root singularities at $\omega_i^2 \cos^2\alpha = k^2$. The simple pole, treated as a principal value, is not necessarily fatal to the Gaussian sum if the discrete α's are arranged symmetrically about the pole, as they are. However, the square-root singularities, which appear at the edges of the integration interval at $k=0$, and move into the interior of the interval for $k>0$, cause the convergence difficulty for bound states near threshold to which we have referred, and are fatal for scattering. What this means is that regardless of the auxiliary functions or methods used in the differential approach, the boundary conditions at $x=\infty$ inherent in that method do not approximate or converge to the true asymptotic boundary conditions. One may expect—and this was the case in a test calculation we performed—that the calculated scattering phase shift oscillates about the correct result without convergence as $N \to \infty$, being either too large or too small depending on how the discrete α's for the relevant N (whether or not the calculation explicitly uses them) lie in the interval $(0,\pi)$ with respect to the singular points. A modification of the differential approach which circumvents this difficulty is given in the following paper.

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Methods for the Bethe-Salpeter Equation. II. Brackets and the N/D Method*

DAVID KERSHAW
University of California, Berkeley, California 94720

AND

CHARLES ZEMACH
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

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A systematic study of the Bethe-Salpeter relativistic two-body equation is continued. The equation is treated in Wick-rotated coordinate space. A bilinear combination of functions, called a bracket, is defined. Its relation to scattering amplitudes and their residues at poles, and to questions of structure of the equation and numerical accuracy of solutions, is developed. N and D matrices similar to the Jost function of potential theory are defined in terms of complete sets of solutions to the equation characterized by appropriate boundary conditions. Scattering and bound-state properties are defined in terms of them, by formulas analogous to the N/D methods of S-matrix theory, and properties of symmetry, unitarity, and behavior at poles are derived. Various methods for computing wave functions and bound-state and scattering data are presented. A generalization of the variable-phase method of potential theory, which substitutes for the Bethe-Salpeter equation a coupled set of ordinary linear first-order differential equations, is given.

I. INTRODUCTION

THIS paper is the second in a series entitled "Methods for the Bethe-Salpeter Equation." We shall assume familiarity with the principal parts of the first paper,[1] hereafter referred to as MBS I, and use its notation without redefinition.

The general objective is to "make friends with the Bethe-Salpeter equation (BSE)," ultimately to exploit it as a familiar and benign tool in a meaningful study of strong interactions. In MBS I, the special functions relevant to a scalar BSE were assembled and the truncation of spherical harmonic expansions was analyzed in terms of various representations of Green's functions. Here, we build a structure for the BSE which parallels both formal and physical aspects of the nonrelativistic problem. Because of the presence of the relative time variable, which is an essential feature of a relativistic theory with retarded interactions, and which signals the coupling of two-body systems to systems of many particles, the parallel is to a multichannel system even though the BSE, outwardly at least, refers only to a one-channel two-body problem.

We treat sets of solutions at a given energy and angular momentum, characterized by regular and singular boundary conditions. In terms of them, N and D matrices similar to the Jost functions of potential theory[2] are defined and dynamical problems are posed in a manner amenable to calculation. The basic properties of symmetry, unitarity, and separability of residues at poles are derived. Several calculational pro-

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[1] D. Kershaw, H. Snodgrass, and C. Zemach, preceding paper, Phys. Rev. D 2, 2806 (1970).

[2] R. Jost, Helv. Phys. Acta 20, 256 (1947).

cedures for the wave functions and physical parameters of interacting systems are given. Another paper will present model calculations.

Our version of the N/D method has a reduction to potential theory, where it is equivalent to the Jost-function formalism when the latter is well defined. But the present method is more general. It applies to a wider class of potentials, including singular ones. This broader applicability is necessary for the relativistic case where, even for the one-variable BSE at $E=0$ and the scalar-exchange potential, the usual Jost approach breaks down.[3]

In Sec. II, an analytic tool is developed which we call the bracket $[\phi,\psi]$ of two functions ϕ and ψ. It is a generalization of the bilinear form that appears in the Lagrange identity[4] in the theory of ordinary differential equations. The BSE, after reduction to partial waves, is still a partial differential equation in two variables; the applicability of this tool depends on the fact that boundary conditions can be expressed in terms of only one variable. A preliminary discussion is given on brackets for the Schrödinger equation, as an orientation, and here the bracket appears as a matrix element of the flux operator, integrated over the surface of a sphere centered at the origin.

In Sec. III, the structure of complete sets of functions and their boundary conditions, brackets, and connecting relations are developed.

Section IV contains the development of the N and D matrices, their connection to bound-state and scattering problems, and a comparison with the formalism of Jost functions.

In Sec. V, certain complete families of functions which solve the free BSE, called channel functions, are defined and their numerical computation through summation of power series is outlined. The channel functions are analogs for the BSE of the spherical Bessel and Hänkel functions of potential theory, and, for computation purposes, are more useful than the vector Bessel functions constructed in MBS I. They enter into the construction of Green's functions, and the N and D matrices, and into the explicit formulation of dynamical problems.

The next three sections present methods for doing dynamics, i.e., for obtaining bound states and scattering once the interaction is specified. Section VI details the N/D method and includes a discussion of bound-state normalizations. Because of the correspondence to the N/D method of S-matrix theory, comparisons between BSE and S-matrix calculations based on the same physical hypotheses are possible not only for output quantities, but also for more basic ingredients such as the D function itself.

[3] R. Haymaker and R. Blankenbecler, Phys. Rev. **186**, 1648 (1969). However, Haymaker and Blankenbecler surmounted the difficulty in this case by a technique for subtracting out divergences.

[4] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).

Section VII describes the use of truncated spherical harmonic expansions. Section VIII shows how to obtain the N and D matrices and the wave function for the dynamical system from a set of coupled first-order differential equations. This procedure can be regarded both as generalization of the variable phase method of potential theory, as developed by Calogero and others, and also as an adaptation of the method of variation of constants of ordinary differential equation theory.

We are not ready to treat all sources of complexity at once; as in MBS I, the particles are scalar and the energy is either in the bound-state region $|m_1^2-m_2^2|^{1/2} \leq E \leq m_1+m_2$, or the scattering region below inelastic threshold. Then the condition of elastic unitarity applies and the Wick rotation is valid for both the differential and integral BSE. We use the BSE in coordinate space in the Wick-rotated form.

The discussion frequently draws the distinction between regular and singular boundary conditions. This follows the extraction of total momentum and angular momentum; i.e., $\Psi(x_1,x_2)=e^{iP_\mu X_\mu}Y_l^m(\theta_3,\phi)\psi(x,\theta)$, and $\psi(x,\theta)$ is the subject of study.

The regular conditions arise from the causality requirement which is incorporated into the Green's functions through the $m \to m-i\epsilon$ rule, and then into the wave functions via the integral version of the BSE:

$$\psi = GV\psi \quad \text{(bound state)} \tag{1.1a}$$

$$= j_l(kr)+GV\psi \quad \text{(scattering)}. \tag{1.1b}$$

One might avoid Green's functions altogether, and connect up the properties of the differential equation with the notion of particle flux, but the above approach is standard and perhaps, for this reason, easier. The bounding behavior of ψ can be read off from one of the representations of the Green's function listed in MBS I.

To obtain a precise statement of regularity in a form which will prove useful, let ψ be expanded in spherical harmonics:

$$\psi(x)=\sum \psi_n(x)R_n(\theta). \tag{1.2}$$

At the origin of x, (1.1) implies the finiteness of each $\psi_n(x)$ and perhaps more, but this is sufficient. Referring to the harmonic expansion of the Green's function [MBS I, Eq. (5.23)] and (1.1), we find each $\psi_n(x)$ represented as a superposition of terms indexed by an angular parameter α, $0\leq\alpha\leq\pi$. As $x\to\infty$, the term in α requires asymptotic behavior of type

$$x^{-3/2}e^{-\lambda_{\alpha,1}x} \quad \text{or} \quad x^{-3/2}e^{-\lambda_{\alpha,2}x}, \tag{1.3}$$

where

$$\lambda_{\alpha,i}=\pm\omega_i\cos\alpha+(\omega_i^2\cos^2\alpha-k^2)^{1/2}, \quad i=1,2. \tag{1.4}$$

The spectrum of the $\lambda_{\alpha,i}$ is limited to a finite range of values, including some in the scattering case with negative imaginary part, as is discussed fully in MBS I. Singular behavior at infinity, acceptable to the differential equation, but not to causality, would encom-

pass terms like

$$x^{-3/2}e^{+\lambda_{\alpha,1}}, \quad x^{-3/2}e^{+\lambda_{\alpha,2}x}. \tag{1.5}$$

Modifications of this framework may be required by ill-behaved interactions, but can be treated as they come up.

II. BRACKETS

A. Nonrelativistic Brackets

In a nonrelativistic context described by

$$(E+\nabla^2/2m)\psi(\mathbf{r})=V(\mathbf{r})\psi(\mathbf{r}), \tag{2.1}$$

the "bracket" $[\phi,\psi]$ between $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$,

$$[\phi,\psi]=\int r^2 d\Omega\left(\phi\frac{\partial\psi}{\partial r}-\psi\frac{\partial\phi}{\partial r}\right), \tag{2.2}$$

is a convenient tool for the analysis of scattering and other properties, particularly if one does not choose to invoke the integral formulation of the Schrödinger equation. For $\phi=\psi^*$, it measures the flux of particles out of a sphere of radius r and so has an immediate physical interpretation. It is a function of r only, linear in both ϕ and ψ, and vanishes at $r=0$ if ϕ and ψ are regular at the origin.

Suppose that

$$(E+\nabla^2/2m)\phi(\mathbf{r})=0. \tag{2.3}$$

Multiply (2.1) by $\phi(\mathbf{r})$, multiply (2.3) by $\psi(\mathbf{r})$, subtract, and integrate over $d\mathbf{r}$, applying Green's theorem. Then

$$2m\int_a^b dr\int\phi(\mathbf{r})V(\mathbf{r})\psi(\mathbf{r})r^2d\Omega = [\phi,\psi]_{r=b}-[\phi,\psi]_{r=a}, \tag{2.4}$$

and, in particular, if both ϕ and ψ are regular at the origin,

$$2m\int\phi(\mathbf{r})V(\mathbf{r})\psi(\mathbf{r})d\mathbf{r}=[\phi,\psi]_{r\to\infty}. \tag{2.5}$$

This argument also shows that if ϕ and ψ satisfy the same equation, i.e., either (2.1) or (2.3), then $[\phi,\psi]$ is a constant independent of r.

We note briefly some uses of this concept in potential theory which will have their generalizations for the BSE.

Firstly, let $\psi_{\mathbf{k}}(\mathbf{r})$ be the outgoing solution of (2.1) for momentum $\mathbf{k}$, i.e., regular at the origin and

$$\psi_{\mathbf{k}}(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+\chi_{\mathbf{k}}(\mathbf{r}), \tag{2.6}$$

where, as $r\to\infty$,

$$\chi_{\mathbf{k}}(\mathbf{r})=(e^{ikr}/r)f+O(r^{-2}). \tag{2.7}$$

Equation (2.7) implies, as $r\to\infty$, that

$$\frac{\partial}{\partial r}\chi_{\mathbf{k}}(\mathbf{r})=ik\chi_{\mathbf{k}}(\mathbf{r})+O(r^{-2}). \tag{2.8}$$

Then Eq. (2.5) provides a formula for the scattering amplitude in terms of a bracket:

$$f(\mathbf{k}'\leftarrow\mathbf{k})=\frac{-2m}{4\pi}\int d\mathbf{r}\,e^{-i\mathbf{k}'\cdot\mathbf{r}}V(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}) = \frac{-1}{4\pi}[e^{-i\mathbf{k}'\cdot\mathbf{r}},\psi_{\mathbf{k}}]_{r\to\infty}, \tag{2.9}$$

or, equivalently,

$$f(\mathbf{k}'\leftarrow\mathbf{k})=-(4\pi)^{-1}[e^{-i\mathbf{k}'\cdot\mathbf{r}},\chi_{\mathbf{k}}]_{r\to\infty}. \tag{2.10}$$

Secondly, consider the inequalities

$$[\psi_{-\mathbf{k}'},\psi_{\mathbf{k}}]=0, \tag{2.11}$$

$$[\psi_{\mathbf{k}'}{}^*,\psi_{\mathbf{k}}]=0. \tag{2.12}$$

The brackets of (2.11) and (2.12) are constant because $\psi_{\mathbf{k}}$, $\psi_{-\mathbf{k}'}$, and $\psi_{\mathbf{k}'}{}^*$ all satisfy (2.1). The equalities are evidently valid at $r=0$, and hence at all r. Now substitute (2.6) and the analogous form for $\psi_{-\mathbf{k}'}$ into (2.11) and apply formula (2.10). One of the terms that appears is $[\chi_{-\mathbf{k}'}(r),\chi_{\mathbf{k}}(r)]$, but this vanishes as $r\to\infty$ in virtue of (2.8). The result is the time-reversal property,

$$f(k'\leftarrow\mathbf{k})=f(-\mathbf{k}\leftarrow-\mathbf{k}'). \tag{2.13}$$

In the same way, (2.12) delivers the unitarity statement

$$\frac{f(\mathbf{k}'\leftarrow\mathbf{k})-f^*(\mathbf{k}\leftarrow\mathbf{k}')}{2i}=\frac{k}{4\pi}\int d\Omega_{\mathbf{k}''} \times f^*(\mathbf{k}''\leftarrow\mathbf{k}')f(\mathbf{k}''\leftarrow\mathbf{k}). \tag{2.14}$$

B. Brackets for BSE

Given a family of functions ψ upon which a partial differential operator acts and a family of functions ϕ upon which the transposed operator acts, one may attempt to construct a bilinear function of ϕ and ψ similar to the form that enters into the "identity of Langrange" in the theory of ordinary differential equations and which may serve similar purposes.

For the Schrödinger equation, the appropriate definition was given in the previous section; the bilinear function is essentially a matrix element of the flux operator.

For the Wick-rotated BSE, it is natural to regard x as the interesting variable in such a construction and to integrate over the angular variables. Thus, we shall define the bracket $[\phi,\psi]$ of functions $\phi(x_\mu)$ and $\psi(x_\mu)$ as a function of x only, in a way which parallels the treatment of the nonrelativistic case above. We shall also write $[\phi,\psi]_x$ when the point x at which the bracket is evaluated deserves emphasis.

As a preliminary, we define "second-order" brackets $(\phi,\psi)^{(1)}$ and $(\phi,\psi)^{(2)}$, associated, respectively, with the second-order operators $\mathfrak{D}_1$ and $\mathfrak{D}_2$,

$$\mathfrak{D}_1=\Box^2+k^2-2\omega_1\partial/(\partial\tau), \tag{2.15a}$$

$$\mathfrak{D}_2=\Box^2+k^2+2\omega_2\partial/(\partial\tau)\,, \tag{2.15b}$$

that enter into the scalar Bethe-Salpeter equation. The transposed operators $\bar{\mathfrak{D}}_1$ and $\bar{\mathfrak{D}}_2$ in rectangular coordinates are obtained from $\mathfrak{D}_1$ and $\mathfrak{D}_2$ simply by reversing the sign of $(\partial/\partial\tau)$. More generally, if the $\mathfrak{D}_i$ are expressed in coordinates $\{\xi_i\}$ which are not rectangular, than the Jacobian $J(\xi)$, where

$$d^4x=J(\xi)d^4\xi\,, \tag{2.16}$$

enters into the definition of transpose. One writes down $J\phi(\mathfrak{D}_i\psi)$, integrates by parts, and carries the differentiations through J to get

$$J(\bar{\mathfrak{D}}_i\phi)\psi=J\phi(\mathfrak{D}_i\psi)+(\text{derivative terms})\,. \tag{2.17}$$

It follows that

$$\int_{\text{closed volume}} Jd^4\xi[(\bar{\mathfrak{D}}_1\phi)\psi-\phi(\mathfrak{D}_1\psi)] =\text{surface terms}. \tag{2.18}$$

This is the essential property to be exploited. We begin by characterizing $(\phi,\psi)^{(1)}$ as

$$\int x^3\sin^2\theta\,d\theta d\Omega[\phi\mathfrak{D}_i\psi-(\bar{\mathfrak{D}}_i\phi)\psi]=\frac{d}{dx}(\phi,\psi)^{(1)}\,, \tag{2.19}$$

but immediately introduce a notational simplification. We shall suppose, as in MBS I, that all functions carry the same angular momentum quantum numbers l, m. Specifically the symbols ψ, ϕ stand for $\psi(x,\theta)Y_l^m(\theta_3,\phi)$ and $\phi(x,\theta)Y_l^m(\theta_3,\phi)^*$ (note the complex conjugate of Y_l^m in the ϕ function). The Y's are normalized in the usual way:

$$\int Y_l^m(\theta_3,\phi)^*Y_l^m(\theta_3,\phi)d\Omega=1\,. \tag{2.20}$$

Then the dependence of $\mathfrak{D}_1$ on θ_3, ϕ can be replaced by dependence on l, and (2.19) can be rewritten

$$\int x^3\sin^2\theta\,d\theta[\phi\mathfrak{D}_1\psi-(\bar{\mathfrak{D}}_1\phi)\psi]=\frac{d}{dx}(\phi,\psi)^{(1)}\,, \tag{2.21}$$

with $\phi\equiv\phi(x,\theta)$, $\psi\equiv\psi(x,\theta)$.

