S Matrix for Yang-Mills and Gravitational Fields

E. S. FRADKIN AND I. V. TYUTIN
Physical Lebedev Institute, Academy of Sciences, Moscow, U.S.S.R.
(Received 19 January 1970)

A method is suggested (and applied to the Yang-Mills and gravitational fields) for the construction of the generating functional (S matrix) for fields possessing an invariance group. The unitarity and gauge independence of the S matrix on the mass shell are seen explicitly.

I. INTRODUCTION

THERE has lately been considerable intensification in the study of theories partially or completely invariant under non-Abelian groups of transformations. This is in connection with the discovery of vector mesons and their classification into multiplets, with the use of vector mesons to account for the form factors of particles, and with the current-algebra approach. Intermediate vector bosons are introduced in many schemes of weak interaction. An important example of a theory with a non-Abelian group of invariance is that of the gravitational field.

In the present paper a procedure for constructing the Feynman rules is proposed for theories possessing a gauge group, such as the theories of the massless Yang-Mills and gravitation fields. It is known that some additional (gauge) condition must be imposed on the dynamical variables in order that a consistent quantum field theory may be formulated on the basis of a Lagrangian density invariant under a local transformation group. In covariant gauges this can conveniently be done by the use of Lagrange multipliers. The basic idea of the method proposed is to choose the Lagrange multiplier in such a way that one is led to free equations of motion for the additional field. This fact guarantees the unitarity of the S matrix in physical space. The Feynman rules obtained coincide with those proposed in Refs. 1–5. The difference of the method under consideration from that of Refs. 2–5 is that we have succeeded in obtaining a set of consistent dynamical equations completely describing the theory. On the one hand, these equations make it possible to elucidate the reason for the additional diagrams to appear, and, on the other hand, guarantee the unitarity of the physical S matrix. Section II is devoted to the construction of the S matrix for the massless Yang-Mills field in arbitrary gauge and to the proof of the gauge invariance of the S matrix. In Sec. III constructing the Feynman rules in the Coulomb and axial gauges is considered on the basis of the canonical quantization procedure. The S matrix obtained coincides with that found in Sec. II, which fact reaffirms its unitarity. It is shown, furthermore, that taking the additional conditions consistently into account makes it possible to obtain self-consistent equations for the massless Yang-Mills field in the presence of an external source.

In Sec. IV the Feynman rules for the gravitational field are constructed in covariant gauges. These rules coincide with those suggested in Refs. 2, 3, and 5. In the framework of our approach we also obtain the S matrix for a noncovariant (Dirac) gauge, for which the Feynman rules have been obtained by Popov and Faddeev[6] using a method closely connected with the canonical formulation of the gravitational field. In addition, by our method, the equivalence of the S matrix in covariant and noncovariant gauges is proved.

We use the following notation. Greek μ, ν, λ, ... and the Latin i, j, k indices take the values 0, 1, 2, 3 and 1, 2, 3, respectively. In Secs. II and III, $g_{\mu\nu}$ means the Minkowski tensor $(+,---)$ and δ_{ik} means the unit tensor. By the summation over repeated indices is everywhere meant $a_\mu b_\mu \equiv a_0 b_0 - a_k b_k$; $\partial_\mu \equiv \partial/\partial x^\mu$; $\Box = \partial_\mu \partial_\mu$; $\nabla \equiv \partial_k \partial_k$. In Sec. IV, $g_{\mu\nu}$ means the metric tensor, and the Minkowski tensor is designated as $\delta_{\mu\nu}$. The usual summation over repeated indices means $a_\mu b^\mu \equiv \sum_{\mu=0}^{3} a_\mu b^\mu$. We use the system of units $\hbar = c = 1$ in Secs. II and III and $c^3/16\pi k = 1$ in Sec. IV (where k is the gravitational constant).

II. GENERAL THEORY OF CONSTRUCTION OF FEYNMAN RULES FOR MASSLESS YANG-MILLS FIELD. GAUGE INVARIANCE OF S MATRIX

In this section the general theory for construction of a unitary S matrix for massless Yang-Mills fields is considered.

The classical Lagrangian for a Yang-Mills[7] field has the form

$$L_0(x) = -\tfrac{1}{4} G_{\mu\nu}{}^a(x) G_{\mu\nu}{}^a(x) . \tag{2.1}$$

Here $G_{\mu\nu}{}^a$ is the field-strength tensor,

$$G_{\mu\nu}{}^a(x) = \partial_\mu A_\nu{}^a(x) - \partial_\nu A_\mu{}^a(x) + \lambda \hat{A}_\mu{}^{ab}(x) A_\nu{}^b(x) , \tag{2.2}$$

$$\hat{A}_\mu{}^{ab}(x) = f^{acb} A_\mu{}^c(x) . \tag{2.3}$$

The f^{abc} are the structure constants of the arbitrary finite-dimensional compact simple Lie group G. The f^{abc}

[1] R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963).
[2] B. S. DeWitt, Phys. Rev. **162**, 1195 (1967); **162**, 1239 (1967).
[3] L. D. Faddeev and V. N. Popov, Phys. Letters **25B**, 30 (1967); V. N. Popov and L. D. Faddeev, ITP report, Kiev, 1967 (unpublished).
[4] S. Mandelstam, Phys. Rev. **175**, 1580 (1968).
[5] S. Mandelstam, Phys. Rev. **175**, 1604 (1968).
[6] V. N. Popov and L. D. Faddeev (to be published).
[7] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).

are real and totally antisymmetric, and satisfy the Jacobi identity

$$f^{abc}f^{cdf}+f^{afc}f^{cfd}+f^{adc}f^{cfb}\equiv 0. \qquad (2.4)$$

The Lagrangian (2.1) is invariant under arbitrary gauge transformations depending on coordinates which for $A_\mu{}^a$ have the form

$$\hat{A}_\mu(x)\to\hat{A}_\mu{}^S(x)\equiv S[u(x)]\hat{A}_\mu(x)S^{-1}[u(x)] + (i/\lambda)\partial_\mu S[u(x)]\cdot S^{-1}[u(x)], \qquad (2.5)$$

and for $G_{\mu\nu}{}^a$ are

$$\hat{G}_{\mu\nu}(x)\to\hat{G}_{\mu\nu}{}^S(x)\equiv S[u(x)]\hat{G}_{\mu\nu}(x)S^{-1}[u(x)]. \qquad (2.6)$$

The matrices $S(u)\equiv S^{ab}(u)$ form the adjoint representation of G, and u^a are the group parameters. In (2.5) and (2.6) these parameters are arbitrary functions of the coordinates.

For infinitesimal transformations,

$$S^{ab}(x)=\delta_{ab}+i\hat{u}^{ab}(x), \qquad (2.7)$$

the gauge transformations have the form

$$A_\mu{}^a(x)\to A_\mu{}^a(x)-(1/\lambda)\nabla_\mu{}^{ab}(x)u^b(x). \qquad (2.5')$$

$\nabla_\mu{}^{ab}$ is the covariant differentiation

$$\nabla_\mu{}^{ab}(x)\equiv\delta_{ab}\partial_\mu+\lambda\hat{A}_\mu{}^{ab}(x). \qquad (2.8)$$

In the case of Abelian groups, formulas (2.5) and (2.6) are the gauge transformations of the electromagnetic field:

$$A_\mu\to A_\mu-\partial_\mu u,\quad F_{\mu\nu}\to F_{\mu\nu}. \qquad (2.9)$$

With the help of the Lagrangian (2.1), the classical field equations are obtained:

$$L_0{}^a{}_{,\mu}(x)\equiv\nabla_\nu{}^{ab}(x)G_{\nu\mu}{}^b(x)=0. \qquad (2.10)$$

According to the invariance of the Lagrangian under gauge transformations, it follows that the gauge variation of the action

$$W_0\equiv\int dx\, L_0(x) \qquad (2.11)$$

must be equal to zero:

$$\delta W_0\equiv\frac{1}{\lambda}\int dx\, u^a(x)\nabla_\mu{}^{ab}(x)L_0{}^b{}_{,\mu}(x)=0. \qquad (2.12)$$

As the functions $u^a(x)$ are arbitrary, an important identity follows from (2.12):

$$\nabla_\mu{}^{ab}(x)L_0{}^b{}_{,\mu}(x)\equiv\nabla_\mu{}^{ab}(x)\nabla_\nu{}^{bc}(x)G_{\mu\nu}{}^c(x)=0. \qquad (2.13)$$

It is necessary to note that the identity (2.13) is valid for arbitrary functions $A_\mu{}^a(x)$. Equation (2.13) may be obtained with the help of covariant differentiation (2.10).

It is known that the theory with Lagrangian (2.1) permits no direct transition to a canonical formalism (both in the classical and in the quantum theories). This is connected with the following. Let us find the canonical momenta:

$$\pi_0{}^a(x)=0,\quad \pi_k{}^a(x)=-G_{0k}{}^a(x). \qquad (2.14)$$

In the canonical theory the following relations must be fulfilled ($x_0=y_0$):

$$[A_i{}^a(x),A_j{}^b(y)]=[\pi_i{}^a(x),\pi_j{}^b(y)]=0,$$
$$[\pi_i{}^a(x),A_j{}^b(y)]=i\delta_{ab}\delta_{ij}\delta^{(3)}(x-y), \qquad (2.15)$$

where[,] is the commutator in the quantum theory and the Poisson bracket in the classical theory.

In canonical variables the equation $L_0{}^a{}_{,0}=0$ is as follows:

$$\partial_k\pi_k{}^a+\lambda\hat{A}_k{}^{ab}\pi_k{}^b=0, \qquad (2.16)$$

and it is in contradiction with (2.15).

A similar situation takes place when constructing the S matrix in the theory with Lagrangian (2.1). Usually one writes the S matrix as an expansion in the normal products of the free fields. The corresponding Lagrangian is equal to the total Lagrangian when the coupling constants vanish. In the present case, we obtain

$$L_0{}^{(0)}=-\tfrac{1}{4}F_{\mu\nu}{}^aF_{\mu\nu}{}^a, \qquad (2.1')$$

$$F_{\mu\nu}{}^a=\partial_\mu A_\nu{}^a-\partial_\nu A_\mu{}^a, \qquad (2.2')$$

$$\partial_\mu F_{\mu\nu}{}^a=0. \qquad (2.9')$$

However, one can prove[8] that in the framework of the axioms of modern quantum field theory the Lorentz-covariant operator $F_{\mu\nu}$ is equal to zero when it satisfies (2.2′) and (2.9′).

For the correct construction of the theory of the Yang-Mills field in the framework of quantum theory or canonical formalism of the classical theory, it is necessary to impose an additional (gauge) condition. For example, in electrodynamics one uses the Lorentz condition

$$\partial_\mu A_\mu=0$$

or the Coulomb gauge

$$\partial_k A_k=0.$$

Now the gauge invariance of the theory consists of the fact that physical observables—in particular, the S matrix—are independent of the choice of gauge conditions.

It is convenient to introduce the gauge conditions in the theory with the help of the Lagrange multiplier $B(x)$. In quantum theory the Lagrange multiplier B must be considered as a new dynamical variable, and it is necessary that the physical observables should not depend on the B field.

For example, the correct formulation of quantum electrodynamics in the Lorentz gauge is obtained with

[8] A. S. Wightman and L. Gårding, Arkiv Physik **28**, 129 (1964); F. Strocchi, Phys. Rev. **162**, 1429 (1967); **166**, 1302 (1968).

the help of the Lagrangian

$$L=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+\partial_\mu A_\mu B+A_\mu J_\mu. \quad (2.17)$$

It turns out that the B field is free. When calculating the S matrix, we must use the transverse photon propagator. The S matrix is unitary in the physical subspace, which is determined with the help of the condition

$$B^{(+)}(x)|\Psi\rangle=0. \quad (2.18)$$

Furthermore, it is known that in electrodynamics the S matrix is gauge invariant, i.e., it is independent of a choice of gauge condition. We shall use a similar procedure in the theories with arbitrary gauge group.

The basic idea of the method proposed for construction of the S matrix in theories with a gauge group consists in a choice of the Lagrange multiplier such that the additional (fictitious) B field is free. It means that the B field is not involved in the scattering and the S matrix is unitary in the physical subspace.

Let us first take the Lagrange multiplier for the Yang-Mills theory in the form (2.17):

$$L=L_0+\partial_\mu A_\mu{}^a B^a. \quad (2.19)$$

Then the B field imposes the Lorentz condition

$$\partial_\mu A_\mu{}^a=0,$$

and we must use the transverse propagator of the A_μ field in a perturbative calculation.

However, as was first noted by Feynman,[1] if one takes $L_0-L_0{}^{(0)}$ as an interaction Lagrangian, the S matrix is nonunitary in the physical subspace. This arises from the fact that in the case of the Lagrangian (2.19), the B fields satisfy the equation

$$\nabla_\mu{}^{ab}\partial_\mu B^b=0. \quad (2.20)$$

The fictitious B field is not free and does take part in the scattering. Thus we conclude that the Lagrange multiplier must depend on the A_μ field.

Now we are in a position to proceed to the concrete construction of the S matrix. Consider a class of gauges which is described by an arbitrary function

$$\psi^a\equiv\psi^a(x;A). \quad (2.21)$$

For example, ψ^a can be equal to $\partial_\mu A_\mu{}^a$ or $\partial_k A_k{}^a$. Later we shall impose a condition on the function ψ^a. Let us choose the action in the form

$$W^\psi=W_0+W_1{}^\psi=W_0+\int dx\, L_1{}^\psi(x). \quad (2.22)$$

The part $W_1{}^\psi$ is added to specify the gauge of the A_μ field:

$$L_1{}^\psi(x)=\psi^a(x;A)\bar{B}_\psi{}^a(x;A)+\tfrac{1}{2}\alpha\bar{B}_\psi{}^a(x;A)\bar{B}_\psi{}^a(x;A), \quad (2.23)$$

$$\bar{B}_\psi{}^a(x;A)=\int dy\,\mathbf{D}_\psi{}^{ab}(x,y;A)B^b(y). \quad (2.24)$$

We shall consider the case $\alpha\neq 0$ only for the gauge function $\psi^a=\partial_\mu A_\mu{}^a$. We choose the function $\mathbf{D}_\psi$ by using the condition that the B^a fields obey the free-field equation.

