PHYSICAL REVIEW D VOLUME 2, NUMBER 12 15 DECEMBER 1970

Gravitational Field of Shells and Disks in General Relativity*

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(Received 29 July 1970)

The problem of obtaining the gravitational field of static, axially symmetric, thin shells is elucidated. In particular, a clear distinction between global and local frames is made. An algorithm is given for obtaining the fields of disks. There are two significant gravitational potentials λ and ϕ. The potential λ is straightforwardly determined from the radial stresses by solving a two-dimensional potential problem. This potential is analytic everywhere except on the disk and, together with its stream function $\bar{z}$, can be used to generate a conformal transformation which brings the equation for ϕ into the form of Laplace's equation. This potential can then be found by solving a Neumann boundary-value problem. However, the surface in the new coordinate system is not a disk since $\bar{z}$ is discontinuous across the disk. This is due to the fact that the Cauchy-Riemann equations imply that if the normal derivative of $\bar{\rho}$ is discontinuous, then the tangential derivative of $\bar{z}$ will be discontinuous.

I. INTRODUCTION

RECENTLY, a general method has been given[1] for finding the gravitational field of static, axially symmetric disks which have only transverse stresses (i.e., $T_\rho{}^\rho=0$). These disks can be supported either by hoop stresses or by the dynamic action of counter-rotating dust. In this paper, a general treatment of static, axially symmetric, thin shells is given. Due to the high degree of symmetry of these shells, it is not necessary to use the elegant methods of Israel[2] in the matching procedure, and it is possible to gain greater insight into the problem.

Weyl and Levi-Civita[3] found the general solution of Einstein's gravitational field equations in a region of empty space for a static, axially symmetric system. This was accomplished by constructing in empty space an intrinsic coordinate system whose only degree of freedom ϕ is a harmonic function. In the Newtonian limit, ϕ becomes the gravitational potential. We shall show that it is not, however, possible to obtain the gravitational field of a thin shell by a simple matching of two of these solutions across a shell with physically realistic sources. The only exceptions to this statement are the solutions given in Ref. 1. The problem is that the intrinsic coordinates of Weyl and Levi-Civita are only defined in empty space and do not match simply at the surface.[4] To illustrate these points we discuss the Schwarzschild and Curzon solutions.

In the last section it is shown that for disks there is a simple algorithm for constructing exact solutions with physically reasonable sources.

The motivation for this work lies in the need for an understanding of the effects of strong gravitational fields in nonspherically symmetric spaces.

II. GRAVITATIONAL FIELD OF STATIC AXIALLY SYMMETRIC BODIES

Consider a static, axially symmetric space which is endowed with an axis of symmetry. There is then a coordinate system t, ρ, χ, z, in which $g_{t\chi}=g_{t\rho}=g_{tz}=g_{\chi\rho}=g_{\chi z}=0$, and in which the remaining components of the metric are functions only of ρ and z.[5] We assume that this coordinate system is quasicylindrical. By this we mean that the coordinate ρ vanishes on the axis of symmetry and, for fixed z, increases monotonically to infinity, while the coordinate z, for fixed ρ, increases monotonically from $-\infty$ to $+\infty$. In other words, the ρ, z half-plane is parametrized in the same way as the usual cylindrical coordinate system. These considerations will play an important role in the following work. The azimuthal angle χ runs from 0 to 2π, as usual.

It is well known that a two-dimensional space whose metric obeys some smoothness condition, for example, a Hölder condition, is piecewise, conformally flat. In other words there exists a (isothermal) coordinate system in which $g_{ij}=\alpha\delta_{ij}$ (i, $j=1$, 2).[6] It can be shown, by using

* Supported in part by National Science Foundation under Grant No. GP-13959 at the University of Nebraska; and National Science Foundation under Grants No. GP-15911, GP-9114, GP-19887, and the Office of Naval Research under Contract No. Nonr-220(47), at the California Institute of Technology.