The operator $\mathfrak{D}_1$ can be separated into parts:

$$\mathfrak{D}_1=D-2\omega_1\partial/(\partial\tau)\,, \tag{2.22}$$

where

$$D=\bar{D}=\frac{d^2}{dx^2}+\frac{3}{x}\frac{d}{dx}-\frac{\mathfrak{L}^2}{x^2}+k^2 \tag{2.23}$$

and $\mathfrak{L}^2$, the square of the operator for four-dimensional angular momentum, is written out in MBS I, Eq. (2.2).

Observe the identities

$$x^3(\phi D\psi-\psi D\phi)=\frac{d}{dx}\left[x^3\left(\phi\frac{d\psi}{dx}-\frac{d\phi}{dx}\psi\right)\right] -x[\phi\mathfrak{L}^2\psi-\psi\mathfrak{L}^2\phi], \tag{2.24}$$

$$x^3\sin^2\theta\left(\phi\frac{\partial}{\partial\tau}\psi-\psi\frac{\tilde{\partial}}{\partial\tau}\phi\right)=x^3\sin^2\theta\frac{\partial}{\partial\tau}(\phi\psi)$$

$$=\frac{d}{dx}(x^3\sin^2\theta\cos\theta\,\phi\psi)$$

$$-\frac{d}{d\theta}(x^2\sin^3\theta\,\phi\psi)\,. \tag{2.25}$$

The $\mathfrak{L}^2$ terms of (2.24) and the $d/d\theta$ term of (2.25) vanish when the integration over angle is carried out.

Thus we infer from (2.21) the explicit definition

$$(\phi,\psi)^{(1)}=\int x^3\sin^2\theta \times d\theta\left[\phi\frac{d\psi}{dx}-\psi\frac{d\phi}{dx}-(2\omega_1\cos\theta)\phi\psi\right]. \tag{2.26}$$

Similarly,

$$\int x^3\sin^2\theta\,d\theta[\phi\mathfrak{D}_2\psi-(\bar{\mathfrak{D}}_2\phi)\psi]=\frac{d}{dx}(\phi,\psi)^{(2)} \tag{2.27}$$

and

$$(\phi,\psi)^{(2)}=\int x^3\sin^2\theta \times d\theta\left[\phi\frac{d\psi}{dx}-\psi\frac{d\phi}{dx}+(2\omega_2\cos\theta)\phi\psi\right]. \tag{2.28}$$

Finally, put

$$\int x^3\sin^2\theta d\theta[\phi\mathfrak{D}_1\mathfrak{D}_2\psi-(\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2\phi)\psi]=\frac{d}{dx}[\phi,\psi]. \tag{2.29}$$

It is easy to see that the left-hand side of (2.29) equals

$$\frac{d}{dx}(\phi,\mathfrak{D}_2\psi)^{(1)}+\frac{d}{dx}(\bar{\mathfrak{D}}_1\phi,\psi)^{(2)}\,, \tag{2.30}$$

so we may define the BSE bracket by

$$[\phi,\psi]=(\phi,\mathfrak{D}_2\psi)^{(1)}+(\bar{\mathfrak{D}}_1\phi,\psi)^{(2)}\,. \tag{2.31a}$$

An equivalent definition is

$$[\phi,\psi]=(\phi,\mathfrak{D}_1\psi)^{(2)}+(\bar{\mathfrak{D}}_2\phi,\psi)^{(1)}\,. \tag{2.31b}$$

Let the spherical harmonic expansions of ψ and ϕ be given by

$$\psi=\sum_n f_n(x)R_n(\theta)\,,\quad \phi=\sum_n g_n(x)R_n(\theta)\,. \tag{2.32}$$

For practical work, we require the explicit expressions for the brackets in terms of the coefficients $f_n(x)$ and $g_n(x)$. Diligent application of the rules for differentia-

tion leads to

$$(\phi,\psi)^{(1)}=x^3\sum\{g_nf_n'-g_n'f_n -2\omega_1A_n(g_nf_{n-1}+g_{n-1}f_n)\}, \quad (2.33a)$$

$$(\phi,\psi)^{(2)}=x^3\sum\{g_nf_n'-g_n'f_n +2\omega_2A_n(g_nf_{n-1}+g_{n-1}f_n)\}, \quad (2.33b)$$

$$\begin{aligned}[\phi,\psi]=x^3\sum\{&g_nf_n'''-g_n'f_n''+g_n''f_n'-g_n'''f_n\\ &+3(g_nf_n''-g_n''f_n)/x\\ &-[4\omega_1\omega_2\beta_n-2k^2+(2n^2+4n+3)/x^2](g_nf_n'-g_n'f_n)\\ &-4\omega_1\omega_2\alpha_n[g_nf_{n-2}'+g_{n-2}f_n'-g_n'f_{n-2}-g_{n-2}'f_n\\ &+2n(g_{n-2}f_n-g_nf_{n-2})/x]+2(\omega_2-\omega_1)A_n\\ &\times[X(g,f)+X(f,g)]\}, \end{aligned} \quad (2.34)$$

where

$$\begin{aligned}X(g,f)=&g_n''f_{n-1}+g_{n-1}''f_n-g_n'f_{n-1}'+k^2g_nf_{n-1}\\ &+[(1-n)g_{n-1}'f_n+(n+2)g_n'f_{n-1}]/x\\ &-(n^2+n+1)g_nf_{n-1}/x^2, \end{aligned} \quad (2.35)$$

and where A_n, α_n, and β_n are given in MBS I, Eq. (2.16).

The general property underlying the use of the bracket is the following. Let

$$\mathfrak{D}_1\mathfrak{D}_2\psi(x,\theta)=A(x,\theta), \quad (2.36a)$$

$$\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2\phi(x,\theta)=B(x,\theta). \quad (2.36b)$$

Multiply (2.36a) by ϕ, (2.36b) by ψ, subtract, and integrate over x and θ. Then, we have

$$\begin{aligned}[\phi,\psi]_{x_2}-[\phi,\psi]_{x_1}=&\int_{x_1}^{x_2}x^3dx\int\sin^2\theta\\ &\times d\theta[\phi(x,\theta)A(x,\theta)-B(x,\theta)\psi(x,\theta)]. \end{aligned} \quad (2.37)$$

In particular, if

$$\mathfrak{D}_1\mathfrak{D}_2\psi=V\psi, \quad (2.38a)$$

$$\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2\phi=0, \quad (2.38b)$$

then

$$[\phi,\psi]_\infty-[\phi,\psi]_0=\int\phi V\psi x^3\sin^2\theta\,dxd\theta. \quad (2.39)$$

C. Some Properties of Brackets

1. Brackets of Solutions

If ψ and ϕ both satisfy the free BSE,

$$\mathfrak{D}_1\mathfrak{D}_2\psi=0, \quad \bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2\phi=0, \quad (2.40)$$

or both satisfy the BSE with interactions,[5]

$$\mathfrak{D}_1\mathfrak{D}_2\psi=V\psi, \quad \bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2\phi=\phi V, \quad (2.41)$$

then by (2.37), $[\phi,\psi]_x$ is a *constant*, characteristic of the solutions ϕ and ψ, but independent of the coordinate variable x.

In numerical work, one calculates families of solutions defined, perhaps, by boundary conditions at $x=0$ or $x\to\infty$, and evaluated for various x. The number of decimals to which the computed brackets are constant as x varies is a measure of the accuracy of the calculation. In our experience, it has proved to be a sufficient measure of accuracy.

2. Complete Sets of Solutions

Suppose we have complete sets $\{\psi^n\}$ and $\{\phi_n\}$ of solutions to a BSE and its transpose. Then arbitrary solutions Ψ and Φ can be expressed as linear combinations (repeated indices are summed):

$$\Psi=\psi^nc_n, \quad \Phi=d^n\phi_n. \quad (2.42)$$

If we work with one of the truncated approximations to the BSE, a complete set of solutions will have finitely many members. But in general, there will be infinitely many terms in (2.42). In MBS I, we have even considered a class of solutions $I^{(\alpha,i)}$ for the exact, free-particle BSE where the index α was a continuous variable.

All the brackets

$$[\phi_n,\psi^m]=B_n{}^m \quad (2.43)$$

will be constant. If arbitrary linear combinations are taken,

$$(\psi^m)'=\psi^qC_q{}^m, \quad (\phi_n)'=A_n{}^p\phi_p, \quad (2.44)$$

then in terms of the transformed functions, we have, dropping the primes,

$$[\phi_n,\psi^m]=A_n{}^pB_p{}^qC_q{}^m. \quad (2.45)$$

If the transformations are managed so that

$$[\phi_n,\psi^m]=\delta_n{}^m/\rho_n \quad \text{(no sum on } n), \quad (2.46)$$

we may say that the solutions to the equation and its transpose have been arranged in conjugate pairs and that $\{\psi^n\}$ and $\{\phi_m\}$ are bi-orthogonal sets. This was done explicitly in MBS I for the $I^{(\alpha,i)}$, $K^{(\alpha,i)}$ of the truncated problem.

Assuming now that the ψ's and ϕ's satisfy (2.46), we can solve for the coefficients c_n of (2.42) by computing $[\phi_n,\Psi]$. Thus, we have

$$\Psi=\psi^m\rho_m[\phi_m,\Psi] \quad (2.47a)$$

and also

$$\Phi=[\Phi,\psi^n]\rho_n\phi_n. \quad (2.47b)$$

Furthermore,

$$[\Phi,\Psi]=[\Phi,\psi^n]\rho_n[\phi_n,\Psi], \quad (2.48)$$

which resembles the procedure for insertion of intermediate states into a quantum-mechanical scalar product.

3. Scattering Amplitude

The formula for the scattering amplitude $f(\mathbf{k}'\leftarrow\mathbf{k})$ is[6]

$$f(\mathbf{k}'\leftarrow\mathbf{k})=\frac{1}{8\pi E}\int e^{-i\mathbf{k}'\cdot\mathbf{r}}V(x_\mu)\psi(x_\mu)d^4x, \quad (2.49)$$

[5] Of course $\phi V=V\phi$ if $V=V(x)$ is a function. The notation of (2.41) anticipates nonlocal interactions.

[6] C. Schwartz and C. Zemach, Phys. Rev. 141, 1454 (1966).

where $\psi(x_\mu)$ satisfies

$$\psi(x_\mu)=e^{i\mathbf{k}\cdot\mathbf{r}}+\int G(x_\mu,x_\mu')V(x_\mu')\psi(x_\mu')d^4x_\mu'. \quad (2.50)$$

The partial-wave reductions for these functions are

$$e^{-i\mathbf{k}'\cdot\mathbf{r}}=\sum(-i)^l(2l+1)P_l(\hat{k}'\cdot\hat{r})j_l(kr), \quad (2.51a)$$

$$\psi(x_\mu)=\sum(i)^l(2l+1)P_l(\hat{r}\cdot\hat{k})\psi_l(x,\theta), \quad (2.51b)$$

$$G(x_\mu,x_\mu')=\sum(4\pi)^{-1}(2l+1)P_l(\hat{r}\cdot\hat{r}')\times G^{(l)}(x,\theta;x',\theta'), \quad (2.51c)$$

$$f(\mathbf{k}'\leftarrow\mathbf{k})=\sum(2l+1)P_l(\hat{k}'\cdot\hat{k})f_l, \quad (2.51d)$$

where the relation of partial-wave amplitude to phase shift (below inelastic threshold) is

$$f_l=(1/k)e^{i\delta_l}\sin\delta_l. \quad (2.52)$$

In reducing (2.49) and (2.50), the projection property of the Legendre polynomials is best expressed by

$$\int(2l+1)P_l(\hat{k}'\cdot\hat{r})d\Omega\,P_l(\hat{r}\cdot\hat{k})=4\pi P_l(\hat{k}'\cdot\hat{k}). \quad (2.53)$$

Then we have

$$f_l=\frac{1}{2E}\int j_l(kr)V(x)\psi_l(x,\theta)x^3\sin^2\theta\,dxd\theta, \quad (2.54)$$

with $\psi_l(x,\theta)$ obeying

$$\psi_l(x,\theta)=j_l(kr)+\int G^{(l)}(x,\theta;x',\theta')\times V(x')\psi(x',\theta')x'^3\sin^2\theta'\,dx'd\theta'. \quad (2.55)$$

An explicit form for $G^{(l)}(x,\theta;x',\theta')$ is given in MBS I, Eq. (5.12).

Since ψ_l and j_l satisfy (2.38a) and (2.38b), respectively, Eq. (2.39) may be applied. Because ψ_l and j_l are regular at the origin, and the definition of bracket carries a factor of x^3, $[j_l,\psi_l]_0=0$. Therefore, the partial-wave scattering amplitude is given by

$$f_l=(1/2E)[j_l,\psi_l]_\infty. \quad (2.56)$$

In practice, the bracket of (2.56) can be evaluated at any x outside the range of the potential.

Of course, f_l and $\psi_l(x,\theta)$ are complex. But the above material can be rearranged to give $\tan\delta_l$ in terms of a real wave function. Let $G^{(l)}$ be divided into real and imaginary parts:

$$G^{(l)}(x,\theta;x'\theta')=G_R{}^{(l)}(x,\theta;x',\theta')+iG_I{}^{(l)}(x,\theta;x',\theta'). \quad (2.57)$$

By MBS I, Eq. (5.56c) we have

$$\mathrm{Im}G(x_\mu,x_\mu')=\frac{1}{8\pi E}\frac{\sin(k|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}=\frac{k}{8\pi E}\sum(2l+1)\times P_l(\hat{r}\cdot\hat{r}')j_l(kr)j_l(kr'), \quad (2.58)$$

and hence, comparing with (2.51c),

$$G_I{}^{(l)}(x,\theta;x',\theta')=(k/2E)j_l(kr)j_l(kr'). \quad (2.59)$$

Substituting this into (2.54), we have[7]

$$\psi_l=j_l(kr)(1+ikf_l)+\int G_R{}^{(l)}V\psi_l. \quad (2.60)$$

Hence, ϕ_l as defined by

$$\phi_l(x,\theta)=\psi_l(x,\theta)/(1+ikf_l) \quad (2.61)$$

satisfies

$$\phi_l=j_l+\int G_R{}^{(l)}V\phi_l \quad (2.62)$$

and is clearly real. It satisfies the same differential BSE as ψ_l, from which it differs by a constant factor. A specification of its asymptotic boundary conditions will be given later.

Replacing ψ_l in (2.56) in terms of ϕ_l and recalling that $(e^{i\delta}\sin\delta)^{-1}=\cot\delta-i$, we obtain

$$\tan\delta_l=(k/2E)[j_l,\phi_l]_\infty. \quad (2.63)$$

In practice, we calculate ϕ_l and $\tan\delta_l$ rather than ψ_l and f_l.

4. *Output Coupling Constant*

Consider a reaction at energy E through the annihilation process

$$1+2\rightarrow 3\rightarrow 1+2. \quad (2.64)$$

Let particle 3 have mass E_0, $E_0<m_1+m_2$. The coupling strength at each vertex is g. Particles 1 and 2 continue to have zero spin. The normalization of g may be standardized by citing the interaction Lagrangian for the three-particle vertex. If the particles belong to *different* fields $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$ and particle 3 also has spin 0, we put

$$\mathfrak{L}_I(x)=g\phi_1(x)\phi_2(x)\phi_3(x). \quad (2.65a)$$

But if two or three of the particles are identical, then

$$\mathfrak{L}_I(x)=(g/2!)\phi_1(x)\phi_2(x)\phi_2(x) \quad (2.65b)$$

or

$$\mathfrak{L}_I(x)=(g/3!)\phi_1(x)\phi_1(x)\phi_1(x), \quad (2.65c)$$

respectively.