The variation of (2.22) over A_μ and B results in the following field equations:

$$\psi^a(x;A)=-\alpha\bar{B}_\psi{}^a(x;A), \quad (2.25)$$

$$\nabla_\nu{}^{ab}(x)G_{\mu\nu}{}^b(x)+(R_\mu{}^{ab}\bar{B}_\psi{}^b)(x)=0. \quad (2.26)$$

In (2.26) $R_\mu{}^a$ is the operator

$$(R_\mu{}^{ab}\varphi^b)(x)\equiv\int dy\frac{\delta\psi^b(y;A)}{\delta A_\mu{}^a(x)}\varphi^b(y). \quad (2.27)$$

With the help of the identity (2.13), one obtains from (2.26)

$$(\nabla_\mu{}^{ab}R_\mu{}^{bc}\bar{B}_\psi{}^c)(x)\equiv(Q_\psi{}^{ac}\bar{B}_\psi{}^c)(x)=0. \quad (2.28)$$

We impose the restriction on ψ^a that the operator

$$Q_\psi{}^{(0)ab}\equiv\partial_\mu R_\mu{}^{ab}(A=0) \quad (2.29)$$

should be a nonsingular differential operator.

If we choose a function $\mathbf{D}_\psi$ as

$$\mathbf{D}_\psi{}^{ab}(x,y;A)=\mathfrak{D}_\psi{}^{ac}(x,y;A)\overleftarrow{Q}_\psi{}^{(0)cb}, \quad (2.30)$$

$$Q_\psi{}^{ab}\mathfrak{D}_\psi{}^{bc}(x,y;A)=\delta_{ac}\delta(x-y), \quad (2.31)$$

or in a symbolic notation

$$\mathbf{D}_\psi{}^{ab}=[Q_\psi{}^{-1}]^{ac}\overleftarrow{Q}_\psi{}^{(0)cb}, \quad (2.32)$$

then the B field satisfies the free-field equation

$$Q_\psi{}^{(0)ab}B^b(x)=0. \quad (2.33)$$

Note that the $\mathfrak{D}_\psi$ function satisfying (2.31) does exist and the determinant $\mathbf{D}_\psi$ is not zero, at least in the framework of perturbation theory.

Since the fictitious B field satisfies the free-field equation (2.33), in the physical subspace

$$B^{(+)a}(x)|\Phi\rangle=0, \quad (2.34)$$

the S matrix is unitary, and the classical field equations for the A_μ field are satisfied:

$$\langle\phi_1|L_0{}^a{}_{,\mu}(x)|\phi_2\rangle=\langle\phi_1|\psi^a(x;A)|\phi_2\rangle=0. \quad (2.35)$$

In order to obtain an expression for the S matrix, we use the connection between tke S matrix and the generating functional of Green's functions, $Z\{J\}$[9]:

$$S=:\exp\left[-i\int dx\, a^{\rm in}(x)\overrightarrow{\Box}\frac{\delta}{i\delta J(x)}\right]:Z\{J\}|_{J=0}. \quad (2.36)$$

Here $a^{\rm in}$ are the set of free-field operators describing the physical system for $t\to-\infty$.

---

[9] E. S. Fradkin, Dokl. Akad. Nauk SSSR **98**, 47 (1954); **100**, 897 (1955); Trudy Lebedev Phys. Inst. **29**, 7 (1965).

For the generating functional we use the representation in the form of a functional integral over the complete set of fields.[9,10]

Thus in the model under consideration we obtain for the generating functional the following expression:

$$Z_\alpha{}^\psi=\int dA_\mu{}^a dB^a \exp\left(iW^\psi+i\int dx\, A_\mu{}^a J_\mu{}^a\right)$$
$$=\int dA_\mu{}^a \exp\left\{i\int dx\left(L_0-\frac{1}{2\alpha}\psi^a\psi^a+A_\mu{}^a J_\mu{}^a\right)\right.$$
$$\left.+\mathrm{Tr}\ln Q_\psi{}^{ab}[Q_\psi{}^{(0)-1}]^{bc}\right\}. \quad (2.37)$$

When calculating (2.37), we passed from the integration over B^a to the integration over $\bar{B}^a$, and for the resulting Jacobian we used the expression

$$\det M\equiv\exp(\mathrm{Tr}\ln M). \quad (2.38)$$

In the gauge $\alpha=0$ (which is an analog of the Lorentz gauge $\partial_\mu A_\mu=0$ or the Coulomb gauge $\partial_k A_k=0$), the expression for the generating functional has the form

$$Z_0{}^\psi=\int dA_\mu{}^a\, \delta\{\psi^a(x;A)\}$$
$$\times\exp\left[i\int dx(L_0+A_\mu{}^a J_\mu{}^a)+\mathrm{Tr}\ln Q_\psi Q_\psi{}^{(0)-1}\right]. \quad (2.39)$$

The expressions (2.37) and (2.39) for the generating functional indicate that the correct Feynman rules for the perturbative calculation of the Green functions are the following:

(a) There exist two usual vertices of the interaction of vector mesons:

$$-\lambda\partial_\mu A_\nu{}^a(x)\hat{A}_\mu{}^{ab}(x)A_\nu{}^b(x)$$
and
$$-\tfrac{1}{4}\lambda^2(\hat{A}_\mu{}^{ab}(x)A_\nu{}^b(x))(\hat{A}_\mu{}^{ac}(x)A_\nu{}^c(x)). \quad (2.40)$$

(b) There exist additional vertices of the interaction of vector mesons with the fictitious B field; its form is determined by Q_ψ.

(c) The fictitious B fields always occur in closed loops, every loop possessing an additional factor $(-)$ and the propagator of the B field being

$$D_B{}^{ab}=[Q_\psi{}^{(0)-1}]^{ab}. \quad (2.41)$$

Let us pass to the proof of the gauge invariance of the S matrix [i.e., to the proof that the S matrix is independent of the α's in (2.37) and independent of the type of gauge condition ψ^a in (2.39)].

First we prove that the S matrix is independent of ψ^a when $\alpha=0$, i.e., independent of the choice of the gauge

$$\psi^a(x;A)=0. \quad (2.42)$$

In this proof we use the method of Ref. 3. Define the function $\Delta_\psi(A)$ by the relation

$$\phi_\psi(A)\equiv\Delta_\psi(A)\int d\mu(S)\,\delta\{\psi^a(x;A^S)\}=1. \quad (2.43)$$

In (2.43), S is an element of the gauge group, $A_\mu{}^{Sa}$ is defined by (2.5), and $d\mu(S)$ is the measure of group integration.[11,12] For compact simple groups (which we consider), $d\mu(S)$ is defined uniquely and has the property (for more details of the gauge group resulting from a given simple Lie group, see Ref. 12):

$$\int d\mu(S)\,f(S)=\int d\mu(S)\,f(SS_1)=\int d\mu(S)\,f(S^{-1}). \quad (2.44)$$

Property (2.44) makes it possible to prove the gauge invariance of Δ_ψ. Indeed,

$$\Delta_\psi{}^{-1}(A^{S_1})=\int d\mu(S)\,\delta\{\psi^a(x;A^{SS_1})\}$$
$$=\int d\mu(S)\,\delta\{\psi^a(x;A^S)\}=\Delta_\psi{}^{-1}(A).$$

It is sufficient to know the function $\Delta_\psi(A)$ only for an $A_\mu{}^a$ field satisfying condition (2.42). In this case the group integral is concentrated in the neighborhood of the unit element of the group. By use of (2.7) and (2.5'), we obtain

$$d\mu(S)\approx\prod_{x,a}du^a(x)\equiv d(u),$$

$$\Delta_\psi{}^{-1}(A)|_{\psi=0}=\int d(u)\,\delta\left\{\frac{1}{\lambda}(Q_\psi{}^{Tab}u^b)(x)\right\}$$
$$=\mathrm{Det}^{-1}Q_\psi{}^{ab}=\exp(-\mathrm{Tr}\ln Q_\psi{}^{ab}). \quad (2.45)$$

In (2.45) we omit the inessential infinite determinant of λ.

One can see from (2.45) that the expression (2.39) for the generating functional can be rewritten in the form

$$Z_0{}^\psi=\int dA_\mu{}^a\delta\{\psi^a\}\Delta_\psi(A)|_{\psi=0}\exp\left[i\int dx(L_0+A_\mu{}^a J_\mu{}^a)\right]$$
$$=\int dA_\mu{}^a\,\delta\{\psi^a\}\Delta_\psi(A)$$
$$\times\exp i\left[\int dx(L_0+A_\mu{}^a J_\mu{}^a)\right]. \quad (2.46)$$

Consider now another gauge:

$$\psi_1{}^a(x;A)=0. \quad (2.42')$$

[10] R. P. Feynman, Phys. Rev. **84**, 108 (1951).

[11] M. A. Naimark, *Normirovannie koltza* (Nauka, Moscow, 1968).

[12] B. S. DeWitt, *Relativity, Groups and Topology* (Gordon and Breach, New York, 1964).

Multiply expression (2.46) by the function $\phi_{\psi_1}(A)$ and perform the gauge transformation

$$A_\mu \to A_\mu{}^{S^{-1}}.$$

Taking into account the gauge invariance of Δ_ψ, Δ_{ψ_1}, and L_0, and the property (2.44), we obtain [the Jacobian $d(A^S)/d(A)$ is equal to 1]

$$Z_0{}^\psi = \int dA_\mu{}^a \int d\mu(S)\, \delta\{\psi^a(x;A^S)\}\Delta_\psi(A) \times\exp\left[i\int dx(L_0+A_\mu{}^{Sa}J_\mu{}^a)\right]\delta\{\psi_1{}^a\}\Delta_{\psi_1}(A). \quad (2.47)$$

Consider the term $A_\mu{}^{Sa}J_\mu{}^a$ in more detail. After passing to the mass shell [i.e., after passing from the generating functional to the S matrix according to (2.36)] this term has the form

$$\int dx\, a_\mu{}^{a\,\mathrm{in}}(x)\overrightarrow{\Box}A_\mu{}^{Sa}(x). \quad (2.48)$$

In terms of diagrams, $\overrightarrow{\Box}$ operating on A_μ means operating on an external line of the diagram. The external line has a pole of the form $1/\Box$, which cancels out $\overrightarrow{\Box}$. $\overrightarrow{\Box}$ operating on the product of the fields in the same point means that $\overrightarrow{\Box}$ operates directly on the vertex. In the framework of perturbation theory the vertex has no pole of the form $1/\Box$. Therefore, we can perform the integration in expressions of the type (2.48) by parts, and (2.48) vanishes.

Thus expression (2.48) can be rewritten on the mass shell (m.s.) as

$$\int dx\, a_\mu{}^{a\,\mathrm{in}}(x)\overrightarrow{\Box}A_\mu{}^a(x) = \int dx\, A_\mu{}^aJ_\mu{}^a|_{\mathrm{m.s.}}. \quad (2.49)$$

Now, according to (2.43), the group integral is equal to 1 and we obtain

$$S^\psi \equiv Z_0{}^\psi|_{\mathrm{m.s.}} = Z_0{}^{\psi_1}|_{\mathrm{m.s.}} \equiv S^{\psi_1}. \quad (2.50)$$

Relation (2.50) means that the S matrix is independent of the type of the gauge condition (2.42).

Below we shall consider three particular gauges and give the proof of the independence of the S matrix on the α's.

A. Axial Gauge

This gauge is defined by the condition

$$\psi_A{}^a(x;A) \equiv A_3{}^a(x) = 0. \quad (2.51)$$

With the help of (2.27), (2.28), and (2.32), we obtain

$$\mathbf{D}_A{}^{ab}(x,y;A) = [\partial_3+\lambda\hat{A}_3]^{-1}\overleftarrow{\partial}_3. \quad (2.52)$$

Because expression (2.39) for the generating functional contains $\delta\{A_3{}^a(x)\}$, $\mathbf{D}_A$ is effectively equal to 1. Thus in the axial gauge the generating functional is

$$Z^A = \int dA_\mu{}^a\, \delta\{A_3{}^a(x)\} \times\exp\left[i\int dx(L_0+A_\mu{}^aJ_\mu{}^a)\right]. \quad (2.53)$$

The Feynman rules have no additional diagrams. The free propagator of the $A_\mu{}^a$ field is

$$D_{\mu\nu}{}^{ab}(p) = \delta_{ab}\left[-ig_{\mu\nu}+\frac{i}{p_3}(g_{\mu 3}p_\nu+g_{\nu 3}p_\mu)+i\frac{p_\mu p_\nu}{p_3{}^2}\right]\frac{1}{p^2}. \quad (2.54)$$

B. Coulomb Gauge

This gauge is defined by the condition[13]

$$\psi_k{}^a(x;A) \equiv \partial_k A_k{}^a(x) = 0. \quad (2.55)$$

With the help of (2.27), (2.28), and (2.32), we obtain

$$\mathbf{D}_k{}^{ab}(x,y;A) = [\nabla+\lambda\hat{A}_k\partial_k]^{-1}\overleftarrow{\nabla}. \quad (2.56)$$

By the use of (2.39) the expression for the generating functional may be calculated to give

$$Z^k = \int dA_\mu{}^a\, \delta\{\partial_k A_k{}^a(x)\}\exp\left[i\int (L_0+A_\mu{}^aJ_\mu{}^a) +\mathrm{Tr}\ln\left(\delta_{ab}+\frac{\lambda}{\nabla}\hat{A}_k{}^{ab}\partial_k\right)\right]. \quad (2.57)$$

The Feynman rules in the Coulomb gauge are the following:

(a) The free propagator of $A_\mu{}^a$ is equal to the well-known propagator of the electromagnetic field in the Coulomb gauge:

$$D_{\mu\nu}{}^{ab}(p) = \delta_{ab}\begin{cases} i/\mathbf{p}^2, & \mu=\nu=0 \\ 0, & \mu=0,\ \nu\neq 0 \text{ or } \mu\neq 0,\ \nu=0 \\ i(\delta_{ik}-p_ip_k/\mathbf{p}^2)p^{-2}, & \mu=i,\ \nu=k. \end{cases} \quad (2.58)$$

(b) Besides the usual vertices (2.40) the vertex $F_\mu{}^{abc}$ of the interaction of $A_\mu{}^a(p)$ with the fictitious $B^b(k)$ and $B^c(q)$ fields exists:

$$F_\mu{}^{abc} = \lambda\delta_{\mu i}k_i f^{abc}, \quad p+k+q=0. \quad (2.59)$$

(c) The lines of the B field occur only in closed loops, every loop possessing an additional factor $(-)$, and the propagator of the B field is

$$\delta_{ab}i/\mathbf{p}^2. \quad (2.60)$$

[13] E. S. Fradkin and I. V. Tyutin, Phys. Letters **30B**, 562 (1969).

In Sec. II C we shall construct generating functionals (2.53) and (2.57) in the axial and Coulomb gauges by use of the canonical quantization procedure.

C. General Feynman Gauges

The general Feynman gauge is defined by the gauge function[13]

$$\psi_F{}^a(x;A)\equiv\partial_\mu A_\mu{}^a(x).\qquad(2.61)$$

We obtain the Lorentz gauge in the additional Lagrangian (2.23) for $\alpha=0$ (Landau gauge) and the Feynman gauge for $\alpha=1$.