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[1] T. Morgan and L. Morgan, Phys. Rev. **183**, 1097 (1969); **188**, 2544(E) (1969).

[2] W. Israel, Nuovo Cimento **44B**, 1 (1966). In this paper Israel expresses the junction conditions of Lanczos, O'Brien, and Synge and of Lichnerowizc in a geometric context by using the extrinsic curvature of the embedding of the surface.

[3] See J. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1964) for a discussion of this work and a complete list of references.

[4] See H. Muller zum Hagen, Proc. Camb. Phil. Soc. **66**, 155 (1969), for a proof that in the case of spherical symmetry one cannot in general extend the usual Weyl cover throughout the interior of the body. This result holds in a much more general context.

[5] P. G. Bergamann, in *Handbuch der Physik IV*, edited by S. Flügge (Springer-Verlag, Berlin, 1962), p. 227.

[6] S. Chern, Comment Math. Helv. **28**, 301 (1954).

the results of Rešetnjak,[7] that if the Gaussian curvature of the ρ, z half-plane is bounded, then there exists an isothermal coordinate system which covers the entire half-plane. It is probably not physically restrictive to assume that there is a quasicylindrical coordinate system as defined above in which $g_{\rho\rho}=g_{zz}$ and $g_{\rho z}=0$ throughout the entire ρ, z half-plane, and in which the metric is everywhere continuous. The metric now has the convenient form

$$ds^2=-e^{2\phi}dt^2+e^{2\sigma-2\phi}(d\rho^2+dz^2)+\lambda^2e^{-2\phi}d\chi^2\,, \quad (2.1)$$

where the potentials λ, ϕ, σ are continuous functions of ρ and z. It is first necessary to investigate the properties of these potentials which are a consequence of only Eq. (2.1) and our assumption that the coordinate system is quasicylindrical. These are

(i) $\lambda=0$ on the axis since the circumference of small circles about the axis must tend to zero as $\rho\rightarrow 0$.

(ii) $\sigma\rightarrow\ln(\lambda/\rho)$ on the axis. This will be shown below. Here we merely note[8] that if it were not so, then the ratio of the circumference of a circle about the axis to its ratio would not approach 2π as $\rho\rightarrow 0$; this means that space would not be locally Euclidean on the axis.

Since we are interested in the gravitational field due to finite bodies, we shall, for definiteness, assume that space is asymptotically flat. We choose the scale of our time coordinate so that $\phi\rightarrow 0$ at spatial infinity, and the scale of our spatial coordinates so that $\lambda\rightarrow\rho$ and $\sigma\rightarrow 0$ at spatial infinity.

We now consider coordinate transformations which preserve the form of Eq. (2.1), the quasicylindrical nature of the coordinate system, the continuity of the metric, and the asymptotic behavior of the potentials. The time coordinate t and the azimuthal angle χ are defined completely, except for their zeros, which play no role in this work. Since we assume that the metric of the ρ, z half-plane is isothermal, the coordinate transformation

$$\rho\rightarrow\bar{\rho}(\rho,z)\,,\quad z\rightarrow\bar{z}(\rho,z) \quad (2.2)$$

must be a conformal transformation, that is $\bar{\rho}$ and $\bar{z}$ obey the Cauchy-Riemann equations

$$\bar{\rho}_{,\rho}=\bar{z}_{,z}\,, \quad (2.3a)$$

$$\bar{\rho}_{,z}=-\bar{z}_{,\rho}\,, \quad (2.3b)$$

the Jacobian of the transformation must be nonzero if the transformation is to be locally one to one, and the partial derivatives of $\bar{\rho}$ and $\bar{z}$ must be continuous, if the potentials, which are continuous in the original coordinate frame, are to be continuous in the new frame. Thus, $\bar{\rho}$ must be a solution to the two-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial\rho_3}+\frac{\partial_3}{\partial z^2}\right)\bar{\rho}=0\,. \quad (2.3c)$$

