Perfect Fluids in General Relativity: Velocity Potentials and a Variational Principle*

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The equations of hydrodynamics for a perfect fluid in general relativity are cast in Eulerian form, with the four-velocity being expressed in terms of six velocity potentials: $U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu})$. Each of the velocity potentials has its own "equation of motion." These equations furnish a description of hydrodynamics that is equivalent to the usual equations based on the divergence of the stress-energy tensor. The velocity-potential description leads to a variational principle whose Lagrangian density is especially simple: $\mathfrak{L}=(-g)^{1/2}(R+16\pi p)$, where R is the scalar curvature of spacetime and p is the pressure of the fluid. Variation of the action with respect to the metric tensor yields Einstein's field equations for a perfect fluid. Variation with respect to the velocity potentials reproduces the Eulerian equations of motion.

I. INTRODUCTION

In this paper we introduce a velocity-potential representation for the four-velocity of a perfect fluid in general relativity. This representation permits a new formulation of relativistic hydrodynamics, in which the velocity potentials themselves have first-order "equations of motion," and in which the changes of the four-velocity with time are expressed in terms of Eulerian[1] changes in the potentials. Einstein's field equations plus the equations of evolution in this new formulation can in turn be obtained from a variational principle whose Lagrangian density is

$$\mathfrak{L}=(-g)^{1/2}(R+16\pi p)\,, \tag{1.1}$$

where R is the scalar curvature and p is the fluid's pressure.

Velocity potentials are not new to Newtonian hydrodynamics, but they have been of limited usefulness. It is well known that irrotational motions can be derived from a single potential, $\mathbf{v}=\nabla\phi$. In 1859, Clebsch[2] proved that *any* (Newtonian) motion can be represented by three potentials:

$$\mathbf{v}=\nabla\phi+\alpha\nabla\beta\,. \tag{1.2}$$

The Clebsch representation had the disadvantage that ϕ, α, and β were not physically useful individually; in particular, there were no individual equations of evolution for ϕ, α, and β that could give changes in $\mathbf{v}$ directly, without reference to the usual equations of hydrodynamics.

By contrast, the Newtonian velocity-potential representation introduced by Seliger and Whitham[3] in 1968 avoids this difficulty. By using five potentials (two more than the minimum necessary), Seliger and Whitham were able to give to each potential an equation of evolution and to some an independent physical interpretation. For example, one potential is the entropy; another is the "thermasy" of van Dantzig.[4]

The representation presented in this paper is a relativistic generalization of the one given by Seliger and Whitham. The six velocity potentials (one more than in the Newtonian case because we have a four-velocity rather than a three-velocity) all have equations of evolution that determine how they change with time. These equations constitute an *alternative* to the usual equations of hydrodynamics (i.e., to those based upon the divergence of a stress-energy tensor), rather than simply an adjunct.

Seliger and Whitham derived their equations from a variational principle. We here generalize their principle to include the effects of a general-relativistic gravitational field. In addition we place the velocity-potential equations of evolution on a firm foundation apart from the variational principle by giving a rigorous proof that they are equivalent to the standard equations of hydrodynamics. If the reader desires a more intuitive feeling for why the fluid's Lagrangian density should be simply the pressure, or for how one originally came to the velocity-potential representation, he is invited to read Seliger and Whitham and the references they cite.

The present paper is divided into two main parts plus four Appendixes. The first part discusses the equations of hydrodynamics, first in their standard form and then in terms of velocity potentials. The proof of equivalence between these two versions of hydrodynamics is left to Appendix B. The second main part presents the variational principle. Appendix A

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[1] History has mercilessly given us half a dozen different uses for the names of Lagrange and Euler. The adjectives *Lagrangian* and *Eulerian* refer, respectively, to observers comoving with the fluid or fixed with respect to some arbitrary reference frame through which the fluid flows (see also Ref. 9). The functional whose integral is extremized in a variational principle is the *Lagrangian density*. Finally the equations that express the extremal conditions are the *Euler-Lagrange* equations. Because we wish to emphasize the Eulerian nature of the velocity potentials, we shall henceforth speak of their equations of *evolution* rather than of their equations of *motion*.

[2] A. Clebsch, J. Riene Agnew. Math. 56, 1 (1859).

[3] R. L. Seliger and G. B. Whitham, Proc. Roy. Soc. (London) A305, 1 (1968). Their representation was based in part on work by C. C. Lin, in *Liquid Helium* (International School of Physics "Enrico Fermi", course 21), edited by G. Careri (Academic, New York, 1963), p. 93.

[4] D. van Dantzig, Physica 6, 693 (1939).

contains Pfaff's theorem, an old theorem in differential forms that is essential to understanding the velocity-potential representation; we include it here (without proof) because it is not well known to physicists in its most general form. Appendixes C and D discuss in detail questions that may interest only the specialist: respectively, the uniqueness of the velocity-potential representation and an initial-value formulation of the equations of evolution.

A word about conventions: We use "geometrized units," with $c=G=1$. Greek indices run from 0 to 3; Latin from 1 to 3. The metric has positive signature, so that timelike intervals are negative. We define proper time τ by

$$d\tau^2=-ds^2=-g_{\mu\nu}dx^\mu dx^\nu, \tag{1.3}$$

so that $d\tau$ is real and positive for a particle moving forward in time. We adopt the notation that $D/D\tau$ means covariant differentiation along a world line, while $d/d\tau$ means partial differentiation; a semicolon denotes a general covariant derivative and a comma denotes a general partial derivative. Thus, the four-velocity is defined as

$$U^\nu=dx^\nu/d\tau, \tag{1.4}$$

so that we have

$$U^\nu U_\nu=-1. \tag{1.5}$$

Then for any function χ,

$$D\chi/D\tau\equiv U^\nu(\chi)_{;\nu} \tag{1.6}$$

and

$$d\chi/d\tau\equiv U^\nu(\chi)_{,\nu}. \tag{1.7}$$

Finally, four-vectors are written in boldface sans serif: **A**. Three-vectors appear in boldface: **A**.

II. RELATIVISTIC THEORY OF ONE-COMPONENT PERFECT FLUID

A. Standard Version

Thermodynamics of One-Component Perfect Fluid

We consider a perfect fluid composed of baryons. Because baryons can undergo transmutation, the true rest mass of a group of baryons may not be conserved; but their baryon number N is conserved. Hence, we *define* the (conserved) rest mass of a sample of matter containing N baryons to be $m_{\rm H}N$, where m_H is the mass of a hydrogen atom in its ground state. The difference between the total mass-energy and m_HN is called the internal energy U. Thus U includes the difference between $m_{\rm H}N$ and the true rest mass of the actual atoms and baryons; and it also includes the energy of electron-positron pairs, of mesons, of photons, of thermal motions, and of "zero-point" Fermi-gas "motions." We denote by ρ_0 the density of rest mass so defined, and by $\Pi\equiv U/m_{\rm H}N$ the specific internal energy, both as measured in a local inertial frame momentarily at rest in the fluid. Then the density of total mass-energy is $\rho=\rho_0(1+\Pi)$.

We assume an equation of state of the form $p=p(\rho_0,\Pi)$. Such a two-parameter expression is sufficient for any one-component fluid.[5] The applicability of the results of this paper to a real baryonic fluid depends in part on how well a two-parameter equation of state characterizes the fluid.

The amount of energy per unit rest mass, δq, added to the fluid in any quasistatic process is (first law of thermodynamics)

$$\delta q=d\Pi+pd(1/\rho_0). \tag{2.1}$$

Because of the two-parameter equation of state, Pfaff's theorem (Appendix A) implies that there exist functions $S(\rho_0,\Pi)$ and $T(\rho_0,\Pi)$, the specific entropy and the temperature, respectively, such that[6]

$$d\Pi+pd(1/\rho_0)=TdS=\delta q. \tag{2.2}$$

If one now defines the specific inertial mass by[7]

$$\mu=(\rho+p)/\rho_0=1+\Pi+p/\rho_0, \tag{2.3}$$

one can use $d\mu$ to eliminate $d\Pi$ in Eq. (2.2) and obtain

$$d\mu-\rho_0^{-1}dp=TdS. \tag{2.4}$$

We will often use this in the form

$$dp=\rho_0 d\mu-\rho_0 TdS. \tag{2.5}$$

Clearly one can express ρ_0 and Π as functions of μ and S, so that one can put the equation of state in the form

$$p=p(\mu,S). \tag{2.6}$$

Stress-Energy Tensor and Equations of Motion

The relativistic one-component perfect fluid is defined by its equation of state, Eq. (2.6), and by the stress-energy tensor

$$\begin{aligned}T^{\mu\nu}&=(\rho+p)U^\mu U^\nu+pg^{\mu\nu}\\&=\rho_0 uU^\mu U^\nu+pg^{\mu\nu}.\end{aligned} \tag{2.7}$$

In a locally comoving inertial frame, $T^{\mu\nu}$ is diag(ρ,p,p,p). Because the fluid is perfect, the stress-energy tensor

[5] E. Fermi, *Thermodynamics* (Dover, New York, 1936), p. 91.

