributed in all possible ways. We will denote the sum of all contributions from these diagrams by $K^{\mu_1\cdots\mu_m}[x;y;z_1\cdots z_m|\mathfrak{A}]$,

$$\frac{\delta^m}{\delta\mathfrak{A}_{\mu_1}(z_1)\cdots\delta\mathfrak{A}_{\mu_m}(z_m)}K_F[x,y|\mathfrak{A}] = iK^{\mu_1\cdots\mu_m}[x;y;z_1\cdots z_m|\mathfrak{A}]. \quad (\text{A3})$$

Taking the divergence of both sides in Eq. (A2), we obtain, with the help of Eq. (A1),

$$-i\partial_\mu{}^{(z)}\frac{\delta}{\delta\mathfrak{A}_\mu(z)}K_F[x,y|\mathfrak{A}] = e[\delta_{(4)}(x-z)-\delta_{(4)}(z-y)]K_F[x,y|\mathfrak{A}]. \quad (\text{A4})$$

This relation may be called the Ward identity for the electron propagator in an external field. From it, by differentiation and with the use of (A3), we derive an analogous relation for the $K^{\mu_1\cdots\mu_m}$,

$$-i\partial_\mu{}^{(z)}K^{\mu\mu_1\cdots\mu_m}[x;y;z,z_1\cdots z_m|\mathfrak{A}] = e[\delta_{(4)}(x-z)-\delta_{(4)}(z-y)] \times K^{\mu_1\cdots\mu_m}[x;y;z_1\cdots z_m|\mathfrak{A}]. \quad (\text{A5})$$

The contribution from $m!$ diagrams having one closed electron loop and $m+1$ photon vertices will be denoted by $L^{\mu\mu_1\cdots\mu_m}[z,z_1\cdots z_m|\mathfrak{A}]$. It can be expressed in terms of the $K^{\mu_1\cdots\mu_m}$ in the following way:

$$L^{\mu\mu_1\cdots\mu_m}[z,z_1\cdots z_m|\mathfrak{A}] = ie\,\mathrm{Tr}\{\gamma^\mu K^{\mu_1\cdots\mu_m}[z;z;z_1\cdots z_m|\mathfrak{A}]\}, \quad (\text{A6})$$

where the trace is with respect to bispinor indices and an appropriate limiting procedure should be used, if necessary, to define the limit $x\rightarrow z\leftarrow y$. The lack of symmetry between z and z_i's in this formula is only apparent. On account of (A5), $L^{\mu\mu_1\cdots\mu_m}$ is divergence free,

$$-i\partial_\mu{}^{(z)}L^{\mu\mu_1\cdots\mu_m}[z_1\cdots z_m|\mathfrak{A}]=0. \quad (\text{A7})$$

Let us denote by

$$H^{\mu_1\cdots\mu_m}[x_1\cdots x_n;y_n\cdots y_1;z_1\cdots z_m|\mathfrak{A}]$$

[19] One can easily extend our proof to the case when an external electromagnetic field $\mathfrak{A}$ is present, since the GWI has the same form also in the presence of $\mathfrak{A}$ (Appendix A).

the sum of the contributions from all connected diagrams having n open electron lines, m external photon lines, and any number of closed electron loops, but truncated in all external photon lines [cf. Eq. (4.6)]. The set of all diagrams contributing to H can be divided into the subsets, each subset representing a product of functions $K^{\mu_1\cdots\mu_m}$, $L^{\nu_1\cdots\nu_m}$ and the free photon propagators representing the internal photon lines. For every contribution to H from such a subset we can use relations (A5) or (A7), and the final relation for H will be

$$-i\partial_\mu{}^{(z)}H^{\mu\mu_1\cdots\mu_m}[x_1\cdots x_n;y_n\cdots y_1;z,z_1\cdots z_m|\mathfrak{A}]$$
$$=e\sum_{i=1}^{n}[\delta_{(4)}(x_i-z)-\delta_{(4)}(z-y_i)]$$
$$\times H^{\mu_1\cdots\mu_m}[x_1\cdots x_n;y_n\cdots y_1;z_1\cdots z_m|\mathfrak{A}]. \quad \text{(A8)}$$

