Neutrino Bremsstrahlung in an Intense Magnetic Field

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In this paper we present the neutrino luminosity due to a completely relativistic electron gas in the presence of an intense magnetic field. The neutrino emission rate is determined rigorously, under the assumption of the $V-A$ theory of universal weak Fermi interaction, by standard field-theoretical methods. It is shown that the neutrino radiation rate can be expressed as a function of temperature, density, and the field-strength parameter $\Theta=H/H_c$, where $H_c=m^2c^3/e\hbar=4.414\times10^{13}$ G. Comparisons are made with previous approximate results due to others, and the reasons for discrepancies are discussed.

1. INTRODUCTION

THE importance of weak interactions in stellar evolution problems has been the production of neutrinos; the reason is that the mean free path of a 1-MeV neutrino in lead is about 10^{18} cm, or one light year, and is much larger than the dimensions of a star ($\sim10^{11}$ cm). A general survey of neutrino processes important in astrophysics in ordinary stellar matter can be found in Chap. 6 of Ref. 1, where an exhaustive list of references is given. In general, neutrino processes are important in later stages of stellar evolution, when the core of the star has contracted considerably. The contraction of the stellar core generally will increase the strength of any magnetic field already present in stellar interiors in earlier stages. The presence of magnetic fields will give rise to new neutrino processes which are normally forbidden in vacuum, such as the emission of a pair of neutrinos by an otherwise free electron moving in a magnetic field. This process is analogous to the ordinary synchrotron radiation by a free electron in a magnetic field.

In this paper we calculate the neutrino production rate by the synchroton process (magnetic bremsstrahlung)

$$e^-\to e^-+\nu+\bar{\nu}. \tag{1}$$

Reaction (1) is forbidden by energy and momentum conservation laws when both electrons are strictly free. The presence of the magnetic field will cause electrons to move in circular orbits; therefore, the electrons will not be free and reaction (1) can take place. As has been extensively studied in a number of papers,[2] the effect of a magnetic field on the electron is to quantize its energy in the direction perpendicular to the field H, which is taken to be in the z direction. This quantization effectively replaces, in the expression for the electron energy, the momenta in the x and y directions by the field strength, i.e., $p_x^2+p_y^2\to n(H/H_c)$, with $n=0,1,2,\ldots$, and $H_c=m^2c^3/e\hbar=4.414\times10^{13}$ G. Such a quantization will leave p_z unaltered. Reaction (1) can therefore be visualized as a bremsstrahlung process or a decay of an electron from an initial state with quantum number n to a new one with n', through the emission of two neutrinos which carry away a certain amount of energy. In this paper we compute the energy-loss rate of process (1) in stellar matter, using the exact solution of the Dirac equation for an electron in an external magnetic field.

2. DIRAC EQUATION WITH MAGNETIC FIELD—TYPICAL INTEGRALS

The Dirac equation describing an electron of mass m, charge $e>0$ in an external electromagnetic potential A_μ has been discussed previously.[2] Here we only give essential details regarding this problem. The Dirac equation is

$$\left[\gamma_\mu\frac{\partial}{\partial x_\mu}+\frac{mc}{\hbar}+\frac{ie}{\hbar c}\gamma_\mu A_\mu\right]\psi_D=0. \tag{2}$$

In an external magnetic field $\mathbf{H}$, we have

$$A_4=0,\quad \mathbf{A}=\tfrac{1}{2}\mathbf{H}\times\mathbf{r}, \tag{3}$$

taking $\mathbf{H}$ in the z direction; then $H_x=H_y=0$, $H_z=H$, and

$$A_4=0,\quad A_x=-\tfrac{1}{2}yH,\quad A_y=\tfrac{1}{2}xH,\quad A_z=0. \tag{4}$$

Because of the nature of the problem, it is more convenient to work with a different ψ gauge transformed in the following way:

$$\psi=\psi_D e^{if(x,y)}, \tag{5}$$

where the function $f(x,y)$ is chosen in such a way that

$$A_4=0,\quad A_x=-yH,\quad A_y=A_z=0. \tag{6}$$

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[1] H. Y. Chiu, *Stellar Physics* (Blaisdell, Waltham, Mass., 1968).
[2] V. Canuto and H. Y. Chiu, Phys. Rev. **173**, 1210 (1968); **173**, 1220 (1968); **173**, 1229 (1968).