[6] For a many-component system (i.e., one whose equation of state has more than two independent parameters), Pfaff's theorem does not suffice to require $\delta q=TdS$, i.e., to ensure an integrating factor for δq. One must then invoke a weak form of the second law of thermodynamics. See S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover, New York, 1939), Chap. 1. For an isolated one-component fluid, Pfaff's theorem makes the second law a mathematical identity.

[7] The quantity $\rho+p$ plays the role of inertial mass per unit volume in a perfect fluid. See Eq. (2.19) of this paper, or the article by K. S. Thorne in *High-Energy Astrophysics*, edited by C. DeWitt, E. Schatzmann, and P. Véron (Gordon and Breach, New York, 1967), Vol. 3. I thank Professor Thorne for pointing out to me that $m_{\rm H}\mu$ is also the injection energy at constant entropy: the energy required to create one baryon and place it in the fluid with the same energy $(m_{\rm H}\Pi)$ as neighboring baryons, doing work $m_{\rm H}p/\rho_0$ to create the same volume $(m_{\rm H}/\rho_0)$ for it as the other baryons have.

contains no viscosity or energy-transport terms.[8] The conservation of baryon number, rewritten in terms of rest mass ρ_0, is embodied in the equation

$$(\rho_0 U^\nu)_{;\nu}=0. \tag{2.8}$$

Normalization of the four-velocity reads

$$U^\nu U_\nu=-1, \tag{2.9}$$

covariant differentiation of which yields the useful equation

$$U^\nu U_{\nu;\sigma}=0. \tag{2.10}$$

The equations of motion obeyed by the fluid are expressed in conservation form by requiring the stress-energy tensor to be divergence-free:

$$T^{\mu\nu}{}_{;\nu}=0. \tag{2.11}$$

These four equations supplemented by Eqs. (2.8) and (2.9) determine the motion of a fluid whose equation of state is known.

The physical meaning of the four equations (2.11) becomes clearer upon separating out their components parallel and perpendicular to the four-velocity. The equation parallel to **U**,

$$U_\mu T^{\mu\nu}{}_{;\nu}=0, \tag{2.12}$$

reduces [by Eqs. (2.8)–(2.10)] to

$$U^\nu p_{,\nu}-\rho_0 U^\nu \mu_{,\nu}=0. \tag{2.13}$$

By Eq. (2.5) this becomes

$$\rho_0 T U^\nu S_{,\nu}=0. \tag{2.14}$$

Thus, the motions of a perfect fluid conserve the entropy per baryon. Because $\delta q=T dS$, this confirms that no heat flows in or out of any element of the perfect fluid during its motions.

One can construct the three independent equations of motion perpendicular to **U** by using the projection tensor

$$P^\sigma{}_\mu=\delta^\sigma{}_\mu+U^\sigma U_\mu. \tag{2.15}$$

The equations are

$$P^\sigma{}_\mu T^{\mu\nu}{}_{;\nu}=0. \tag{2.16}$$

By using Eqs. (2.8)–(2.10), one can reduce this to

$$-P_\sigma{}^\nu p_{,\nu}=\mu\rho_0 U_{\sigma;\nu}U^\nu \tag{2.17}$$

$$=\mu\rho_0 D U_\sigma/D\tau. \tag{2.18}$$

In a locally comoving intertial frame, $P_\sigma{}^\nu$ picks out the spatial gradient of p. If $\mathbf{v}$ is the (instantaneously zero) spatial part of **U**, Eq. (2.18) becomes

$$-\nabla p=(\rho+p)d\mathbf{v}/dt. \tag{2.19}$$

This is the familiar force law; it justifies calling $(\rho+p)$ the inertial mass per unit volume.

B. Velocity-Potential Version

Velocity-Potential Representation and Equations of Motion

One usually interprets the equations of motion in the "standard version" in a Lagrangian sense. One regards the four-velocity as vector representing the change of a particle's position in proper time. It is a vector "field" only in the continuum approximation, in which one overlooks the fact that the fluid is "really" composed of discrete particles packed very closely together. Because one tends to regard the four-velocity as a little arrow carried along by the particles, one also tends to interpret the equations of motion in terms of what happens to little fluid elements. Thus, Eq. (2.19) describes the response of a fluid element to a pressure gradient, and Eqs. (2.8) and (2.14) require the conservation of the number of baryons and the amount of entropy contained in a fluid element.

The "velocity-potential version" of hydrodynamics, by contrast, lends itself most naturally to an Eulerian interpretation. One regards the four-velocity as a vector field over spacetime. As such it can be represented in terms of scalar fields and their gradients. While the particles move through space, the scalars at a given point of space simply change their amplitudes with time.[9]

According to Pfaff's theorem (Appendix A), four potentials are sufficient to describe the four-velocity:

$$U_\nu=AB_{,\nu}+CD_{,\nu}. \tag{2.20}$$

While four such potentials are guaranteed to exist, they may not be physically useful. In this paper we introduce instead a six-potential representation that has a ready and important physical interpretation. This representation is[10]

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}). \tag{2.21}$$

The potentials μ and S are just the specific inertial mass and the specific entropy as defined above. The physical significance of the remaining potentials ϕ, α, β, and θ[11] will be explored below.

[8] Newtonian perfect fluids permit heat conduction. In relativity, however, conduction leads to a nonzero momentum density and to anisotropic stresses in the rest frame of the baryons; it must therefore be excluded from *perfect* fluids in relativity.

[9] The distinction between Eulerian and Lagrangian coordinates, while useful, is not rigid in general relativity, because all equations are independent of coordinate system. Lagrangian interpretations are valid only in comoving frames. The "Eulerian" equations for the velocity potentials are good in *any* reference frame; in fact, however, they are most easily interpreted in a comoving frame. (See the section on Physical Interpretation below.)

[10] Seliger and Whitham replace the term $\theta S_{,\nu}$ with $-S\theta_{,\nu}$ in their Newtonian representation, and thus achieve the nonrelativistic version of Schmid's representation (Ref. 26). Note also that Eq. (2.21) is a local equation; the existence of a global set of potentials is not guaranteed.

[11] To my knowledge, D. van Dantzig (Ref. 4) was the first to define θ. He called it the "thermasy."

The equations of evolution in this representation are

$$(\rho_0 U^\nu)_{;\nu}=0, \tag{2.22a}$$

$$U^\nu S_{,\nu}=dS/d\tau=0, \tag{2.22b}$$

$$U^\nu \alpha_{,\nu}=d\alpha/d\tau=0, \tag{2.22c}$$

$$U^\nu \beta_{,\nu}=d\beta/d\tau=0, \tag{2.22d}$$

$$U^\nu \phi_{,\nu}=d\phi/d\tau=-\mu, \tag{2.22e}$$

$$U^\nu \theta_{,\nu}=d\theta/d\tau=T. \tag{2.22f}$$

From Eqs. (2.21), (2.22b), (2.22d), and (2.22e) follows the result

$$U^\nu U_\nu=-1. \tag{2.23}$$

There is no equation for μ. Its evolution can be computed from Eqs. (2.22a), (2.22b), and the equation of state.

Appendix B contains the proof that these velocity-potential equations are equivalent to the standard version of the equations of motion.