With the help of (2.27), (2.28), (2.32), and (2.39), we obtain

$$\mathbf{D}_F{}^{ab}(x,y;A)=[\square+\lambda\hat{A}_\mu\partial_\mu]^{-1}\overleftarrow{\square},\qquad(2.62)$$

$$Z_\alpha{}^F=\int dA_\mu{}^a\times\exp\left[i\int dx\left(L_0-\frac{1}{2\alpha}\partial_\mu A_\mu{}^a\partial_\nu A_\nu{}^a+A_\mu{}^aJ_\mu{}^a\right)+\mathrm{Tr}\ln\left(\delta_{ab}+\frac{\lambda}{\square}\hat{A}_\mu{}^{ab}\partial_\mu\right)\right].\qquad(2.63)$$

In the transverse gauge we have

$$Z_0{}^F=\int dA_\mu{}^a\,\delta\{\partial_\mu A_\mu{}^a(x)\}\exp\left[i\int dx(L_0+A_\mu{}^aJ_\mu{}^a)+\mathrm{Tr}\ln\left(\delta_{ab}+\frac{\lambda}{\square}\hat{A}_\mu{}^{ab}\partial_\mu\right)\right].\qquad(2.64)$$

Now let us prove that the S matrix (2.63) is independent of the α's. Replace the variables in functional integrals (2.63)[2]

$$A_\mu{}^a(x)\to\bar{A}_\mu{}^a(x)\equiv A_\mu{}^a(x)-\frac{\delta\alpha}{2\alpha}\nabla_\mu{}^{ab}(x)(\mathfrak{D}^{bc}\partial_\nu A_\nu{}^c)(x),\qquad(2.65)$$

$$\mathfrak{D}^{bc}=[\square+\lambda\hat{A}_\mu\partial_\mu]^{-1},\qquad(2.66)$$

$\delta\alpha$ being infinitesimal. Retaining the terms of first order in $\delta\alpha$ we have

$$\frac{1}{2\alpha}\partial_\mu A_\mu{}^a\partial_\nu A_\nu{}^a\to\frac{1}{2(\alpha+\delta\alpha)}\partial_\mu A_\mu{}^a\partial_\nu A_\nu{}^a,\qquad(2.67)$$

$$L_0-\frac{1}{2\alpha}\partial_\mu A_\mu{}^a\partial_\nu A_\nu{}^a+A_\mu{}^aJ_\mu{}^a\to L_0-\frac{1}{2(\alpha+\delta\alpha)}\partial_\mu A_\mu{}^a\partial_\nu A_\nu{}^a+\bar{A}_\mu{}^aJ_\mu{}^a,\qquad(2.68)$$

$$\ln\det\frac{\partial\bar{A}_\mu{}^a(x)}{\partial A_\nu{}^b(y)}\equiv\mathrm{Tr}\ln\frac{\partial\bar{A}_\mu{}^a(x)}{\partial A_\nu{}^b(y)}=(\delta\alpha/2\alpha)\,\mathrm{Tr}[\nabla_\mu{}^{ac}(x)\mathfrak{D}^{cb}(x,y)\overleftarrow{\partial}_\nu+\lambda\nabla_\mu{}^{ac}(x)\mathfrak{D}^{cc'}(x,y)f^{c'bd}(\mathfrak{D}^{dd'}\partial_\lambda A_\lambda{}^{d'})(y)],\qquad(2.69)$$

$$\mathrm{Tr}\ln\mathfrak{D}^{-1}\overleftarrow{\square}^{-1}\to\mathrm{Tr}\ln\mathfrak{D}^{-1}\overleftarrow{\square}^{-1}-\lambda(\delta\alpha/2\alpha)\,\mathrm{Tr}[\partial_\mu\mathfrak{D}^{ac}(x,y)f^{cc'b}\nabla_\nu{}^{c'd}(y)\times(\mathfrak{D}^{dd'}\partial_\lambda A_\lambda{}^{d'})(y)].\qquad(2.70)$$

Remember that $\mathrm{Tr}f(x,t)$ means

$$\mathrm{Tr}f(x,y)\equiv\int dxdy\,f(x,y)\delta(y-x).\qquad(2.71)$$

Let us integrate by parts in (2.69) and (2.70) so that no derivative operates on the expression $(\mathfrak{D}^{dd'}\partial_\lambda A_\lambda{}^{d'})(y)$. Taking into account the antisymmetry of f^{abc} and the relations

$$\square\,\mathfrak{D}^{ab}(x,y)=\delta_{ab}\delta(x-y)-\lambda\hat{A}_\mu{}^{ab}(x)\partial_\mu\mathfrak{D}^{cb}(x,y),\qquad(2.72)$$

$$\mathfrak{D}^{ab}(x,y)\overleftarrow{\square}=\delta_{ab}\delta(x-y)+\lambda[\mathfrak{D}^{ac}(x,y)\hat{A}_\mu{}^{cb}(y)]\overleftarrow{\partial}_\mu,\qquad(2.73)$$

we obtain

$$(2.69)+(2.70)=\lambda 2(\delta\alpha/2\alpha)(f^{abc}f^{cdf}+f^{afc}f^{cbd}+f^{adc}f^{cfb})\times\mathrm{Tr}\{[\mathfrak{D}^{df}(x,y)A_\mu{}^a(y)]\overleftarrow{\partial}_\mu(\mathfrak{D}^{bb'}\partial_\lambda A_\lambda{}^{b'})(y)\}-(\delta\alpha/2\alpha)\,\mathrm{Tr}\delta_{ab}\delta(x-y).\qquad(2.74)$$

Using the Jacobi identity (2.4), we find that the change of the expression Tr ln in (2.69) compensates the Jacobian $d(\bar{A})/d(A)$. Thus we have

$$Z_\alpha{}^F=\int dA_\mu{}^a\exp\left[i\int dx\left(L_0-\frac{1}{2(\alpha+\delta\alpha)}\partial_\mu A_\mu{}^a\partial_\nu A_\nu{}^a+\bar{A}_\mu{}^aJ_\mu{}^a\right)+\mathrm{Tr}\ln\left(1+\frac{\lambda}{\square}\hat{A}_\mu\partial_\mu\right)\right].\qquad(2.75)$$

We can prove again that the following replacement is true on the mass shell:

$$\bar{A}_\mu{}^aJ_\mu{}^a\to A_\mu{}^aJ_\mu{}^a.\qquad(2.76)$$

Finally, we obtain

$$S_\alpha{}^F\equiv Z_\alpha{}^F|_{\mathrm{m.s.}}=Z_{\alpha+\delta\alpha}{}^F\equiv S_{\alpha+\delta\alpha}{}^F.\qquad(2.77)$$

Q.E.D.

The Feynman rules for calculating the generating functional (2.63) in the framework of perturbation theory are the following:

(a) The free propagator of the vector mesons has the form

$$D_{\mu\nu}{}^{ab}(p)=\delta_{ab}[-ig_{\mu\nu}-i(\alpha-1)p_\mu p_\nu/p^2]p^{-2}.\qquad(2.78)$$

(b) Besides the usual vertices (2.40), there exists the interaction $F_\mu{}^{abc}$ of $A_\mu{}^a(p)$ with the fictitious $B^b(k)$ and

$B^c(q)$ fields:

$$F_\mu{}^{abc}=\tfrac{1}{2}\lambda f^{abc}(k_\mu-q_\mu)\,,\quad p+k+q=0\,. \qquad (2.79)$$

(c) The lines of the B field occur only in closed loops, and the propagator of the B field is

$$\delta^{ab}i/p^2\,. \qquad (2.80)$$

The Feynman rules for the massless Yang-Mills field were obtained also by DeWitt,[2] Faddeev and Popov,[3] and Mandelstam[4] by other methods.

III. CONSTRUCTION OF THE S MATRIX IN CANONICAL FORMALISM

In this section we construct tke S matrix for the massless Yang-Mills field in the axial and Coulomb gauges in the framework of the canonical quantization procedure and the interaction representation. The corresponding Feynman rules coincide with those found in Sec. II.

Consider the Lagrangian

$$L=L_0+A_\mu{}^a J_\mu{}^a\,. \qquad (3.1)$$

Here $J_\mu{}^a$ is the external current, on which no restriction of the type of the conservation laws is imposed.

Besides the difficulty with the canonical formalism, there is also another problem for the Lagrangian (3.1).

Let us write the field equations obtained from (3.1) by varying over all $A_\mu{}^a$:

$$\nabla_\nu{}^{ab}G_{\nu\mu}{}^b+J_\mu{}^a=0\,. \qquad (3.2)$$

Using the identity (2.13) leads to

$$\partial_\mu J_\mu{}^a(x)+\lambda\hat{A}_\mu{}^{ab}(x)J_\mu{}^b(x)=0\,. \qquad (3.3)$$

Obviously we cannot satisfy (3.3) since the external source $J_\mu{}^a$ does not depend on $A_\mu{}^a$.

Below we shall show that the gauge condition gives a possibility of avoiding this difficulty as well.

A. Coulomb Gauge

The Yang-Mills field in the Coulomb gauge is defined both by the Lagrangian and by the gauge condition

$$\partial_k A_k{}^a(x)=0\,. \qquad (3.4)$$

The gauge condition (3.4) can be introduced into the theory with the help of the Lagrange multiplier just as has been done in Sec. II. It can be proved that the corresponding field equations are consistent in the presence of the external source as well. However, the method of Lagrange multipliers does not permit canonical quantization of the theory.

For the purpose of canonical quantization, we consider (3.4) as a constraint excluding one of the dynamical variables. This exclusion must be made before finding the field equations.

Thus let us choose

$$A_0{}^a,\; A_1{}^a,\;\text{ and }\; A_2{}^a \qquad (3.5)$$

as independent variables. With the help of (3.4) we obtain the expression for $A_3{}^a$:

$$A_3{}^a(x)=-\partial_3{}^{-1}(\partial_i A_i{}^a(x))\,,\quad i=1,\,2\,. \qquad (3.6)$$

Expression (3.6) should be substituted in (3.1) and the Lagrangian should be varied only over $A_0{}^a$, $A_1{}^a$, and $A_2{}^a$. The corresponding field equations are

$$\nabla_k{}^{ab}G_{0k}{}^b+J_0{}^a=0\,, \qquad (3.7)$$

$$\nabla_\mu{}^{ab}G_{\mu i}{}^b+J_i{}^a-\partial_3{}^{-1}\partial_i(\nabla_\mu{}^{ab}G_{\mu 3}{}^b+J_3{}^a)=0\,,\quad i=1,\,2\,. \qquad (3.8)$$

When quantizing, just as in any essentially nonlinear theory, the question of the order of noncommutative variables arises.

Later we shall consider this question and show that, with the help of the corresponding symmetrization procedure of multipliers, quantum expressions can always be represented in a form coinciding with the corresponding classical expressions.

Therefore, for the time being, we assume that the multipliers in quantum theory are commutative, as they are in classical theory.

Let us find the canonical momenta

$$\pi_0{}^a=0\,,\quad \pi_i{}^a=-G_{0i}{}^a+\partial_3{}^{-1}\partial_i G_{03}{}^a\,, \qquad (3.9)$$

$$[\pi_i{}^a(x),A_{i'}{}^b(y)]_{x^0=y^0}=i\delta_{ab}\delta^{(3)}(x-y)\,. \qquad (3.10)$$

(3.7) is the constraint equation and it should be used for excluding $A_0{}^a$ when constructing the Hamiltonian.

Let us decompose $G_{0k}{}^a$ into transverse and longitudinal components:

$$G_{0k}{}^a\equiv G_{0k}{}^{Ta}+\partial_k\Psi^a\,,\quad \partial_k G_{0k}{}^{Ta}=0\,, \qquad (3.11)$$

$$G_{0k}{}^{Ta}=-(\delta_{ki}-\partial_k\partial_i/\nabla)\pi_i{}^a\,,\quad k=1,\,2,\,3\,. \qquad (3.12)$$

Then Eq. (3.7) has the form[14]

$$\nabla_k{}^{ab}(x)\partial_k\Psi^b(x)=-\lambda\hat{A}_k{}^{ab}(x)G_{0k}{}^{Tb}(x)-J_0{}^a(x)\,. \qquad (3.13)$$

This can be solved with the help of the $\mathfrak{D}$ function:

$$\nabla_k{}^{ab}(x)\partial_k\mathfrak{D}^{bc}(x,y;A)=\delta_{ac}\delta^{(3)}(x-y)\,,\quad x^0=y^0\,. \qquad (3.14)$$

The $\mathfrak{D}$ function was considered first by Schwinger.[14] Thus

$$\Psi^a(x)=-\lambda(\mathfrak{D}^{ab}\hat{A}_k{}^{bc}G_{0k}{}^{Tc})(x)-(\mathfrak{D}^{ab}J_0{}^b)(x)\,. \qquad (3.15)$$

Rewriting (3.7) in the form

$$-\nabla_k{}^{ab}\partial_k A_0{}^b+\lambda\hat{A}_k{}^{ab}G_{0k}{}^{Tb}+\lambda\hat{A}_k{}^{ab}\partial_k\Psi^b+J_0{}^a=0 \qquad (3.16)$$

and using (3.14) and (3.15), we obtain

$$\nabla_k{}^{ab}\partial_k A_0{}^b(x)=\lambda\nabla(\mathfrak{D}^{ab}\hat{A}_k{}^{bc}G_{0k}{}^{Tc})(x)+\nabla(\mathfrak{D}^{ab}J_0{}^b)(x)\,, \qquad (3.17)$$

$$A_0{}^a(x)=\lambda(\mathfrak{D}^{ab}\nabla\mathfrak{D}^{bc}\hat{A}_k{}^{cd}G_{0k}{}^{Td})(x)+(\mathfrak{D}^{ab}\nabla\mathfrak{D}^{bc}J_0{}^c)(x)\,. \qquad (3.18)$$

[14] J. Schwinger, Phys. Rev. **125**, 1043 (1962); **127**, 324 (1962).