From Eq. (2), we see that

$$f(x,y)=-\tfrac{1}{2}(e/\hbar c)xyH. \quad (7)$$

As shown before,[2,3] the spinor ψ is given by

$$\psi(\mathbf{r},t)=e^{-i\eta Et/\hbar}\psi(\mathbf{r}),$$
$$\psi(\mathbf{r})=e^{i(k_x x+k_z z)}e^{-\xi^2/2}u_n(\xi),$$
$$u_n(\xi)=\begin{bmatrix} C_1\bar{H}_n(\xi)\\ C_2\bar{H}_{n-1}(\xi)\\ C_3\bar{H}_n(\xi)\\ C_4\bar{H}_{n-1}(\xi)\end{bmatrix}, \quad (8)$$

with

$$E=mc^2(1+x^2+2n\Theta)^{1/2}=mc^2\epsilon(x), \quad (9)$$

$$\begin{aligned} &n=0,1,2,\ldots, \quad x=p_z/mc,\\ &H_c=m^2c^3/e\hbar=4.414\times10^{13}\text{ G}, \quad \Theta=H/H_c,\\ &\xi=y\gamma^{1/2}+k_x/\gamma^{1/2}, \quad \gamma=\Theta\lambda_c^{-2}, \quad \lambda_c=\hbar/mc,\\ &C_1=aA, \quad C_2=SaB, \quad C_3=\eta SbA, \quad C_4=\eta bB,\\ &a=[\tfrac{1}{2}(1+\eta\epsilon^{-1})]^{1/2}, \quad b=[\tfrac{1}{2}(1-\eta\epsilon^{-1})]^{1/2}, \end{aligned} \quad (10)$$

$$A=\left[\frac{1}{2}\left(1+S\frac{x}{(x^2+2n\Theta)^{1/2}}\right)\right]^{1/2},$$
$$B=\left[\frac{1}{2}\left(1-S\frac{x}{(x^2+2n\Theta)^{1/2}}\right)\right]^{1/2}. \quad (11)$$

The two indices $\eta=\pm1$, $S=\pm1$ stand for positive and negative energies and the sign of the projection of the momentum component along the spin. $\bar{H}_n(x)$ are the Hermite polynomials normalized to 1:

$$\bar{H}_n(x)=\left(\frac{\gamma^{1/2}}{\pi^{1/2}2^n n!}\right)^{1/2}H_n(x). \quad (12)$$

For future reference, we will study here the integral

$$R_\mu=\int d^3r\,\psi_f^\dagger(\mathbf{r})\Gamma_\mu\psi_i(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}, \quad (13)$$

where $\Gamma_\mu=\gamma_4\gamma_\mu(1+\gamma_5)$. Substituting Eq. (8) into Eq. (13), we easily obtain

$$R_\mu=\frac{\delta[(p_x-p_x'-q_x)/\hbar]}{L_x}\frac{\delta[(p_z-p_z'-q_z)/\hbar]}{L_z}I_\mu, \quad (14)$$

$$I_\mu=\int_{-\infty}^{\infty}e^{-(\xi^2+\xi'^2)/2}u_{n'}^\dagger(\xi')\Gamma_\mu u_n(\xi)e^{iq_y y/\hbar}dy. \quad (15)$$

This integral is of the form

$$\langle n|n'\rangle\equiv\int_{-\infty}^{\infty}e^{-(\xi^2+\xi'^2)/2}\bar{H}_{n'}(\xi')\bar{H}_n(\xi)e^{iq_y y/\hbar}dy. \quad (16)$$

[3] N. P. Klepikov, Zh. Eksperim. i Teor. Fiz. 26, 19 (1954).

For $n\geqslant n'$, we denote $\langle n|n'\rangle$ by $I(n'|n)$. It can be shown that[2]

$$I(n'|n)=\Phi(n|n')(-1)^{n'}e^{-i(n-n')\phi}, \quad n\geqslant n' \quad (17)$$

with

$$\Phi(n|n')=\Phi(n'|n)=(n!n'!)^{-1/2}e^{-t/2}t^{(n+n')/2}\times{}_2F_0(-n',-n;-t^{-1}), \quad (18)$$

$$\phi=\tan^{-1}(q_y/q_x), \quad t=(q_x^2+q_y^2)/2\gamma\hbar^2, \quad (19)$$

$${}_2F_0(a,b;x)=1+\sum_{k=1}^{\infty}(a)_k(b)_k x^k/k!,$$
$$(a)_k=\prod_{s=1}^{k}(a+s-1). \quad (20)$$

The variable q_x enters in (16) because it has made use of the first of the δ functions in Eq. (14). For any n and n', the obvious generalization of Eq. (17) is $(n-n'=N)$

$$\langle n|n'\rangle=\Phi(n|n')[e^{-iN\phi}(-1)^{n'}\theta(n-n')+e^{iN\phi}(-1)^n\theta(n'-n)-(-1)^n\delta_n^{n'}]. \quad (21)$$