Physical Significance of Velocity-Potential Version

Circulation. The representation Eq. (2.21) is well suited to Taub's[12] Eulerian analysis of circulation. Taub defines a current vector $\mathbf{V}=\mu\mathbf{U}$,[13] which in our representation is

$$V_\sigma=\mu U_\sigma=\phi_{,\sigma}+\alpha\beta_{,\sigma}+\theta S_{,\sigma}. \tag{2.24}$$

He then defines the circulation tensor $\Omega_{\sigma\lambda}=2V_{[\sigma;\lambda]}$, where square brackets denote the antisymmetric part. In our representation this becomes

$$\Omega_{\sigma\lambda}=2V_{[\sigma;\lambda]}=2\alpha_{[,\lambda}\beta_{,\sigma]}+2\theta_{[,\lambda}S_{,\sigma]}. \tag{2.25}$$

Taub then defines circulation C in the following manner. Consider a spacelike hypersurface Σ through the world lines of the fluid's particles. A closed curve Λ in Σ may in general enclose some circulating fluid. If λ is the ordinary length parameter along Λ, and $t^\alpha=dx^\alpha/d\lambda$ is the tangent vector to Λ in Σ, then the circulation C is defined as the integral

$$C=\oint_\Lambda V_\alpha t^\alpha d\lambda \tag{2.26}$$

around the closed curve. From Eq. (2.24) we see that

$$C=\oint_\Lambda \alpha d\beta_\Lambda+\oint_\Lambda \theta dS_\Lambda, \tag{2.27}$$

where the subscript Λ on $d\beta$ and dS means that the differentials are directed along the curve. Clearly, if α is a function only of β, and if either θ or S is a constant,[14] then C will be zero for any choice of Σ and Λ. In this case, $\Omega_{\sigma\kappa}$ is also zero. One can easily see, then, that C will vanish for *every* curve Λ in *every* hypersurface Σ if and only if $\Omega_{\sigma\kappa}=0$, which is a result Taub also mentions. This establishes the significance of $\Omega_{\sigma\kappa}$ in circulation.

In order to see the roles of α, β, θ, and S more clearly, let us look at the circulation in a momentarily comoving local Lorentz frame, with $g_{\alpha\beta}=\mathrm{diag}(-1, 1, 1, 1)$. Then $V^0=-V_0=\mu$, $V^i=V_i=0$. Define the vorticity vector

$$\begin{aligned} v^\nu &= \tfrac{1}{2}(-g)^{-1/2}\epsilon^{\nu\sigma\lambda\kappa}U_\sigma U_{\lambda;\kappa} \\ &= \tfrac{1}{2}\mu^{-2}(-g)^{-1/2}\epsilon^{\nu\sigma\lambda\kappa}V_\sigma\Omega_{\lambda\kappa}, \end{aligned} \tag{2.28}$$

where $(-g)^{-1/2}\epsilon^{\nu\sigma\lambda\kappa}$ is the completely antisymmetric contravariant tensor. In the comoving frame, because $V_i=0$, v^ν has vanishing time component. In fact, we have

$$\mu v^i=\tfrac{1}{2}\epsilon^{ijk}\Omega_{jk}, \tag{2.29}$$

$$\mu\mathbf{v}=\nabla\alpha\times\nabla\beta+\nabla\theta\times\nabla S. \tag{2.30}$$

That is, if $\nabla S=0$, surfaces of constant α and β intersect along vortex lines, which are carried along with the fluid because $d\alpha/d\tau=d\beta/d\tau=0$.[15] If initially $\alpha=$const, $\beta=$const, but $S\neq$const, then surfaces of constant θ and S determine vortex lines whose orientation with respect to the fluid's particles changes in time because $d\theta/d\tau=T$.

Uniqueness of the velocity-potential representation. Formulation of Eqs. (2.22) in terms of initial values will give us more insight into the velocity-potential representation. The first-order nature of Eqs. (2.22) makes an initial-value approach especially simple for the restricted case of no self-gravitation, i.e., the case where the fluid does not disturb the background geometry of spacetime. The case with self-gravitation, although important, is more difficult and would not add substantially to our understanding of the potential representation itself, so we ignore it here.

The first question—which has nothing to do with self-gravitation—concerns the uniqueness of the representation. Given a physical situation, how much "gauge freedom" does one have to choose the initial values of the potentials without changing the physical situation they describe? If any such freedom exists, Eqs. (2.22) clearly imply that it lies only in the choice of initial values: The evolution of the potentials away from their initial values is fully determined by the physical situation ($\mathbf{U}$, μ, and T). Before we can deal

[12] A. H. Taub, Arch. Ratl. Mech. Anal. **3**, 312 (1959).

[13] By Ref. 7, $m_H\mathbf{V}$ is the four-momentum a baryon must have to be injected into the fluid. In the nonrelativistic limit ($\mu\to 1$), we have $\mathbf{V}\to\mathbf{U}$. Thus, both $\mathbf{U}$ and $\mathbf{V}$ are relativistic generalizations of the three-velocity $\mathbf{v}$. In circulation it is more useful to deal with $\mathbf{V}$ (see Ref. 12). For example, Bernoulli's equation for nonsteady irrotational isentropic flow generalizes using $\mathbf{V}$ because in that case it can be derived from a single potential ϕ. As another example, the tangential component of $\mathbf{V}$ is conserved across shock fronts. The simple form the circulation equations assume in terms of velocity potentials is another indication of the utility of $\mathbf{V}$.

[14] If θ and S are not both constant, neither can be a function only of the other, because $d\theta/d\tau=T$ while $dS/d\tau=0$. If either of them is constant, the second integral in Eq. (2.27) is zero. Only if $T=0$ can θ be constant.

[15] Circulation due to α and β will not change in a comoving frame. Such circulation may, however, emit gravitational radiation and be damped out as seen by a distant, noncomoving observer.

with this evolution we must resolve the question of gauge freedom among the potentials.

There is no gauge freedom in μ and S because changing them changes the physical situation. The question is whether there are two sets of potentials, $(\phi,\alpha,\beta,\theta)$ and $(\phi',\alpha',\beta',\theta')$, differing in initial values, which give the same $\mathbf{U}$ when substituted into Eq. (2.21) using the same μ and S. Two such sets are said to be equivalent. The equivalence transformations by which one set is obtained from another are discussed in detail in Appendix C. These transformations are essentially contact transformations. The result of interest here is: *The initial value of any one potential may be chosen arbitrarily; the remaining initial values are then constrained by the physical condition of the fluid (by* $\mathbf{U}$, μ, S, *and the equation of state).*[16]

Let us discuss the physical meaning of the equivalence transformations. Circulation in the fluid is an observable and hence must be preserved by the transformation. In the isentropic case ($S_{,\nu}=0$), circulation proceeds around intersections of surfaces of constant α and β. The effect of the equivalence transformation Eqs. (C21) is to preserve these intersections while changing α and β.

Intersecting surfaces of constant θ and S determine a kind of thermal circulation. Because physical conditions fix S, equivalence transformation on θ but not on α and β must leave $\nabla\theta$ unchanged except for parts parallel to ∇S. This is why requiring any equivalence transformation to leave α and β unchanged leads to the equation $\theta'=\theta+f(S)$.

A general equivalence transformation changes α and β as well as θ, but it keeps the sum $\nabla\alpha\times\nabla\beta+\nabla\theta\times\nabla S$ constant by transferring some circulation from one term to the other. The two types of circulation cannot therefore be separated from each other uniquely on any given spacelike hypersurface; they can be distinguished, however, by the way they change as the fluid moves off that hypersurface.