It is important to realize that the only solution to this equation, which is everywhere regular in the ρ, z half-plane and which obeys the boundary conditions $\bar{\rho}=0$ for $\rho=0$ and $\bar{\rho}=\rho$ at infinity, is the trivial solution $\bar{\rho}=\rho$. Consequently, the only coordinate transformation which preserves the above conditions everywhere is $\bar{\rho}=\rho$, $\bar{z}=\pm z+z_0$, $t=t+t_0$, $\chi=\pm\chi+\chi_0$. Hence, we have the important result that this coordinate system is an intrinsic coordinate system. We shall call this the global Weyl coordinate system.

It is, however, possible to perform conformal coordinate transformations in regions of the ρ, z half-plane. In these regions $\bar{\rho}$ and $\bar{z}$ will be analytic functions of ρ and z, but the coordinate system will not be extendable to the entire half-plane. The potentials ϕ and λ transform as scalars; however, $g_{\rho\rho}$ transforms as a density of weight 1 so that in the $\bar{\rho}$, $\bar{z}$ coordinate system, $\bar{\sigma}$ is given by

$$\bar{\sigma}=\sigma-\tfrac{1}{2}\ln J\,, \quad (2.4)$$

where $J=\bar{\rho}_{,\rho}\bar{z}_{,z}-\bar{\rho}_{,z}\bar{z}_{,\rho}$ is the Jacobian of the transformation. One can now see in a simple way why $\sigma\rightarrow\ln(\lambda/\rho)$ on the axis. First, note that since the analytic function $\bar{\rho}$ vanishes on the axis, $\bar{\rho}_{,\rho}\rightarrow\bar{\rho}/\rho$ and $J\rightarrow(\bar{\rho}/\rho)^2$ as $\rho\rightarrow 0$. In a neighborhood of any point on the axis one can treat ϕ as a constant, and one can introduce a Cartesian-like coordinate system in which the metric is Euclidean, i.e., $g_{ij}=e^{-2\phi}\delta_{ij}$, i, $j=1$, 2, 3. Transforming to quasicylindrical coordinates in the usual manner we then have that $g_{\rho\rho}=g_{zz}=e^{-2\phi}$ and $g_{\chi\chi}=\rho^2e^{-2\phi}$. The metric now has the form of Eq. (1) with $\sigma=0$ and $\lambda=\rho$. If we perform any conformal coordinate transformation in this neighborhood which satisfies the condition $\bar{\rho}=0$ when $\rho=0$, we find that in the new coordinate frame $\bar{\sigma}=-\ln(\bar{\rho}/\rho)=\ln(\lambda/\bar{\rho})$ on the axis. Since λ is a scalar, we obtain the result that $\lambda\rightarrow 0$ and $\sigma\rightarrow\ln\lambda/\rho$ on the axis in any coordinate system whose metric has the form of Eq. (2.1).

The field equations of general relativity are

$$G_\mu{}^\nu=-8\pi GT_\mu{}^\nu\,, \quad (2.5)$$

where $G_\mu{}^\nu=R_\mu{}^\nu-\frac{1}{2}\delta_\mu{}^\nu R$. Our notation is essentially that of Synge.[3] The field equations naturally separate into two groups. We first consider those equations which transform as scalars under ρ, z transformations. One finds from Eqs. (2.1) and (2.5) that

$$(\partial_\rho{}^2+\partial_z{}^2)\lambda=8\pi G(-g)^{1/2}(T_\rho{}^\rho+T_z{}^z) \quad (2.6)$$

and

$$\lambda(\partial_\rho{}^2+\partial_z{}^2)\phi+\nabla\lambda\cdot\nabla\phi=-4\pi G(-g)^{1/2}(T_0{}^0-T_i{}^i)\,. \quad (2.7)$$

If the right-hand side of Eq. (2.6) is given, we can solve for λ directly. Note that the boundary conditions on λ

[7] Yu. G. Rešetnyak, Dokl. Akad. Nauk SSSR (N. S.) **94**, 631 (1954); see H. Buseman's reviews in Math. Rev. **16**, 167 (1955) and *ibid.* **23**, 778 (1962). See also A. Huber, Comment Math. Helv. **34**, 99 (1960), Math. Rev. **22**, 1005 (1961).