The Fourier transform $\tilde{H}$,

$$\tilde{H}^{\mu_1\cdots\mu_m}[p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m|\mathfrak{A}]$$
$$\equiv\int d^{4n}xd^{4n}yd^{4m}z\exp(i\textstyle\sum p_i\cdot x_i-i\sum q_j\cdot y_j-i\sum k_l\cdot z_l)$$
$$\times H^{\mu_1\cdots\mu_m}[x_1\cdots x_n;y_n\cdots y_1;z_1\cdots z_m|\mathfrak{A}], \quad \text{(A9)}$$

obeys the relations

$$k_\mu\tilde{H}^{\mu\mu_1\cdots\mu_m}[p_1\cdots p_n;q_n\cdots q_1;k,k_1\cdots k_m|\mathfrak{A}]$$
$$=e\sum_{i=1}^{n}(\tilde{H}^{\mu_1\cdots\mu_m}[p_1\cdots,p_i-k,\cdots p_n;$$
$$q_n\cdots q_1;k_1\cdots k_m|\mathfrak{A}]$$
$$-\tilde{H}^{\mu_1\cdots\mu_m}[p_1\cdots p_n;q_n\cdots,$$
$$q_i+k,\cdots q_1;k_1\cdots k_m|\mathfrak{A}]). \quad \text{(A10)}$$

On account of (A7), relations (A8) and (A10) have the same form no matter whether we include or disregard the vacuum polarization corrections to the external photon lines.

In the absence of the external field we can separate the δ function, which expresses the conservation of the total four-momentum,

$$\tilde{H}^{\mu_1\cdots\mu_m}[p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m|\mathfrak{A}]|_{\mathfrak{A}=0}$$
$$=(2\pi)^4\delta(\textstyle\sum p_i-\sum q_j-\sum k_l)$$
$$\times H^{\mu_1\cdots\mu_m}(p_1\cdots p_n;q_n\cdots q_1;k_1\cdots k_m). \quad \text{(A11)}$$

Equations (A10) and (A11) give the GWI (33) for H. Finally, from (33) and relation (46) between F and H, we obtain the GWI for F as given by formula (53).

APPENDIX B

It follows from the definition of the free photon propagator $D_{\mu\nu}(z_1,z_2)$ that it is a symmetric function of its arguments:

$$D_{\mu\nu}(z_1,z_2)=D_{\nu\mu}(z_2,z_1). \quad \text{(B1)}$$

In addition, in every theory which is invariant under TP, this propagator obeys the relation

$$D_{\mu\nu}(z_1,z_2)=D_{\nu\mu}(-z_2,-z_1). \quad \text{(B2)}$$

Thus in a TP-invariant theory

$$D_{\mu\nu}(z_1,z_2)=D_{\mu\nu}(-z_1,-z_2). \quad \text{(B3)}$$

In terms of the Fourier transforms (assuming translational invariance), relations (B1) and (B3) read

$$D_{\mu\nu}(k)=D_{\nu\mu}(-k), \quad \text{(B4)}$$

$$D_{\mu\nu}(k)=D_{\mu\nu}(-k). \quad \text{(B5)}$$

The gauge transformations of the photon propagator (25) clearly do not destroy these relations. Since all the remaining ingredients (free-electron propagators, vertices, etc.) which enter into the definition of the complete propagators also have the correct transformation properties under the TP transformation, we can use the TP invariance to derive some useful relations for the complete propagators.

For the one-photon propagator $\mathcal{G}_{\mu\nu}(k)$, the one-electron propagator $G(p)$, and the vertex function $\Gamma^\mu(p;q;k)$, TP invariance leads to the following relations:

$$\mathcal{G}_{\mu\nu}(k)=\mathcal{G}_{\mu\nu}(-k), \quad \text{(B6)}$$

$$\bar{G}(p)=G^{(*)}(p), \quad \text{(B7)}$$

$$\bar{\Gamma}^\mu(p;q;k)=\Gamma^{\mu(*)}(q;p;-k), \quad \text{(B8)}$$

where the bar denotes bispinor conjugation,

$$\bar{G}(p)\equiv\gamma^0G^\dagger(p)\gamma^0, \quad \text{(B9)}$$

and the symbol $(^*)$ denotes the complex conjugation of all the complex numbers but not of the γ matrices,

$$\gamma^{\mu(*)}=\gamma^\mu. \quad \text{(B10)}$$

We will use now the relations (B6) and (B7) to prove that below the threshold

$$\bar{G}(p)=G(p), \quad \text{(B11)}$$

$$\overline{\delta N(p)}=\delta\bar{N}(p). \quad \text{(B12)}$$

First let us consider $G(p)$ as represented by the sum of Feynman diagrams. Every contribution to $G(p)$ has the form of a multiple integral in momentum space.