$\theta(x)$ is a step function such that

$$\theta(x)=1 \quad \text{for } x\geqslant0$$
$$=0 \quad \text{for } x<0.$$

3. PROCESS $e^-\to e^-+\nu+\bar{\nu}$—EQUATION FOR THE ENERGY LOSS

From the theory of weak interactions, we know that the Lagrangian for this process has the form[4]

$$\mathfrak{L}=\sum[\bar{\psi}_e(x)O_i\psi_e(x)][\bar{\psi}_\nu(x)F_i\psi_\nu(x)], \quad (22)$$

where

$$F_i=(1/\sqrt{2})O_i g_i(1+\gamma_5). \quad (23)$$

In agreement with the $V-A$ theory, the index i can be only V (vectorial, $O_V=\gamma_\mu$) and A (axial, $O_A=i\gamma_\mu\gamma_5$). The g_i's are the weak interaction constant, to be specified later. The wave function ψ_e is given by Eq. (8), while ψ_ν is simply the free-particle wave function

$$\psi_\nu(x)=(1/\sqrt{\Omega})e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}e^{-iE_\nu t/\hbar}u_\nu(p), \quad (24)$$

where

$$(\gamma_\mu p_\mu+m_\nu c/\hbar)u_\nu(p)=0.$$

Using Eq. (8), with $\eta_i=\eta_f=1$ and (24) in the expression for the S matrix

$$S=\frac{1}{\hbar c}\int d^4x\,\mathfrak{L}(x) \quad (25)$$

and integrating over t, we obtain

$$\begin{aligned} S_{fi}&=\Omega^{-1}\delta(E_i-E_f-E_\nu-E_{\bar{\nu}})\sum_i\bar{u}_\nu(p)\\ &\quad\times F_i u_{\bar{\nu}}(q)\int d^3r\,\bar{\psi}_i(r)O_i\psi_f(r)e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}/\hbar}\\ &=\Omega^{-1}\delta(E_i-E_f-E_\nu-E_{\bar{\nu}})\sum_i\bar{u}_\nu(p)F_i u_{\bar{\nu}}(q)\langle O_i\rangle, \end{aligned} \quad (26)$$

where p_μ and q_μ are the four-momenta of ν and $\bar{\nu}$. The $|S_{fi}|^2$ is easily seen to be

$$|S_{fi}|^2=\frac{T\Omega^{-2}}{2\pi\hbar}\delta(E_i-E_f-E_\nu-E_{\bar{\nu}})\times\sum_{ij}[\bar{u}_\nu(p)F_iu_{\bar{\nu}}(q)\bar{u}(q)\tilde{F}_ju(p)]N_{ij}, \quad (27)$$

with

$$\tilde{F}_j\equiv\gamma_4F_j^\dagger\gamma_4,\quad N_{ij}=O_iO_j^\dagger.$$

The usual rule $\delta^2(E)\to(T/2\pi\hbar)\delta(E)$ has been used[4] to derive Eq. (27). The next step is to perform the summation over the spinorial index $r=1,2$, which has been omitted in Eq. (27) for clarity reasons. Applying the usual procedure to handle these problems,[5] we have after some algebra

$$\sum_{rr'}[u^{(r)}(p)F_iu^{(r')}(q)\bar{u}^{(r')}(q)\tilde{F}_ju^{(r)}(p)]=\tfrac{1}{2}g_ig_j\frac{cp_\alpha cq_\beta}{4E_pE_q}\mathrm{Tr}M_{ij}, \quad (28)$$

$$\mathrm{Tr}\{i\gamma_\alpha O_i(1+\gamma_5)i\gamma_\beta(1-\gamma_5)\gamma_4O_j^\dagger\gamma_4\}=\mathrm{Tr}M_{ij}, \quad (29)$$

where $E_p\equiv E_\nu=cp$, $E_q\equiv E_{\bar{\nu}}=cq$, $p\equiv|\mathbf{p}|$ because for neutrinos $p_\mu^2=\mathbf{p}^2-p_0^2=-m_\nu^2=0$. The next step is to integrate over the neutrino and antineutrino momenta p,q. This is obtained by using

$$\sum_p\to\frac{\Omega}{(2\pi\hbar)^3}\int d^3p,\quad \sum_q\to\frac{\Omega}{(2\pi\hbar)^3}\int d^3q. \quad (30)$$

Again calling expression (27) after the sum over polarizations and momenta have been performed $|S_{fi}|^2$, we obtain

$$|S_{if}|^2=\frac{T}{(2\pi\hbar)}\frac{1}{h^6}\int d^3p\int d^3q\frac{cp_\alpha cq_\beta}{8E_pE_q}\delta(E_i-E_f-E_p-E_q)\times\sum_{ij}g_ig_jN_{ij}\,\mathrm{Tr}M_{ij}. \quad (31)$$

We now introduce a four-vector Q_μ such that

$$Q_\mu=(\mathbf{Q},iQ_0)=[\mathbf{p}+\mathbf{q},(i/c)(E_p+E_q)]. \quad (32)$$

Multiplying Eq. (31) by the identity

$$\int d^3Q\,\delta^3(\mathbf{Q}=\mathbf{p}+\mathbf{q})=1,$$

we obtain

$$|S_{if}|^2=\frac{T}{2\pi\hbar h^6}\frac{1}{2c}\int d^3Q\sum_{ij}g_ig_jN_{ij}\,\mathrm{Tr}M_{ij}\int d^3p\int d^3q\times\frac{cp_\alpha cq_\beta}{4E_pE_q}\delta^4(Q_\mu=p_\mu+q_\mu). \quad (33)$$