Restricted initial-value formulation. Suppose one chooses initial values of the velocity potentials on some initial hypersurface; what kind of initial-value information is necessary to determine a unique fluid motion in the background metric? Are the initial values of the six velocity potentials μ, S, ϕ, α, β, and θ sufficient; or are their derivatives off the hypersurface also necessary? Once the set of initial values is chosen, the equivalence transformation of Appendix C can lead to other sets that give the same fluid motion. Nevertheless, each set can be so chosen that it determines one and only one fluid motion. Appendix D presents two different initial-value schemes whereby the four-velocity and thermodynamic state of the fluid are determined throughout spacetime by the specification of certain data on an initial hypersurface. The first scheme shows that specifying values of all six potentials and the equation of state is sufficient. The second scheme shows that specifying the thermodynamic condition (μ and S) is not essential: The equation of state, the initial values of ϕ, α, β, and θ, and the derivatives of any two of those four potentials normal to the hypersurface will fully determine μ, S, and $\mathbf{U}$. Appendix D also leads to an obvious consistency condition on the initial values: *The initial values of* μ, S, ϕ, α, β, *and* θ *must be so chosen that the three-space velocity of the fluid parallel to the initial hypersurface nowhere exceeds the velocity of light.*

Once sufficient Cauchy data have been specified, the subsequent evolution of the velocity potentials is most easily discussed from a Lagrangian point of view. From Eqs. (2.22) one can see that the initial values of α, β, S, and baryon number N are carried along by the fluid: Each fluid element sees no change in these four functions. They are therefore "initial-value parameters." By contrast, the functions θ and ϕ are "dynamical variables": Their evolution is determined by the thermodynamic condition of the fluid. Changes in them cause the changes in the motion of the fluid seen in a comoving (Lagrangian) frame. They are dynamical in the sense that the complete history of a fluid element can be given by a plot of θ against ϕ, along which the given values of S, α, β, and N are constant. That there are only two dynamical variables in this sense does not imply that there are only two "degrees of freedom" in the fluid's motion. The question of degrees of freedom is taken up at the end of Appendix C.

III. EULERIAN VARIATIONAL PRINCIPLE

In 1954, Taub[17] gave a variational principle whose Euler-Lagrange equations were the general-relativistic field equations plus the equations of motion for a perfect fluid in what we have called the standard version. An essential feature of any such variational principle is that the world lines of the fluid's particles be among the quantities varied. Consistent with the Lagrangian interpretation of the standard version that we discussed in Sec. II A, in Taub's principle one varies the world lines in a Lagrangian manner: One attaches a label to every particle and directly changes the particle's path by changing the position of its label in spacetime.

The variational principle given in this section uses the velocity-potential version of hydrodynamics and hence is Eulerian. The independent coordinates with respect to which the Lagrangian density is varied are the velocity potentials themselves. Varying the potentials varies the four-velocity and thence implicitly the world lines.

[16] The remaining initial values are constrained but not fully determined by the physics. See Ref. 35. Moveover, the arbitrary choice of the initial value of one potential *may* lead to divergences in others. These divergences will not affect any observables like $\mathbf{U}$ or the circulation. For example, if the term $\alpha\beta_{,\nu}$ is nonzero in one representation, choosing $\alpha'=0$ will not generally eliminate this term; it will only force β' to diverge in order to keep $\alpha'\beta'_{,\nu}$ nonzero and finite.

[17] A. H. Taub, Phys. Rev. 94, 1468 (1954).

The action principle. In step-by-step form,

(1) Select an equation of state for the one-component perfect fluid. Express it in the form

$$p=p(\mu,S). \tag{3.1}$$

Then Eq. (2.5) follows from basic thermodynamics:

$$dp=\rho_0 d\mu-\rho_0 T dS. \tag{3.2}$$

(2) Define the four-velocity vector field in terms of six scalar velocity-potential fields:

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}). \tag{3.3}$$

Normalization of $\mathbf{U}$ implies

$$\mu^2=-g^{\mu\nu}(\phi_{,\mu}+\alpha\beta_{,\mu}+\theta S_{,\mu})(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}), \tag{3.4}$$

which defines the functional dependence of μ upon the independent variables of our variational principle, ϕ, α, β, θ, S, and $g^{\mu\nu}$.

(3) Define the action I as

$$I=\int (R+16\pi p)(-g)^{1/2}d^4x, \tag{3.5}$$

where R is the scalar curvature, taken as a function of $g^{\mu\nu}$ and its derivatives, and where p is the ordinary pressure, which depends on all the independent variables through Eqs. (3.1), (3.2), and (3.4).

(4) Extremize I to obtain the following Euler-Lagrange equations[18]:

$$\delta g^{\mu\nu}: \quad G_{\mu\nu}-8\pi[(\rho+p)U_\mu U_\nu+pg_{\mu\nu}]=0, \tag{3.6a}$$

$$\delta\phi: \quad (\rho_0 U^\nu)_{;\nu}=0, \tag{3.6b}$$

$$\delta\theta: \quad U^\nu S_{,\nu}=0, \tag{3.6c}$$

$$\delta S: \quad U^\nu\theta_{,\nu}=T, \tag{3.6d}$$

$$\delta\alpha: \quad U^\nu\beta_{,\nu}=0, \tag{3.6e}$$

$$\delta\beta: \quad U^\nu\alpha_{,\nu}=0. \tag{3.6f}$$

Equations (3.3), (3.4), (3.6c), and (3.6e) imply

$$U^\nu\phi_{,\nu}=-\mu. \tag{3.6g}$$

We have thus reproduced Eqs. (2.22) of the velocity-potential representation. This establishes the validity of the variational principle.

Comparison with other action principles. Our variational principle is equivalent to Taub's 1954 principle.[17] To prove this we use a procedure taken from Seliger and Whitham.[3] Taub extremizes the action

$$I_T=\int [R-16\pi(\rho-\rho_0 TS+\lambda g_{\mu\nu}U^\mu U^\nu)] \times(-g)^{1/2}d^4x, \tag{3.7}$$

where λ is a Lagrange multiplier that ensures normalization of $\mathbf{U}$. Since we impose that normalization explicitly in our principle, we can drop the λ term and work with

$$I_T'=\int [R-16\pi(\rho-\rho_0 TS)](-g)^{1/2}d^4x. \tag{3.8}$$

Taub imposes two explicit constraints upon variations of I_T'. The first is conservation of baryons, and the second is that there exist a field θ such that $U^\nu\theta_{,\nu}=T$ (Taub uses α rather than θ). The second is not a physical constraint, of course, since θ exists for all $\mathbf{U}$ and T. Nevertheless, it is a mathematical constraint. We can eliminate both constraints by using Lagrange multipliers:

$$I_T''=\int \{R-16\pi[\rho-\rho_0 TS-\phi(U^\nu\rho_0)_{;\nu}-\theta(\rho_0 U^\nu S)_{;\nu}]\} \times(-g)^{1/2}d^4x. \tag{3.9}$$

Variations of ϕ and θ give the equations of conservation of baryons and entropy. Variation of S gives $T=U^\nu\theta_{,\nu}$. Variation with respect to ρ_0 gives [noting that $(\partial\rho/\partial\rho_0)_S=\mu$]

$$U^\nu\phi_{,\nu}=-\mu. \tag{3.10}$$

To complete the identification of Taub's principle with ours, we add to the Lagrangian density the divergence

$$Y^\nu{}_{,\nu}=16\pi[(-g)^{1/2}(U^\nu\rho_0\phi+U^\nu\rho_0\theta S)]_{,\nu}. \tag{3.11}$$

We obtain

$$I_T'''=\int [R-16\pi(\rho-\rho_0 TS+\rho_0 U^\nu\phi_{,\nu}+\rho_0 SU^\nu\theta_{,\nu})] \times(-g)^{1/2}d^4x, \tag{3.12}$$

which reduces to

$$I_T'''=\int (R+16\pi p)(-g)^{1/2}d^4x. \tag{3.13}$$

The modified version of Taub's principle is thus equivalent to ours, except for the technical point that Taub's variations are Lagrangian and do not use velocity potentials, while ours are Eulerian and rely on the velocity potentials.[19]

More recently, Taub published a variational principle expressed in comoving coordinates, in which the action

[18] See, e.g., L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1962), Sec. 93.

[19] These calculations give the potentials θ and ϕ richer meaning; one might ask if α and β have similar meanings. They do, in a formal way (see Seliger and Whitham, Ref. 3): One can make the transition from Taub's variables to the Eulerian variables complete by requiring "conservation of Lagrangian coordinates" (i.e., once a fluid element is labeled with a comoving coordinate, that coordinate never changes). Let β be such a coordinate and α be its Lagrange multiplier. Then one adds the term $\rho_0\alpha U^\nu\beta_{,\nu}$ into Eq. (3.9); variation of α and β then gives the appropriate equations, without changing either Eq. (3.10) or Eq. (3.13). This device, due orginally to Lin (Ref. 3), is somewhat mysterious, especially since only one Lagrangian coordinate is required, and not all three.

is the same as the present one.[20,21] In fact, by specializing the calculations of this section to a comoving coordinate system, one can show that variation of $g^{\mu\nu}$ is equivalent to Taub's first variation of the action. The price paid for working in a comoving system is that the potential representation is rendered useless while the equations for conservation of rest mass and for normalization of the four-velocity have to be assumed *ad hoc*, because the "comoving" constraint is nonholonomic in the variables ϕ, α, β, θ, and S.