[7] The element of integration, when not explicitly indicated, is always $x^3\sin^2\theta\,dxd\theta$, or, when appropriate, the same form with primed variables.

Any one of these leads, by Feynman's rules, to a vertex factor of simply g and a scattering amplitude for the annihilation process which is

$$f^{\text{ann}}(\mathbf{k}'\leftarrow\mathbf{k})=\frac{1}{8\pi E}\frac{g^2}{E_0^2-E^2}. \tag{2.66}$$

If particle 3 has spin l, we suppose, without supplying details, that (2.65) is adjusted so that the result is

$$f^{\text{ann}}(\mathbf{k}'\leftarrow\mathbf{k})=\frac{1}{8\pi E}\frac{g^2}{E_0^2-E^2}(2l+1)P_l(\hat{k}'\cdot\hat{k}). \tag{2.67}$$

The partial-wave amplitude is

$$f_l^{\text{ann}}=\frac{1}{8\pi E}\frac{g^2}{E_0^2-E^2}. \tag{2.68}$$

Now suppose that a certain interaction between 1 and 2 defines amplitudes which are computed by (2.56) and, further, that a bound state at E_0 is produced in the lth wave. For E below threshold, f_l is real (even though the argument of j_l is $i\kappa r$) and has a pole at $E=E_0$.

According to the bootstrap philosophy, the interaction has simulated the annihilation process (2.64) and the residue of f_l at $E=E_0$ is related to the output coupling constant g, i.e., the coupling to the particle of mass E_0 which emerges as part of the output of the calculation with this interaction. Comparing the form (2.68) valid at the pole with the general formula (2.56), we have

$$(g^2/4\pi)_{\text{out}}=\lim_{E\to E_0}(E_0^2-E^2)[j_l(i\kappa r),\psi_l]_\infty \tag{2.69}$$

as the prescription for the output coupling constant in rationalized form. An alternative formulation is given in (5.6).

Consider now that the mass of the intermediate particle is above threshold. Then, we have a resonance rather than a bound state and replace E_0 by $E_0-\frac{1}{2}i\Gamma$. Assuming $\Gamma\ll E_0$, we can approximate

$$(E_0-\tfrac{1}{2}i\Gamma)^2-E^2\approx E_0^2-E^2-iE_0\Gamma. \tag{2.70}$$

Then, in the neighborhood of $E=E_0$,

$$\frac{1}{k}e^{i\delta_l}\sin\delta_l=\frac{g^2}{8\pi E}\frac{1}{E_0^2-E^2-iE_0\Gamma}, \tag{2.71}$$

which implies the connection between width and coupling constant,

$$\Gamma=\frac{k_0}{2E_0^2}\frac{g^2}{4\pi}, \tag{2.72}$$

and the rule for obtaining $g^2/4\pi$,

$$\left(\frac{g^2}{4\pi}\right)_{\text{out}}=\lim_{E\to E_0}\frac{2E_0}{k}(E_0^2-E^2)\tan\delta_l$$
$$=\lim_{E\to E_0}(E_0^2-E^2)[j_l,\phi_l]_\infty, \tag{2.73}$$

where ϕ_l is the function of (2.62).

III. COMPLETE SETS OF SOLUTIONS

A. Boundary Conditions for Free BSE

The Bethe-Salpeter equation, with or without interaction, has infinitely many solutions regular for $0<x<\infty$. Because the equation is linear, one may construct a complete set or basis of solutions, an arbitrary linear combination of which represents the general solution.[8] The basis members may be conveniently delineated in terms of boundary conditions at $x=0$ or $x=\infty$. In either case, there will be a natural division between singular and regular conditions at the boundary points.

Let there be given a set of functions $r^{(\nu)}(x,\theta)$ regular at the origin indexed by (ν) and a second set $s^{(\nu)}(x,\theta)$ singular at the origin which solve the free BSE at a definite energy and angular momentum, such that the combined set is a basis. An enumerable family of functions of this type will be constructed explicitly in Sec. V by means of expansions in spherical harmonics and powers of x. They, and certain derived functions, will be called channel functions because the lead term in the power series occurs in a distinct spherical harmonic channel.

If $\psi(x,\theta)$ satisfies the BSE, the *transposed* function $\tilde{\psi}(x,\theta)$ satisfies the transposed BSE. The transposition is obtained by $\tau\to-\tau$, or equivalently, $\theta\to\pi-\theta$:

$$\tilde{\psi}(x,\theta)=\psi(x,\pi-\theta). \tag{3.1}$$

This means that if

$$\psi=\sum f_n(x)R_n(\theta), \tag{3.2a}$$

then

$$\tilde{\psi}=\sum f_n(x)R_n(\pi-\theta)=\sum(-1)^{n-l}f_n(x)R_n(\theta). \tag{3.2b}$$

Hence we may define a transposed basis for the transposed free BSE, namely,

$$r_{(\nu)}(x,\theta)=\tilde{r}^{(\nu)}(x,\theta)=r^{(\nu)}(x,\pi-\theta), \tag{3.3a}$$

$$s_{(\nu)}(x,\theta)=\tilde{s}^{(\nu)}(x,\theta)=s^{(\nu)}(x,\pi-\theta), \tag{3.3b}$$

with the regularity or singularity of the function again identified by its name.

The input wave $j_l(kr)$ satisfies both equations of (2.36). Thus, there must exist expansions (summation

[8] We are treating a BSE at definite E and l and the basis spans the family of solutions for those quantum members, not the whole function space.

convention again)

$$j_l(kr)=r^{(\nu)}(x,\theta)C_\nu(k)=C^\nu(k)r_{(\nu)}(x,\theta)\,. \tag{3.4}$$

Since $j_l=\tilde{j}_l$, therefore $C_\nu(k)=C^\nu(k)$. We have used two notations for one quantity to keep the summation indices fluent. In general, the $r^{(\nu)}(x,\theta)$ alone are a complete basis for solutions regular for all finite x including $x=0$.

Now consider a complete set of functions $h^{(\nu)}(x,\theta)$ regular at $x=\infty$. These may be defined by

$$h^{(\nu)}(x,\theta)=s^{(\nu)}+r^{(\mu)}X_\mu{}^\nu\,. \tag{3.5a}$$

The matrix $X_\mu{}^\nu$, to be obtained when $r^{(\nu)}$ and $s^{(\nu)}$ are known explicitly, tells how much of the $r^{(\mu)}$ must be added to each $s^{(\nu)}$ to cancel off the part of $s^{(\nu)}$ which is singular at infinity. Taking transposes, we have

$$h_{(\nu)}(x,\theta)=\tilde{h}^{(\nu)}(x,\theta)=s_{(\nu)}+\tilde{X}_\nu{}^\mu r_{(\mu)}\,, \tag{3.5b}$$

where, for the sake of fluent indices, we put

$$\tilde{X}_\nu{}^\mu=X_\mu{}^\nu\,. \tag{3.6}$$

The BSE under consideration is real. We can, and shall, suppose that the boundary conditions at the origin are real. Hence the $s^{(\nu)}$ and $r^{(\nu)}$ are real. But above threshold the regular boundary condition at $x=\infty$ is not real, as it involves outgoing waves. We separate X and $\tilde{X}$ into real and imaginary parts:

$$X_\mu{}^\nu=\chi_\mu{}^\nu+i\zeta_\mu{}^\nu\,,\quad \tilde{X}_\nu{}^\mu=\tilde{\chi}_\nu{}^\mu+i\tilde{\zeta}_\nu{}^\mu\,, \tag{3.7}$$

where

$$\tilde{\chi}_\nu{}^\mu=\chi_\mu{}^\nu\,,\quad \tilde{\zeta}_\nu{}^\mu=\zeta_\mu{}^\nu\,. \tag{3.8}$$

The last in our list of functions are $z^{(\nu)}(x,\theta)$ and $z_{(\nu)}(x,\theta)=\tilde{z}^{(\nu)}(x,\theta)$; they are the real parts of the h's,

$$z^{(\nu)}(x,\theta)=s^{(\nu)}+r^{(\mu)}\chi_\mu{}^\nu\,, \tag{3.9a}$$

$$z_\nu(x,\theta)=s_{(\nu)}+\tilde{\chi}_\nu{}^\mu r_{(\mu)}\,, \tag{3.9b}$$

and are equal to the h's below threshold where $\zeta_\mu{}^\nu$ vanishes.

B. Bracket Relations

For arbitrary functions ϕ and ψ, we find, by putting $\theta\to\pi-\theta$ in (2.29), that

$$[\phi,\psi]=-[\tilde{\psi},\tilde{\phi}]. \tag{3.10}$$

This tells us, for example, that $[s_{(\mu)},s^{(\nu)}]$ is antisymmetric in ν and μ and that $[s_{(\mu)},r^{(\nu)}]=-[r_{(\nu)},s^{(\mu)}]$.

We shall, in fact, add to our specifications of the function basis the requirement

$$[s_{(\mu)},r^{(\nu)}]=-[r_{(\mu)},s^{(\nu)}]=\delta_\mu{}^\nu\,. \tag{3.11}$$

The channel functions, whose construction has been promised, will satisfy this condition. It is generally true that

$$[r_{(\mu)},r^{(\nu)}]=0\,, \tag{3.12}$$

because the bracket may be evaluated at $x=0$ where the functions and their derivatives are finite and the extra factor of x^3 secures the vanishing. We shall not suppose that $[s_{(\mu)},s^{(\nu)}]$ vanishes, because our channel functions will not have this property. If, however, the property were thought especially desirable, one could define new singular functions by

$$\sigma^{(\nu)}=s^{(\nu)}-\tfrac{1}{2}r^{(\rho)}[s_{(\rho)},s^{(\nu)}]\,, \tag{3.13a}$$

$$\sigma_{(\mu)}=s_{(\mu)}+\tfrac{1}{2}[s_{(\mu)},s^{(\lambda)}]r_{(\lambda)}\,, \tag{3.13b}$$

which by (3.10) are transposes of one another. Then

$$[\sigma_{(\mu)},\sigma^{(\nu)}]=[s_{(\mu)},s^{(\nu)}]-\tfrac{1}{2}\delta_\mu{}^\rho[s_{(\rho)},s^{(\nu)}]-\tfrac{1}{2}[s_{(\mu)},s^{(\lambda)}]\delta_\lambda{}^\nu=0\,, \tag{3.14}$$

and, moreover,

$$[\sigma_{(\mu)},r^{(\nu)}]=-[r_{(\mu)},\sigma^{(\nu)}]=\delta_\mu{}^\nu\,. \tag{3.15}$$

Thus, the families $r^{(\nu)}$, $\sigma^{(\nu)}$ and $\sigma_{(\mu)}$, $r_{(\mu)}$ would be bi-orthogonal sets with the simplest bracket relations.

It is also true that

$$[h_{(\mu)},h^{(\nu)}]=0\,, \tag{3.16}$$

in virtue of the property of regularity at ∞ which the h's share. This is less easy to establish from the bracket definition. The family of $K^{(\alpha,i)}$, $K_{(\alpha,i)}$ which satisfied a truncated BSE had the vanishing bracket property, as shown in MBS I. These functions behaved like $x^{-3/2}e^{-\lambda_{\alpha,i}x}$ as $x\to\infty$, and since there was no possibility of canceling when $[K_{(\alpha,i)},K^{(\beta,i)}]$ was formed, the constancy of the bracket for all x implied its vanishing. Now the h's that we shall construct as channel functions will have truncated versions which will be explicit combinations of the K's. Hence the brackets among truncated h's will vanish. As the truncation parameter goes to infinity, the truncated h's will converge to the true h's, even though the K's themselves do not converge to anything. Thus (3.16) is established for these h's. It is then established generally, for any function regular at infinity is a linear combination of these h's.

By (3.5) and (3.9), we have the decomposition of h into real and imaginary parts:

$$h^{(\nu)}=z^{(\nu)}+ir^{(\mu)}\zeta_\mu{}^\nu\,, \tag{3.17a}$$

$$h_{(\nu)}=z_{(\nu)}+i\tilde{\zeta}_\nu{}^\mu r_{(\mu)}\,. \tag{3.17b}$$

This resembles the Bessel relation

$$ih_l^{(1)}(kr)=-n_l(kr)+ij_l(kr)\,. \tag{3.18}$$

Next, substitute (3.5) into (3.16) to get

$$[s_{(\mu)},s^{(\nu)}]=X_\mu{}^\nu-\tilde{X}_\mu{}^\nu=X_\mu{}^\nu-X_\nu{}^\mu\,. \tag{3.19}$$

The real and imaginary parts of (3.19) read

$$[s_{(\mu)},s^{(\nu)}]=\chi_\mu{}^\nu-\tilde{\chi}_\mu{}^\nu=\chi_\mu{}^\nu-\chi_\nu{}^\mu \tag{3.20}$$

and

$$0=\zeta_\mu{}^\nu-\tilde{\zeta}_\mu{}^\nu=\zeta_\mu{}^\nu-\zeta_\nu{}^\mu\,; \tag{3.21}$$

i.e., $\zeta_\mu{}^\nu$ is symmetric in μ and ν.

Hence, using (3.20) we have, from (3.9), that

$$[z_{(\mu)},z^{(\nu)}]=0. \tag{3.22}$$

Also, it is clear from (3.9), (3.5), and (3.11) that

$$[h_{(\mu)},r^{(\nu)}]=-[r_{(\nu)},h^{(\mu)}]=\delta_\mu{}^\nu, \tag{3.23}$$

$$[z_{(\mu)},r^{(\nu)}]=-[r_{(\nu)},z^{(\mu)}]=\delta_\mu{}^\nu. \tag{3.24}$$

Finally, observing that taking complex conjugates

$$h_{(\mu)}{}^*=z_{(\mu)}-i\zeta_\mu{}^\nu r_{(\nu)}=h_{(\mu)}-2i\zeta_\mu{}^\nu r_{(\nu)}, \tag{3.25}$$

and noting the symmetry of ζ, (3.21), we have the last bracket relations of this series,

$$[h_{(\mu)}{}^*,h^{(\nu)}]=-[h_{(\nu)},h^{(\mu)*}]=2i\zeta_\mu{}^\nu. \tag{3.26}$$

We have then three different bases of functions to work with, the (r,s) basis, the (r,z) basis, and the (r,h) basis, of which the first two are real and the last two satisfy the simplest bracket relations for bi-orthogonal sets. The ingredients of the theory so far are the functions themselves, the coefficients C_μ of (3.4), and the matrix X. The imaginary part of X, namely ζ, is simply expressible in terms of the C_μ, as will be seen below.

C. Free Green's Functions

The partial-wave Green's function[9] obeys

$$\mathfrak{D}_1\mathfrak{D}_2G(x,\theta;x'\theta')=\frac{\delta(x-x')\delta(\theta-\theta')}{x^3\sin^2\theta}. \tag{3.27}$$

It has already been exhibited in several representations, namely, MBS I, Eqs. (5.12) and (5.23).

Its expansion in terms of basis functions is done as follows: Because of the boundary conditions, we have

$$\begin{aligned}G(x,\theta;x',\theta')&=r^{(\nu)}(x,\theta)a_\nu(x',\theta'), \quad x<x'\\ &=h^{(\nu)}(x,\theta)b_\nu(x',\theta'), \quad x>x'\end{aligned} \tag{3.28}$$

where $a_\nu(x',\theta')$ and $b_\nu(x',\theta')$ are coefficients to be determined. We see that

$$[r_{(\nu)},G]_0=0, \quad [h_{(\nu)},G]_\infty=0. \tag{3.29}$$

Here, and in the brackets to follow, G always means $G(x,\theta;x'\theta')$, and the primed variables are merely parameters. There are two ways to calculate $[r_{(\nu)},G]_\infty$ and $[h_{(\nu)},G]_0$. Firstly, from (3.28), we get

$$[r_{(\nu)},G]_\infty=-b_\nu(x',\theta'), \quad [h_{(\nu)},G]_0=a_\nu(x',\theta). \tag{3.30}$$

Or, multiply (3.27) by $r_{(\nu)}(x)$; multiply $\mathfrak{D}_1\mathfrak{D}_2r_{(\nu)}(x)=0$ by G and apply (2.37) and (3.29) to get

$$[r_{(\nu)},G]_\infty=r_{(\nu)}(x') \tag{3.31}$$

and, similarly,

$$[h_{(\nu)},G]_0=h_{(\nu)}(x'). \tag{3.32}$$

[9] In this section, the superscript (l) carried by the partial-wave Green's function, and its real and imaginary parts in Sec. II and MBS I, is dropped.