The expression for $\dot{A}_k{}^a$ will also be needed:

$$\dot{A}_k{}^a(x)=G_{0k}{}^{Ta}(x)+\partial_k\Psi^a(x)+\partial_k A_0{}^a(x)-\lambda\hat{A}_0{}^{ab}(x)A_k{}^b(x)$$
$$=G_{0k}{}^{Ta}(x)+\hat{A}_k{}^{ab}(x)[\lambda^2(\mathfrak{D}^{bc}\nabla\mathfrak{D}^{cd}\hat{A}_j{}^{df}G_{0j}{}^{Tf})(x)$$
$$+\lambda(\mathfrak{D}^{bc}\nabla\mathfrak{D}^{cd}J_0{}^d)(x)]$$
$$-\partial_k\{\lambda^2[\mathfrak{D}^{ab}(\hat{A}_j{}^{bc}\partial_j\mathfrak{D}^{cd}\hat{A}_l{}^{df}G_{0l}{}^{Tf})](x)$$
$$+\lambda[\mathfrak{D}^{ab}(\hat{A}_j{}^{bc}\partial_j\mathfrak{D}^{cd}J_0{}^d)](x)\}. \quad (3.19)$$

Now let us find the expression for $\nabla_\nu{}^{ab}G_{\mu\nu}{}^b$. Using the field equations, we obtain

$$L_0{}^a{}_{,0}(x)\equiv\nabla_\nu{}^{ab}(x)G_{\nu0}{}^b(x)=-J_0{}^a(x), \quad (3.20)$$

$$L_0{}^a{}_{,k}(x)=-J_k{}^a(x)+\partial_3{}^{-1}\partial_k[\nabla_\mu{}^{ab}(x)G_{\mu3}{}^b(x)+J_3{}^a(x)]$$
$$=-J_k{}^a(x)+\partial_k\dot{\Psi}^a(x)$$
$$+(\partial_k\partial_j/\nabla)[\lambda\hat{A}_0{}^{ab}(x)G_{0j}{}^b(x)+J_j{}^a(x)]. \quad (3.21)$$

When deducing (3.21), we expressed $\dot{G}_{0k}{}^{Ta}$ with the help of Eqs. (3.8) and (3.12). Expressing $\dot{\Psi}^a$ in terms of $A_k{}^a$ and $G_{0k}{}^{Ta}$ using (3.19) for $\dot{A}_k{}^a$ and (3.8) for $\dot{G}_{0k}{}^{Ta}$, we obtain, after some tedious algebra,

$$\nabla_\mu{}^{ab}(x)G_{\mu k}{}^b(x)=-J_k{}^a(x)-\partial_k(\mathfrak{D}^{ab}\nabla_\mu{}^{bc}J_\mu{}^c)(x). \quad (3.22)$$

It is not difficult to prove that Eqs. (3.20) and (3.22) are consistent:

$$0=\nabla_\mu{}^{ab}\nabla_\nu{}^{bc}G_{\mu\nu}{}^c=-\nabla_\mu{}^{ab}J_\mu{}^b$$
$$+\nabla_k{}^{ab}\partial_k(\mathfrak{D}^{bc}\nabla_\mu{}^{cd}J_\mu{}^d)=0. \quad (3.23)$$

Thus we can see that consistent use of the gauge condition enables one to obtain consistent field equations in the presence of an external source as well. The correct form of the field equations was obtained by Schwinger[15] with the help of a different method.

Note that in the case $J_\mu{}^a\equiv0$ or in the case when $J_\mu{}^a$ is a current of matter (i.e., when the equation $\nabla_\mu{}^{ab}J_\mu{}^b=0$ is true), the field equations for the Yang-Mills field written in four-dimensional form have the usual form.

The Hamiltonian in the Coulomb gauge is (to within a total space derivative)

$$H(x)=-\pi_i{}^a(x)\dot{A}_i{}^a(x)-L=\tfrac{1}{2}G_{0k}{}^{Ta}(x)G_{0k}{}^{Ta}(x)$$
$$+\tfrac{1}{4}G_{jk}{}^a(x)G_{jk}{}^a(x)+\tfrac{1}{2}\partial_k\Psi^a(x)\partial_k\Psi^a(x)$$
$$+A_k{}^a(x)J_k{}^a(x). \quad (3.24)$$

In (3.24) it is assumed that $A_3{}^a$ is expressed with the help of (3.6) and $G_{0k}{}^{Ta}$ is expressed with the help of (3.12).

Let us pass to the interaction representation. For this purpose H should be split into a free Hamiltonian H_0 and an interaction Hamiltonian H_{int}. We choose the total H with $\lambda=J_\mu{}^a=0$ for H_0:

$$H_0=\tfrac{1}{2}F_{0k}{}^{Ta}F_{0k}{}^{Ta}+\tfrac{1}{4}F_{jk}{}^aF_{jk}{}^a, \quad (3.25)$$

$$F_{jk}{}^a=\partial_jU_k{}^a-\partial_kU_j{}^a,\quad U_3{}^a=-\partial_3{}^{-1}\partial_iU_i{}^a, \quad (3.26)$$

[15] J. Schwinger, *Theoretical Physics* (IAEA, Vienna, 1963).

$$F_{0k}{}^{Ta}=-(\delta_{ki}-\partial_k\delta_i/\nabla)\pi_i{}^a=\dot{U}_k{}^a, \quad (3.27)$$

$$\pi_i{}^a=-\dot{U}_i{}^a-\partial_3{}^{-2}\partial_i\partial_{i'}\dot{U}_{i'}{}^a,\quad [\pi_i{}^a(x),U_{i'}{}^b(y)]_{x^0=y^0}$$
$$=i\delta_{ab}\delta_{ii'}\delta^{(3)}(x-y). \quad (3.28)$$

In (3.25)–(3.28) we use the designation $U_k{}^a$ instead of $A_k{}^{(0)a}$ for the free Yang-Mills field in the interaction representation, keeping the designation $A_\mu{}^a$ for the Heisenberg fields.

With the help of (3.25)–(3.28) the field equations for $U_k{}^a$ and the propagator can be obtained:

$$\Box U_k{}^a=0,\quad \partial_kU_k{}^a=0, \quad (3.29)$$

$$D_{kj}{}^{ab}(x-y)\equiv\langle0|T_DU_k{}^a(x)U_j{}^b(y)|0\rangle$$
$$=\frac{1}{(2\pi)^4}\int dp\, e^{-ip(x-y)}D_{kj}{}^{ab}(p), \quad (3.30)$$

$$D_{kj}{}^{ab}(p)=i\delta_{ab}\left(\delta_{kj}-\frac{p_kp_j}{\mathbf{p}^2}\right)p^{-2}. \quad (3.31)$$

In (3.30), T_D means the Dyson T chronological product:

$$T_DA(t_1)B(t_2)\equiv\theta(t_1-t_2)A(t_1)B(t_2)$$
$$+\theta(t_2-t_1)B(t_2)A(t_1). \quad (3.32)$$

The interaction Hamiltonian is

$$H_{\text{int}}=H-H_0=\tfrac{1}{2}\lambda F_{jk}{}^a\hat{U}_j{}^{ab}U_k{}^b$$
$$+\tfrac{1}{4}\lambda^2\hat{U}_j{}^{ab}U_k{}^b\hat{U}_j{}^{ac}U_k{}^c+U_k{}^aJ_k{}^a$$
$$-\tfrac{1}{2}[\lambda(\mathfrak{D}^{ab}\hat{U}_k{}^{bc}F_{0k}{}^{Tc})\times(\mathfrak{D}^{ab}J_0{}^b)]$$
$$\times\nabla[\lambda(\mathfrak{D}^{ab'}\hat{U}_{k'}{}^{b'c'}F_{0k'}{}^{Tc'})+(\mathfrak{D}^{ab'}J_0{}^{b'})], \quad (3.33)$$

$$(\nabla\delta_{ab}+\lambda\hat{U}_k{}^{ab}\partial_k)\mathfrak{D}^{bc}(x,y;U)$$
$$=\delta_{ac}\delta^{(3)}(x-y),\quad x^0=y^0. \quad (3.34)$$

The relations between $U_i{}^a$ and $A_i{}^a$ are given by the usual formulas connecting interaction and Heisenberg representations. Let us define

$$\langle Q\rangle\equiv S^\dagger T_DQS, \quad (3.35)$$

where S means the S matrix in the interaction representation

$$S=T_D\exp\left[-i\int dxH_{\text{int}}(x)\right]. \quad (3.36)$$

Then we have

$$A_k{}^a(x)=\langle U_k{}^a(x)\rangle, \quad (3.37)$$

$$G_{0k}{}^{Ta}(x)=\langle F_{0k}{}^{Ta}(x)\rangle. \quad (3.38)$$

When passing from the classical expressions of the type (3.15) and (3.18), or from the field equations, to the corresponding quantum expressions, we assume that all operators are to be expressed in terms of the canonical variables $A_k{}^a$ and $G_{0k}{}^{Ta}$, and the product of operators is to be understood as the T_D product, the exact meaning of which is defined by (3.35). For example,

$$A_0{}^a(x)=\langle\lambda(\mathfrak{D}^{ab}\nabla\mathfrak{D}^{bc}\hat{U}_k{}^{cd}F_{0k}{}^{Td})(x)\rangle +\langle(\mathfrak{D}^{ab}\nabla\mathfrak{D}^{bc}J_0{}^c)(x)\rangle\,. \quad (3.39)$$

One can prove, using the commutation relations (3.27) and the field equations (3.29), that the Heisenberg operators defined by (3.35)–(3.39) satisfy the field equations (3.7), (3.8), and (3.22).

One can also show (but we do not dwell on this) that the symmetrization procedure of the Heisenberg operators $A_k{}^a$ and $G_{0k}{}^{Ta}$ can be suggested in such a way that the expressions of type (3.15) and (3.18) in the quantum case should formally coincide with the corresponding classical expressions.[14]

Let us pass to constructing the Feynman rules. Calculate for this purpose the generating functional[16]

$$Z^k\equiv\langle 0|\,T_D\exp\left[-i\int dx\,H_{\rm int}(x)\right]|0\rangle$$

$$=\exp\left[-iQ(U_k)+\tfrac{1}{2}i\left(\mathfrak{D}\frac{\delta}{\delta\Lambda}\right)\nabla\left(\mathfrak{D}\frac{\delta}{\delta\Lambda}\right)\right] \times\exp[i(\lambda\hat{U}_jF_{0j}{}^T+J_0)\Lambda]Z_0|_{\Lambda=\xi=0}\,, \quad (3.40)$$

$$Q(U_k)\equiv\int dx[\tfrac{1}{2}\lambda F_{jk}{}^a\hat{U}_j{}^{ab}U_k{}^b +\tfrac{1}{4}\lambda^2\hat{U}_j{}^{ab}U_k{}^b\hat{U}_j{}^{ac}U_k{}^c]\,. \quad (3.41)$$

In (3.40), the exponent operates on Z_0 according to the rules

$$U_k{}^a\to\delta/i\delta J_k{}^a\,,\quad F_{0j}T^a\to\delta/i\delta\xi_j{}^a\,. \quad (3.42)$$

Z_0 is the generating functional of the free Green's functions:

$$Z_0=\langle 0|\,T_D\exp\left\{i\int dx(\xi_k{}^aF_{0k}{}^{Ta}-U_k{}^aJ_k{}^a)\right\}|0\rangle\,. \quad (3.43)$$

Let us pass in (3.43) from the T_D product to Wick's T product; i.e., instead of the Dyson propagator, we shall use the Wick propagator.

$$\langle 0|\,T_WU_k{}^aU_j{}^b|0\rangle=D_{jk}{}^{ab}=\langle 0|\,T_DU_k{}^aU_j{}^b|0\rangle\,, \quad (3.44)$$

$$\langle 0|\,T_WF_{0j}{}^{Ta}U_k{}^b|0\rangle=\partial_0D_{jk}{}^{ab}=\langle 0|\,T_DF_{0j}{}^{Ta}U_k{}^b|0\rangle\,, \quad (3.45)$$

$$\langle 0|\,T_WF_{0j}{}^{Ta}F_{0k}{}^{Tb}|0\rangle=-\partial_0{}^2D_{jk}{}^{ab}=\langle 0|\,T_DF_{0j}{}^{Ta}F_{0k}{}^{Tb}|0\rangle -i\delta_{ab}(\delta_{jk}-\partial_j\partial_k/\nabla)\,. \quad (3.46)$$

Thus

$$Z_0=\exp[\tfrac{1}{2}i\xi_j{}^a(\delta_{jk}-\partial_j\partial_k/\nabla)\xi_k{}^a]\langle 0|\,T_W \times\exp(i\xi_k{}^aF_{0k}{}^{Ta}-iU_k{}^aJ_k{}^a)|0\rangle\,. \quad (3.47)$$

[16] E. S. Fradkin, *Problems of Theoretical Physics* (Nauka Moscow, 1969).

Perform in (3.40) the functional differentiation $\delta/\delta\xi_k{}^a$ only in the first factor in (3.47). Then one can substitute

$$F_{0k}{}^{Ta}\to\dot{U}_k{}^a\to\partial_0\delta/i\delta J_k{}^a \quad (3.48)$$

and put $\xi_k{}^a=0$.

For the functional expression involving Λ, we use the relation

$$F\left(\frac{\delta}{\delta\Lambda}\right)e^{\Lambda J}\phi(\Lambda)|_{\Lambda=0}=\phi\left(\frac{\delta}{\delta J}\right)F(J)\,. \quad (3.49)$$

Then (3.40) transforms into

$$Z^k=\exp\left\{i\int dx\,L_{\rm int}(U_\mu)+{\rm Tr}\ln\nabla_k\partial_k\overleftarrow{\nabla}^{-1}\right\}Z_0'\,. \quad (3.50)$$

In (3.50) the $U_0{}^a$ means

$$U_0{}^a\to\delta/i\delta J_0{}^a$$

and

$$Z_0'=\exp(-\tfrac{1}{2}J_0{}^a(-i/\nabla)J_0{}^a)\langle 0|\,T_W\exp(-iU_k{}^aJ_k{}^a)|0\rangle$$

$$=\exp(-\tfrac{1}{2}J_\mu{}^aD_{\mu\nu}{}^{ab}J_\nu{}^b)\equiv\int dA_\mu{}^a\,\delta\{\partial_kA_k{}^a(x)\} \times\exp\left[i\int dx(-\tfrac{1}{4}F_{\mu\nu}{}^aF_{\mu\nu}{}^a+A_\mu{}^aJ_\mu{}^a)\right], \quad (3.51)$$

$$L_{\rm int}\equiv-\tfrac{1}{2}\lambda F_{\mu\nu}{}^a\hat{U}_\mu{}^{ab}U_\nu{}^b-\tfrac{1}{4}\lambda^2\hat{U}_\mu{}^{ab}U_\nu{}^b\hat{U}_\mu{}^{ac}U_\nu{}^c\,. \quad (3.52)$$

The Green's function $D_{\mu\nu}{}^{ab}$ in (3.51) coincides with the free propagator in the Coulomb gauge (2.58).

Combining (3.50) and (3.51), we obtain the following expression for the generating functional:

$$Z^k=\int dA_\mu{}^a\,\delta\{\partial_kA_k{}^a(x)\}\exp\left[i\int dx(L_0+A_\mu{}^aJ_\mu{}^a) +{\rm Tr}\ln\left(\delta_{ab}+\frac{\lambda}{\nabla}\hat{A}_k{}^{ab}\partial_k\right)\right]. \quad (3.53)$$

Expression (3.53) coincides with expression (2.57) for the generating functional in the Coulomb gauge obtained in Sec. II.