The integration over p and q can be performed by using the Lenard relation,[6] i.e.,

$$c^2\int\frac{d^3p}{2E_p}\int\frac{d^3q}{2E_q}p_\alpha q_\beta\delta^4(Q_\mu=p_\mu+q_\mu)=\frac{\pi}{24}[Q_\mu^2\delta_{\alpha\beta}+2Q_\alpha Q_\beta]\theta(Q_0)\theta(Q_\mu^2); \quad (34)$$

we finally obtain

$$|S_{if}|^2=\frac{T}{2\pi\hbar}\frac{1}{h^6}\frac{1}{2c}\frac{\pi}{24}\int d^3Q\,F(Q)\theta(Q_0)\theta(Q_\mu^2),$$
$$F(Q)=\sum_{ij}g_ig_j(Q_\mu^2\delta_{\alpha\beta}+2Q_\alpha Q_\beta)\langle O_i\rangle\langle O_j^\dagger\rangle\,\mathrm{Tr}M_{ij}, \quad (35)$$

and

$$M_{ij}\equiv i\gamma_\alpha O_i(1+\gamma_5)i\gamma_\beta(1-\gamma_5)\gamma_4O_j^\dagger\gamma_4. \quad (36)$$

Summing over the indices $i=V$, $O_V=\gamma_\rho$ and $i=A$, $O_A=i\gamma_\rho\gamma_5$ (and analogously for index j, $O_V=\gamma_\tau$, $O_A=i\gamma_\tau\gamma_5$), we obtain

$$\begin{aligned}F(Q)=(Q_\mu^2\delta_{\alpha\beta}+2Q_\alpha Q_\beta)\{&g_V^2\langle\gamma_\rho\rangle\langle\gamma_\tau\rangle^\dagger(-1)^{1+\delta_{\tau4}}\\&\times\mathrm{Tr}[i\gamma_\alpha\gamma_\rho(1+\gamma_5)i\gamma_\beta(1-\gamma_5)\gamma_\tau]\\&+g_Ag_V(-1)^{\delta_{\tau4}}\langle\gamma_\rho\rangle\langle i\gamma_\tau\gamma_5\rangle^\dagger\\&\times\mathrm{Tr}[i\gamma_\alpha\gamma_\rho(1+\gamma_5)i\gamma_\beta(1-\gamma_5)i\gamma_\tau\gamma_5]\\&+g_Ag_V(-1)^{1+\delta_{\tau4}}\langle i\gamma_\rho\gamma_5\rangle\langle\gamma_\tau\rangle^\dagger\\&\times\mathrm{Tr}[i\gamma_\alpha i\gamma_\rho\gamma_5(1+\gamma_5)i\gamma_\beta(1-\gamma_5)\gamma_\tau]\\&+g_A^2(-1)^{\delta_{\tau4}}\langle i\gamma_\rho\gamma_5\rangle\langle i\gamma_\tau\gamma_5\rangle^\dagger\\&\times\mathrm{Tr}[i\gamma_\alpha i\gamma_\rho\gamma_5(1+\gamma_5)i\gamma_\beta(1-\gamma_5)i\gamma_\tau\gamma_5]\},\end{aligned} \quad (37)$$

where we have used

$$\gamma_\tau^T=\gamma_4\gamma_\tau^\dagger\gamma_4=(-1)^{1+\delta_{\tau4}}\gamma_\tau,$$
$$(i\gamma_\tau\gamma_5)^T=i\gamma_\tau\gamma_5(-1)^{\delta_{\tau4}}.$$

The general expression

$$S^{ij}=-g_ig_jQ_\mu^2\,\mathrm{Tr}[\gamma_\lambda O_i(1+\gamma_5)\gamma_\lambda(1-\gamma_5)O_j]+2g_ig_j\,\mathrm{Tr}[i\gamma_\mu Q_\mu O_i(1+\gamma_5)i\gamma_\beta Q_\beta(1-\gamma_5)O_j]$$

is given in Table 14-1 of Ref. 5. Using these results, Eq. (37) becomes

$$F(Q)=32g^2(Q_\mu^2\delta_{\rho\tau}-Q_\rho Q_\tau)(-1)^{1+\delta_{\tau4}}\times\langle\gamma_\rho(1+\gamma_5)\rangle\langle\gamma_\tau(1+\gamma_5)\rangle^\dagger, \quad (38)$$

where we have taken $g_V=-g_A=g$. With a little change

[4] H. Muirhead, *The Physics of Elementary Particles* (Pergamon, New York, 1965), pp. 279–283.

[5] G. Källén, *Elementary Particle Physics* (Addison-Wesley, Reading, Mass., 1964), p. 190.