Hence,

$$\begin{aligned}G(x,\theta;x',\theta')&=-r^{(\nu)}(x,\theta)h_{(\nu)}(x',\theta'), \quad x<x'\\ &=-h^{(\nu)}(x,\theta)r_{(\nu)}(x',\theta'), \quad x>x'.\end{aligned} \tag{3.33}$$

We define the transpose Green's function $\tilde{G}$ following (3.1);

$$\tilde{G}(x,\theta;x',\theta')=G(x,\pi-\theta;x',\pi-\theta'), \tag{3.34}$$

that is, *without* transposing the order of the arguments. Then $\tilde{G}$ satisfies the same boundary conditions as G and

$$\tilde{\mathfrak{D}}_1\tilde{\mathfrak{D}}_2G=\frac{\delta(x-x')\delta(\theta-\theta')}{x^3\sin^2\theta}. \tag{3.35}$$

We have the explicit form

$$\begin{aligned}\tilde{G}(x,\theta;x',\theta')&=-r_{(\nu)}(x,\theta)h^{(\nu)}(x',\theta'), \quad x<x'\\ &=-h_{(\nu)}(x,\theta)r^{(\nu)}(x',\theta'), \quad x>x'.\end{aligned} \tag{3.36}$$

and note the relevant bracket relations

$$[\tilde{G},r^{(\nu)}]_0=0, \quad [\tilde{G},h^{(\nu)}]_\infty=0, \tag{3.37}$$

and

$$[\tilde{G},r^{(\nu)}]_\infty=-r^{(\nu)}(x',\theta'), \quad [\tilde{G},h^{(\nu)}]_0=h^\nu(x',\theta'). \tag{3.38}$$

The symmetry of the Green's function

$$\tilde{G}(x',\theta';x,\theta)=G(x,\theta;x'\theta') \tag{3.39}$$

can be obtained by bracketing the differential equations or by inspection of (3.33) and (3.36). In physical terms, this symmetry reflects time reversal invariance.

The imaginary part of G is seen to be

$$G_I(x,\theta;x',\theta')=-r^{(\nu)}(x,\theta)\zeta_\nu{}^\mu r_{(\mu)}(x',\theta'). \tag{3.40}$$

But G_I is already given in terms of $j_l(kr)$ and $j_l(kr')$ in (2.58). Comparing with (3.4), we get, for E above threshold

$$\zeta_\nu{}^\mu=-\frac{k}{2E}C_\nu(k)C^\mu(k). \tag{3.41}$$

The calculation of $\chi_\mu{}^\nu$ depends more directly upon knowledge of the basis functions and will be done for the channel functions in Sec. V E and the Appendix.

Let $G_V(x,\theta;x'\theta')$ be the (real) interaction-dependent Green's function generated by the real free Green's function G_R, that is,

$$G_V=G_R+\int G_RVG_V. \tag{3.42a}$$

For reference purposes, we list some further properties:

$$G_V=G_R+\int G_VVG_R, \tag{3.42b}$$

$$(\mathfrak{D}_1\mathfrak{D}_2-V)G_V=1, \tag{3.43a}$$

$$(\tilde{\mathfrak{D}}_1\tilde{\mathfrak{D}}_2)'G_V-G_VV=1. \tag{3.43b}$$

The primed operator $(\mathfrak{D}_1\mathfrak{D}_2)'$ differentiates the *second* pair of coordinates of G_V in (3.43b).

The proofs of these relations are routine, as the ingredients for them have already been laid out.

IV. *N* AND *D* MATRICES

A. Definitions

Let $r_V{}^{(\nu)}(x,\theta)$ be solutions of the BSE with interaction

$$(\mathfrak{D}_1\mathfrak{D}_2 - V)r_V{}^{(\nu)} = 0 \tag{4.1}$$

indexed by ν, which comprise a basis for the solutions of (4.1) regular for all finite x including $x=0$. If V is sufficiently well behaved at the origin, one may also require that each $r_V{}^{(\nu)}$ satisfy the same condition at $x=0$ as the corresponding free solution $r^{(\nu)}$, but this is not essential for the present argument. Let us assume that the matrices

$$d_\mu{}^\nu(x) = [h_{(\mu)}, r_V{}^{(\nu)}]_x, \quad n_\mu{}^\nu(x) = [r_{(\mu)}, r_V{}^{(\nu)}]_x \tag{4.2}$$

converge to finite limits as $x \to \infty$. Then we define

$$D_\mu{}^\nu = \lim_{x\to\infty} d_\mu{}^\nu(x) = [h_{(\mu)}, r_V{}^{(\nu)}]_\infty, \tag{4.3a}$$

$$N_\mu{}^\nu = \lim_{x\to\infty} n_\mu{}^\nu(x) = [r_{(\mu)}, r_V{}^{(\nu)}]_\infty. \tag{4.3b}$$

The existence of a basis of $r_V{}^{(\nu)}$ functions with these properties is a limiation on the pathology of V at the origin and at infinity. An infinite range (e.g., Coulomb-type) interaction is excluded, but interactions which are both repulsive and singular at the origin need not be. As a practical matter, one may evaluate the brackets of (4.3) at a point x_∞ "outside" the range of V. That is, the part of V beyond x_∞ is supposed to be too small to affect the system to the desired order of accuracy.

Sometimes, one replaces V with λV, where λ is a varying parameter. The matrices may be designated as $D(E)$, $D(E,\lambda)$, etc., according to which parameters need to be emphasized.

The regularity of $r_V{}^{(\nu)}$ at the origin implies

$$[r_{(\mu)}, r_V{}^{(\nu)}]_0 = 0. \tag{4.4}$$

Invoking (2.39) and (4.4), we have another expression for N:

$$N_\mu{}^\nu = \int r_{(\mu)} V r_V{}^{(\nu)}. \tag{4.5}$$

One may attempt to write, in similar fashion,

$$D_\mu{}^\nu \overset{?}{=} [h_{(\mu)}, r_V{}^{(\nu)}]_0 + \int h_{(\mu)} V r_V{}^{(\nu)}, \tag{4.6}$$

but the terms on the right side of (4.6) may be separately infinite, at least for some values of μ and ν. Thus, the most regular interaction a one-particle-exchange model can deliver is that defined by scalar exchange. This interaction becomes the Yukawa potential in the nonrelativistic limit. But in the BSE, the resulting interaction goes like x^{-2} at $x \to 0$. There will also be functions $r_V{}^{(\nu)}$ that go like 1 and functions $h_{(\mu)}$ that go like x^{-2} in this limit, in a typical case. Then the integral in (4.6), which is over x^3dx, will diverge logarithmically at the origin, although $D_\mu{}^\nu$, of course, is well defined. This point underlies a subtle distribution between our approach to these matrices and that which is conventional in potential theory. See the discussion below in Sec. IV C under item (4).

The transposed function $r_{V(\nu)}(x,\theta) = r_V{}^{(\nu)}(x,\theta)$ obeys

$$(\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2 - \tilde{V})r_{V(\nu)}(x,\theta) = 0. \tag{4.7}$$

The transpose $\tilde{V}$—where V may be a nonlocal operator—is defined as in (3.1) or (3.34); i.e., $\theta \to \pi - \theta$ for all angles, but no transposition of the order of coordinates.

We define V as *symmetric* if

$$\tilde{V}\psi = \psi V \tag{4.8}$$

and *self-adjoint* if under complex conjugation

$$(\tilde{V}\psi)^* = \psi^* V. \tag{4.9}$$

Symmetry and self-adjointness correspond to the physical conditions of time reversal invariance and unitarity, as they do for the Schrödinger equation. Now both $[r_{V(\mu)}, r_V{}^{(\nu)}]_x$ and $[r_{V(\mu)}{}^*, r_V{}^{(\nu)}]_x$ vanish at $x=0$, because they are regular there. If V is symmetric, we have

$$\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2 r_{V(\mu)} = r_{V(\mu)} V. \tag{4.10}$$

Then by the argument following Eq. (2.41),

$$[r_{V(\mu)}, r_V{}^{(\nu)}] = 0 \tag{4.11}$$

for all x. Likewise, if V is self-adjoint,

$$\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2 r_{V(\mu)}{}^* = r_{V(\mu)}{}^* V, \tag{4.12}$$

and then

$$[r_{V(\mu)}{}^*, r_V{}^{(\nu)}] = 0 \tag{4.13}$$

for all x.

The N matrix is real if, as is convenient, we take the boundary conditions for $r^{(\nu)}$ and $r_V{}^{(\nu)}$, and hence the functions themselves, to be real. Let D be separated into real and imaginary parts:

$$D_\mu{}^\nu = (D_R)_\mu{}^\nu + i(D_I)_\mu{}^\nu. \tag{4.14}$$

Then

$$(D_R)_\mu{}^\nu = [z_{(\mu)}, r_V{}^{(\nu)}]_\infty. \tag{4.15}$$

Below threshold, $h_{(\mu)} = z_{(\mu)}$ so $D = D_R$ and $D_I = 0$. But above threshold, the imaginary part of $h_{(\mu)}$ is $\zeta_\mu{}^\nu r_{(\nu)}$, so

$$(D_I)_\mu{}^\nu = \zeta_\mu{}^\lambda N_\lambda{}^{(\nu)}. \tag{4.16}$$

B. Some Integral Equations

What integral equation does $r_V{}^{(\nu)}(x,\theta)$ satisfy? Notice, from (3.36), (4.4), and the definition of D, that

$$[\tilde{G}, r_V{}^{(\nu)}]_0 = 0, \quad [G, r_V{}^{(\nu)}]_\infty = -r^{(\mu)}(x')D_\mu{}^\nu. \tag{4.17}$$

Then, multiply (4.1) by $\tilde{G}(x,\theta;x',\theta')$, multiply (3.35) by $r_V{}^{(\nu)}(x)$, subtract, and apply (2.37) and (4.7) to get

$$r_V{}^{(\nu)}(x',\theta')=r^{(\mu)}(x',\theta')D_\mu{}^\nu + \int \tilde{G}(x,\theta x',\theta')V(x)r_V{}^{(\nu)}(x)x^3\sin^2\theta\,dxd\theta. \quad (4.18)$$

After interchange of x, θ and x', θ', the result reads

$$r_V{}^{(\nu)}=r^{(\mu)}D_\mu{}^\nu+\int GVr_V{}^{(\nu)}. \quad (4.19)$$

If the matrix $D_\mu{}^\nu$ is *not* singular, we can introduce

$$\phi^{(\nu)}(x,\theta)=r_V{}^{(\mu)}(x,\theta)(D^{-1})_\mu{}^\nu. \quad (4.20)$$

This function satisfies a more familiar type of integral equation:

$$\phi^{(\nu)}=r^{(\nu)}+\int GV\phi^{(\nu)}. \quad (4.21)$$

C. Elementary Properties of N and D Matrices

1. Dynamics

First suppose that the determinant

$$\Delta(E)=\det D_\mu{}^\nu(E) \quad (4.22)$$

vanishes at $E=E_0$. Then there exists a right eigenvector a_ν of $D_\mu{}^\nu(E_0)$ with zero eigenvalue, i.e.,

$$D_\mu{}^\nu(E_0)a_\nu=0. \quad (4.23)$$

Then by (4.16), the wave function $\psi^B=\sum r_V{}^{(\nu)}a_\nu$ satisfies the homogeneous equation

$$\psi^B=\int GV\psi^B \quad (4.24)$$

and is regular at ∞ as well as at the origin. Thus ψ^B is a bound state for energy E_0.

Second, it is clear from the representation of $j_l(kr)$ and the integral equation for $\phi^{(\nu)}$ that $\psi^{(l)}=\phi^{(\nu)}C_\nu(k)$ solves the scattering equation (2.55). Hence the scattering amplitude for the lth wave is

$$\begin{aligned} f_l&=(2E)^{-1}[j_l,\psi_l]_\infty \\ &=\frac{1}{2E}[C^{(\lambda)}(k)r_{(\lambda)},r_V{}^{(\mu)}]_\infty(D^{-1})_\mu{}^\nu C_\nu(k) \\ &=(2E)^{-1}C^\mu(k)(ND^{-1})_\mu{}^\nu C_\nu(k). \end{aligned} \quad (4.25)$$

Thus, the questions of dynamic interest are phrased in terms of the functions $r^{(\nu)}$, $h^{(\nu)}$, $r_V{}^{(\nu)}$, and the N and D matrices derived from them. We shall consider these questions more fully in Sec. VI.

2. Symmetry and Unitarity. B Matrix

In the asymptotic region, we have

$$\begin{aligned} r_V{}^{(\nu)}(x,\theta)&=-h^{(\lambda)}[r_{(\lambda)},r_V{}^{(\nu)}]_{x\to\infty}+r^{(\lambda)}[h_{(\lambda)},r_V{}^{(\nu)}]_{x\to\infty} \\ &=-h^{(\lambda)}N_\lambda{}^\nu+r^{(\lambda)}D_\lambda{}^{(\nu)}, \quad x\to\infty \end{aligned} \quad (4.26a)$$

and also

$$r_{V(\mu)}(x,\theta)=-h_{(\rho)}N_\rho{}^\mu+r_{(\rho)}D_\rho{}^\mu, \quad x\to\infty. \quad (4.26b)$$

Now set

$$B_\mu{}^\nu=(ND^{-1})_\mu{}^\nu. \quad (4.27)$$

Remembering (4.20), we have

$$\phi^{(\nu)}(x,\theta)=-h^{(\lambda)}B_\lambda{}^\nu+r^{(\nu)} \quad (x\to\infty), \quad (4.28a)$$

$$\phi_{(\mu)}(x,\theta)=-h_{(\rho)}B_\rho{}^\mu+r_{(\mu)} \quad (x\to\infty). \quad (4.28b)$$

Let us follow the route that in the nonrelativistic case led to (2.13) and (2.14). Equations (4.11) and (4.13) can be converted into

$$[\phi_{(\mu)},\phi^{(\nu)}]=0, \quad (4.29a)$$

$$[\phi_{(\mu)}{}^*,\phi^{(\nu)}]=0, \quad (4.29b)$$

in view of (4.20). If V is symmetric, we substitute (4.28) into (4.29a) to get the symmetry of B:

$$B_\mu{}^\nu=B_\nu{}^\mu. \quad (4.30)$$

If V is self-adjoint, substitute into (4.29b) and use (3.26) to get

$$(2i)^{-1}(B_\mu{}^\nu-B_\nu{}^{\mu*})=-B_\rho{}^{\mu*}\zeta_\rho{}^\lambda B_\lambda{}^\nu. \quad (4.31)$$

Or, if we combine (4.30) and (4.31) and express ζ in terms of the C's, Eq. (3.41), we get

$$\mathrm{Im}(B_\mu{}^\nu)=\frac{k}{2E}(B_\mu{}^\rho C_\rho(k))^*C^\lambda(k)B_\lambda{}^\nu. \quad (4.32)$$

As we see from (4.28a) with the significance of $\phi^{(\nu)}$ supplied by (4.21), $B_\mu{}^\nu$ is a kind of T matrix for scattering from channel ν to channel μ. Even though there is only one physical process permitted under the condition that E be below inelastic threshold, namely, two particles in and two particles out, our N/D formalism is that of a multichannel problem. But, according to (4.32) only one channel is "open," namely, the free two-particle state with wave function $j_l(kr)$ in the coordinate representation and wave function $C^\mu(k)$ in the representation by basis functions. If we set $t_l=kf_l$, we have, from (4.25),

$$t_l=\frac{k}{2E}C^\mu(k)B_\mu{}^\nu C_\nu(k). \quad (4.33)$$

Then, multiplying (4.32) by $C^\mu(k)C_\nu(k)$, we obtain

$$\mathrm{Im}t_l=|t_l|^2. \quad (4.34)$$

Thus the usual elastic unitarity statement for $t_l=e^{i\delta_l}\times\sin\delta_l$ is recovered.

3. Separability of $B=ND^{-1}$ at a Singularity

As we shall argue later, if the basis functions are set up wisely, only a finite number of them actually will be affected by the potential, and so, dynamical calculations can be made using not the infinite matrices N and D, but only certain finite dimensional submatrices. Thus, it is meaningful, even in a general discussion, to manipulate N and D as if they were finite dimensional.