B. Axial Gauge

The axial gauge is defined by the conditions

$$A_3{}^a(x)=0\,. \quad (3.54)$$

Choosing $A_0{}^a$ and $A_i{}^a$, $i=1,2$, as independent variables, we obtain the field equations

$$\nabla_k{}^{ab}G_{0k}{}^b+J_0{}^a=0\,, \quad (3.55)$$

$$\nabla_\mu{}^{ab}G_{\mu i}{}^b+J_i{}^a=0\,, \quad (3.56)$$

and the canonical momenta

$$\pi_0{}^a=0\,,\quad \pi_i{}^a=-G_{0i}{}^a\,, \tag{3.57}$$

$$[\pi_i{}^a(x),A_{i'}{}^b(y)]_{x^0=y^0}=i\delta_{ab}\delta_{ii'}\delta^{(3)}(x-y)\,. \tag{3.58}$$

Equation (3.55) is the constraint equation from which we find the expression for $A_0{}^a$:

$$A_0{}^a=-\partial_3{}^{-2}\nabla_i{}^{ab}\pi_i{}^b+\partial_3{}^{-2}J_0{}^a\,. \tag{3.59}$$

After a calculation analogous to that in the Coulomb gauge, we obtain the field equations in the four-dimensional form

$$\nabla_\nu{}^{ab}G_{\mu\nu}{}^b+J_\mu{}^a-\delta_{\mu3}\partial_3{}^{-1}\nabla_\nu{}^{ab}J_\nu{}^b=0\,. \tag{3.60}$$

Equations (3.60) are self-consistent.

The Hamiltonian in the axial gauge is (to within a total space derivative)

$$H=\tfrac{1}{2}\pi_i{}^a\pi_i{}^a+\tfrac{1}{4}G_{jk}{}^aG_{jk}{}^a+A_i{}^aJ_i{}^a+\tfrac{1}{2}\partial_3A_0{}^a\partial_3A_0{}^a\,. \tag{3.61}$$

Let us pass to the interaction representation:

$$H_0=\tfrac{1}{2}\pi_i{}^a\pi_i{}^a-\tfrac{1}{2}\partial_i\pi_i{}^a\partial_3{}^{-2}\partial_{i'}\pi_{i'}{}^a+\tfrac{1}{4}F_{jk}{}^aF_{jk}{}^a\,, \tag{3.62}$$

$$[\pi_i{}^a(x),U_{i'}{}^b(y)]_{x^0=y^0}=i\delta_{ab}\delta_{ii'}\delta^{(3)}(x-y)\,. \tag{3.63}$$

With the help of (3.62), we find

$$\pi_i{}^a=-F_{0i}{}^a\,, \tag{3.64}$$

where we use the definition

$$U_0{}^a\equiv(1/\nabla)\partial_i\hat{U}_i{}^a\,, \tag{3.65}$$

$$\langle 0|\,T_DU_\mu{}^a(x)U_\nu{}^b(y)\,|0\rangle =D_{\mu\nu}{}^{ab}(x-y)+ig_{\mu0}g_{\nu0}\partial_3{}^{-2}\delta(x-y)\,, \tag{3.66}$$

$$D_{\mu\nu}{}^{ab}(p) =-i\delta_{ab}\left(g_{\mu\nu}-\frac{1}{p_3}g_{\mu3}p_\nu-\frac{1}{p_3}g_{\nu3}p_\mu-\frac{p_\mu p_\nu}{p_3{}^2}\right)\frac{1}{p^2}. \tag{3.66$'$}$$

Expression (3.66′) coincides with the free propagator (2.54), and we will call it the Wick propagator.

$$\langle 0|\,T_DF_{0i}{}^aU_\mu{}^b\,|0\rangle=(-\partial_0g_{\lambda i}-\partial_ig_{\lambda0})D_{\lambda\mu}{}^{ab}. \tag{3.67}$$

Expression (3.67) coincides with the Wick propagator.

$$\langle 0|\,T_DF_{0i}{}^a(x)F_{0i'}{}^b(y)\,|0\rangle =(\partial_0g_{\lambda i}+\partial_ig_{\lambda0})(-\partial_0g_{\sigma i'}-\partial_{i'}g_{\sigma0})D_{\lambda\sigma}{}^{ab}(x-y) -i\delta_{ab}\delta_{ii'}\delta(x-y)\,. \tag{3.68}$$

Expression (3.68) differs from the Wick propagator (the first term) by the contact term. The interaction Hamiltonian is

$$H_{\rm int}=\tfrac{1}{2}\lambda F_{ii'}{}^a\hat{U}_i{}^{ab}U_{i'}{}^b+\tfrac{1}{4}\lambda^2\hat{U}_i{}^{ab}U_{i'}{}^b\hat{U}_i{}^{ac}U_{i'}{}^c -\lambda U_0{}^a\hat{U}_i{}^{ab}F_{0i}{}^b-\tfrac{1}{2}\lambda^2(\hat{U}_i{}^{ab}F_{0i}{}^b)\partial_3{}^{-2}(\hat{U}_{i'}{}^{ab'}F_{0i'}{}^{b'}) -\lambda\partial_3{}^{-2}J_0{}^a(\hat{U}_i{}^{ab}F_{0i}{}^b)-\tfrac{1}{2}J_0{}^a\partial_3{}^{-2}J_0{}^a-J_\mu{}^aU_\mu{}^a\,. \tag{3.69}$$

(Note that $U_3{}^a=0$.) The generating functional is equal to

$$Z^A=\langle 0|\,T_D\exp\left[-i\int dx\,H_{\rm int}(x)\right]|0\rangle\,. \tag{3.70}$$

Let us pass from the T_D product to the T_W product.

$$Z_0\equiv\langle 0|\,T_D\exp\{i(\xi_i{}^aF_{0i}{}^a+U_\mu{}^aJ_\mu{}^a)\}\,|0\rangle =\exp\{\tfrac{1}{2}i(\xi_i{}^a\xi_i{}^a-J_0{}^a\partial_3{}^{-2}J_0{}^a)\}Z_{0,W}\,, \tag{3.71}$$

$$Z_{0,W}=\langle 0|\,T_W\exp[i(\xi_i{}^aF_{0i}{}^a+iU_\mu{}^aJ_\mu{}^a)]\,|0\rangle\,, \tag{3.72}$$

$$\exp(\tfrac{1}{2}iJ_0{}^a\partial_3{}^{-2}J_0{}^a)\exp(i\lambda\hat{U}_i{}^{ab}F_{0i}{}^b\partial_3{}^{-2}J_0{}^a) \times\exp\{i[\lambda U_0{}^a\hat{U}_i{}^{ab}F_{0i}{}^b +\tfrac{1}{2}\lambda^2(\hat{U}_i{}^{ab}F_{0i}{}^b)\partial_3{}^{-2}(\hat{U}_{i'}{}^{ab'}F_{0i'}{}^{b'})]\}Z_0 =\exp(\tfrac{1}{2}i\lambda^2\hat{U}_0{}^{ab}U_k{}^b\hat{U}_0{}^{ac}U_k{}^c +i\lambda F_{0k}{}^a\hat{U}_0{}^{ab}U_k{}^b)Z_{0,W}\,. \tag{3.73}$$

In (3.73) the exponents operate on Z_0 and $Z_{0,W}$ according to the rules (3.42) and (3.51). Making the substitution

$$\frac{\delta}{i\delta\xi_i{}^a}\to\partial_0\frac{\delta}{i\delta J_i}-\partial_i\frac{\delta}{i\delta J_0} \tag{3.74}$$

and using the relation

$$\langle 0|\,T_W\exp(iU_\mu{}^aJ_\mu{}^a)\,|0\rangle=\int dA_\mu{}^a\,\delta\{A_3{}^a(x)\} \times\exp\left[i\int dx(-\tfrac{1}{4}F_{\mu\nu}{}^aF_{\mu\nu}{}^a+A_\mu{}^aJ_\mu{}^a)\right],$$

we obtain finally the expression for the generating functional

$$Z^A=\int dA_\mu{}^a\,\delta\{A_3{}^a(x)\}\exp\left[i\int dx(L_0+A_\mu{}^aJ_\mu{}^a)\right], \tag{3.75}$$

which coincides with (2.53).

The results of this section can be regarded as additional proof of the fact that the method for constructing the Feynman rules developed in Sec. II results in a unitary S matrix.

IV. CONSTRUCTION OF FEYNMAN RULES FOR GRAVITATIONAL FIELD

In this section we shall obtain the general rules for construction of the S matrix for the gravitation field, prove the gauge invariance of the S matrix, and in more detail consider two covariant gauges (the harmonic condition and its linearized form) and the Dirac[17] noncovariant gauge. The Feynman rules found by us coincide with those of Refs. 2, 3, and 5.

[17] P. A. M. Dirac, Phys. Rev. **114**, 924 (1959).

The classical gravitational field is described by the action[18]

$$W_0=\int dx\, L_0(x)\,, \tag{4.1}$$

$$L_0=\tfrac{1}{2}(\sqrt{-g})g^{\mu\nu}R_{\mu\nu}. \tag{4.2}$$

Here $g_{\mu\nu}$ is the metric tensor, $g\equiv\det g_{\mu\nu}$, $g^{\mu\nu}g_{\nu\lambda}=\delta^\mu{}_\lambda$, and $R_{\mu\nu}$ is the curvature tensor of second rank (Ricci tensor)

$$R_{\mu\nu}=\partial_\nu\Gamma^\sigma{}_{\mu\sigma}-\partial_\sigma\Gamma^\sigma{}_{\mu\nu}+\Gamma^\rho{}_{\mu\sigma}\Gamma^\sigma{}_{\nu\rho}-\Gamma^\rho{}_{\mu\nu}\Gamma^\sigma{}_{\rho\sigma}. \tag{4.3}$$

$\Gamma^\mu{}_{\nu\lambda}$ is the Christoffel symbol

$$\Gamma^\mu{}_{\nu\lambda}=\tfrac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\sigma\lambda}+\partial_\lambda g_{\sigma\nu}-\partial_\sigma g_{\nu\lambda}). \tag{4.4}$$

As we shall show later, the most convenient choice of the concrete form of the dynamical variables depends on the gauge condition.

The variables we shall use belong to the following class:

$$g^{(\beta)}{}_{\mu\nu}\equiv g^\beta g_{\mu\nu} \quad\text{and}\quad g_{(\beta)}{}^{\mu\nu}\equiv g^\beta g^{\mu\nu}. \tag{4.5}$$

In the general case we shall designate the variables $g_{(\beta)}{}^{\mu\nu}$ and $g^{(\beta)}{}_{\mu\nu}$ as $\kappa_{\mu\nu}$. The Einstein equations for the gravitational field can be obtained from (4.1) by two methods:

In the first-order formalism the expression (4.1) should be varied with respect to $\kappa_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}$, considered as independent variables. From the equations obtained by varying (4.1) with respect to $\Gamma^\mu{}_{\nu\lambda}$, the expression (4.4) for $\Gamma^\mu{}_{\nu\lambda}$ can be deduced.

In the second-order formalism the variation should be made only with respect to $\kappa_{\mu\nu}$. It is assumed in this case that (4.1) is expressed only in terms of $\kappa_{\mu\nu}$ with the help of (4.4).

Both methods lead, of course, to the same field equations.

As is well known, (4.1) and (4.2) are invariant under the gauge transformations of $\kappa_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}$, the infinitesimal form of which is

$$g_{(\beta)}{}^{\mu\nu}\to g_{(\beta)}{}^{S\mu\nu}\equiv g_{(\beta)}{}^{\mu\nu}-\xi^\gamma\partial_\gamma g_{(\beta)}{}^{\mu\nu}+g_{(\beta)}{}^{\mu\gamma}\partial_\gamma\xi^\nu +g_{(\beta)}{}^{\nu\gamma}\partial_\gamma\xi^\mu-2\beta g_{(\beta)}{}^{\mu\nu}\partial_\gamma\xi^\gamma, \tag{4.6}$$

$$g^{(\beta)}{}_{\mu\nu}\to g^{(\beta)S}{}_{\mu\nu}\equiv g^{(\beta)}{}_{\mu\nu}-\xi^\gamma\partial_\gamma g^{(\beta)}{}_{\mu\nu}-g^{(\beta)}{}_{\mu\gamma}\partial_\nu\xi^\gamma -g^{(\beta)}{}_{\nu\gamma}\partial_\mu\xi^\gamma-2\beta g^{(\beta)}{}_{\mu\nu}\partial_\gamma\xi^\gamma, \tag{4.6'}$$

$$\Gamma^\mu{}_{\nu\lambda}\to\Gamma^{S\mu}{}_{\nu\lambda}\equiv\Gamma^\mu{}_{\nu\lambda}-\xi^\gamma\partial_\gamma\Gamma^\mu{}_{\nu\lambda}+\Gamma^\sigma{}_{\nu\lambda}\partial_\sigma\xi^\mu -\Gamma^\mu{}_{\nu\sigma}\partial_\lambda\xi^\sigma-\Gamma^\mu{}_{\lambda\sigma}\partial_\nu\xi^\sigma-\partial_\nu\partial_\lambda\xi^\mu. \tag{4.7}$$

$\xi^\mu(x)$ are arbitrary infinitesimal functions of x^μ. In the first-order formalism, gauge transformations of both $\kappa_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}$ should be made; in the second-order formalism, only gauge transformations of $\kappa_{\mu\nu}$ should be made.

The gauge variation of (4.1) has the form

$$\delta W_0=\int dx\,\xi^\mu(x)[\mathbf{R}_{\mu\nu\lambda}G^{\nu\lambda}(x)+\mathbf{R}_{\mu\nu\lambda}{}^\sigma G_\sigma{}^{\nu\lambda}(x)]. \tag{4.8}$$

[18] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, London, 1962), 2nd ed.