Bardeen[22] has recently obtained an Eulerian action principle for axially symmetric, differentially rotating configurations; we will show that one version of his action principle is equivalent to ours, specialized to such configurations. Bardeen extremizes the action

$$I_2=2\pi\int\int[-R/16\pi-T^0{}_0-\Phi\rho_0U^0-\Lambda(\rho+p)U_\varphi U^0]\times(-g)^{1/2}dx^2dx^3, \quad (3.14)$$

where x^2 and x^3 are any two coordinates such that $\partial/\partial x^2$ and $\partial/\partial x^3$ are both orthogonal to the Killing vectors $\partial/\partial t$ and $\partial/\partial\varphi$. The independent variables are the nonzero components of $g_{\alpha\beta}$ and four "internal" variables governing changes in the fluid and its motions: ρ_0, U^φ/U^0, ξ, and η. The variables $\xi(x^2,x^3)$ and $\eta(x^2,x^3)$ are Lagrangian coordinates giving the position of a fluid element in the x^2-x^3 subspace, and are actually redundant: Only one of them is needed to extract the full physical content of the principle.[23] Consequently there are really only three internal variables. The only constraint on the variations is that $\mathbf{U}$ be normalized. The two Lagrange multipliers $\Phi(\xi,\eta)$ and $\Lambda(\xi,\eta)$ ensure that the baryon number and angular momentum, respectively, of a fluid element be unaffected by variations of $g_{\alpha\beta}$. When the actual values of Φ and Λ are put in ($\Phi=\mu/U^0$, $\Lambda=U^\varphi/U^0$), I_2 reduces to

$$I_2'=-\frac{1}{8}\int\int(R+16\pi p)(-g)^{1/2}dx^2dx^3. \quad (3.15)$$

This is the same action as in our principle. Moreover, our principle also has three internal variables: The five variables ϕ, α, β, θ, and S are reduced to three by the relations $U_2=U_3=0$. These three may differ from Bardeen's three, but their Euler-Lagrange equations will be equivalent to his because they are a complete set of variables: a one-component fluid constrained to mover in only the φ direction has three degrees of freedom—two thermodynamic and one kinetic. Since the only constraint on our variational principle is also the normalization of $\mathbf{U}$, the two principles are equivalent.

IV. CONCLUDING REMARKS

The work reported in this paper was originally undertaken in the hope of finding stability criteria for self-gravitating masses of fluid. Although that goal is still far off, the existence of an Eulerian variational principle may be a beginning.

What is needed, I believe, is a Hamiltonian principle in a minimum number of variables. The present action principle seems to have "too many" free variables: Witness the existence of equivalence transformations among ϕ, α, β, and θ; witness also the fact that variations of the Lagrangian violate the conservation of ρ_0. Perhaps the methods of Arnowitt, Deser, and Misner[24] or of Dirac[25] can be applied to isolate the "true variables" of the principle. Then one might be able to obtain a self-adjoint variational principle that could lead to stability criteria.

It may also be possible to extend this work to viscous fluids and charged fluids. The key step would be the extension of Theorem 1 of Appendix B to the appropriate case.

Note added in proof. An equivalent set of velocity potentials and a similar variational principle have been obtained independently by Schmid from a very different approach.[26] His potentials nicely illustrate a symmetry of the velocity-potential formulation. He defines ϕ differently: $d\phi/d\tau=-\mu+TS$. Then all the results of this paper carry through if one replaces $\theta S_{,\nu}$ by $-S\theta_{,\nu}$.

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I am deeply indebted to Dr. E. D. Fackerell for introducing me to the paper of Seliger and Whitham, upon which so much of this paper is based, and for many helpful discussions. I would also like to thank Professor Whitham himself for discussing his paper with me, Professor A. H. Taub for his correspondence and reprints, and Professor Jürgen Ehlers for many helpful suggestions. Finally I wish to thank Professor K. S. Thorne for his numerous suggestions and criticisms during the writing of this paper that have made it clearer than it would otherwise have been.

[20] A. H. Taub, in *Fluides et Champ Gravitationnel en Relativité Générale* (Centre National de la Recherche Scientifique, Paris, 1969), pp. 57–72.

[21] A. H. Taub, Commun. Math. Phys. **15**, 235 (1969).

[22] James M. Bardeen, Astrophys. J. **162**, 71 (1970).

[23] Variations of ξ and η give the ξ and η components of the (vector) equation of hydrostatic equilibrium. Since the Jacobian $\partial(\xi,\eta)/\partial(x^2,x^3)$ is assumed well behaved, hydrostatic equilibrium in ξ-η space implies equilibrium in x^2-x^3 space. However, since ξ and η are arbitrary functions of x^2 and x^3, the Euler-Lagrange equation for either ξ *or* η is sufficient to guarantee hydrostatic equilibrium everywhere in x^2-x^3 space. This ξ-η redundancy seems closely related to the problem mentioned at the end of Ref. 19, namely, that requiring conservation of only *one* Lagrangian coordinate is sufficient to complete the transformation from Taub's first principle to ours.

[24] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962), Chap. 7. See also the references cited therein.

[25] P. A. M. Dirac, Proc. Roy. Soc. (London) **A246**, 326 (1958).

[26] L. A. Schmid, in *A Critical Review of Thermodynamics*, edited by E. B. Stuart, B. Gal-Or, and A. J. Brainard (Mono, Baltimore, 1970); also L. A. Schmid, in Proceedings of the International Conference on Thermodynamics, 1970 [Pure Appl. Chem. (to be published)].

APPENDIX A: PFAFF'S THEOREM

We have occasion to use Pfaff's theorem several times in this paper; we state it here without proof. An application of the theorem familiar to physicists concerns criteria for the integrability of a so-called "Pfaffian form,"

$$\sum_{i=1}^{N} f_i(x^k)dx^i.$$

These criteria are closely related to the second law of thermodynamics and lead to definitions of entropy and temperature for many-component systems.[6] Pfaff's theorem is much more general than the second law, however. It says[27] that if $f_i(x^k)$ are N functions of N independent variables x^k, then there exist functions $A_\alpha(x^k)$, $B_\alpha(x^k)$, and $C(x^k)$ such that

$$\begin{aligned}\sum_{i=1}^{N} f_i dx^i &= \sum_{\alpha=1}^{N/2} A_\alpha dB_\alpha \quad \text{if } N \text{ is even} \\ &= dC + \sum_{\alpha=1}^{(N-1)/2} A_\alpha dB_\alpha \quad \text{if } N \text{ is odd}.\end{aligned} \tag{A1}$$

Consequently we have

$$f_i = \sum_{\alpha=1}^{N/2} A_\alpha \frac{\partial B_\alpha}{\partial x^i}$$

or (A2)

$$f_i = \frac{\partial C}{\partial x_i} + \sum_{\alpha=1}^{(N-1)/2} A_\alpha \frac{\partial B_\alpha}{\partial x^i},$$

respectively. The number of functions remains the same, but the number of differentials is cut essentially in half. Pfaff's theorem sets a least upper bound on the number of differentials required: One might need fewer but one never needs more. This least upper bound depends only on the number of independent variables. For example, if $\alpha_i(x^k)$ and $\beta_i(x^k)$ are $2N$ functions ($i=1, \ldots, N$) of $n<N$ independent variables, then

$$\sum_{i=1}^{N} \alpha_i d\beta_i = \sum_{i=1}^{N} \sum_{k=1}^{n} \alpha_i \frac{\partial \beta_i}{\partial x^k} dx^k.$$

The expressions

$$\sum_{i=1}^{N} \alpha_i \frac{\partial \beta_i}{\partial x^k}$$

are n functions of n variables; from Pfaff's theorem we therefore obtain (if, for example, n is even)

$$\sum_{i=1}^{N} \alpha_i d\beta_i = \sum_{\sigma=1}^{n/2} A_\sigma dB_\sigma. \tag{A3}$$

For $N=2$, Eq. (A1) becomes the familiar statement that every differential form in two variables has an integrating factor.