Let us write the inverse of $D(E)$ as

$$\{D^{-1}(E)\}_\mu{}^\nu=\bar{D}_\mu{}^\nu(E)/\Delta(E)\,, \tag{4.35}$$

where $\Delta(E)$ is the determinant of $D_\mu{}^\nu(E)$, as already defined in (4.22) and $\bar{D}(E)$ is the cofactor matrix. Then

$$D_\mu{}^\lambda\bar{D}_\lambda{}^\nu=\bar{D}_\mu{}^\lambda D_\lambda{}^\nu=\delta_\mu{}^\nu\Delta\,. \tag{4.36}$$

Let us pass to the limit $E\to E_0$, where

$$\Delta(E_0)=0\,. \tag{4.37}$$

Then

$$D_\mu{}^\nu(E_0)\bar{D}_\nu{}^\rho(E_0)=\bar{D}_\nu{}^\rho(E_0)D_\rho{}^\mu(E_0)=0\,. \tag{4.38}$$

That is, for each ρ, the $\bar{D}_\nu{}^\rho(E_0)$ are the components of a right eigenvector of $D(E_0)$, and for each ν, the $\bar{D}_\nu{}^\rho(E_0)$ are the components of a left eigenvector of $D(E_0)$, always with eigenvalue zero. Let $\{f_\nu{}^{(n)}\}$ and $\{g_{(n)}{}^\rho\}$, $n=1, 2, \ldots$, be complete sets of such eigenvectors. The $\bar{D}_\nu{}^\rho(E_0)$ is a superposition in two ways:

$$\bar{D}_\nu{}^\rho(E_0)=\sum_n a(\rho,n)f_\nu{}^{(n)}=\sum_n g_{(n)}{}^\rho b(n,\nu)\,. \tag{4.39}$$

Taking the $\{f^{(n)}\}$ as orthonormal, we may project out to find that $a(\rho,n)$ is also a left eigenvector, so it can also be designated $g_{(n)}{}^\rho$ and $\bar{D}_\nu{}^\rho(E_0)$ is represented as $\sum_n f_\nu{}^{(n)}g_{(n)}{}^\rho$. Hence $(N\bar{D})_\mu{}^\nu$, which is symmetric in the limit $E\to E_0$, as it is symmetric for $E\neq E_0$, also has a separable form of this kind. We expect, in general, that $\Delta(E)$ has a simple zero for a single partial wave without any special symmetry, and then $D_\mu{}^\nu(E_0)$ has only one eigenvector of each kind with eigenvalue zero. Thus, the form of $B=ND^{-1}$ near $E=E_0$ would be

$$B_\mu{}^\nu(E)\approx\epsilon\gamma_\mu\gamma_\nu/(E_0-E)\,, \tag{4.40}$$

where ϵ is a positive or negative constant.

4. Comparison with Jost Functions of Potential Theory

We consider the matrix Q given by

$$Q_\mu{}^\nu=[h_{(\mu)},r_V{}^{(\nu)}]_0\,. \tag{4.41}$$

We have already remarked that the integral in (4.6) is expected to diverge, at least for some (μ), (ν); hence some components of Q will be infinite since the left-hand side of (4.6) is finite. But suppose this were not true. Then we could define a new function $\psi^{(\mu)}=r_V{}^{(\nu)}\times(Q^{-1})_\nu{}^\mu$ and new matrices $\dot{N}=NQ^{-1}$ and $\dot{D}=DQ^{-1}$. Equation (4.6) converts to

$$\dot{D}_\nu{}^\mu=\delta_\nu{}^\mu+\int h_{(\nu)}V\psi^{(\mu)} \tag{4.42}$$

and the important quantity ND^{-1} is identical to $\dot{N}\dot{D}^{-1}$. Also, the bound-state condition $\det D=0$ is equivalent to $\det\dot{D}=0$, assuming no pathology in Q.

We are now ready to compare our structure with that of the Jost functions in potential theory. We refer to the excellent review article by Newton,[10] in particular to Newton's Secs. 3 and 4 and especially his equations (3.6), (3.7), (4.2), (4.3), and (4.4). In the reduction of our formalism to a single nonrelativistic partial wave, there is only one function each of type $h(x)$, $r(x)$, and $\psi(x)$. These correspond to Newton's $w_l(kr)^*$, $v_l(kr)$, and $\phi_l(kr)$, respectively. His Eq. (4.3) defines the numerator Jost function $f_l(k)$ as $k^lW[f_l(kr),\phi_l(kr)]$, which is termed a Wronskian, perhaps unwisely.[11] By taking $r\to\infty$ in this "Wronskian" and also the complex conjugate, one gets the denominator Jost function $f_l(-k)=f_l(k)^*$ as, essentially,

$$f_l(-k)=[h,\psi]\,. \tag{4.43}$$

This identifies it with our $\dot{D}$, which is no longer a matrix. Then the complex conjugate of his (4.4) is the same as our (4.42). Newton's normalization for the Jost functions, or, equivalently, the D and N, was set by the definition of $\phi_l(kr)$ [$\equiv$ our $\psi(x)$] from an integral equation whose Green's function was not $-r(x<)h(x>)$, but rather

$$\begin{aligned} g(x,x')&=0\,, && x<x'\\ &=r(x)h(x')-h(x)r(x')\,, && x>x'\,. \end{aligned} \tag{4.44}$$

Then the existence of $\phi_l(kr)$ was proved by the Born series for this integral equation which converges, for regular potentials, regardless of the potential strength. Our approach, on the other hand, relies on the theory of differential equations to prove the existence of the $r_V{}^{(\nu)}$, once the boundary condition at $x=0$ is made clear.

In potential theory r_V and ψ go like x^{l+1} at the origin and h goes like x^{-l}, and one may easily verify that the "Wronskian" (4.41) depends only on these leading terms. If the potential is no more singular than x^{-1}, then one may also show that the x^l term of the wave function is unaffected by the potential. Thus, with care for normalization, Q defined by (4.41) is unity and the two theories are identical.

[10] R. Newton, J. Math. Phys. 1, 319 (1960); reprinted in *Quantum Scattering Theory*, edited by Marc Ross (Indiana U. P., Bloomington, Ind., 1963).

[11] Given an nth-order ordinary differential equation, one may distinguish two constructs: the Wronskian, which is an $n\times n$ determinant depending on a complete set of solutions, and a "bracket," formed from one solution of the equation and one solution of the transposed equation. For the radial Schrödinger equation, which is second order and symmetric, the two constructs have the same definition. Clearly, the bracket rather than the Wronskian is the relevant one for the present considerations.

The difference appears when one goes to more singular potentials, but where the quantum-mechanical structure is still valid, as it often is if the potential is repulsive. Then our version of Jost theory seems to remain valid—admittedly, we are not presenting a full discussion here—while the conventional theory breaks down because the integral equation with $g(x,x')$ contains divergent integrals. When one passes to the BSE, the conventional theory breaks down even for nonsingular potentials as observed by Haymaker and Blankenbecler.[3]

V. CHANNEL FUNCTIONS

A. Introduction

We shall now map out a classification of solutions to the BSE in terms of power-series expansions about $x=0$. The solutions will be called channel functions because the leading power of x in the expansion multiplies a single spherical harmonic; that is, the power series is initiated in a specific channel.

Because of the variety of functions and boundary conditions encountered, it pays to organize notation carefully and to indicate in advance the principal features of the channel functions to be defined.

First, we shall have "regular" functions $r^{(n,1)}(x,\theta)$ and "very regular" functions $r^{(n,2)}(x,\theta)$ for $n=l, l+1, l+2, \ldots$. Their leading power-series terms will be proportional to $x^n R_n(\theta)$ and $x^{n+2}R_n(\theta)$, respectively. Then there are "singular" functions and "very singular" functions $S^{(n,1)}$ and $S^{(n,2)}$ with leading terms $x^{-n}R_n(\theta)$ and $x^{-n-2}R_n(\theta)$. In the special case $n=0$, which occurs only for $l=0$, $S^{(0,1)}$ will begin with $(\ln x)\,R_0(\theta)$. The functions so defined will not contain, among their higher terms, any term which initiates one of the other functions.

Thus, the channel functions are labeled by index pairs, of which the first index lables the channel and the second takes on the values 1, 2 only. For compactness, we also use the single-index notation, (ν) for (n,i), (μ) for (m,j), etc., interchangeably with the double-index notation. Moreover, when (ν) stands for (n,j), $[\nu]$ will stand for the following:

$$\begin{aligned}[\nu]&=n \quad \text{if} \quad i=1\\ &=n+2 \quad \text{if} \quad i=2.\end{aligned} \tag{5.1}$$

For example, one may say that the leading powers of $r^{(\nu)}$ and $S^{(\nu)}$ are $x^{[\nu]}$ and $x^{-[\nu]}$, respectively.

Transpose solutions $r_{(\nu)}$ and $S_{(\nu)}$ are obtained from $r^{(\nu)}$ and $S^{(\nu)}$ as described in Sec. III. The brackets $[S_{(\mu)},r^{(\nu)}]$, etc., will be easy to compute because in the limit $x\to 0$ only the leading power-series terms contribute.

Then, a new set of singular solutions, $s^{(\nu)}$ and $s_{(\nu)}=\tilde{s}^{(\nu)}$, will be formed from linear combinations of the $S^{(\nu)}$ and the $S_{(\nu)}$, respectively. The object will be to have singular functions which form the simplest possible bracket relations with the regular functions, namely, those of (3.11).

We remark in passing that since $\mathfrak{D}_1 j_l(kr)=0$ and $\mathfrak{D}_2 j_l(kr)=0$, it follows that $\psi=(\partial/\partial k)\,j_l(kr)$ solves the free BSE. This example shows that a complete set of solutions to $\mathfrak{D}_1\mathfrak{D}_2\psi=0$ cannot be obtained solely from linear combinations of solutions to $\mathfrak{D}_1\psi=0$ and $\mathfrak{D}_2\psi=0$.

As a preliminary example, consider

$$(\Box^2-m_1^2)(\Box^2-m_2^2)\psi=0, \tag{5.2}$$

obtained from the BSE by setting $\omega_1=\omega_2=0$. Then the channels are uncoupled, and in the nth channel, there are four independent solutions, e.g.,

$$\mathcal{J}_n(m_1x,\theta),\quad \mathcal{J}_n(m_2x,\theta),\quad \mathcal{K}_n(m_1x,\theta),\quad \mathcal{K}_n(m_2x,\theta), \tag{5.3}$$

as indicated in Sec. II of MBS I. In the equal-mass case, these functions are equal in pairs, and the additional solutions may be taken as

$$I_{n+1}'(m_1x)R_n(\theta),\quad (-1)^n K_{n+1}'(m_1x)R_n(\theta). \tag{5.4}$$

These are found by considering the limit

$$\lim_{m_2\to m_1}\frac{1}{x}\frac{I_{n+1}(m_1x)-I_{n+1}(m_2x)}{m_1-m_2}=I_{n+1}'(m_1x) \tag{5.5}$$

and the similar limit for the K functions.

For the special case (5.2), the regular channel functions are given explicitly by

$$r^{(n,1)}=\tfrac{1}{2}[m_1^{-n}\mathcal{J}_n(m_1x,\theta)+m_2^{-n}\mathcal{J}_n(m_2x,\theta)], \tag{5.6a}$$

$$r^{(n,2)}=(m_1^2-m_2^2)^{-1}[m_1^{-n}\mathcal{J}_n(m_1x,\theta)-m_2^{-n}\mathcal{J}_n(m_2x,\theta)]. \tag{5.6b}$$

Their power series begin

$$r^{(n,1)}=\frac{x^n}{2^{n+1}(n+1)!}R_n(\theta)+O(x^{n+4}), \tag{5.7a}$$

$$r^{(n,2)}=\frac{x^{n+2}}{2^{n+3}(n+2)!}R_n(\theta)+O(x^{n+4}). \tag{5.7b}$$

Similar combinations of the $\mathcal{K}$ functions yield explicit formulas for the $S^{(\nu)}$:

$$S^{(n,1)}=\frac{(-1)^{n-l+1}2^{n-2}(n-1)!}{x^n}R_n(\theta)+O(x^{-n+2}),\quad n>0 \tag{5.7c}$$

$$S^{(0,1)}=\tfrac{1}{2}\ln x R_0(\theta)+O(x^2), \tag{5.7d}$$

$$S^{(n,2)}=\frac{(-1)^{n-l}2^n n!}{x^{n+2}}R_n(\theta)+O(x^{-n+2}). \tag{5.7e}$$

Equations (5.7) specify these solutions completely and may be taken as their defining boundary conditions, in preference to the definitions like (5.6a) and (5.6b). No special consideration for the equal-mass case is required.

The conditions for $r_{(\nu)}$ and $S_{(\nu)}$ are the same as (5.7) except that $R_n(\theta)$ becomes $R_n(\pi-\theta)=(-1)^{n-l}R_n(\theta)$. The $S^{(\nu)}$ contain, in addition to powers of x, terms proportional to $(\ln x)r^{(\nu)}$ because of the logarithms in the K functions.

B. Recursion Relations

Suppose that a solution to

$$\mathfrak{D}_1\mathfrak{D}_2\psi=0 \tag{5.8}$$

has the expansion

$$\psi=\sum\psi_n(x)R_n(\theta). \tag{5.9}$$

Substitute (5.9) into (5.8) and apply the recursion relations for the spherical harmonics enumerated in MBS I. Then (5.8) reduces to the following matrix equation for the ψ_n:

$$\mathfrak{D}_n{}^2\psi_n(x)=\sum_i \mathfrak{F}_{n,i}\psi_{n+i}(x), \tag{5.10}$$

where $\mathfrak{D}_n$ and $\mathfrak{F}_i$ are derivative operators and the sum goes over $i=0,\ \pm1,\ \pm2$. We have

$$\mathfrak{D}_n=\frac{d^2}{dx^2}+\frac{3}{x}\frac{d}{dx}-\frac{n(n+2)}{x^2}, \tag{5.11a}$$

$$\mathfrak{F}_{n,0}=(4\omega_1\omega_2\beta_n-2k^2)\mathfrak{D}_n-k^4, \tag{5.11b}$$

$$\mathfrak{F}_{n,+1}=2(\omega_1-\omega_2)A_{n+1}(\mathfrak{D}_n+k^2)\left[\frac{d}{dx}+\frac{(n+3)}{x}\right], \tag{5.11c}$$

$$\mathfrak{F}_{n,-1}=2(\omega_1-\omega_2)A_n(\mathfrak{D}_n+k^2)\left[\frac{d}{dx}+\frac{(1-n)}{x}\right], \tag{5.11d}$$

$$\mathfrak{F}_{n,+2}=4\omega_1\omega_2\alpha_{n+2}\times\left[\frac{d^2}{dx^2}+\frac{2n+7}{x}\frac{d}{dx}+\frac{(n+2)(n+4)}{x^2}\right], \tag{5.11e}$$

$$\mathfrak{F}_{n,-2}=4\omega_1\omega_2\alpha_n\left[\frac{d^2}{dx^2}+\frac{3-2n}{x}\frac{d}{dx}+\frac{n(n-2)}{x^2}\right]. \tag{5.11f}$$

For each n, write

$$\psi_n(x)=\sum_{a=a_0}^{\infty}[C(n,a)x^a+C_L(n,a)x^a\ln x] \tag{5.12}$$

and substitute this into (3.10) to obtain recursion relations for the coefficients $C(n,a)$ and $C_L(n,a)$. The procedure is straightforward and clearly explained in Ref. 3, Chap. 4. There will be a family of solutions regular at the origin. These will not have any terms in $(\ln x)$. Then there will be a family of singular solutions, each having first powers in $\ln x$.

The indicial equation comes from the fourth-order terms (e.g., d^4/dx^4, $x^{-1}d^3/dx^3$) in (5.10) as applied to the lead term of the series (5.10); it is

$$(a_0-n)(a_0+n)(a_0-n-2)(a_0+n+2)=0. \tag{5.13}$$

This fixes the possible powers x^{a_0} which may initiate a series solution in the nth channel. It matches the prescriptions of (5.7) exactly because the indicial equation is independent of ω_1 and ω_2.

Hence, the defining conditions for the channel functions $r^{(\nu)}$ and $S^{(\nu)}$ as $x\to0$ are given correctly—and with factorical coefficients which will prove to be reasonable by Eq. (5.7).