In the case $\kappa_{\mu\nu}=g_{(\beta)}{}^{\mu\nu}$ the first term in (4.8) should be written as

$$\int dx\,\xi^\mu(x)\mathbf{R}_\mu{}^{\nu\lambda}G_{\nu\lambda}(x). \tag{4.9}$$

The second term in (4.8) is absent in the second-order formalism. We use the following designations:

$$G^{\mu\nu}(x)\equiv G_{(\beta)}{}^{\mu\nu}(x)\equiv\delta W_0/\delta g^{(\beta)}{}_{\mu\nu}(x)\,, \tag{4.10}$$

$$G_{\mu\nu}(x)\equiv G^{(\beta)}{}_{\mu\nu}(x)\equiv\delta W_0/\delta g_{(\beta)}{}^{\mu\nu}(x)\,, \tag{4.11}$$

$$G_\mu{}^{\nu\lambda}(x)\equiv\delta W_0/\delta\Gamma^\mu{}_{\nu\lambda}(x). \tag{4.12}$$

Equation (4.12) exists in the second-order formalism only. We find the differential operators R with the help of (4.6) and (4.7):

$$\mathbf{R}_{\mu\nu\lambda}\equiv\mathbf{R}^{(\beta)}{}_{\mu\nu\lambda}\equiv(2\beta-1)\partial_\mu g^{(\beta)}{}_{\nu\lambda} +2\beta g^{(\beta)}{}_{\nu\lambda}\partial_\mu+2\partial_\nu g^{(\beta)}{}_{\mu\lambda}+2g^{(\beta)}{}_{\mu\lambda}\partial_\nu \tag{4.13}$$

for the case $\kappa_{\mu\nu}=g^{(\beta)}{}_{\mu\nu}$;

$$\mathbf{R}_\mu{}^{\nu\lambda}\equiv\mathbf{R}^{(\beta)}{}_\mu{}^{\nu\lambda}\equiv(2\beta-1)\partial_\mu g_{(\beta)}{}^{\nu\lambda}+2\beta g_{(\beta)}{}^{\nu\lambda}\partial_\mu -2\delta_\mu{}^\nu\partial_\gamma g_{(\beta)}{}^{\gamma\lambda}-2\delta_\mu{}^\nu g_{(\beta)}{}^{\gamma\lambda}\partial_\gamma \tag{4.14}$$

for the case $\kappa_{\mu\nu}=g_{(\beta)}{}^{\mu\nu}$;

$$\mathbf{R}_{\mu\nu\lambda}{}^\sigma\equiv-\partial_\mu\Gamma^\sigma{}_{\nu\lambda}-\delta^\sigma{}_\mu\partial_\gamma\Gamma^\gamma{}_{\nu\lambda}-\delta^\sigma{}_\mu\Gamma^\gamma{}_{\nu\lambda}\partial_\gamma +2\partial_\lambda\Gamma^\sigma{}_{\mu\nu}+2\Gamma^\sigma{}_{\mu\nu}\partial_\lambda-\delta^\sigma{}_\mu\partial_\nu\partial_\lambda. \tag{4.15}$$

Since the $\xi^\mu(x)$ are arbitrary, we obtain an important identity from (4.8):

$$\mathbf{R}_{\mu\nu\lambda}G^{\nu\lambda}(x)+\mathbf{R}_{\mu\nu\lambda}{}^\sigma G_\sigma{}^{\nu\lambda}(x)\equiv0. \tag{4.16}$$

Note that (4.16) is satisfied by arbitrary $\kappa_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}$. According to (4.16), four identities exist among the Einstein equations. As noted in Sec. II, this means that four additional (gauge) conditions should be imposed on $\kappa_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}$ for consistent construction of the quantum theory.

As in Sec. II, we shall use the method of Lagrange multipliers. Consider the class of gauges determined by the function

$$\psi_\mu\equiv\psi_\mu(x;\kappa,\Gamma). \tag{4.17}$$

Three concrete forms of the gauge functions will be considered later. The Lagrangian is

$$L=L_0+\psi_\mu\bar{B}_\psi{}^\mu+\tfrac{1}{2}\alpha\bar{B}_\psi{}^\mu\delta_{\mu\nu}\bar{B}_\psi{}^\nu, \tag{4.18}$$

where $\delta_{\mu\nu}$ is the Minkowski tensor. The case $\alpha\neq0$ will be considered only for these gauge functions (4.17) which depend linearly on $\kappa_{\mu\nu}$ and are independent of $\Gamma^\mu{}_{\nu\lambda}$.

$$\bar{B}_\psi{}^\mu(x)=\int dy\,\mathbf{D}_{\psi\nu}{}^\mu(x,y;\kappa,\Gamma)B^\nu(y). \tag{4.19}$$

Varying (4.18) with respect to $\kappa_{\mu\nu}$, $\Gamma^\mu{}_{\nu\lambda}$, and B^μ, we obtain the field equations

$$\psi_\mu=-\alpha\delta_{\mu\nu}\bar{B}_\psi{}^\nu,\quad G^{\mu\nu}+\frac{\delta\psi_\sigma}{\delta\kappa_{\mu\nu}}\bar{B}^\sigma=0, \tag{4.20}$$

in the case of the second-order formalism. In the first-order formalism the following equation should be added:

$$G_\mu{}^{\nu\lambda}+\frac{\delta\psi_\sigma}{\delta\Gamma^\mu{}_{\nu\lambda}}\bar{B}^\sigma=0. \quad (4.21)$$

With the help of the identity (4.16), we obtain the following consequences of the field equations:

$$(Q^\psi{}_{\mu\nu}\bar{B}^\nu)(x)=0, \quad (4.22)$$

$$Q^\psi{}_{\mu\nu}\equiv Q^\psi{}_{\mu\nu}(x,y;\kappa,\Gamma) =\mathbf{R}_{\mu\lambda\gamma}\frac{\partial\psi_\nu(y;\kappa,\Gamma)}{\partial\kappa_{\lambda\gamma}(x)}+\mathbf{R}_{\mu\lambda\gamma}{}^\sigma\frac{\partial\psi_\nu(y;\kappa,\Gamma)}{\partial\Gamma^\sigma{}_{\lambda\gamma}(x)}. \quad (4.23)$$

We impose the restriction on ψ_μ that in the limit

$$\kappa_{\mu\nu}=\delta_{\mu\nu}, \quad \Gamma^\mu{}_{\nu\lambda}=0,$$

the operator $Q^\psi{}_{\mu\nu}$ should be a nonsingular differential operator $Q^{(0)}{}_{\psi\mu\nu}$. Choose the $\mathbf{D}_\psi$ function in the form

$$\mathbf{D}_\psi{}^\mu{}_\nu=[Q_\psi{}^{-1}]^{\mu\nu}\overleftarrow{Q}_{\psi\lambda\nu}{}^{(0)}. \quad (4.24)$$

Then the B field satisfies the free equation

$$Q_\psi{}^{(0)}{}_{\mu\nu}B^\nu=0. \quad (4.25)$$

Therefore, the S matrix is unitary, and the Einstein equations are valid in the physical subspace.

The generating functional is equal to

$$Z_\alpha{}^\psi=\int d(\kappa,\Gamma)dB^\mu\exp\left[i\int dx(L_0+\psi_\mu\bar{B}_\psi{}^\mu+\tfrac{1}{2}\alpha\bar{B}^\mu\delta_{\mu\nu}\bar{B}^\nu+\kappa_{\mu\nu}J^{\mu\nu})\right]$$

$$=\int d(\kappa,\Gamma)\exp\left[i\int dx\left(L_0-\frac{1}{2\alpha}\psi_\mu\delta^{\mu\nu}\psi_\nu+\kappa_{\mu\nu}J^{\mu\nu}\right)+\mathrm{Tr}\ln Q_\psi(\overleftarrow{Q}_\psi{}^{(0)})^{-1}\right]. \quad (4.26)$$

In the gauge

$$\psi_\mu=0,$$

i.e., for $\alpha=0$, we have

$$Z_0{}^\psi=\int d(\kappa,\Gamma)\delta\{\psi_\mu(x;\kappa,\Gamma)\}\times\exp\left\{i\int dx(L_0+\kappa_{\mu\nu}J^{\mu\nu})+\mathrm{Tr}\ln Q_\psi(\overleftarrow{Q}_\psi{}^{(0)})^{-1}\right\}. \quad (4.27)$$

In (4.26) and (4.27), $d(\kappa,\Gamma)$ is equal to

$$\prod_x\prod_{\mu\le\nu}d\kappa_{\mu\nu}(x) \quad (4.28)$$

in the second-order formalism, and to

$$\prod_x\prod_{\mu\le\nu}d\kappa_{\mu\nu}(x)\prod_{\mu;\nu\le\lambda}d\Gamma_{\nu\lambda}{}^\mu(x) \quad (4.29)$$

in the first-order formalism.

Note that the generating functionals (4.26), written in terms of different variables $g^{(\beta_1)}{}_{\mu\nu}$ or $g^{(\beta_2)}{}_{\mu\nu}$, do not coincide in general. In this connection there arises the question of a choice of "the true variables" ("the true measure") in terms of which the generating functional can be written as the functional integral of $\exp(iL)$ over "the true variables." The proposed method does not permit the value of the β to be determined. It could be found in principle with the help of the correctly formulated canonical formalism for the gravitational field. However, we shall not investigate this problem in the present work.

We now give some arguments which show that the S matrix corresponding to the generating functional (4.26) and (4.27) is independent of the choice of the variables of the functional integration belonging to class (4.5). Suppose we integrate over $\kappa^1{}_{\mu\nu}$ in (4.26). The corresponding Jacobian has in general the form

$$d(\kappa^1,\Gamma)/d(\kappa,\Gamma)=\mathrm{Det}^{4\gamma}\kappa(x)\delta(x-y), \quad (4.30)$$

where γ is some number and

$$\kappa(x)\equiv\det\kappa_{\mu\nu}(x). \quad (4.31)$$

Then we have

$$\tilde{Z}_\alpha{}^\psi\equiv\int d(\kappa^1,\Gamma)\exp\left[i\int dx\left(L_0-\frac{1}{2\alpha}\psi_\mu\delta^{\mu\nu}\psi_\nu+\kappa_{\mu\nu}J^{\mu\nu}\right)+\mathrm{Tr}\ln Q_\psi(\overleftarrow{Q}_\psi{}^{(0)})^{-1}\right]$$

$$=\int d(\kappa,\Gamma)\exp\left[i\int dx\left(L_0-\frac{1}{2\alpha}\psi_\mu\delta^{\mu\nu}\psi_\nu+\kappa_{\mu\nu}J^{\mu\nu}\right)+\mathrm{Tr}\ln Q_\psi(-\kappa)^\gamma(\overleftarrow{Q}_\psi{}^{(0)})^{-1}\right]$$

$$=\int d(\kappa,\Gamma)dB^\mu\exp\left[i\int dx(L_0+\psi_\mu\tilde{B}_\psi{}^\mu+\tfrac{1}{2}\alpha\tilde{B}_\psi{}^\mu\delta_{\mu\nu}\tilde{B}_\psi{}^\nu+\kappa_{\mu\nu}J^{\mu\nu})\right], \quad (4.26')$$

$$\tilde{B}_\psi{}^\mu(x)=(Q_\psi{}^{-1\mu\nu}(-\kappa)^{-\gamma}\overleftarrow{Q}_{\psi\nu\lambda}{}^{(0)}B^\lambda)(x). \quad (4.19')$$

We can see from (4.26′) that $\tilde{Z}_\alpha{}^\psi$ corresponds to the field equations for $\kappa_{\mu\nu}$, $\Gamma^\mu{}_{\nu\lambda}$, and B^μ which are obtained from (4.20)–(4.22) by the substitution $\bar{B}_\psi{}^\mu\to\tilde{B}_\psi{}^\mu$.

$$\psi_\mu=-\alpha\delta_{\mu\nu}\tilde{B}_\psi{}^\nu, \quad G^{\mu\nu}+\frac{\delta\psi_\sigma}{\delta\kappa_{\mu\nu}}\tilde{B}_\psi{}^\sigma=0, \quad (4.20')$$

$$G_\mu{}^{\nu\lambda}+\frac{\delta\psi_\sigma}{\delta\Gamma^\mu{}_{\nu\lambda}}\tilde{B}_\psi{}^\sigma=0\,,\qquad(4.21')$$

$$(Q_{\mu\nu}{}^\psi\tilde{B}_\psi{}^\nu)(x)=0\,.\qquad(4.22')$$

It is seen from (4.19′) and (4.22′) that B^ν satisfies the free equation

$$Q_\psi{}^{(0)}{}_{\mu\nu}B^\nu=0\,.\qquad(4.25')$$

Let us write $[-\kappa(x)]^{-\gamma}$ in the form

$$[-\kappa(x)]^{-\gamma}=1+a(x)\,.\qquad(4.32)$$

The expression (4.19′) then acquires the form

$$\tilde{B}_\psi{}^\mu(x)=\bar{B}_\psi{}^\mu(x)+(Q_\psi{}^{-1\mu\lambda}a\overleftarrow{Q}_{\psi\lambda\nu}{}^{(0)}B^\nu)(x)\,.\qquad(4.33)$$

It is clear that in the second term of (4.33), integrating by parts can change the direction of the $Q_\psi{}^{(0)}$ operation (at least in the perturbation theory) and one can use (4.25′).

Finally we observe that the field equations (4.20′)–(4.22′) coincide with (4.20)–(4.22).

Thus the generating functional (4.26′) leads to the same field equations for $\kappa_{\mu\nu}$, $\Gamma_{\nu\lambda}{}^\mu$, B^μ, and consequently to the same S matrix, as the generating functional (4.26) does.

Note also that all the Heisenberg operators belonging to class (4.5) must lead to the same S matrix according to the Borchers theorem. (We ignore the question of the meaning of $\kappa^1{}_{\mu\nu}$ as an operator function of $\kappa_{\mu\nu}$. See also the analogous statement for the case of nonlinear chiral Lagrangians in Ref. 19.)

Now we pass to the proof of gauge invariance. We first prove that the S matrix is independent of the type of gauge condition

$$\psi_\mu(x;\kappa,\Gamma)=0\,,\qquad(4.34)$$

i.e., that the S matrix corresponding to the generating functional (4.27) is independent of the form of the function ψ_μ.

Define the function $\Delta_\psi(\kappa,\Gamma)$ by the relation

$$\phi_\psi(\kappa,\Gamma)\equiv\Delta_\psi(\kappa,\Gamma)\int d\mu(S)\;\delta\{\psi_\mu(x;\kappa^S,\Gamma^S)\}=1\,.\qquad(4.35)$$

Here S is an element of the coordinate gauge transformation group, and $d\mu(S)$ is the measure of group integration. (For more details on the coordinate gauge transformation group see Ref. 12.)