APPENDIX B: EQUIVALENCE OF STANDARD VERSION AND VELOCITY-POTENTIAL VERSION

The proof of equivalence between the equations of the standard version and those of the velocity-potential version rests upon Theorem 1 below. Once the theorem is established it will allow us to show that the equations of each version imply those of the others. Theorem 1 should be regarded as an algebraic identity: No equations are assumed other than those explicitly stated in the theorem.

Theorem 1. Let $\mathbf{U}$ be the four-velocity of a one-component fluid. Define a tensor $T_\nu{}^\sigma$ with components

$$T_\nu{}^\sigma \equiv \rho_0 \mu U_\nu U^\sigma + p\delta_\nu{}^\sigma. \tag{B1}$$

Define the scalar functions ϕ and θ by the differential equations

$$d\phi/d\tau = -\mu, \tag{B2}$$

$$d\theta/d\tau = T. \tag{B3}$$

Define the entropy by the equation

$$TdS = d\mu - \rho_0^{-1} dp. \tag{B4}$$

Require conservation of entropy[28] and baryons during motions of the fluid:

$$dS/d\tau = 0, \tag{B5}$$

$$(\rho_0 U^\nu)_{;\nu} = 0. \tag{B6}$$

Do not impose any other equations of motion. Then the following is an identity:

$$\mathfrak{L}_U(\mu U_\nu - \phi_{,\nu} - \theta S_{,\nu}) = \rho_0^{-1} T_\nu{}^\sigma{}_{;\sigma}, \tag{B7}$$

where $\mathfrak{L}_U$ denotes the Lie derivative[29] with respect to $\mathbf{U}$.

We note that Theorem 1 is true even if $T_\nu{}^\sigma{}_{;\sigma} \neq 0$, i.e., when $T_\nu{}^\sigma$ as defined by Eq. (B1) is not the complete stress-energy tensor of the fluid. For example, in magnetohydrodynamics Eqs. (B5) and (B6) still hold, so Theorem 1 is still valid.

Proof. The proof of Theorem 1 is an elementary exercise in Lie derivations, whose properties can be found in many references.[30] We simply note that the definitions of θ and ϕ and Eqs. (B4) and (B5) yield

$$\mathfrak{L}_U(\mu U_\nu - \phi_{,\nu} - \theta S_{,\nu}) = U^\sigma(\mu U_\nu)_{;\sigma} + \rho_0^{-1} p_{,\nu}. \tag{B8}$$

Similarly, application of Eq. (B6) to the divergence of

[27] See Seliger and Whitham (Ref. 3) or A. R. Forsythe, *Theory of Differential Equations* (Cambridge U. P., London, 1900), Vol. I.

[28] According to Ref. 8, *perfect* fluids must have $\delta q = TdS = 0$ during their motions.

[29] I am indebted to Professor K. S. Thorne for suggesting the use of Lie derivatives in proving equivalence between the two versions.

[30] See, e.g., K. Yano, *The Theory of Lie Derivatives and its Applications* (North-Holland, Amsterdam, 1955), Chap. 1.

Eq. (B1) gives

$$\rho_0^{-1}T_\nu{}^\sigma{}_{;\sigma}=U^\sigma(\mu U_\nu)_{;\sigma}+\rho_0^{-1}p_{,\nu}. \tag{B9}$$

Q.E.D.

Let us now turn to the first half of the proof of equivalence: the proof that the equations of the velocity-potential version imply those of the standard version. The velocity-potential representation of **U**, Eq. (2.21), gives

$$\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu}=\alpha\beta_{,\nu}.$$

Therefore, we have

$$\mathfrak{L}_U(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})=\mathfrak{L}_U(\alpha\beta_{,\nu})=0, \tag{B10}$$

where the last equality follows from $d\alpha/d\tau=d\beta/d\tau=0$. Then Theorem 1 gives

$$T_\nu{}^\sigma{}_{;\sigma}=0, \tag{B11}$$

which is the standard version of the equation of motion.

The second half of the equivalence proof is the proof that the equations of the standard version imply those of the velocity-potential version. We already have the three equations $d\phi/d\tau=-\mu$, $d\theta/d\tau=T$, and $dS/d\tau=0$ from the requirements of Theorem 1. We need only show that the velocity-potential representation of **U**,

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}), \tag{B12a}$$

and the two remaining equations of evolution,

$$d\alpha/d\tau=0, \tag{B12b}$$

$$d\beta/d\tau=0, \tag{B12c}$$

follow from Theorem 1 and the standard version's equations of motion,

$$T_\nu{}^\sigma{}_{;\sigma}=0. \tag{B13}$$

Equation (B13) and Theorem 1 imply

$$\mathfrak{L}_U(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})=0. \tag{B14}$$

This leads to the following theorem.

Theorem 2. There exist functions α, β, and γ such that

$$\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu}=\alpha\beta_{,\nu}+\gamma_{,\nu} \tag{B15}$$

and

$$d\alpha/d\tau=d\beta/d\tau=d\gamma/d\tau=0. \tag{B16}$$

Proof.[31] Define

$$W_\nu=\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu}. \tag{B17}$$

Then W_ν is orthogonal to and Lie-dragged by U^ν (i.e., its Lie derivative along U^ν is zero); expressed in comoving coordinates (τ, y^i such that $U^\nu=\delta_0{}^\nu$) this means $W_0=0$, $W_{i,0}=0$. Then Pfaff's theorem (Appendix A) for $N=3$ implies

$$W_i dy^i=\alpha d\beta+d\gamma, \tag{B18}$$

with α, β, and γ functions only of y^i. Consequently, Eqs. (B15) and (B16) are valid in any coordinate system. Q.E.D.

We now note that ϕ was defined only by the differential equation $d\phi/d\tau=-\mu$, so that any function independent of τ can be added to ϕ without changing any of the previous results. Such a function is γ. Consequently we can "absorb" γ into ϕ and obtain from Theorem 2 the velocity-potential representation

$$U_\nu=\mu^{-1}(\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}). \tag{B19}$$

This completes the proof that the equations of the standard version imply the equations of the velocity-potential version. The two versions are equivalent.

By way of relating Theorem 1 to results more familiar in Newtonian hydrodynamics, we establish a corollary that is a generalization of Weber's transformation.[32] Define the spacelike vector separating two neighboring particles in the fluid, δx^ν, in the following manner. Let $(\delta x^\nu)_0$ be their separation on some arbitrary initial spacelike hypersurface. Then let δx^ν be the vector that results when $(\delta x^\nu)_0$ is Lie-dragged off the initial hypersurface by the fluid's four-velocity; i.e., let δx^ν be the separation between the particles after they have advanced equal proper times off the initial hypersurface. Then by construction we have

$$\mathfrak{L}_U(\delta x^\nu)=0. \tag{B20}$$

Consequently, Theorem 1 implies (with $T_\nu{}^\sigma{}_{;\sigma}=0$)

$$\mathfrak{L}_U[(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})\delta x^\nu]=0. \tag{B21}$$

But the quantity inside the square brackets in Eq. (B21) is a scalar, and Lie differentiation of a scalar is simply differentiation in proper time:

$$\frac{d}{d\tau}[(\mu U_\nu-\phi_{,\nu}-\theta S_{,\nu})\delta x^\nu]=0. \tag{B22}$$

Define $\delta\chi$, the change in any scalar field χ along the vector δx^ν, by

$$\delta\chi=\chi_{,\nu}\delta x^\nu.$$

Then Eq. (B22) implies the following corollary of Theorem 1.

Corollary (generalized Weber's transformation). Let the subscript 0 denote the value of a quantity on some initial spacelike hypersurface, and let the subscript τ denote its value on some hypersurface advanced a proper time τ from the initial hypersurface. Then the equations of hydrodynamics are equivalent to

$$(\mu U_\nu\delta x^\nu)_\tau-(\mu U_\nu\delta x^\nu)_0=(\delta\phi)_\tau-(\delta\phi)_0+(\theta\delta S)_\tau-(\theta\delta S)_0. \tag{B23}$$

[31] This proof was kindly suggested by J. Ehlers (private communication).

[32] See H. Lamb, *Hydrodynamics* (Cambridge U. P., London, 1932), Sec. 15, for the Newtonian version of Weber's transformation in the restricted case $p=p(\rho)$. For the general $p=p(\rho,S)$, see J. Serrin, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1959), Vol. 8, Sec. 29A.