Substituting (5.12) into (5.10) and equating coefficients of powers of x^{a-4} and $x^{a-4}\ln x$ yields the recursion relations

$$\begin{aligned}f_{00}C(n,a)=&-h_{00}C_L(n,a)\\&+\sum_{i=0,\pm2}[f_iC(n+i,a-2)+h_iC_L(n+i,a-2)]\\&-k^4C(n,a-4)\\&+\sum_{i=\pm1}[f_iC(n+i,a-1)+h_iC_L(n+i,a-1)\\&+f_i'C(n+i,a-3)+h_i'C_L(n+i,a-3)]\end{aligned} \tag{5.14a}$$

and

$$\begin{aligned}f_{00}C_L(n,a)=&\sum_{i=0,\pm2}f_iC_L(n+i,a-2)\\&+\sum_{i=\pm1}f_iC_L(n+i,a-1)-k^4C_L(n,a-4).\end{aligned} \tag{5.14b}$$

The f and h coefficients, dependent on n and a, are

$$f_{00}=(a^2-n^2)[a^2-(n+2)^2], \tag{5.15a}$$

$$h_{00}=4a(a^2-n^2-2n-2), \tag{5.15b}$$

$$f_2=4\omega_1\omega_2\alpha_{n+2}(a+n+2)(a+n), \tag{5.15c}$$

$$f_1=2(\omega_1-\omega_2)A_{n+1}(a+n)(a-n-2)(a+n+2), \tag{5.15d}$$

$$f_0=(4\omega_1\omega_2\beta_n-2k^2)(a-n-2)(a+n), \tag{5.15e}$$

$$f_{-1}=2(\omega_1-\omega_2)A_n(a-n)(a+n)(a-n-2), \tag{5.15f}$$

$$f_2=4\omega_1\omega_2\alpha_n(a-n-2)(a+n), \tag{5.15g}$$

$$f_1'=2(\omega_1-\omega_2)k^2A_{n+1}(a+n+2), \tag{5.16a}$$

$$f_{-1}'=2(\omega_1-\omega_2)k^2A_n(a-n), \tag{5.16b}$$

$$h_2=4\omega_1\omega_2\alpha_{n+2}(2a+2n+2), \tag{5.16c}$$

$$h_1=2(\omega_1-\omega_2)A_{n+1}(3a^2-n^2+2an-4n-4), \tag{5.16d}$$

$$h_0=(4\omega_1\omega_2\beta_n-2k^2)(2a-2), \tag{5.16e}$$

$$h_{-1}=2(\omega_1-\omega_2)A_n(3a^2-n^2-2an-4a), \tag{5.16f}$$

$$h_{-2}=4\omega_1\omega_2\alpha_n(2a-2n-2), \tag{5.16g}$$

$$h_1'=2(\omega_1-\omega_2)k^2A_{n+1}, \tag{5.16h}$$

$$h_{-1}'=2(\omega_1-\omega_2)k^2A_n. \tag{5.16i}$$

To compute the series for $r^{(n,1)}$, assign the lead coefficient $C(n,a_0)$ with $a_0=n$ equal to $[2^{n+1}(n+1)!]^{-1}$ in accordance with (5.7a). The coefficient $C(n,a_0+2)$, as well as all other coefficients which can initiate other

solutions, e.g., $C(n+1, a_0+1)$, are zero for $r^{(n,1)}$. All the C_L are zero for the regular solutions. Then (5.14a) prescribes the values of all $C(n',a)$ for a $>a_0$ and all n'. The procedure for $r^{(n,2)}$ begins with $C(n,a_0)$, where $a_0=n+2$ and is similar.

The calculation of the singular functions follows the same lines. The lead coefficients for $S^{(n,1)}$ and $S^{(n,2)}$ are $C(n,a_0)$ with $a_0=-n$ and $a_0=-(n+2)$, respectively, and they are defined by (5.7a). In the exceptional case $n=0$, the lead coefficient is $C_L(0,0)$, as indicated in (5.7d). The recursion process (5.14a) for the singular functions eventually brings us to index values n', a', where $a'=n'$ or $a'=n'+2$. These are the lead positions for the regular functions and $f_{00}=0$. Then $C(n',a')$ is taken as zero and (5.14a) defines instead the earliest logarithm coefficient $C_L(n',a')$ for channel n'. There is such an earliest logarithm coefficient for each n'. All further C_L's are determined recursively from (5.41b). After they are known, the remaining C's are determined from (5.14a).

The $r^{(\nu)}$ and $S^{(\nu)}$ have now been defined by recursion relations for their series coefficients. Computing time for numerical calculations in typical cases is of the order of seconds or less. Details of convergence and illustrative functional values will be presented elsewhere.

C. Brackets for Channel Functions

The transposed channel functions $r_{(\nu)}$ and $S_{(\nu)}$, defined as in (3.3) and satisfying the transposed free BSE, satisfy the same boundary conditions (5.7), except that each entry has an extra factor of $(-1)^{n-l}$. The recursion relations above may be taken over directly for the transposed functions provided that the factors $(\omega_1-\omega_2)$ that appear in (5.11), (5.15), and (5.16) are reversed in sign.

Now, we compute brackets via (3.24). The brackets among regular solutions vanish, of course.

The evaluation of $[S_{(\nu)},r^{(\mu)}]$ is simplified by the observation that in the limit $x\to 0$, only the initial terms of the power series, as given in (5.7), can contribute to a nonzero result. Direct calculation shows that

$$[S_{(\nu)},r^{(\nu)}]=1 \quad \text{(no sum on } \nu). \tag{5.17}$$

The only other nonvanishing brackets are

$$[S_{(n,2)},r^{(n-2,2)}]=[S_{(n,1)},r^{(n-2,1)}]=-4\omega_1\omega_2\alpha_n, \tag{5.18a}$$

$$\begin{aligned}[S_{(n,1)},r^{(n-1,1)}]&=[S_{(n-1,2)},r^{(n,1)}]=[S_{(n,2)},r^{(n-1,2)}]\\&=k^{-2}[S_{(n,2)},r^{(n-1,1)}]\\&=2(\omega_2-\omega_1)A_n,\end{aligned} \tag{5.18b}$$

$$[S_{(n,2)},r^{(n,1)}]=-4\omega_1\omega_2\beta_n+2k^2. \tag{5.18c}$$

Now we define

$$s_{(\nu)}=S_{(\nu)}-\sum_{\mu,\mu\neq\nu}[S_{(\nu)},r^{(\mu)}]s_{(\mu)}, \tag{5.19a}$$

$$s^{(\nu)}=\tilde{s}_{(\nu)}. \tag{5.19b}$$

The sum over μ is, of course, limited to terms where $[S_{(\nu)},r^{(\mu)}]$ is nonvanishing, as indicated in (5.18). Equation (5.19a) is to be used recursively. One begins by evaluating $s_{(n,1)}$ in terms of $s_{(m,1)}$ for m values less then n; then goes on to $s_{(n,2)}$ in terms of already known functions $s_{(m,1)}$ and $s_{(p,2)}$ for $p<n$. Then one can verify, by mathematical induction, that

$$[s_{(\nu)},r^{(\mu)}]=-[r_{(\nu)},s^{(\mu)}]=\delta_\nu{}^\mu. \tag{5.20}$$

In the equal-mass case, $\omega_1-\omega_2=0$ and

$$r_{(\nu)}=(-1)^{[\nu]}r^{(\nu)}; \quad s_{(\nu)}=(-1)^{[\nu]}s^{(\nu)}. \tag{5.21}$$

D. Completeness

Suppose that $\psi=\sum f_n(x)R_n(\theta)$ solves the free BSE. Then the coefficients $f_n(x)$ are subject to the indicial conditions and recursion relations treated above. Hence we may put

$$f_n(x)=d_{n,1}\frac{x^n}{2^{n+1}(n+1)!}+d_{n,2}\frac{x^{n+2}}{2^{n+3}(n+2)!}+O(x^{n+4}). \tag{5.22}$$

It follows then that ψ has an expansion in the channel functions:

$$\psi(x,\theta)=r^{(\nu)}(x,\theta)d_\nu. \tag{5.23}$$

In other words, the family $r^{(\nu)}$ is a complete basis for the regular solutions of the BSE. A general formula for the coefficients in (3.28) is

$$d_\nu=[s_{(\nu)},\psi]. \tag{5.24}$$

As an example, consider the spherical Bessel function $j_l(kr)$:

$$j_l(kr)=\pi\sum_{n\geq l}(i)^{n-l}\frac{J_{n+1}(kx)}{kx}R_n(\tfrac{1}{2}\pi)R_n(\theta), \tag{5.25}$$

which, because of $R_n(\frac{1}{2}\pi)$, contains only terms with $n-l$ even. Therefore, the expansion of $j_l(kr)$ in channel functions is

$$j_l(kr)=\pi i^{-l}\sum_n R_n(\tfrac{1}{2}\pi)\times[(ik)^n r^{(n,1)}(x,\theta)+(ik)^{n+2}r^{(n,2)}(x,\theta)]. \tag{5.26}$$

In terms of the compressed notation with $(\nu)=(n,i)$, we have formula (3.4) with the coefficients evaluated as

$$C_\nu(k)=C^\nu(k)=\pi i^{-l}R_n(\tfrac{1}{2}\pi)(ik)^{[\nu]}. \tag{5.27}$$

A more general relation is [compare with MBS I, Eq. (2.19)]

$$j_l(-i\lambda_{\alpha,i}\sin\alpha\, r)e^{\lambda_{\alpha,i}\cos\alpha\, \tau}=\pi\sum R_n(\alpha)(\lambda_{\alpha,i})^{[\nu]}r^{(\nu)}(x,\theta). \tag{5.28}$$

We remark that the truncated equation $\langle\mathfrak{D}_1\rangle^N\times\langle\mathfrak{D}_2\rangle^N\psi=0$, to which we found explicit solutions

$I^{(\alpha,i)}$ and $K^{(\alpha,i)}$ in MBS I, can also be solved by truncated channel functions. There are only $2N$ of each type, as there are only N channels, and the recursion relations must be adjusted so as not to couple to the higher channels. Then there is an expansion of the type

$$K^{(\alpha,i)}=s^{(\nu)}c_\nu{}^{\alpha,i}+r^{(\nu)}d_\nu{}^{\alpha,i}, \tag{5.29}$$

where all the functions solve the truncated BSE.

Inverting the c matrix, we have

$$K^{(\alpha,i)}(c^{-1})_{\alpha,i}{}^{\nu}=s^{(\nu)}+r^{(\nu)}d_\nu{}^{\alpha,i}(c^{-1})_{\alpha,i}{}^{\nu}. \tag{5.30}$$

Equation (5.30) expresses a truncated $h^{(\nu)}$ in terms of a linear combination of $K^{(\alpha,i)}$. It is the expression referred to in Sec. III B and used to argue for the vanishing of the bracket $[h_{(\mu)},h^{(\nu)}]$ between the exact h functions.

E. Matrix $\chi_\mu{}^\nu$

The channel functions $r^{(\nu)}$ and $s^{(\nu)}$ and their transposes have been given explicit definitions and their brackets follow the scheme of Sec. III. The other functions $h^{(\nu)}$, $z^{(\nu)}$ and $h_{(\nu)}$, $z_{(\nu)}$ can then be calculated if $X_\mu{}^\nu=\chi_\mu{}^\nu+i\zeta_\mu{}^\nu$ is determined. In fact, $\zeta_\mu{}^\nu$ is already known in terms of the coefficients $C^\nu(k)$ by formula (3.41) and these coefficients have been evaluated in (5.23).

To obtain $\chi_\mu{}^\nu$, we return to the Green's function $G^l(x,\theta;x',\theta')$ and its real part, $G_R{}^{(l)}(x,\theta;x',\theta')$. Take the case $x>x'$. From (3.33), we have

$$G_R{}^l(x,\theta;x',\theta')=-s^{(\nu)}(x,\theta)r_{(\nu)}(x',\theta') -r^{(\nu)}(x,\theta)\chi_\nu{}^\mu r_{(\mu)}(x',\theta'). \tag{5.31}$$

Suppose, also, that the Green's function is known in the form

$$G^l(x,\theta;x',\theta')=\sum_{n,n'}g_{nn}(x,x')R_n(\theta)R_{n'}(\theta'). \tag{5.32}$$

Then, with $(\nu)=(n,i)$ and $(\mu)=(n',i')$, as before, we have $-\chi_\nu{}^\mu=$ real part of the coefficient of

$$\frac{x^{[\nu]}}{2^{[\nu]+1}(n+i)!}\frac{x'^{[\mu]}}{2^{[\mu]+1}(n'+i)!}(-1)^{n'-l} \tag{5.33}$$

in the power-series expansion of $g_{nn'}(x,x')$. For example, Eqs. (5.23) and (2.11) of MBS I give us

$$g_{nn'}(x,x')=-\sum_{j=1,2}\int\sin^2\alpha\,d\alpha \times R_n(\alpha)R_{n'}(\alpha)\frac{K_{n+1}(\lambda_{\alpha,j}x)}{\lambda_{\alpha,j}x}\frac{I_{n+1}(\lambda_{\alpha,j}x')}{\lambda_{\alpha,j}x'}\sigma_{\alpha,j}. \tag{5.34}$$

Let $\gamma=0.577\cdots$ denote the Euler constant and set

$$\eta(\nu)=\sum_{q=1}^{n+1}\frac{1}{q} \quad \text{if } i=1$$
$$=1+\sum_{q=1}^{n+2}\frac{1}{q} \quad \text{if } i=2. \tag{5.35}$$

Then, invoking the known power-series expansions of $I_{n+1}(z)$ and $K_{n+1}(z)$, we get $\chi_\nu{}^\mu=$ real part of

$$(-1)^{n'-l}\sum_{\omega=1,2}\int_0^\pi\sin^2\alpha\,d\alpha \times R_n(\alpha)R_m(\alpha)\sigma_{\alpha,j}(\lambda_{\alpha,j})^{[\nu]+[\mu]} \times[\ln(\tfrac{1}{2}\lambda_{\alpha,j})+\gamma-\tfrac{1}{2}\eta(\nu)]. \tag{5.36}$$

This formula is quite usable for the bound-state case. The integral is replaced by a Gaussian quadrature,

$$\int_0^\pi\sin^2\alpha\,d\alpha\,R_n(\alpha)R_m(\alpha)\to\sum_\alpha\bar{R}_n(\alpha)\bar{R}_m(\alpha), \tag{5.37}$$

as described in MBS I. The sum converges to the correct $\chi_\mu{}^\nu$ quite rapidly as the number of intervals is increased, provided that the energy is not extremely close to threshold. Equation (5.36) exhibits the same phenomenon seen for the Green's function in the previous paper. That is, Gaussian quadrature is not applicable for scattering because of singularities in the integrand, i.e., in $\sigma_{\alpha,j}$.

A more general method for calculating $\chi_\mu{}^\nu$ begins with the representation of $G^{(l)}(x,\theta;x',\theta')$ derived in MBS I and given in Eq. (A1) of the Appendix of this paper. This is converted into the form (5.32) and then the prescription (5.33) is applied. This technique is more general than the preceding one, and still practicable. Some calculational details are given in the Appendix. We have not been able to find a form for $\chi_\mu{}^\nu$ which is both compact and explicit.

VI. N/D METHOD IN COORDINATE SPACE

A. Scattering Wave Functions and Amplitudes

A natural approach to the calculation of wave functions, with or without the help of N and D matrices, is to match up a general solution of the BSE having the proper boundary conditions at the origin with a general solution satisfying boundary conditions at infinity. It is convenient to take the matching point $x=x_\infty$ outside the range of potential so that the "outside" solution obeys the free BSE. This is tantamount to using a "cutoff" interaction which vanishes for $x>x_\infty$. The calculation may be done for a series of increasing values of x_∞ until the results have converged to the order of accuracy desired.[12]

This approach to the Wick-rotated BSE cannot be applied routinely to scattering above inelastic threshold as evidenced by the unitarity relation (4.32) which accounts only for elastic scattering in intermediate states. We hope to generalize the present approach later on.

[12] All components of N and D are necessarily finite in a cutoff calculation, but some, depending on the asymptotic behavior of V, may diverge as the cutoff goes to infinity. This does not affect the use of the N and D matrices for the calculation of the phase shift or other physical quantities below inelastic threshold.