[6] H. Y. Chiu and M. H. Zaidi, *High Energy Astrophysics* (Gordon and Breach, New York, 1967), Vol. II, p. 62.

of notation, Eq. (35) now reads ($q^2 \equiv q_\mu^2$)

$$|S_{if}|^2 = \frac{T}{2\pi\hbar}\frac{1}{h^6}\frac{1}{2c}\frac{32g^2\pi}{24}\int d^3q(q_\mu^2\delta_{\alpha\beta}-q_\alpha q_\beta)\times\theta(q_0)\theta(q_\mu^2)(-1)^{1+\delta_{\beta 4}}R_\alpha R_\beta^\dagger. \quad (39)$$

Using Eq. (14) for R_α and $R_\beta^\dagger$, we obtain

$$|S_{if}|^2 = \frac{T}{2\pi\hbar}\frac{32g^2\pi}{2ch^6\times 24}\frac{hL_x^2L_z}{L_x^2L_z^2}\int_{-\infty}^{\infty}dq_x\int_{-\infty}^{\infty}dq_y\int_{-\infty}^{\infty}dq_z \times\delta(p_z-p_z'-q_z)\delta_{p_x=p_x'+q_x}\theta(q_0)\theta(q_\mu^2)\times[q_\mu^2\delta_{\alpha\beta}-q_\alpha q_\beta](-1)^{1+\delta_{\beta 4}}I_\alpha I_\beta^\dagger, \quad (40)$$

where we have used the fact that $\delta^2(x)=L^{-1}\delta(x)$ if x has the dimension of length, and we have changed the δ function on the variable p_x into a Kronecker δ function using the relation $\delta(x)=L^{-1}\delta_x$. Integration on q_z can be immediately performed. Introducing polar coordinates $q_x=q\cos\varphi$, $q_y=q\sin\varphi$, we obtain

$$|S_{if}|^2 = \frac{T}{2\pi\hbar}\frac{32\pi g^2}{2\times 24ch^6}\frac{hL_x^2L_z}{L_x^2L_z^2}\int_0^{\infty}qdq\int_0^{2\pi}d\phi\times\theta(q_0)\theta(q_0^2-q^2-q_z^2)F(q,\varphi), \quad (41)$$

with

$$F(q,\varphi)=[q_\mu^2\delta_{\alpha\beta}-q_\alpha q_\beta](-1)^{1+\delta_{\beta 4}}\times I_\alpha I_\beta^\dagger|_{q_x=q\cos\varphi,\ q_y=q\sin\phi,\ q_z=p_z-p_z'}. \quad (42)$$

The θ function indicates that integration on q^2 can go only up to $q_M^2=q_0^2-q_z^2$. Introducing a new variable $\rho=1-(q^2/q_M^2)$, Eq. (41) reduces to

$$|S_{if}|^2=S_0T\frac{L_x^2L_z}{L_x^2L_z^2}\int_0^1 d\rho I(\rho,x,x'), \quad (43)$$

$$S_0=\frac{32\pi g^2\hbar}{48ch^6}\frac{(mc)^4}{2}, \quad (44)$$

$$I(\rho,x,x')=\frac{q_M^2}{2\pi}\int_0^{2\pi}d\phi\, F(q_M(1-\rho)^{1/2},\phi). \quad (45)$$

A factor $(mc)^4$ has been taken out of the integral in the definition of S_0 of Eq. (44), so that the integral I in (45) is dimensionless. All the momenta entering in (45) are now measured in units of mc, i.e., $p_z/mc=x$, $p_z'/mc=x'$, etc. The probability per unit time and volume, $\mathcal{P}$, is given by

$$\mathcal{P}=\frac{|S_{if}|^2}{L_xL_yL_zT}=\frac{1}{L_xL_yL_z^2}S_0\int_0^1 d\rho\, I(\rho,x,x'). \quad (46)$$

The neutrino luminosity l is defined as

$$l=\sum_i\sum_f(E_i-E_f)f(E_i)[1-f(E_f)]\mathcal{P}, \quad (47)$$

where $E_i-E_f=mc^2[\epsilon(x)-\epsilon(x')]$ is the energy carried away by the neutrinos and $f(E)$ is the Fermi distribution

$$f(E)=(1+e^{(\epsilon-\mu)/\phi})^{-1}, \quad \phi=kT/mc^2. \quad (48)$$

The summations are given by[2]

$$\sum_i\to\sum_n\frac{L_z}{2\pi\hbar}\int dp_z=\frac{L_z}{2\pi\lambda_c}\sum_n\int_{-\infty}^{\infty}dx, \quad (49)$$

$$\sum_f\to\sum_{n'}\sum_{p_z'}=\sum_{n'}\frac{L_z}{2\pi\hbar}\int dp_z'N(p_z') =\frac{L_z}{2\pi\hbar}\frac{1}{2\pi}\frac{H}{H_c}\frac{L_xL_y}{\lambda_c^2}\sum_{n'}\int dp_z' =\frac{L_xL_yL_z}{(2\pi)^2\lambda_c^3}\frac{H}{H_c}\sum_{n'}\int_{-\infty}^{\infty}dx'. \quad (50)$$