Below the threshold in P we can perform the simultaneous Wick rotation in all integration variables. For every vector integration variable l_μ, it consists in rotating the integration contour in the complex l_0 plane,

$$l_0\to il_0. \quad \text{(B13)}$$

It follows from the Feynman $i\epsilon$ prescription that the rotating contours will not encounter any singularities. After the Wick rotation, the integrand is regular and we can take $\epsilon\to0$ in all denominators. Thus the imaginary numbers $i\epsilon$ disappear in all propagators and we are only left with a product of i's in front of the integral. From the general structure of diagrams contributing to $G(p)$, one finds that the total number of i's coming from the propagators, the vertices, and the Wick rotation is always even. Hence, below the threshold

$$G^{(*)}(p)=G(p)\,, \tag{B14}$$

and Eq. (B11) follows.

In the proof of Eq. (B12) we follow the same line of reasoning. We use Eqs. (43a), (B7), and (B8) to find

$$\overline{\delta N(p)} =\left[ie^2\int(dk)\,\delta\tilde{g}_\nu(k)\Gamma^\nu(p;p-k;k)G(p-k)\right]^{(*)}. \tag{B15}$$

Then we change the integration variable $k\to -k$ and use the property that $\delta\tilde{g}_\nu(k)$ is an odd function of k. Finally, we employ the argument involving the Wick rotation to show that the expression in the bracket in (B15) does not change under (*) conjugation below the threshold. On account of (43b), this gives the desired relation (B12).

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Field-Theory Realization of the Droplet Model*

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We demonstrate with two field-theoretic models that the operator droplet model proposed by Chou and Yang can be understood as due to the contribution from a class of leading Feynman diagrams at $s=\infty$. In the first model, we consider a theory which consists of both a strong and an electromagnetic interaction. The purpose of introducing two types of interactions is to supply a natural division between the production of particles and the interaction between the jets. In this model, the sum of pure photon-exchange diagrams leads automatically to an expression identical to that of the operator droplet model. Limitations and generalizations of the model are investigated. The second model is derived from pure quantum electrodynamics. We find that an operator-droplet-model formulation can reproduce leading amplitudes in quantum electrodynamics, including those which give rise to $(\ln s)^N$ behavior. This confirms and generalizes an earlier result of Lee. We demonstrate explicitly how the N-bubble diagrams should be treated in this calculation. By including diagrams related to one another by covariance, a reference-frame-independent result always emerges. These frame-independent results coincide with earlier calculations based on the usual Feynman rules.

I. INTRODUCTION

A DROPLET model has been proposed by Chou, Yang,[1] and their co-workers in order to understand qualitatively high-energy scattering. In particular, they conjectured that hadron production processes can be understood through an operator version of the droplet model. In a recent article,[2] the operator droplet model was put into an elegant and useful form by Lee. Lee then applied the operator droplet model to quantum electrodynamics (QED), with the identification of the matter density ρ and the charge density. He demonstrated that this model can reproduce the field-theoretic results of Cheng and Wu[3] as to impact factors. It is interesting to know whether or how more complicated $(\ln s)^N$-dependent terms can be obtained in Lee's formulation. It is also important to find out if his conclusion can be generalized to high-order processes.

The purpose of this paper is to show that the operator-droplet-model results can indeed be obtained by summing a proper set of diagrams. By establishing the connection between a physical model and a category of Feynman diagrams, one can hope to gain some insights and understanding of the model, such as its possible limitations and generalizations. This is one of the important reasons for carrying out a systematical analysis of Feynman diagrams.

The first model that we shall study is a combination of a pseudoscalar-meson theory and the electromagnetic (EM) interaction. We first analyze a diagram by

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