Suppose then that we seek the real scattering wave function $\phi(x,\theta)$ which satisfies

$$(\mathfrak{D}_1\mathfrak{D}_2-V)\phi=0. \tag{6.1}$$

Then ϕ also satisfies the real integral equation

$$\phi(x,\theta)=j(x,\theta)+\int G_R{}^{(l)}V\phi, \tag{6.2}$$

where the input wave $j(x,\theta)$ is any solution of the free BSE which is regular in the finite plane. In terms of solutions regular at the origin, we have the expansion

$$\phi(x,\theta)=r_V{}^{(\nu)}(x,\theta)a_\nu \tag{6.3}$$

with coefficients a_ν to be determined. The asymptotic form is

$$\phi(x,\theta)=-z^{(\nu)}(x,\theta)b_\nu+j(x,\theta)\quad(x\geq x_\infty), \tag{6.4}$$

with coefficients b_ν to be determined. In a practical computation, we approximate (6.3) and (6.4) as finite series, retaining the channel functions with the lower channel indices, then increase the number of functions until the results converge to the accuracy desired. Expressions for b_ν in terms of ϕ can be obtained from (6.2):

$$b_\nu=\int r_{(\nu)}V\phi=[r_{(\nu)},\phi]_{x_\infty}. \tag{6.5}$$

Alternatively, in terms of the real interacting Green's function $G_V{}^{(l)}$ defined by (3.42a) we have

$$b_\mu=\int r_\mu V_j+\int r_{(\mu)}VG_V{}^{(l)}Vj. \tag{6.6}$$

The coefficients a_ν and b_ν are calculated by matching (6.3) and (6.4) at $x=x_\infty$. That is, we have continuity of the wave function,

$$r_V{}^{(\nu)}(x,\theta)a_\nu=-z^{(\nu)}(x,\theta)b_\nu+j(x,\theta), \tag{6.7}$$

at $x=x_\infty$ plus continuity of all its derivatives as well. Thus, bracketing (6.7) successively with $z_\mu(x,\theta)$ and $r_\mu(x,\theta)$ at the matching point, we get

$$(D_R)_\mu{}^\nu a_\mu=[z_{(\mu)},j], \tag{6.8a}$$

$$N_\mu{}^\nu a_\nu=b_\mu. \tag{6.8b}$$

Then the coefficients and hence $\phi(x,\theta)$ for all x is determined from

$$a_\mu=(D_R{}^{-1})_\mu{}^\nu[z_{(\nu)},j], \tag{6.9a}$$

$$b_\mu=(ND_R{}^{-1})_\mu{}^\nu[z_{(\nu)},j]. \tag{6.9b}$$

For the physical scattering process, we take $j(x,\theta)=j_l(kr)=r^{(\nu)}C_\nu(k)$ and obtain the phase shift via (2.63):

$$(2E/k)\tan\delta_l=[j_l,z^\nu b_\nu+j_l]_\infty=C^\mu(k)b_\mu$$
$$=C^\mu(k)(ND_R{}^{-1})_\mu{}^\nu C_\nu(k). \tag{6.10}$$

B. Bound States

Calculation of the wave function $\psi(x,\theta)$ for a bound state at $E=E_0$ can be approached in two ways. Firstly, we can set up the representations

$$\psi(x,\theta)=r_V{}^{(\nu)}(x,\theta)\alpha_\nu \tag{6.11}$$

and

$$\psi(x,\theta)=-z^{(\nu)}(x,\theta)\beta_\nu \tag{6.12}$$

and match at $x=x_\infty$. Proceeding as above, we get, with $D=D_R$ below threshold,

$$D_\mu{}^\nu\alpha_\nu=0, \tag{6.13a}$$

$$N_\mu{}^\nu\alpha_\nu=\beta_\mu. \tag{6.13b}$$

Equation (6.13a) supplies the eigenvalue condition

$$\Delta(E_0)=\det D_\mu{}^\nu(E_0)=0 \tag{6.14}$$

to determine E_0. When E_0 is found, the solution of the homogeneous equation (6.13a) for the α_ν is obtained by standard algebraic methods. One may also replace V by λV, where λ is a variable, keep E constant, and use

$$\Delta(E,\lambda)=0 \tag{6.15}$$

to determine $\lambda=\lambda(E)$. There is a slight calculational advantage to obtaining λ as a function of E rather than the other way around as the functions $z^{(\nu)}(x,\theta)$ are independent of λ.

A second approach to the bound-state problem is via the solutions $\phi(x,\theta)$ to the inhomogeneous problem (6.12) for energies E in the neighborhood of E_0. This method also yields the normalization of the bound state and the residues of the scattering amplitudes at their pole at $E=E_0$, including, for example, the output coupling constant.

Let us suppose that the bound state at E_0 is nondegenerate. The interacting Green's function $G_V{}^{(l)}$ will have a pole at $E=E_0$ with residue proportional to $\psi(x,\theta)\hat{\psi}(x',\theta')$, where

$$\mathfrak{D}_1\mathfrak{D}_2\psi=V\psi,\quad \bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2\hat{\psi}=\hat{\psi}V, \tag{6.16}$$

or, equivalently,

$$\psi=GV\psi,\quad \hat{\psi}=\hat{\psi}VG. \tag{6.17}$$

This can be seen by multipling (3.42) and (3.43) by (E_0-E) and taking the limit $E\to E_0$. Let us also assume $\hat{\psi}V=\bar{V}\hat{\psi}$ (time reversal invariance) so that $\hat{\psi}$ is proportional to the transpose $\tilde{\psi}$ of ψ. Then we can write, for the region of the pole,

$$G_V{}^{(l)}(x,\theta;x',\theta')\approx\epsilon\frac{\psi(x,\theta)\tilde{\psi}(x',\theta')}{E_0{}^2-E^2},\quad E_0\approx E \tag{6.18}$$

where $\epsilon=\pm1$. This fixes the normalization of the bound-state wave function. Let $\psi(x,\theta)$ still be represented by (6.11) and (6.12).

Consider the function $\phi(x,\theta)$ defined in Sec. VI A above for an arbitrary energy E, and suppose E is near E_0. Consider the two formulas for b_ν given in (6.6) and (6.9b). These also have a pole at $E=E_0$.

Noting that

$$\int r_{(\mu)} V\psi = [r_{(\mu)},\psi] = \beta_\mu , \tag{6.19}$$

$$\int \bar{\psi} V j = -[\bar{\psi},j] = \beta_\nu [z_{(\nu)},j], \tag{6.20}$$

we have for the pole part of b_ν, by (6.6),

$$b_\mu \approx \frac{\epsilon \beta_\mu \beta_\nu [z_{(\nu)},j]}{E_0{}^2 - E^2}. \tag{6.21}$$

And, by (6.9b) and (4.35), the pole part is also

$$b_\mu \approx \frac{(N\bar{D})_\mu{}^\nu [z_\nu,j]}{(E-E_0)(d\Delta/dE)_{E_0}}. \tag{6.22}$$

Hence

$$\beta_\mu \beta_\nu = -\epsilon(2E_0)\left(\frac{d\Delta}{dE}\right)^{-1}_{E_0} (N\bar{D})_\mu{}^\nu , \tag{6.23}$$

which establishes the normalization of the β coefficients and hence of ψ. In order that (6.19) be meaningful, it is necessary that the right-hand side be symmetric in μ, ν, and have the product form. This was already established in Sec. IV; see especially (4.40).

The derivative of Δ is found conveniently enough by calculating Δ for values of E near E_0 and taking differences. As a practical matter, Δ is a smoother function of the imaginary momentum κ than of E, particularly near threshold. It is better to compute $d\Delta/d\kappa$ by differences and use

$$\frac{d\Delta}{dE} = \frac{d\Delta}{d\kappa}\left(\frac{dE}{d\kappa}\right)^{-1}, \tag{6.24}$$

where

$$\frac{dE}{d\kappa} = \frac{d\omega_1}{d\kappa} + \frac{d\omega_2}{d\kappa} = -\frac{E\kappa}{\omega_1\omega_2}. \tag{6.25}$$

To obtain the output coupling constant, put $j(x,\theta) = j_l(ikr)$ in (6.4) and (6.18) and apply (2.69):

$$\left(\frac{g^2}{4\pi}\right)_{\text{out}} = \lim_{E\to E_0} (E_0{}^2 - E^2)[j_l(ikr),\phi]_\infty$$
$$= 2E_0 C^\mu(i\kappa)(N\bar{D})_\mu{}^\nu C_\nu(i\kappa)\left(\frac{d\Delta}{dE}\right)^{-1}_{E_0}. \tag{6.26}$$

An alternative is, from (6.6),

$$g^2/4\pi = \epsilon\left(\int j_l V\psi\right)\left(\int \bar{\psi} V j_l\right) = \epsilon\{C^\mu(i\kappa)\beta_\mu\}^2. \tag{6.27}$$

The form factor $F(k)$,

$$F(k) = \int j_l(kr) V(x) \psi(x,\theta), \tag{6.28}$$

depending on the variable k and the bound-state wave function $\psi(x,\theta)$ for an energy E_0, can be obtained by integration once $\psi(x,\theta)$ is known.

Many papers on the structure of Bethe-Salpeter equations, including the earliest ones,[13] contained bound-state normalization conditions, different in form from one another and from the above, but equivalent in content.

We can make contact with some of these forms beginning with the identity (for E below threshold)

$$\int G_V{}^{(l)}\left[\frac{\partial}{\partial E}(\mathfrak{D}_1\mathfrak{D}_2 - V)\right] G_V{}^{(l)} = -\frac{\partial}{\partial E} G_V{}^{(l)}. \tag{6.29}$$

This can be verified by differentiating

$$G_V{}^{(l)}(\mathfrak{D}_1\mathfrak{D}_2 - V) G_V{}^{(l)} = G_V{}^{(l)} \tag{6.30}$$

with respect to E and pursuing the consequences. The key point in the verification is to note that

$$\int G_V{}^{(l)}(\mathfrak{D}_1\mathfrak{D}_2 - V)\frac{\partial G_V{}^{(l)}}{\partial E}$$
$$= \int\left[(\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2) G_V{}^{(l)} - G_V{}^{(l)} V\right]\frac{\partial G_V{}^{(l)}}{\partial E}, \tag{6.31}$$

which, by (3.43b), reduces to $\partial G_V{}^{(l)}/\partial E$. Equation (6.31) depends, in turn, on the integration-by-parts formula

$$\int G_V{}^{(l)}\mathfrak{D}_1\mathfrak{D}_2\frac{\partial G_V{}^{(l)}}{\partial E} = \int (\bar{\mathfrak{D}}_1\bar{\mathfrak{D}}_2) G_V{}^{(l)}\frac{\partial G_V{}^{(l)}}{\partial E}. \tag{6.32}$$

This holds because the difference between the two sides of (6.32) is expressible in terms of brackets at $x=0$ and $x=\infty$, each of which vanishes on account of the behavior of the Green's functions at these limits.

Substituting (6.18) into (6.29) and equating the double-pole terms, we get one of the canonical normalization expressions

$$\int \bar{\psi}\frac{\partial}{\partial E}(\mathfrak{D}_1\mathfrak{D}_2 - V)\psi = -\epsilon. \tag{6.33}$$

Another form is produced by introducing a variable coupling strength $\lambda = \lambda(E)$ so that

$$(\mathfrak{D}_1\mathfrak{D}_2 - \lambda(E) V)\psi = 0 \tag{6.34}$$

[13] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **14**, 175 (1955); S. Mandelstam, Proc. Roy. Soc. (London) **233**, 248 (1955); A. Klein and C. Zemach, Phys. Rev. **108**, 126 (1957).

is an identity for the bound-state wave function $\psi=\psi_E$ for a range of E values including E_0.

Taking the E derivative of

$$\int\tilde{\psi}(\mathfrak{D}_1\mathfrak{D}_2-\lambda V)\psi=0 \tag{6.35}$$

and integrating by parts where necessary, we get

$$\int\tilde{\psi}\left[\frac{\partial}{\partial E}(\mathfrak{D}_1\mathfrak{D}_2)-\lambda\frac{\partial V}{\partial E}\right]\psi-\frac{\partial\lambda}{\partial E}\int\tilde{\psi}V\psi=0.$$

Therefore the normalization (6.33) at $E=E_0$ is equivalent to

$$\left(\frac{\partial\lambda}{\partial E}\right)_{E_0}\int\tilde{\psi}V\psi=-\epsilon. \tag{6.36}$$

Equation (6.36) is easier to apply numerically than (6.33) because the integral is cut off by V, but (6.23), which has no integration, is easier still.

For an attractive interaction, $V>0$ and $\lambda>0$, and also $(\partial\lambda/\partial E)<0$. In the equal-mass case, $m_1=m_2$, the bound state has a definite parity under $t\to-t$ (time parity). Thus (6.36) tells us that $\epsilon=+1$ if $\tilde{\psi}=+\psi$ and $\epsilon=-1$ if $\tilde{\psi}=-\psi$. A bound state in the case $m_1\neq m_2$ can be imagined as continuously evolved from an $m_1=m_2$ problem, with λ and for E also evolving continuously. In this evolution ϵ cannot flip sign. Thus the notion of time parity can be assigned to states when $m_1\neq m_2$, not as a symmetry quantum number, but either by this evolution process or by the sign of ϵ. Another method, relating to the nodal lines of the wave function, will be discussed elsewhere.

VII. TRUNCATION METHOD

There is another way to handle the matching problem which, at first sight, appears more direct and simple than the N/D way, and deserves to be noted on that account.

Suppose that $r_V{}^{(\nu)}$, $z^{(\nu)}$, and j are approximated by spherical harmonic expansions which contain only N terms:

$$r_V{}^{(\nu)}(x,\theta)=\sum_{n=l}^{l+N-1}f_n{}^{(\nu)}(x)R_n(\theta), \tag{7.1a}$$

$$z^{(\nu)}(x,\theta)=\sum_{n=l}^{l+N-1}g_n{}^{(\nu)}(x)R_n(\theta), \tag{7.1b}$$

$$j(x,\theta)=\sum_{n=l}^{l+N-1}j_n(x)R_n(\theta). \tag{7.1c}$$

It is appropriate to consider only the $2N$ functions $r_V{}^{(\nu)}$ and the $2N$ functions $z^{(\nu)}$ whose power series are initiated in the first N channels. The matching condition at $x=x_\infty$ applies to the continuity of the coefficient functions and their first three derivatives in the relevant N channels. That is,

$$\sum_\nu\left[a_\nu\frac{d^i}{dx^i}f_n{}^{(\nu)}(x)+b_\nu\frac{d^i}{dx^i}g_n{}^{(\nu)}(x)\right]=\frac{d^i}{dx^i}j_n(x)\quad(x=x_\infty) \tag{7.2}$$

for $i=0,1,2,3$, and $l\leq n\leq l+N+1$. Thus there are $4N$ linear equations to determine $4N$ unknowns, the coefficients a_ν and b_ν. The calculation is done for a series of increasing values of N, until the results have converged to the desired order of accuracy.

One may formulate this program in such a way that the approximations (7.1) are exact solutions to the truncated equations $\langle\mathfrak{D}_1\mathfrak{D}_2\rangle^N\psi=V\psi$ and $\langle\mathfrak{D}_1\mathfrak{D}_2\rangle^N\psi=0$. Thus, $r_V{}^{(\nu)}$, $r^{(\nu)}$, and $s^{(\nu)}$ can be taken from the beginning as solutions to the truncated equations. If the power-series method is used to calculate them, the recursion relations themselves can be truncated. Then $z^{(\nu)}$ is computed from the $s^{(\nu)}$ and the $r^{(\nu)}$ with the aid of the matrix $\chi_\mu{}^\nu$. To find the approximation to $j(x,\theta)$ which solves the truncated free BSE, expand it in a series of $r^{(\nu)}$, and then replace the $r^{(\nu)}$ by their truncated versions.

In this way, one arrives at an approximation to a BSE result as an exact result of the truncated BSE. We found it perfectly feasible for calculation. The essential point, which is not obvious in a naive approach to the truncation approximation, is that the asymptotic boundary conditions which determine $z^{(\nu)}$ cannot be developed within the truncation approximation itself; rather the properties of the exact BSE, as contained in its Green's function, and the matrix $\chi_\mu{}^\nu$, must be exploited. Otherwise, one lands in the difficulties for the scattering problems which were exhibited in MBS I.

To treat the bound-state problem via (7.2), one puts the j_n equal to zero. Then the eigenvalue condition is

$$W(E)=0, \tag{7.3}$$

where $W(E)$ is the Wronskian of the set of solutions. That is, $W(E)$ is the determinant of the $4N\times4N$ matrix formed from the array of function coefficients $f_n{}^{(\nu)}(x_\infty)$, $g_n{}^{(\nu)}(x_\infty)$, and their first three derivatives.