Substituting now Eqs. (46), (49), and (50) into Eq. (47), we obtain the final form

$$l=l_0\frac{H}{H_c}\sum_n\sum_{n'}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'[\epsilon(x)-\epsilon(x')]\times f(x)[1-f(x')]\int_0^1 d\rho\, I(x,x',\rho), \quad (51)$$

$$l_0=\frac{1}{(2\pi)^3}\frac{1}{\lambda_c^4}\frac{1}{2\pi\hbar}\frac{32\pi g^2\hbar}{48ch^6}\frac{(mc)^4 2\pi}{2}mc^2 =\frac{1}{6}\frac{1}{(2\pi)^8}\frac{g^2mc^2}{c\hbar^2\lambda_e^8}=1.74748\times10^{18}\ \text{erg/cm}^3\ \text{sec}, \quad (52)$$

with

$$g=1.4149\times10^{-49}\ \text{erg cm}^3.$$

4. FUNCTION $I(x,x',\varrho)$

Using the Pauli-Dirac representation for the γ matrices, Eq. (42) can be rewritten in the following way:

$$F(q,\phi)=(q_\mu^2\delta_{\alpha\beta}-q_\alpha q_\beta)I_\alpha L_\beta, \quad (53)$$

where I_α is given by Eq. (15) and $L_\alpha=I_\alpha(\xi\to\xi', q_\varphi\to -q_\varphi)$. It is a matter of a very lengthy and tedious algebra to show that

$$I(x,x',\rho)=q_M^4(A+\rho B)+q_M^2(q_3C+q_0D)^2, \quad (54)$$

where

$$\begin{aligned}A&=\omega_1^2\phi_1^2+\omega_2^2\phi_2^2-4\omega_3\omega_4\phi_3\phi_4,\\ B&=\omega_1^2\phi_1^2+\omega_2^2\phi_2^2+4\omega_3\omega_4\phi_3\phi_4,\\ C&=\omega_3\phi_3-\omega_4\phi_4,\\ D&=\omega_3\phi_3+\omega_4\phi_4\end{aligned} \quad (55)$$

and

$$\phi_1=\Phi(n|n'-1),\quad \phi_3=\Phi(n|n'),$$
$$\phi_2=\Phi(n-1|n'),\quad \phi_4=\Phi(n-1|n'-1). \tag{56}$$

The general expression for the function $\Phi(n|n')$ has been given in Eq. (18), where the argument t is now

$$t=(1/2\Theta)q_M^2(1-\rho). \tag{57}$$

The parameters ω_k, after having performed the summation over the spins of the initial and final states, are given by

$$\begin{aligned}
\omega_1^2&=\left[1-\frac{x}{\epsilon(x)}\right]\left[1+\frac{x'}{\epsilon(x')}\right], & \omega_3\omega_4&=-\omega_1\omega_2=\frac{2\Theta(nn')^{1/2}}{\epsilon(x)\epsilon(x')},\\
\omega_2^2&=\left[1+\frac{x}{\epsilon(x)}\right]\left[1-\frac{x'}{\epsilon(x')}\right], & \omega_1\omega_3&=\left[1-\frac{x}{\epsilon(x)}\right]\frac{(2n'\Theta)^{1/2}}{\epsilon(x')},\\
\omega_3^2&=\left[1-\frac{x}{\epsilon(x)}\right]\left[1-\frac{x'}{\epsilon(x')}\right], & \omega_2\omega_3&=\left[1-\frac{x'}{\epsilon(x')}\right]\frac{(2n\Theta)^{1/2}}{\epsilon(x)},\\
\omega_4^2&=\left[1+\frac{x}{\epsilon(x)}\right]\left[1+\frac{x'}{\epsilon(x')}\right], & \omega_1\omega_4&=\left[1+\frac{x'}{\epsilon(x')}\right]\frac{(2n\Theta)^{1/2}}{\epsilon(x)},\\
& & \omega_2\omega_4&=\left[1+\frac{x}{\epsilon(x)}\right]\frac{(2n'\Theta)^{1/2}}{\epsilon(x')}.
\end{aligned} \tag{58}$$

The integration over the variable ρ, which appears in Eq. (51), can be performed by using the expression for the hypergeometric function, given in Eq. (20). The result is

$$\begin{aligned}
\int_0^1 I(x,x',\rho)d\rho=&4\Theta q_M^2[\omega_1^2K(n|n'-1)+\omega_2^2K(n-1|n')+\omega_3\omega_4M_1(n,n'|n-1,n'-1)]\\
&-4\Theta^2[\omega_1^2L(n|n'-1)+\omega_2^2L(n-1|n')+4\omega_3\omega_4M_2(n,n'|n-1,n'-1)]\\
&+2\Theta\{\omega_3^2[(\epsilon-\epsilon')+(x-x')]^2K(n|n')+\omega_4^2[(\epsilon-\epsilon')-(x-x')]^2K(n-1|n'-1)\}.
\end{aligned} \tag{59}$$