APPENDIX C: PHYSICALLY EQUIVALENT REPRESENTATIONS

Two sets of velocity potentials are said to be equivalent if they give the same four-velocity for the same thermodynamic state of the fluid. The purpose of this appendix is to derive the equations of transformation whereby one set of velocity potentials may be obtained from an equivalent one and thereby to determine how much "gauge freedom" one has to choose the potentials arbitrarily.[33]

Equivalent sets by definition have the same μ and S. We therefore seek transformations between two sets of potentials $(\phi,\alpha,\beta,\theta)$ and $(\phi',\alpha',\beta',\theta')$ such that [from Eq. (2.21)]

$$\phi_{,\nu}+\alpha\beta_{,\nu}+\theta S_{,\nu}=\phi'_{,\nu}+\alpha'\beta'_{,\nu}+\theta' S_{,\nu}. \tag{C1}$$

The potentials must individually satisfy these equations:

$$d\phi/d\tau=d\phi'/d\tau=-\mu, \tag{C2a}$$

$$d\theta/d\tau=d\theta'/d\tau=T, \tag{C2b}$$

$$dS/d\tau=d\alpha/d\tau=d\alpha'/d\tau=d\beta/d\tau=d\beta'/d\tau=0. \tag{C2c}$$

We write Eq. (C1) in a more useful form:

$$\phi_{,\nu}-\phi'_{,\nu}=\alpha'\beta'_{,\nu}-\alpha\beta_{,\nu}+(\theta'-\theta)S_{,\nu}. \tag{C3}$$

In general, ϕ and ϕ' will differ by some scalar field F:

$$\phi-\phi'=F. \tag{C4}$$

By Eq. (C2a) we have

$$dF/d\tau=F_{,\nu}U^{\nu}=0. \tag{C5}$$

Equation (C3) becomes

$$F_{,\nu}=\alpha'\beta'_{,\nu}-\alpha\beta_{,\nu}+(\theta'-\theta)S_{,\nu}. \tag{C6}$$

As we shall see, each different choice of F generates a different equivalence transformation. The only restriction on the choice of F is Eq. (C5). Accordingly, we can take F to be some arbitrary function of any three functions that are independent of τ. Equation (C6) suggests the choice

$$F=F(\beta,\beta',S). \tag{C7}$$

Differentiation of F gives

$$F_{,\nu}=\frac{\partial F}{\partial\beta}\beta_{,\nu}+\frac{\partial F}{\partial\beta'}\beta'_{,\nu}+\frac{\partial F}{\partial S}S_{,\nu}. \tag{C8}$$

Treating α and α' as independent variables for the moment, we have

$$\partial F/\partial\alpha=0, \tag{C9a}$$

$$\partial F/\partial\alpha'=0. \tag{C9b}$$

Having chosen some F and found its derivatives, we see that F will generate an equivalence transformation if and only if it satisfies Eq. (C6). Comparison with Eq. (C8) reveals the equations[34]

$$\alpha'=\partial F/\partial\beta', \tag{C9c}$$

$$\alpha=-\partial F/\partial\beta, \tag{C9d}$$

$$\theta'-\theta=\partial F/\partial S. \tag{C9e}$$

Thus the function F generates a transformation from $(\phi,\alpha,\beta,\theta)$ to $(\phi',\alpha',\beta',\theta')$. We include Eqs. (C9a) and (C9b) as a formal device that will enable us to obtain other equivalence transformations in the following paragraphs. Equations (C2b) and (C2c) are clearly fulfilled.

The restriction of F to functions of β, β', and S can be relaxed by a device called the Legendre transformation. For example, define

$$F=F_2(\alpha,\beta',S)-\alpha\beta. \tag{C10}$$

The subscript 2 distinguishes this form of F from Eq. (C7). Then Eqs. (C9) become, in terms of F_2,

$$\alpha'=\partial F_2/\partial\beta', \tag{C11a}$$

$$0=\partial F_2/\partial\beta, \tag{C11b}$$

$$\theta'-\theta=\partial F_2/\partial S, \tag{C11c}$$

$$\beta=\partial F_2/\partial\alpha, \tag{C11d}$$

$$0=\partial F_2/\partial\alpha'. \tag{C11e}$$

From Eq. (C4) we find

$$\phi'-\phi=-F_2+\alpha\partial F_2/\partial\alpha. \tag{C11f}$$

Notice that these equations would also follow directly from Eqs. (C6) and (C10). One special case of this type is the identity transformation, generated by

$$F_2=\alpha\beta'. \tag{C12}$$

Then Eqs. (C11) give

$$\alpha'=\alpha, \tag{C13a}$$

$$\beta'=\beta, \tag{C13b}$$

$$\theta'=\theta, \tag{C13c}$$

$$\phi'=\phi. \tag{C13d}$$

Infinitesimal transformations can be generated by a function G added to the identity generator:

$$F_2=\alpha\beta'+\epsilon G(\alpha,\beta',S), \tag{C14}$$

where ϵ is the infinitesimal parameter. The resulting transformation is

$$\alpha'=\alpha+\epsilon\partial G/\partial\beta', \tag{C15a}$$

[33] For a brief but similar analysis of the Clebsch representation, see C. Eckart, Phys. Fluids **3**, 421 (1960), Appendix. For a review of contact transformations and their use in classical mechanics, see H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass. 1950), Chap. 8.

[34] The physical interpretation of these and other equations of transformation is discussed more fully in Sec. II B.

$$\beta' = \beta - \epsilon \partial G/\partial \alpha , \tag{C15b}$$

$$\theta' = \theta + \epsilon \partial G/\partial S , \tag{C15c}$$

$$\phi' = \phi + \epsilon(\alpha \partial G/\partial \alpha - G) . \tag{C15d}$$

By analogy with F_2 we can define two other types of generating functions:

$$F = F_3(\alpha, \alpha', S) - \alpha\beta + \alpha'\beta' \tag{C16}$$

and

$$F = F_4(\beta, \alpha', S) + \alpha'\beta' . \tag{C17}$$

The nontrivial equations of transformation generated by F_3 are

$$\beta' = -\partial F_3/\partial \alpha' , \tag{C18a}$$

$$\beta = \partial F_3/\partial \alpha , \tag{C18b}$$

$$\theta' - \theta = \partial F_3/\partial S , \tag{C18c}$$

$$\phi' - \phi = -F_3 + \alpha \partial F_3/\partial \alpha + \alpha' \partial F_3/\partial \alpha' . \tag{C18d}$$

The corresponding set for F_4 is

$$\beta' = -\partial F_4/\partial \alpha' , \tag{C19a}$$

$$\alpha = -\partial F_4/\partial \beta , \tag{C19b}$$

$$\theta' - \theta = \partial F_4/\partial S , \tag{C19c}$$

$$\phi' - \phi = -F_4 + \alpha' \partial F_4/\partial \alpha' . \tag{C19d}$$

The generator $F_4 = -\alpha'\beta$ also generates the identity transformation and can serve as a starting point for infinitesimal transformations. A special case of F_4 is

$$F_4 = -\alpha' g(\beta) , \tag{C20}$$

which generates

$$\beta' = g(\beta) , \tag{C21a}$$

$$\alpha' = \alpha (dg/d\beta)^{-1} , \tag{C21b}$$

$$\theta' = \theta , \tag{C21c}$$

$$\phi' = \phi . \tag{C21d}$$

This is the simplest equivalence transformation; it just reshuffles α and β without touching ϕ and θ.

Notice that if β' is not a monotonic function of β in Eqs. (C21), α' will be infinite wherever $d\beta'/d\beta = 0$. This divergence is not of course physically observable. In fact, it ensures that the term $\alpha'\beta'_{,\nu}$ in the velocity-potential representation will equal $\alpha\beta_{,\nu}$. This example is an omen: Ill-chosen transformations will introduce divergences into some of the velocity potentials in order to keep the observables of the fluid's motion unchanged under the transformation.