Let us explain some peculiarities of the coordinate group transformations. Under the transformation

$$x^\mu\longrightarrow\bar{x}^\mu(x)\,,\qquad(4.36)$$

the corresponding transformation of the metric tensor is

$$g_{\mu\nu}{}^{(\beta)}(x)\longrightarrow g_{\mu\nu}{}^{(\beta)\prime}(x)=J_z{}^{2\beta}\frac{\partial z^\alpha(x)}{\partial x^\mu}\frac{\partial z^\rho(x)}{\partial x^\nu}g_{\alpha\rho}{}^{(\beta)}(z)\,,\qquad(4.37)$$

$$z^\mu(x)=\bar{x}^{-1\mu}(x)\,,\quad J_z(x)=\det\partial z^\mu(x)/\partial x^\nu\,,\qquad(4.38)$$

and the Jacobian of the transformation of the integration measure $d(\kappa)$ is

$$D_\beta\equiv d(g_{\mu\nu}{}^{(\beta)\prime})/d(g_{\mu\nu}{}^{(\beta)})=\mathrm{Det}J_z{}^{5+20\beta}(x)\delta^{10}(z(x)-y)\,.\qquad(4.39)$$

In order to calculate the D_β, we note that the matrix inverse to the $\delta(z(x)-y)$ is

$$\delta(x-z(y))J_z(y)\,.\qquad(4.40)$$

If we formally use the rule of the calculation of the determinant of the product of the matrices, then

$$\mathrm{Det}\delta(z(x)-y)=\mathrm{Det}^{-1/2}J_z(x)\delta(x-y)\,.\qquad(4.41)$$

If

$$\mathrm{Det}J_z(x)\delta(x-y)\neq1\,,\qquad(4.42)$$

then the invariant gravitation measure is

$$\prod_x\prod_{\mu\leq\nu}dg_{\mu\nu}(x)\,.\qquad(4.43)$$

This result does not agree with the form of the invariant measure

$$\prod_x\prod_{\mu\leq\nu}g^{(-5/2)}(x)dg_{\mu\nu}(x)=\prod_x\prod_{\mu\leq\nu}dg_{\mu\nu}{}^{(-5/20)}(x)\,,\qquad(4.44)$$

which is proposed by a number of authors.[20]

Note that the integration measure over the group of the gauge coordinate transformations has a form[12] which is analogous to (4.39):

$$d\mu_L=\mathrm{Det}^4J_{\bar{x}}(x)\delta(\bar{x}(x)-y)\prod_{x,\mu}d\bar{x}^\mu(x)\,,\qquad(4.45)$$

$$d\mu_R=\mathrm{Det}^{-1}J_{\bar{x}}(x)\delta(x-y)\prod_{x,\mu}d\bar{x}^\mu(x)\,.\qquad(4.46)$$

If (4.42) is true, then the left and the right measures are different. When proving the gauge independence of the S matrix, this fact should be taken into account [in particular, in (4.35) one should use the $d\mu_L$].

The formal proof of the invariance of the S matrix can be made in the general case $\mathrm{Det}J(x)\delta(x-y)\neq1$ (then it is necessary to assume the β-independence of the S matrix). However, taking into account that the arbitrary functions $z^\mu(x)$ are the coordinates themselves, we can expect that

$$\mathrm{Det}J(x)\delta(x-y)=1\,.\qquad(4.47)$$

Indeed, the $\delta(z(x)-y)$ can be considered as the matrix

[19] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969).

[20] C. W. Misner, Rev. Mod. Phys. **29**, 497 (1957); J. R. Klauder, Nuovo Cimento **19**, 1059 (1961); B. Laurent, Arkiv Fysik **16**, 279 (1959); B. S. DeWitt, J. Math. Phys. **3**, 1073 (1962).

of the permutation of the points. The corresponding finite-dimensional matrix has determinant 1. Therefore, the $\delta(z(x)-y)$ has determinant 1 if it is calculated by the use of the finite-dimensional approximations. In this case (4.47) follows from (4.41) and

$$d\mu_L = d\mu_R = d\mu = \prod_{x,\mu} d\bar{x}^\mu(x). \tag{4.48}$$

Furthermore, $d\mu$ has the property (2.44), and the measures $d(\kappa)$ and $d(\kappa,\Gamma)$ for arbitrary β's are invariant under the gauge coordinate transformations. Below we assume that (4.47) is true.

As in Sec. II the property (2.44) enables one to prove the gauge invariance of $\Delta_\psi(\kappa,\Gamma)$. We must know the function $\Delta_\psi(\kappa,\Gamma)$ only for $\kappa_{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}$ satisfying the condition (4.34). In this case the group integral is concentrated in the neighborhood of the unit element. The gauge transformations have the form (4.6), (4.7), and

$$d\mu(S) \approx \prod_{x,\mu} d\xi^\mu(x). \tag{4.49}$$

As in Sec. II we obtain

$$\Delta_\psi^{-1}(\kappa,\Gamma)|_{\psi_\mu=0} = \int d\xi^\mu(x)\, \delta\{(Q_{\mu\nu}{}^{\psi T}\xi^\nu)(x)\} = \mathrm{Det}^{-1} Q_\psi. \tag{4.50}$$

Keeping (4.50) in mind, the expression (4.27) can be rewritten in the form

$$Z_0{}^\psi = \int d(\kappa,\Gamma)\delta\{\psi_\mu(x;\kappa,\Gamma)\}\Delta_\psi(\kappa,\Gamma) \times \exp\left[i\int dx(L_0+\kappa_{\mu\nu}J^{\mu\nu})\right]. \tag{4.51}$$

Consider another gauge condition:

$$\psi_\mu{}^1(\kappa,\Gamma)=0. \tag{4.52}$$

Multiply (4.51) by $\phi_{\psi^1}(\kappa,\Gamma)$ and perform the gauge transformation

$$\kappa_{\mu\nu} \to \kappa_{\mu\nu}{}^{S^{-1}}, \quad \Gamma^\mu{}_{\nu\lambda} \to \Gamma^{S^{-1}\mu}{}_{\nu\lambda}.$$

The quantities L_0, Δ_ψ, and Δ_{ψ^1} are invariant. Furthermore, the following substitution can be made on the mass shell:

$$\kappa^{S^{-1}}{}_{\mu\nu}J^{\mu\nu} \to \kappa_{\mu\nu}J^{\mu\nu} \tag{4.53}$$

as discussed in Sec. II.

Then we obtain

$$Z_0{}^\psi|_{\mathrm{m.s.}} = \int d(\kappa,\Gamma)\int d\mu(S)\, \delta\{\psi_\mu(x;\kappa,\Gamma)\}\Delta_\psi(\kappa,\Gamma)\delta\{\psi_\mu{}^1\} \times \Delta_{\psi^1} \exp\left[i\int dx(L_0+\kappa_{\mu\nu}J^{\mu\nu})\right] = Z_0{}^{\psi^1}|_{\mathrm{m.s.}}. \tag{4.54}$$

Thus we prove that the S matrix of the gravitation field is independent of the type of gauge condition (4.34).

The proof that the S matrix is independent of the α's for the case when the gauge function is fixed can be made similarly to Sec. II. For this purpose substitution of (4.6) or (4.6′) should be made in expression (4.26) with

$$\xi^\mu(x) = -(\delta\alpha/2\alpha)([Q_\psi{}^{T-1}]^{\mu\nu}\psi_\nu)(x). \tag{4.55}$$

Then

$$L_0 - \frac{1}{2\alpha}\psi_\mu\delta^{\mu\nu}\psi_\nu + \kappa_{\mu\nu}J^{\mu\nu} \to L_0 - \frac{1}{2(\alpha+\delta\alpha)}\psi_\mu\delta^{\mu\nu}\psi_\nu + \kappa_{\mu\nu}J^{\mu\nu}, \tag{4.56}$$

and the variation of the term Tr ln in (4.26) is compensated by the resulting Jacobian. We omit the corresponding cumbersome calculations.[2]

Let us pass to the consideration of some particular gauges.

A. Harmonic Condition

Consider the class of gauges determined by the function

$$\psi_1{}^\nu(x) \equiv \partial_\mu \hat{g}^{\mu\nu}(x), \quad \hat{g}^{\mu\nu} = (\sqrt{-g})g^{\mu\nu}. \tag{4.57}$$

The harmonic condition corresponds to

$$\psi_1{}^\mu = 0. \tag{4.58}$$

We shall use $\hat{g}^{\mu\nu}$ as independent variables. By means of (4.14) (with $\beta=\frac{1}{2}$), (4.23), and (4.24), we find

$$Q_1{}^\mu{}_\nu = \delta^\mu{}_\nu(\hat{g}^{\lambda\sigma}\partial_\lambda\partial_\sigma + \partial_\lambda\hat{g}^{\lambda\sigma}\partial_\sigma) + \partial_\lambda\hat{g}^{\lambda\mu}\partial_\nu. \tag{4.59}$$

The generating functional is equal to[21]

$$Z_\alpha{}^1 = \int d(\hat{g})\, \exp\left[i\int dx\left(L_0 - \frac{1}{2\alpha}\psi_1{}^\mu\delta_{\mu\nu}\psi_1{}^\nu + \hat{g}^{\mu\nu}J_{\mu\nu}\right) + \mathrm{Tr}\ln Q_1\overleftarrow{\Box}^{-1}\right]. \tag{4.60}$$

In transverse gauge ($\alpha=0$), we have

$$Z_0{}^1 = \int d(\hat{g})\, \exp\left[i\int dx(L_0+\hat{g}^{\mu\nu}J_{\mu\nu}) + 4\,\mathrm{Tr}\ln \hat{g}^{\mu\nu}\partial_\mu\partial_\nu\overleftarrow{\Box}^{-1}\right]\delta\{\partial_\mu\hat{g}^{\mu\nu}(x)\}. \tag{4.61}$$

The Feynman rules for calculation of the generating functional (4.60) in powers of

$$\hat{h}^{\mu\nu} \equiv \hat{g}^{\mu\nu} - \delta^{\mu\nu} \tag{4.62}$$

[21] E. S. Fradkin and I. V. Tyutin, CNR Laboratorio di Cibernetica report, Napoli, 1969 (unpublished).

are

$$\text{(a)}\ L^1{}_{\text{int}}=L_0-(1/2\alpha)\partial_\mu\hat{h}^{\mu\nu}\delta_{\nu\lambda}\partial_\sigma\hat{h}^{\lambda\sigma}-L_{(0)}{}^1 -i\,\text{Tr}\ln\{\delta_\nu{}^\mu+[(\hat{h}^{\lambda\sigma}\partial_\lambda\partial_\sigma+\partial_\lambda\hat{h}^{\lambda\sigma}\partial_\sigma)\delta_\nu{}^\mu+\partial_\lambda\hat{h}^{\lambda\mu}\partial_\nu]\Box\}. \quad (4.63)$$

These are taken as the interaction Lagrangian.

(b) $L_{(0)}{}^1$ is the Lagrangian of the linearized theory

$$L_{(0)}{}^1=\tfrac{1}{4}\partial^\mu\hat{h}^{\nu\lambda}\partial_\mu\hat{h}_{\nu\lambda}-[(\alpha+1)/2\alpha]\partial_\mu\hat{h}^{\mu\nu}\partial^\lambda\hat{h}_{\lambda\nu}-\tfrac{1}{8}\partial_\mu\hat{h}\partial^\mu\hat{h},\quad \hat{h}\equiv\delta_{\mu\nu}\hat{h}^{\mu\nu}. \quad (4.64)$$

The lowering and raising of indices in (4.64) are done by the Minkowski tensors $\delta_{\mu\nu}$ and $\delta^{\mu\nu}$.

(c) The free propagator of $\hat{h}^{\mu\nu}$ is calculated from (4.64) to be

$$D_1{}^{\mu\nu,\lambda\sigma}(p)=-i\bigg[(2+\alpha)\delta^{\mu\nu}\delta^{\lambda\sigma}-\delta^{\mu\lambda}\delta^{\nu\sigma}-\delta^{\mu\sigma}\delta^{\nu\lambda}+\frac{2(\alpha+1)}{p^2}(\delta^{\mu\nu}p^\lambda p^\sigma+\delta^{\lambda\sigma}p^\mu p^\nu)+\frac{\alpha+1}{p^2}(\delta^{\mu\lambda}p^\nu p^\sigma+\delta^{\mu\sigma}p^\nu p^\lambda+\delta^{\nu\lambda}p^\mu p^\sigma+\delta^{\nu\sigma}p^\mu p^\lambda)\bigg]\frac{1}{p^2}. \quad (4.65)$$

The Feynman rules for the gravitational field in the gauge $\partial_\mu\hat{g}^{\mu\nu}=0$ were also obtained by Fadeev and Popov.[3]

B. Linearized Form of Harmonic Condition

Consider the class of gauges described by the function

$$\psi_\mu{}^2\equiv\delta^{\sigma\lambda}(\partial_\sigma g_{\lambda\mu}-\tfrac{1}{2}\partial_\mu g_{\lambda\sigma}). \quad (4.66)$$

We choose $g_{\mu\nu}$ as independent variables. We then obtain with the help of (4.13) (with $\beta=0$), (4.32), and (4.24)

$$Q_{\mu\nu}{}^2=g_{\mu\nu}\Box+(\partial_\alpha g_{\mu\beta}-\tfrac{1}{2}\partial_\mu g_{\alpha\beta})\times(\delta^{\sigma\alpha}\partial_\sigma\delta_\nu{}^\beta+\delta^{\sigma\beta}\partial_\sigma\delta_\nu{}^\alpha-\delta^{\alpha\beta}\partial_\nu). \quad (4.67)$$

The generating functional is[21]

$$Z_\alpha{}^2=\int d(g)\exp\bigg[i\int dx\bigg(L_0-\frac{1}{2\alpha}\psi_\mu{}^2\delta^{\mu\nu}\psi_\nu{}^2+g_{\mu\nu}J^{\mu\nu}\bigg)+\text{Tr}\ln Q^2\overleftarrow{\Box}^{-1}\bigg]. \quad (4.68)$$

The Feynman rules for the perturbation calculation of (4.68) in powers of

$$h_{\mu\nu}\equiv g_{\mu\nu}-\delta_{\mu\nu} \quad (4.69)$$

are

$$\text{(a)}\ L_{\text{int}}{}^2=L_0-(1/2\alpha)\psi_\mu{}^2\delta^{\mu\nu}\psi_\nu{}^2-L_{(0)}{}^2-i\,\text{Tr}\ln\{\delta_{\mu\nu}+[h_{\mu\nu}\Box+(\partial_\alpha h_{\beta\mu}-\tfrac{1}{2}\partial_\mu h_{\beta\alpha})\times(\delta^{\sigma\alpha}\partial_\sigma\delta_\nu{}^\beta+\delta^{\sigma\beta}\partial_\sigma\delta_\nu{}^\alpha-\delta^{\alpha\beta}\partial_\nu)]\Box\}. \quad (4.70)$$

These should be taken as the interaction Lagrangian.

(b) $L_{(0)}{}^2$ is the Lagrangian of the linearized theory

$$L_{(0)}{}^2=\tfrac{1}{4}\partial_\mu h_{\nu\lambda}\partial^\mu h^{\nu\lambda}-\frac{\alpha+1}{2\alpha}\partial_\mu h^{\mu\nu}\partial^\lambda h_{\nu\lambda}+\frac{\alpha+1}{2\alpha}\partial_\mu h\partial_\nu h^{\mu\nu}-\frac{2\alpha+1}{8\alpha}\partial_\mu h\partial^\mu h,\quad h\equiv\delta^{\mu\nu}h_{\mu\nu}. \quad (4.71)$$

The raising of indices in (4.71) is accomplished by the Minkowski tensor $\delta^{\mu\nu}$.