The general expressions for the functions K, L, M_1, and M_2 are the following ($\alpha=q_M^2/2\Theta$, $\Theta=H/H_c$):

$$\begin{aligned}
K(N|M)=(N!M!)^{-1}\Bigg\{&(N+M)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M}\frac{\alpha^m}{m!}\right]\\
&+2\sum_{k=1}^{\infty}\frac{(-N)_k(-M)_k(-1)^k}{k!}(N+M-k)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M-k}\frac{\alpha^m}{m!}\right]\\
&+\sum_{l=2}^{\infty}(-1)^lC_l(N+M-l)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M-l}\frac{\alpha^m}{m!}\right]\Bigg\},
\end{aligned} \tag{60}$$

$$\begin{aligned}
L(N|M)=(N!M!)^{-1}\Bigg\{&(N+M+1)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M+1-k}\frac{\alpha^m}{m!}\right]\\
&+2\sum_{k=1}^{\infty}\frac{(-N)_k(-M)_k(-1)^k}{k!}(N+M+1-k)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M+1-k}\frac{\alpha^m}{m!}\right]\\
&+\sum_{l=2}^{\infty}(-1)^lC_l(N+M+1-l)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M+1-l}\frac{\alpha^m}{m!}\right]\Bigg\},
\end{aligned} \tag{61}$$

$$M_1(N,M|N-1,M-1)=[(N-1)!(M-1)!]^{-1}\left\{(N+M-1)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M-1}\frac{\alpha^m}{m!}\right]\right.$$
$$+\sum_{k=1}^{\infty}\frac{(-N)_k(-M)_k+(-N+1)_k(-M+1)_k}{k!}(-1)^k(N+M-k-1)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M-1-k}\frac{\alpha^m}{m!}\right]$$
$$\left.+\sum_{l=2}^{\infty}(-1)^l C_l'(N+M-l-1)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M-1-l}\frac{\alpha^m}{m!}\right]\right\}, \quad (62)$$

$$M_2(N,M|N-1,M-1)=[(N-1)!(M-1)!]^{-1}\left\{(N+M)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M}\frac{\alpha^m}{m!}\right]\right.$$
$$+\sum_{k=1}^{\infty}\frac{(-N)_k(-M)_k+(-N+1)_k(-M+1)_k}{k!}(-1)^k(N+M-k)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M-k}\frac{\alpha^m}{m!}\right]$$
$$\left.+\sum_{l=2}^{\infty}(-1)^l C_l'(N+M-l)!\left[1-e^{-\alpha}\sum_{m=0}^{N+M-l}\frac{\alpha^m}{m!}\right]\right\}, \quad (63)$$

with

$$C_l=\sum_{k+k'=l;\,k>0;\,k'>0}\frac{(-N)_k(-M)_k(-N)_{k'}(-M)_{k'}}{k!k'!},$$
$$C_l'=\sum_{k+k'=l;\,k>0;\,k'>0}\frac{(-N)_k(-M)_k(-N+1)_{k'}(-M+1)_{k'}}{k!k'!}, \quad (64)$$
$$(\alpha)_k=\prod_{S=1}^{k}(\alpha+S-1).$$

5. NUMERICAL RESULTS

In Tables I–III we list for each density ρ_6/μ_e ($10^{-6}\rho/\mu_e$ in g/cm^3) the neutrino luminosities for $H=H_c$, $H_c=m^2c^3/e\hbar=4.414\times10^{13}$ G ($\Theta=1$) at temperatures $T=2.5\times10^8$, 5×10^8, and 10^9 °K, respectively.

The dependence of the neutrino luminosity on temperature and density can be seen qualitatively from Fig. 1. In general the neutrino luminosity increases with density for a particular temperature, until a maximum is reached. We also note that the neutrino luminosity decreases rapidly toward zero at high densities corresponding to the regime of complete degeneracy. The reason is that the neutrino bremsstrahlung process considered here cannot occur in a completely degenerate electron gas, regardless of whether the electron plasma is magnetized. Mathematically it is clear that the product $\theta(\epsilon-\mu)\theta(\mu-\epsilon)$ vanishes. Furthermore, the maximum energy-loss rate shifts to higher value at higher temperatures.

TABLE I. Neutrino luminosities $\log_{10}l$ (erg/cm^3 sec) as a function of the degeneracy parameter η and density $\log_{10}(\rho_6/\mu_e)$ ($10^{-6}\rho/\mu_e$ in g/cm^3) at temperature $T=2.5\times10^8$ °K.