The method outlined here combines two approximations in one. The single parameter N determines how many basis functions are used to approximate the wave function and how many channels are employed in the calculation of each basis function. The N/D method separates these two problems, and if used efficiently, requires substantially less computer time. For a given level of accuracy, the number of basis functions required in the N/D method can be much less than the number required in solving the truncated BSE.

VIII. PHASE METHOD

The N and D matrices have been defined in (4.3) in terms of basis functions for both the free and the interacting BSE. We now outline a method for computing N and D which does not require prior solution of the interacting BSE, but replaces it with a system of first-order ordinary differential equations. We call it the phase method because it generalizes the method of variable phase which Calogero has developed as a powerful tool in the analysis of nonrelativistic potential theory.[14]

It also may be regarded as an adaptation of the method of variation of constants in ordinary differential equations.[4] The assertion is sometimes made that there is no method of variation of constants in partial differential equations. But this does not hold in our case, where the boundary conditions are expressed in terms of one variable.

Let the interaction $V(x)$ be replaced by a cutoff interaction $V_y(x)$ which vanishes for $x\geq y$. Let the associated functions of the basis regular at the origin be $r_V{}^{(\nu)}(x,\theta;y)$. We are discussing here only local interactions. Then for $x\leq y$, we have

$$r_V{}^{(\nu)}(x,\theta,y)=r_V{}^{(\nu)}(x,\theta) \tag{8.1}$$

and for $x\geq y$,

$$r_V{}^{(\nu)}(x,\theta;y)=-h^\lambda(x,\theta)n_\lambda{}^\nu(y)+r^\lambda(x,\theta)d_\lambda{}^\nu(y). \tag{8.2}$$

The $r_V{}^{(\nu)}(x,\theta;y)$ have jumps at $x=y$ in their fourth-order derivatives in x, but will be continuous in their third- and lower-order derivatives. The coefficients $n_\lambda{}^\nu$ and $d_\lambda{}^\nu$ can be evaluated by bracketing the $r_V{}^{(\nu)}(x,\theta;y)$ with $h^{(\lambda)}$ and $r^{(\lambda)}$ for any x in the cutoff region, in particular, for $x=y$, where $r_V{}^{(\nu)}(x,\theta;y)$ coincides with $r_V{}^{(\nu)}(x,\theta)$.

Thus, we can drop the notion of cutoff and have, for all x,

$$r_V{}^{(\nu)}(x,\theta)=-h^\lambda(x,\theta)n_\lambda{}^\nu(x)+r^\lambda(x,\theta)d_\lambda{}^\nu(x), \tag{8.3}$$

where $n_\lambda{}^\nu(x)$ and $d_\lambda{}^\nu(x)$ have the definitions already given in (4.2). We also see that [compare with (2.37)]

$$n_\mu{}^\nu(x)-n_\mu{}^\nu(x_0)=\int_{x_0}^{x}x^2dx\int\sin^2\theta\,d\theta\,r_{(\mu)}Vr_V{}^{(\nu)}, \tag{8.4}$$

whence

$$\frac{d}{dx}n_\mu{}^\nu(x)=x^2\int\sin^2\theta\,d\theta\,r_{(\mu)}Vr_V{}^{(\nu)}. \tag{8.5a}$$

The corresponding equation for $d_\mu{}^\nu$ is

$$\frac{d}{dx}d_\mu{}^\nu(x)=x^2\int\sin^2\theta\,d\theta\,h_{(\mu)}Vr_V{}^{(\nu)}. \tag{8.5b}$$

Equations (8.3), (8.5), and (8.6) are a complete system. Let us define certain matrix functions of x, depending only on the basis of free solutions as follows:

$$E_\mu{}^\nu(x)=x^2\int\sin^2\theta\,d\theta\,r_{(\mu)}r^{(\nu)}, \tag{8.6a}$$

$$F_\mu{}^\nu(x)=x^2\int\sin^2\theta\,d\theta\,r_{(\mu)}h^{(\nu)}, \tag{8.6b}$$

$$G_\mu{}^\nu(x)=x^2\int\sin^2\theta\,d\theta\,h_{(\mu)}r^{(\nu)}=F_\nu{}^\mu(x), \tag{8.6c}$$

$$H_\mu{}^\nu(x)=x^2\int\sin^2\theta\,d\theta\,h_{(\mu)}h^{(\nu)}. \tag{8.6d}$$

Then, combining the above equations, we get

$$\frac{d}{dx}n_\mu{}^\nu(x)=V(x)[-F_\mu{}^\lambda(x)n_\lambda{}^\nu(x)+E_\mu{}^\lambda(x)d_\lambda{}^\nu(x)], \tag{8.7a}$$

$$\frac{d}{dx}d_\mu{}^\nu(x)=V(x)[-H_\mu{}^\lambda(x)n_\lambda{}^\nu(x)+G_\mu{}^\lambda(x)d_\lambda{}^\nu(x)]. \tag{8.7b}$$

The integration of this coupled, linear, first-order system is an interesting alternative to the solution of the fourth-order BSE. The functions $n_\mu{}^\nu(x)$ and $d_\mu{}^\nu(x)$ are defined from boundary conditions at $x=0$. These can be inferred from the early terms of the power series for the functions $r_{(\mu)}$, $h_{(\mu)}$, and $r_V{}^{(\nu)}$ once the behavior of $V(x)$ at the origin is specified. Above threshold, the equations can be made real, of course, by replacing $h_{(\mu)}$ with $z_{(\mu)}$ and $d_\mu{}^\nu$ with its real part. The values of N, D, and D_R are obtained as indicated in (4.3). The integration of (8.7) does not have to be carried beyond the range of the potential. We have not explored this system numerically except in the trivial case $m_1=m_2$, $E=0$, where it worked well.

To see the parallel to Calogero's method in potential theory, let us start from the integral equation for the Schrödinger wave function in the partial wave l:

$$\psi(r)=j_l(kr)+2mk\int_0^\infty j_l(kr_<)n_l(kr_>)\times V(r')\psi(r')r'^2dr'. \tag{8.8}$$

Then in terms of $u(r)=kr\psi(r)$ and the capped Bessel functions $\hat{\jmath}_l(z)=zj_l(z)$ and $\hat{n}_l(z)=zn_l(z)$, we have

$$u(r)=A(r)\hat{\jmath}_l(kr)-B(r)\hat{n}_l(kr), \tag{8.9}$$

[14] F. Calogero, *Variable Phase Approach to Potential Scattering* (Academic, New York, 1967).

where

$$A(r)=1+\frac{2m}{k}\int_r^\infty \hat{n}_l(r')V(r')u(r')dr', \quad (8.10a)$$

$$B(r)=\frac{-2m}{k}\int_0^r \hat{j}_l(r')V(r')u(r')dr'. \quad (8.10b)$$

Thus,

$$\frac{dA}{dr}=-\left(\frac{2m}{k}\right)V\hat{n}_l(A\hat{j}_l-B\hat{n}_l), \quad (8.11a)$$

$$\frac{dB}{dr}=-\left(\frac{2m}{k}\right)V\hat{j}_l(A\hat{j}_l-B\hat{n}_l), \quad (8.11b)$$

which are the analogs of (8.7). Finally, put

$$A(r)=R(r)\cos\delta(r), \quad (8.12a)$$

$$B(r)=R(r)\sin\delta(r), \quad (8.12b)$$

so that

$$\frac{d\delta}{dr}=\left(-\sin\delta\frac{dA}{dr}+\cos\delta\frac{dB}{dr}\right)R^{-1}$$

$$=-(2m/k)V(\cos\delta\,\hat{j}_l-\sin\delta\,\hat{n}_l)^2, \quad (8.13)$$

which is Calogero's equation for the variable phase shift.

IX. SUMMARY

The structure of the Bethe-Salpeter equation in the bound-state region and the scattering region below inelastic threshold has been analyzed in terms of brackets, N and D matrices, and complete bases of functions characterized by boundary conditions at the origin and at infinity. The calculation of bound-state and scattering data and wave functions has been outlined by several methods. The work provides the ingredients for the elementary kind of bootstrap where the forces are produced by one-particle exchange and self-consistency is required between input and output values of masses and couplings. More sophisticated bootstraps must await development of more sophisticated calculational techniques. The angular momentum quantum number l, which enters as a parameter in the definition of the coefficients A_n, α_n, and β_n, and delimits the range of the channel index n ($n=l+$integer), can be adjusted in value to define outgoing wave states for complex l, Regge trajectories, etc., by the methods described. Model calculations using these methods and further properties of Bethe-Salpeter systems will be described in a paper to follow.

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APPENDIX

From (5.12) of MBS I, we have

$$G^l(x,\theta;x',\theta')$$

$$=\int_{-\omega_2}^{+\omega_1}\frac{d\beta}{4E}e^{\beta(\tau-\tau')}\sum_{n=l}^{\infty}(n+1)^{-1}R_n(\theta)R_n(\theta')$$

$$\times[I_n(Qx_<)K_n(Qx_>)-I_{n+2}(Qx_<)K_{n+2}(Qx_>)], \quad (A1)$$

where $Q=(\beta^2-k^2)^{1/2}$. Also,

$$e^{\beta\tau}=\sum_{r=0}^{\infty}\frac{(\beta\cos\theta)^r}{r!}, \quad (A2)$$

$$\cos^r\theta\,R_n(\theta)=\sum_{m=\max\{l,(n-r)\}}^{(n+r)}\xi_{nm}{}^rR_m(\theta). \quad (A3)$$

The quantities $\xi_{nm}{}^r$ are defined recursively by

$$\xi_{nm}{}^0=\delta_{nm} \quad (A4a)$$

and from MBS I

$$\xi_{nm}{}^r=\xi_{n(m+1)}{}^{r-1}A_{(m+1)}+\xi_{n(m-1)}{}^{r-1}A_m. \quad (A4b)$$

Substituting these into (A1) and using the relation

$$\sum_{n=0}^{\infty}\sum_{m=\max\{l,(n-r)\}}^{(n+r)}=\sum_{m=l}^{\infty}\sum_{r=|n-m|}^{\infty}, \quad (A5)$$

it follows, if $g_{mm'}(x,x')$ is defined as in (5.31), that

$$g_{mm'}(x,x')=\sum_{n=l}^{\infty}\sum_{r=|n-m|}^{\infty}\sum_{r'=|n-m'|}^{\infty}\int_{-\omega_2}^{+\omega_1}\frac{d\beta}{4E}(n+1)^{-1}$$

$$\times\frac{(\beta x)^r(-\beta x')^{r'}}{r!r'!}\xi_{nm}{}^r\xi_{nm'}{}^{r'}(I_nK_n-I_{n+2}K_{n+2}). \quad (A6)$$

Consider the case $x>x'$. We have

$$K_n(Qx)I_n(Qx')-K_{n+2}(Qx)I_{n+2}(Qx')$$

$$=\sum_{q=0}^{\infty}\sum_{q'=0}^{\infty}[(x^{(n+2q)}\ln x)(x'^{(n+2q')})\rho_{qq'}{}^n(Q)$$

$$+(x^{(-n-2+2q)})(x'^{(n+2q')})\sigma_{qq'}{}^n(Q)]. \quad (A7)$$

The relation defines ρ and σ in terms of the coefficients in the power-series expansion of the Bessel functions. Substituting this in (A2) and using the relation

$$\sum_{n=l}^{\infty}\sum_{r=|n-m|}^{\infty}\sum_{r'=|n-m'|}^{\infty}\sum_{q=0}^{\infty}\sum_{q'=0}^{\infty}$$

$$=\sum_{i=(-m-2)}^{\infty}\sum_{i'=m'}^{\infty}\sum_{n=\mathrm{MIN}}^{\mathrm{MAX}}\sum_{q=0}^{\max}\sum_{q'=0}^{\max'}, \quad (A8)$$

$$
\begin{aligned}
i &= -n-2+r+2q,\\
i' &= n+r'+2q',\\
\max &= \tfrac{1}{2}(i+n+2-|n-m|)+\delta,\\
\max' &= \tfrac{1}{2}(i'-n-|n-m'|)+\delta,\\
\mathrm{MAX} &= \tfrac{1}{2}(i'+m')+\delta,\\
\mathrm{MIN} &= \text{the greater of } l,\ \tfrac{1}{2}(m-2-i)-\delta,
\end{aligned}
\tag{A9}
$$

where δ is either zero or $\frac{1}{2}$, depending on which choice makes the above limits integral, we find that

$$
\begin{aligned}
g_{mm'}(x,x') &= \text{terms in } (x^i \ln x)(x'^{i'})\\
&+ \sum_{i=-m-2}^{\infty}\sum_{i'=m'}^{\infty}(x^i)(x'^{i'})\sum_{n=\mathrm{MIN}}^{\mathrm{MAX}}\sum_{q=0}^{\max}\sum_{q'=0}^{\max'}\\
&\times \xi_{nm}{}^{r}\xi_{nm'}{}^{r'}\frac{(-1)^{r'}}{(n+1)}\int_{-\omega_1}^{+\omega_2}\frac{d\beta}{4E}\frac{\beta^{(r+r')}}{r!r'!}\sigma_{qq'}{}^{n}(Q).
\end{aligned}
\tag{A10}
$$

Now $-\chi_{(m,j)}{}^{(m',j')}$ is the real part of the coefficient of

$$
\frac{(\frac{1}{2}x)^{m+2(j-1)}(\frac{1}{2}x')^{m'+2(j'-1)}(-1)^{m'-l}}{2(m+j)!2(m'+j')!}
$$

in the above expansion, so (Re means real part)

$$
\begin{aligned}
-\chi_{(m,j)}{}^{(m',j')} &= \mathrm{Re}N_{mj}{}^{m'j'}\sum_{n=\mathrm{MIN}}^{\mathrm{MAX}}\sum_{q=0}^{\max}\sum_{q'=0}^{\max'}\\
&\times \xi_{nm}{}^{r}\xi_{nm'}{}^{r'}\frac{(-1)^{r'}}{(n+1)}\int_{-\omega_2}^{+\omega_1}\frac{d\beta}{4E}\frac{\beta^{r+r'}}{r!r'!}\sigma_{qq'}{}^{n}(Q),
\end{aligned}
\tag{A11}
$$

where

$$
\begin{aligned}
N_{mj}{}^{m'j'} &= (-1)^{m'-l}2^{(m+2j-1)}\\
&\times (m+j)!2^{(m'+2j'-1)}(m'+j')!
\end{aligned}
\tag{A12}
$$

and

$$
\begin{aligned}
i &= m+2(j-1),\quad i' = m'+2(j-1),\\
r &= m+2(j-1)+n+2-2q,\\
r' &= m'+2(j'-1)-n-2q',\\
\mathrm{MIN} &= l,\quad \mathrm{MAX} = m'+j'-1,\\
\max' &= \tfrac{1}{2}(m'-n-|n-m'|+2j-2),\\
\max &= \tfrac{1}{2}(m+n+2-|n-m|+2j-2).
\end{aligned}
\tag{A13}
$$

One further point is to be noted. All the integrals in (A11) are of the form

$$
\int_{-\omega_2}^{+\omega_1}d\beta\,\beta^i(\beta^2-k^2)^j \tag{A14}
$$

or

$$
\mathrm{Re}\int_{-\omega_2}^{+\omega_1}d\beta\,\beta^i(\beta^2-k^2)^j\ln(\beta^2-k^2), \tag{A15}
$$

where $r,\ j\geq 0$.

These integrals, after integration by parts and algebraic manipulation, can be reduced to three elementary integrals:

$$
\int_{-\omega_2}^{+\omega_1}\beta^i d\beta = \omega_1{}^{i+1}-(-\omega_2)^{i+1} \tag{A16}
$$

and

$$
\mathrm{Re}\int_{-\omega_2}^{+\omega_1}\frac{d\beta}{(\beta^2-k^2)} = \frac{1}{k}\ln\left[\frac{(\omega_1-k)(\omega_2-k)}{m_1m_2}\right], \tag{A17}
$$

$$
\mathrm{Re}\int_{-\omega_2}^{+\omega_1}\frac{\beta d\beta}{(\beta^2-k^2)} = \ln\left(\frac{m_2}{m_1}\right). \tag{A18}
$$

Below threshold, $k=i\kappa$ and the left-hand side of (A17) can be expressed as

$$
\frac{1}{\kappa}\tan^{-1}\left(\frac{\omega_1}{\kappa}\right)+\frac{1}{\kappa}\tan^{-1}\left(\frac{\omega_2}{\kappa}\right). \tag{A19}
$$