(c) The free propagator of $h_{\mu\nu}$ is calculated from (4.71) to be

$$D_{\mu\nu,\lambda\sigma}{}^2(p)=-i[\delta_{\mu\nu}\delta_{\lambda\sigma}-\delta_{\mu\lambda}\delta_{\nu\sigma}-\delta_{\mu\sigma}\delta_{\nu\lambda}+[(\alpha+1)/p^2](\delta_{\mu\lambda}p_\nu p_\sigma+\delta_{\mu\sigma}p_\nu p_\lambda+\delta_{\nu\lambda}p_\mu p_\sigma+\delta_{\nu\sigma}p_\mu p_\lambda)]p^{-2}. \quad (4.72)$$

The Feynman rules for $\alpha=-1$ were also given by Mandelstam.[5]

C. Dirac Gauge

We give the arguments which show that the S matrix obtained by Popov and Faddeev[6] in the Dirac gauge[17] coincides with the S matrix in the covariant gauges. Consider the following set of gauge conditions:

$$\psi_3{}^0\equiv(\sqrt{-g})e^{ik}\Gamma_{ik}{}^0=0,$$
$$\psi_3{}^k\equiv\partial_i[(-g^{(3)})^{1/3}e^{ik}]=0. \quad (4.73)$$

Here

$$g^{(3)}\equiv\det g_{ik},\quad g=(1/g^{00})g^{(3)},\quad e^{ik}g_{kj}=\delta_j{}^i. \quad (4.74)$$

In gauge (4.73) it is natural to use the first-order formalism. We choose $g^{\mu\nu}$ and $\Gamma^\mu{}_{\nu\lambda}$ as independent variables. With the help of (4.14) (with $\beta=0$), (4.15), and (4.23) one finds

$$Q^0{}_{30}=[-2\overleftarrow{\partial}_ie^{ik}\Gamma_{0k}{}^0-2\overleftarrow{\partial}_i(g^{0k}/g^{00})e^{il}\Gamma_{lk}{}^0+\overleftarrow{\partial}_ie^{lk}\Gamma_{lk}{}^i-\overleftarrow{\partial}_i\overleftarrow{\partial}_ke^{ik}]\sqrt{-g}, \quad (4.75)$$

$$Q_3{}^0{}_j=0, \quad (4.76)$$

$$Q^i{}_{30}=-2\overleftarrow{\partial}_iT^{ij}-e^{il}(-g^{(3)})^{1/3}(\tfrac{1}{3}g_{il}e^{kj}\partial_k-\delta_i{}^j\partial_l), \quad (4.77)$$

$$T^{ij}=(1/g^{00})g^{0m}e^{il}(-g^{(3)})^{1/3}\times(\tfrac{1}{3}g_{lm}e^{kj}\partial_k-\tfrac{1}{2}\delta_m{}^j\partial_l-\tfrac{1}{2}\delta_l{}^j\partial_m), \quad (4.78)$$

$$Q^i{}_{3j}=-\tfrac{1}{3}\overleftarrow{\partial}_j(-g^{(3)})^{1/3}e^{li}\partial_l+\delta_j{}^i(-g^{(3)})^{1/3}e^{lm}\partial_l\partial_m. \quad (4.79)$$

From (4.75)–(4.79), we obtain

$$Q_3{}^{(0)}{}_0{}^0=\nabla,\quad Q_3{}^{(0)}{}_j{}^i=-\delta_j{}^i\nabla-\tfrac{1}{3}\delta^{il}\partial_l\partial_j,$$
$$Q_3{}^{(0)}{}_i{}^0=Q_3{}^{(0)}{}_0{}^i=0. \quad (4.80)$$

The generating functional is

$$Z^3=\int d(g^{\mu\nu},\Gamma)\delta\{\psi_3{}^\mu\}\exp\Big[i\int dx(L_0+g^{\mu\nu}J_{\mu\nu})$$
$$+\mathrm{Tr}\ln Q_{30}{}^0\overleftarrow{\nabla}^{-1}+\mathrm{Tr}\ln B_j{}^i(Q_3{}^{(0)-1})_k{}^j\Big],\quad (4.81)$$

$$B_j{}^i\equiv Q_3{}^T{}_j{}^i=(-g^{(3)})^{1/3}(\delta_j{}^ie^{lm}\partial_l\partial_m+\tfrac{1}{3}e^{li}\partial_l\partial_j)\,.$$

According to the general arguments given in this section, the generating functional (4.81) on the mass shell is equivalent to (4.60), (4.61), and (4.68).

Now we transform expression (4.75). Using (4.25) and (4.80) the field equations for B can be obtained:

$$\nabla B_0=0\,,\quad (\delta_j{}^i\nabla+\tfrac{1}{3}\delta^{il}\partial_l\partial_j)B_i=0\,.\qquad (4.82)$$

The only physical solution of (4.82) is $B_\mu\equiv 0$. Thus $g^{\mu\nu}$ and $\Gamma_{\nu\lambda}{}^\mu$ satisfy (4.20) and (4.21) with $\bar{B}_\mu{}^{(3)}\equiv 0$; i.e., relations (4.4) for $\Gamma_{\nu\lambda}{}^\mu$ are valid, and $g^{\mu\nu}$ satisfies the usual Einstein equations.

Substituting (4.4) into (4.75), one obtains

$$Q_{30}{}^0=(\sqrt{-g^{(3)}})e^{ik}\nabla_i\partial_k(\sqrt{\alpha})$$
$$-(\sqrt{\alpha})(\sqrt{-g^{(3)}})e^{ik}\nabla_i\partial_k\,,\quad (4.83)$$

$$\nabla_i\partial_k\equiv\partial_i\partial_k-\gamma^l{}_{ik}\partial_l\,.\qquad (4.84)$$

Here $\gamma_{ik}{}^l$ is the three-dimensional Christoffel symbol, and $\alpha=(g^{00})^{-1}$. Let us find the expression for $\Gamma^0{}_{ik}$ with the help of the Einstein equations for $g^{\mu\nu}$, and substitute it into the relation

$$e^{ik}\dot{\Gamma}^0{}_{ik}+\dot{e}^{ik}\Gamma^0{}_{ik}=\dot{\psi}_3{}^0=0\,,\qquad (4.85)$$

which must be true according to (4.73). From (4.85) one obtains

$$(\sqrt{-g^{(3)}})e^{ik}\nabla_i\partial_k(\sqrt{\alpha})+(\sqrt{\alpha})(\sqrt{-g^{(3)}})R^{(3)}=0\,.\quad (4.86)$$

Here $R^{(3)}=e^{ik}R^{(3)}{}_{ik}$, and $R^{(3)}{}_{ik}$ is the three-dimensional curvature tensor of second rank.

Finally, the expression for $Q_{30}{}^0$ takes the form

$$Q_{30}{}^0=-(\sqrt{\alpha})A\,,\qquad (4.87)$$

$$A=(\sqrt{-g^{(3)}})R^{(3)}+(\sqrt{-g^{(3)}})e^{ik}\nabla_i\partial_k\,.\qquad (4.88)$$

Thus we can see that the expression

$$Z'=\int d(g^{\mu\nu},\Gamma)\delta\{\psi_3{}^\mu\}\exp\Big[i\int dx(L_0+g^{\mu\nu}J_{\mu\nu})$$
$$+\mathrm{Tr}\ln A\sqrt{\alpha}+\mathrm{Tr}\ln B_j{}^i-\mathrm{Tr}\ln Q_3{}^{(0)}\Big]\quad (4.89)$$

for the generating functional with gauge condition (4.73) can be used instead of (4.81). The generating functional (4.89) leads to the same field equations for $g^{\mu\nu}$ and $\Gamma_{\nu\lambda}{}^\mu$ and consequently to the same S matrix as (4.81) does.

Now let us integrate in (4.89) over all the $\Gamma^\mu{}_{\nu\lambda}$ except $\Gamma^0{}_{ik}$. One can show[6] that L_0 takes the form

$$L_0\to\pi^{ik}\dot{g}_{ik}-H(\pi,g)\,.\qquad (4.90)$$

Here $H(\pi,g)$ is the Hamiltonian of the gravitation field, the explicit form of which we do not need. π^{ik} are the canonical momenta for g_{ik}:

$$\pi^{ik}=[\sqrt{(-g^{(3)})}/\sqrt{g^{00}}](e^{ik}e^{lm}-e^{il}e^{km})\Gamma^0{}_{lm}\,.\quad (4.91)$$

With the help of (4.91) the gauge condition (4.73) can be rewritten exactly in the form given by Dirac[17]:

$$\psi_3{}^0\equiv g_{ik}\pi^{ik}=0\,,\quad \psi_3{}^i\equiv\partial_i[(-g^{(3)})^{1/3}e^{ik}]=0\,.\quad (4.92)$$

Let us pass from the integration over $\Gamma^0{}_{ik}$ and $g^{\mu\nu}$ in (4.89) to the integration over π^{ik} and $g_{\mu\nu}$. The resulting Jacobian can be omitted. The proof of this fact is analogous to that of the possibility of arbitrary choice of the functional integration variables belonging to class (4.5).

The final expression for the generating functional in gauge (4.73) or (4.92) is

$$Z''=\int d(g,\pi)\delta\{\psi_3{}^\mu\}$$
$$\times\exp\Big[i\int dx(\pi^{ik}\dot{g}_{ik}-H(\pi,g)+g_{ik}J^{ik})$$
$$+\mathrm{Tr}\ln A\overleftarrow{\nabla}^{-1}+\mathrm{Tr}\ln B_j{}^i(Q_3{}^{(0)-1})_k{}^j\Big].\quad (4.93)$$

Expression (4.93) has been obtained by Popov and Faddeev[6] with the help of another method closely connected with the canonical quantization procedure.

Note once more that the S matrix corresponding to (4.93) is equal to that in covariant gauges.

V. CONCLUSION

The present paper has been devoted to constructing the S matrix in theories invariant under gauge groups. Though the only cases considered were those of the Yang-Mills field and gravitation, the method developed can in principle be applied to arbitrary theories (the theories of the Yang-Mills and the gravitational fields are apparently the only gauge theories of physical interest[12]), particularly in the cases where no connection with the canonical scheme can be traced. Furthermore, the method suggested proves to be convenient for constructing the perturbation expansion of the S matrix in theories partially invariant under a gauge group, the power of divergence in the S matrix being considerably reduced.

In this paper no attention was paid to possible interactions with other particles. The latter would not affect our considerations, however.

We would like to discuss briefly the problems which have not yet been solved.

(1) Owing to divergences, there is an important problem of introducing a regularization which will not affect the group properties of the theory. Recall that non-gauge-invariant regularization in electrodynamics creates the photon mass. From the more recent view, the resulting photon mass is due to Schwinger terms or, in the end, to the singular character of products of field operators at coincident points. In nonlinear theories this problem becomes even more complicated. The Schwinger terms affect even the renormalization constant, as for instance in the case of the Yang-Mills field.

(2) There is an interesting question whether the gravitation field is renormalizable in the framework of perturbation theory. (We mean here the usual perturbation expansion with respect to a coupling constant and not the method of Fradkin and Efimov.[22]) It is convenient to treat this problem using the variables $\hat{h}^{\mu\nu}$ and $\Gamma^{\mu}{}_{\nu\lambda}$ in the first-order formalism where there are two vertices: a vertex $\Gamma\Gamma\hat{h}$ and the vertex responsible for the interaction of $\hat{h}^{\mu\nu}$ with the fictitious B field. The formal estimate of degrees of growth leads to the conclusion that the theory is of unrenormalizable type.

ACKNOWLEDGMENTS

The authors are grateful to the participants in the theoretical seminar at the P. N. Lebedev Institute for useful discussions. One of the authors (E. F.) thanks Dr. Popov and Dr. Faddeev for being so kind as to acquaint him with Ref. 6 prior to publication, and Professor E. R. Caianiello for his kind hospitality. He is also grateful to Professor Abdus Salam for hospitality at the International Centre for Theoretical Physics, Trieste.

[22] E. S. Fradkin, Nucl. Phys. **49**, 624 (1963); **76**, 588 (1966); G. V. Efimov, Zh. Eksperim. i Teor. Fiz. **44**, 2107 (1963) [Soviet Phys. JETP **17**, 1417 (1963)]; Nuovo Cimento **32**, 1046 (1964).

PHYSICAL REVIEW D VOLUME 2, NUMBER 12 15 DECEMBER 1970

Formulation of Dual Theory in Terms of Functional Integrations*

C. S. Hsue, B. Sakita, and M. A. Virasoro
Department of Physics, University of Wisconsin, Madison, Wisconsin 53706
(Received 27 March 1970)

A formulation of dual-symmetric theory is given in terms of functional integrations. Known formulas of the theory, such as the N-particle Veneziano amplitude and both planar and nonplanar one-loop amplitudes, are explicitly given in terms of the new formulation. The new formulation is also shown to be equivalent to the other formulations such as the harmonic-oscillator formulation.

I. INTRODUCTION

With the invention of the harmonic-oscillator formalism[1] of the dual-symmetric model, considerable advances have been made in the calculational technique of dual scattering amplitudes[2] and in their renormalization.[3] As shown by Nambu,[4] a hadron in this model may be described in terms of a master wave function which depends on an infinite number of space-time coordinates (we may simply call it a wave functional). In order to incorporate unitarity into the theory, one must face the problem of second quantization of this master wave functional. Since the master wave functional is super-nonlocal, the standard canonical quantization method meets considerable difficulties. It would be much easier to follow the space-time approach, such as the one Nielsen[5] has proposed for the N-particle Veneziano amplitudes.

As a first step in this direction, we try to formulate some of the known formulas of the theory, such as N-particle Veneziano amplitudes and one-loop amplitudes[6] (both planar and nonplanar) in terms of Feynman's path (functional) integrals. There are several advantages in this formulation. (i) The N-particle Veneziano amplitude is manifestly symmetric with

* Supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-881, COO-881-277.

[1] S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 679 (1969); Y. Nambu, Enrico Fermi Institute Report No. COO-264-507, 1969 (unpublished); L. Susskind, Phys. Rev. D (to be published).

[2] D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters **2**, 399 (1969).

[3] A. Neveu and J. Scherk, Phys. Rev. D **1**, 2355 (1970); T. H. Burnett, David J. Gross, A. Neveu, J. Scherk, and John H. Schwarz, Princeton report (unpublished).

[4] See Ref. 1.

[5] H. B. Nielsen, Nordita report, 1969 (unpublished). We gratefully acknowledge the access of this paper from which many of the ideas of the present paper were gotten. See also L. Susskind, Yeshiva report (unpublished).

[6] K. Kikkawa, B. Sakita, and M. A. Virasoro, Phys. Rev. **184**, 1701 (1969); K. Kikkawa, *ibid.* **187**, 2249 (1969).