η	$\log_{10}(\rho_6/\mu_e)$ (g/cm^3)	$\log_{10}l$ (erg/cm^3 sec)
−8	−6	−4.2518
−2.8	−3	−2.0000
10	−0.268	3.5682
15	−0.315	5.7324
20	0.204	7.8976
25	0.447	9.6721
30	0.602	10.3222
35	0.663	10.5798
40	0.771	10.9685
45	0.978	11.6435
50	1.079	11.7993
55	1.114	12.0414
60	1.176	12.1461
65	1.230	12.2552
70	1.300	12.3979
75	1.672	11.9494
115	1.699	8.0792

TABLE II. Same as Table I for $T=5\times10^8$ °K.

η	$\log_{10}(\rho_6/\mu_e)$ (g/cm^3)	$\log_{10}l$ (erg/cm^3 sec)
−8	−5.602	4.7924
−2.8	−2.509	7.0414
2.2	−0.398	9.2304
10	0.114	12.5682
25	0.748	14.2788
34	1.000	14.3222
43	1.301	14.2788
50	1.653	14.2788
55	1.000	13.3802
60	1.000	11.3010
65	2.000	9.6721
70	2.000	7.7076
75	2.000	6.1461

For purposes of comparison, we quote here the relevant approximate expressions for the neutrino luminosities from Landstreet's thesis [the criterion for using $l_\nu{}^{\text{I}}$, namely, Eq. (20) in the published version of Landstreet's thesis,[7] contains an error]; these are

$$l_\nu{}^{\text{I}} = 10^{-44} H_8{}^6 T_7 \rho^4 \;\; (\text{if } H_8 \rho^{2/3} \lesssim 8\times 10^6 T_7),$$
$$l_\nu{}^{\text{II}} = 2\times 10^{-7} T_7{}^{19/3} \rho^{4/9} H_8{}^{2/3}.$$

It is interesting to note that the discrepancies between Landstreet's results and ours are large at low densities. At high densities, Landstreet's results are essentially in agreement with our work. The reason for the discrepancies at low densities is due apparently to the various approximations introduced by Landstreet. The most serious one of the simplifying assumptions made by Landstreet is his replacement of sums over states by integrals with the densities of a free electron gas in the absence of the magnetic field, namely,

$$\sum_n \rightarrow \int \frac{4\pi p^2 dp}{(2\pi\hbar)^3} \simeq \int \frac{E^2 dE}{2\pi^2 \hbar^2 c^3}.$$

In doing so he has neglected the effects of the magnetic-field energy levels. This is a reasonable approximation for high density and large quantum numbers. It is clear

TABLE III. Same as Table I for $T = 10^9$ °K.

η	$\log_{10}(\rho_6/\mu_e)$ (g/cm³)	$\log_{10} l$ (erg/cm³ sec)
−8	−4.698	9.491
−2.8	−1.698	11.813
2.2	0.0	13.949
10	1.114	15.886
25	2.079	16.579
50	2.897	15.845
55	3.000	14.973
60	3.000	12.643
65	3.000	10.785
70	3.204	8.544
75	3.000	6.361
85	3.000	2.000

[7] J. D. Landstreet, Phys. Rev. **153**, 1372 (1967).

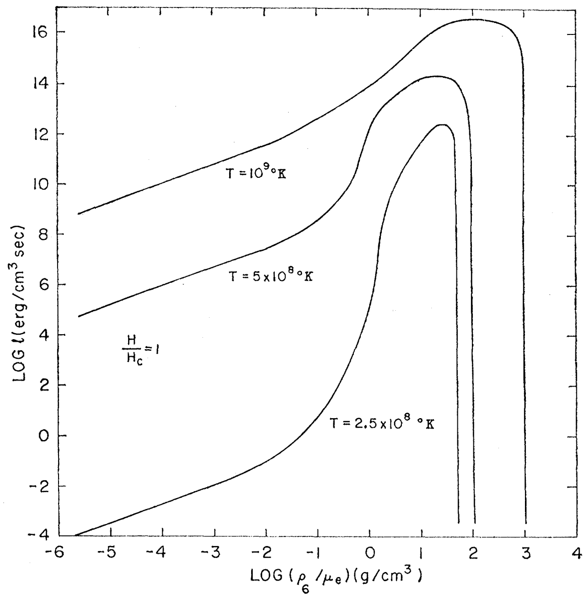

FIG. 1. Neutrino luminosities ($\log_{10} l$ in erg/cm³) as a function of density [$\log_{10}(\rho_6/\mu_e)$ in g/cm³, $\rho_6 = 10^{-6}\rho$] at temperatures 2.5×10^8, 5×10^8, and 10^9 °K, respectively. The field-strength parameter $\Theta = H/H_c$, where $H_c = m^2c^3/e\hbar = 4.414\times 10^{13}$ G, is taken to be unity.

that the replacement of the sums $\sum_n$ by integrals is not really valid for small quantum numbers and at low densities. Considering the large number of approximations made by Landstreet, the general agreement of his results with ours at not too low densities is satisfactory.

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