Inconvenient as such divergences are, they do not fundamentally affect the gauge freedom in ϕ, α, β, and θ. Suppose one has a set of velocity potentials that determines the thermodynamic condition and motion of a fluid. An equivalent set can be obtained by choosing the value of any one potential arbitrarily at each point on the initial hypersurface. The equations of transformation then show how the initial values of the other three potentials must be changed in compensation. (Only initial values are affected because $dF/d\tau = 0$.) It is not possible to choose a second potential arbitrarily at every point of the initial hypersurface without affecting the value of the first one. None of the transformations that leave one potential invariant have enough freedom to permit choice of a second one arbitrarily at every point. A simple example is Eq. (C21), which transforms α and β but leaves ϕ and θ alone. It permits only transformations that leave surfaces of constant β invariant: Choosing β at one point fixes its value on a whole two-dimensional subspace of the hypersurface. We therefore conclude that the initial value of one and only one potential is completely arbitrary. The remaining initial values are constrained (but not fully determined) by the physical condition of the fluid.[35]

I thank Professor Kip S. Thorne for pointing out that the arbitrariness of one potential is consistent with intuitive ideas of the number of degrees of freedom in a fluid. That is, it should be possible to describe a fluid completely with five functions at each point: two thermodynamic variables (μ and S) and three independent components of velocity. Because we use six potentials to describe the fluid, one and only one of them must be completely arbitrary.

APPENDIX D: RESTRICTED INITIAL-VALUE FORMULATION

Whereas in Appendix C we began with a physical situation and asked what sets of potentials could describe that situation equally well, in this appendix we begin with the potentials and ask what physical situation they determine. Accordingly we present here two different prescriptions for constructing fluid motions from knowledge of the potentials on some initial hypersurface, under the restriction that the background metric remain unchanged by the fluid's motions.

The first prescription requires knowledge only of the potentials on an initial hypersurface, and not of their derivatives off that hypersurface:

(1) Choose an initial spacelike hypersurface Σ with future timelike normal $\mathbf{N}$. On Σ specify the thermodynamic state of the fluid by giving μ and S. Also specify the initial values of ϕ, α, β, and θ on Σ. Say nothing about their derivatives normal to Σ.

(2) From these initial values, find the three components of $\mathbf{U}$ parallel to Σ from Eq. (2.21). Then the equation $\mathbf{U}\cdot\mathbf{U} = -1$ yields a quadratic equation for

[35] The same situation exists in electromagnetism. Choice of the Lorentz gauge (which corresponds to our choosing one potential arbitrarily) does not completely fix the gauge. Other Lorentz gauges may be generated by any function Λ that satisfies the homogeneous wave equation, $\Box\Lambda = 0$. Such transformations do not establish an arbitrary gauge at every point because of the restriction on Λ, but they do modify the gauge without changing the physics.

$\mathbf{U}\cdot\mathbf{N}$. If this equation has imaginary solutions anywhere on Σ, then α, β, ϕ, and θ have been chosen wrong: They have yielded a three-space velocity parallel to Σ greater than the speed of light. This is the only consistency requirement on ϕ, α, β, and θ. If the quadratic equation has real solutions for $\mathbf{U}\cdot\mathbf{N}$ everywhere, choose the sign of $\mathbf{U}\cdot\mathbf{N}$ negative. One now has determined $\mathbf{U}$ on Σ.

(3) Using this value for $\mathbf{U}$, proceed to calculate the condition of the fluid on a hypersurface Σ' slightly advanced in time from Σ. Construct this second hypersurface by advancing off the first a proper time $d\tau$ in the direction of $\mathbf{U}$. Points of Σ and Σ' joined by $\mathbf{U}$ we shall call "corresponding points." The values of S, α, and β at corresponding points are equal. The value of θ has increased from any point in Σ to the corresponding point of Σ' by the amount $Td\tau$, while that of ϕ has decreased by $\mu d\tau$.

(4) Finally, use the equation $(\rho_0 U^\nu)_{;\nu}=0$ to relate the (as yet unknown) values of ρ_0 and $\mathbf{U}\cdot\mathbf{N}'$ on Σ' (where $\mathbf{N}'$ is the future timelike normal to Σ'). Use the equation of state to express ρ_0 in terms of S and μ; because S is known on Σ', one now has a relation between μ and $\mathbf{U}\cdot\mathbf{N}'$ there. Equation (2.21) yields a relation between μ and the *spatial part* of $\mathbf{U}$ on Σ', since only derivatives of ϕ, β, and S parallel to Σ' are known. Use the equation $\mathbf{U}\cdot\mathbf{U}=-1$ to get a third relation, this one among μ, the spatial part of $\mathbf{U}$, and $\mathbf{U}\cdot\mathbf{N}'$. Solve these relations simultaneously for μ and the four components of $\mathbf{U}$ on Σ'. One now has enough information to advance to a third hypersurface, and so on.

In step (2) we imposed the consistency requirement that the spatial velocity of the fluid on the initial hypersurface be less than that of light. Are we guaranteed that the solutions in step four for μ and $\mathbf{U}$ on the new hypersurface will satisfy this requirement: Will μ and all the components of $\mathbf{U}$ be real? It is not hard to show that if the initial conditions are so chosen that the spatial part of $\mathbf{U}$ is zero, and if there are no infinite gradients of p, then the relations of step (4) imply that, on the new hypersurface, $\mathbf{U}\cdot\mathbf{N}'=-1+O(d\tau^2)$, the spatial part of $\mathbf{U}$ is $O(d\tau)$, and μ has changed to order $d\tau$: i.e., that the new condition of the fluid is physically acceptable. Moreover, any physical situation that satisfies the consistency requirement of step (2) admits of a choice of initial spacelike hypersurface on which the spatial part of $\mathbf{U}$ is zero. Since the physics cannot be affected by such a choice, and since the equations of motion in the potential representation are not affected by such a choice, we conclude that if the potentials are constructed to be self-consistent on some initial hypersurface, then they will remain self-consistent throughout spacetime if infinite gradients of p do not develop.

The second prescription for constructing the fluid motions from the potentials is more complex. It does not require knowledge of the initial thermodynamic state of the fluid but does require knowledge of the derivatives of ϕ and θ off the initial hypersurface:

(1) On Σ specify $\alpha, \beta, \phi, \phi_{,\nu}, \theta, \theta_{,\nu}$ and the equation of state. Note that μ, S, and the derivatives of α, β, and S normal to Σ are unnecessary.

(2) From the known data, determine $\mathbf{U}$ and the thermodynamic state of the fluid on Σ in the following manner. The equation $\mathbf{U}\cdot\mathbf{U}=-1$ gives a relation between $\mathbf{U}$ (the part of $\mathbf{U}$ parallel to Σ) and $\mathbf{U}\cdot\mathbf{N}$; let us write this as $A(\mathbf{U}\cdot\mathbf{N}, \mathbf{U})=0$. The equation $U^\nu\theta_{,\nu}=T$ similarly gives a relation of the form $B(\mu,S,\mathbf{U}\cdot\mathbf{N},\mathbf{U})=0$ after the equation of state has been used to express T in terms of μ and S. The equation $U^\nu\phi_{,\nu}=-\mu$ gives a third relation: $C(\mu,\mathbf{U}\cdot\mathbf{N},\mathbf{U})=0$. We therefore have three relations in six unknowns. They can be solved[36] to express three of the unknowns in terms of the other three. Thus we can write $\mu=f(\mathbf{U})$, $S=g(\mathbf{U})$, and $\mathbf{U}\cdot\mathbf{N}=h(\mathbf{U})$. Finally, we use the potential representation, Eq. (2.21), to determine $\mu\mathbf{U}-\theta\nabla S$, a three-vector parallel to Σ. Because we know μ and S in terms of $\mathbf{U}$, we can solve for the three components of $\mathbf{U}$. From these we determine μ, S, and $\mathbf{U}\cdot\mathbf{N}$.

(3) We now have as much information as at the end of step (2) of the first prescription. To find the condition of the fluid on Σ', follow steps (3) and (4) of the first prescription.

The second prescription distinguishes between what we refer to in the text as initial-value parameters and dynamical variables. The initial data for the dynamical variables ϕ and θ were their values on Σ plus their derivatives off it. By contrast, only the initial values of α and β are required. This breakup of initial data is not unique, however. One could have specified the derivatives of, say, ϕ and α normal to Σ; the calculations would in fact have been easier.

[36] As in the first prescription, if these equations have complex solutions, the initial data have been chosen inconsistently.