[8] This argument is given by H. Bondi, Rev. Mod. Phys. **29**, 423 (1957).

on the half-plane are $\lambda \rightarrow \rho$ at spatial infinity and $\lambda=0$ on the axis. Using the logarithmic Green's function of two-dimensional potential theory, and image sources on the unphysical ($\rho<0$) half-plane to ensure that $\lambda=0$ on the axis, we find that

$$\lambda=\rho+\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\ln[(\rho-\rho')^2+(z-z')^2] \times\Theta(\rho',z')d\rho'dz', \quad (2.8)$$

where $\Theta(\rho,z)=8\pi G(-g)^{1/2}(T_\rho{}^\rho+T_z{}^z)$ for $\rho>0$ and $\Theta(-\rho,z)=-\Theta(\rho,z)$ for $\rho<0$.

If the right-hand side of Eq. (2.7) and λ are both known, then ϕ is, in principle, determined (it obeys the boundary condition $\phi\rightarrow 0$ at infinity). In a region of empty space, Eq. (2.7) can be written in a particularly simple form in a special coordinate system. In empty space λ is an analytic function so we may set $\bar{\rho}=\lambda$ and find $\bar{z}$ (up to a constant) from the Cauchy-Riemann equations. In this new coordinate system Eq. (2.7) becomes the three-dimensional Laplace equation in empty space. We shall refer to this coordinate system as "the local Weyl coordinate system." Note that it will only cover the entire space if $T_\rho{}^\rho+T_z{}^z$ vanishes everywhere.

There are only three remaining field equations which have to be satisfied. It is most convenient to use the two combinations of field equations

$$(-g)^{1/2}(G_z{}^z-G_\rho{}^\rho)=-8\pi G(-g)^{1/2}(T_z{}^z-T_\rho{}^\rho), \quad (2.9a)$$

$$(-g)^{1/2}G_\rho{}^z=-8\pi G(-g)^{1/2}T_\rho{}^z, \quad (2.9b)$$

and to write them as a single complex equation by introducing the operator $D\equiv\partial_\rho+i\partial_z$. Equations (2.9a) and (2.9b) are then equivalent to

$$2(D\sigma)(D\lambda)-D^2\lambda-2\lambda(D\phi)^2 =8\pi G(T_\rho{}^\rho-T_z{}^z+2iT_\rho{}^z) \quad (2.9c)$$

or to

$$D\sigma=\frac{1}{2D\lambda}[D^2\lambda+2\lambda(D\phi)^2 +8\pi G(T_\rho{}^\rho-T_z{}^z+2iT_\rho{}^z)]. \quad (2.9d)$$

The real and imaginary parts of this equation give $\sigma_{,\rho}$ and $\sigma_{,z}$, respectively, in terms of ϕ, λ, $(-g)^{1/2}(T_\rho{}^\rho-T_z{}^z)$, and $(-g)^{1/2}T_\rho{}^z$. If these two expressions are consistent, then σ can be found by straightforward integration. Note that the constant of integration is determined by the condition $\sigma\rightarrow\ln(\lambda/\rho)$ on the axis. The consistency of the two expressions in a simply connected[9] region of space requires that $(\sigma_{,\rho})_{,z}=(\sigma_{,z})_{,\rho}$ or equivalently that the imaginary part of $D^*D\sigma$ vanishes. It is easy to show that in empty space the Bianchi identities $G_\mu{}^\nu{}_{;\nu}\equiv 0$ ensure that this condition is satisfied as long as λ and ϕ are solutions of the field equations. Inside matter one must use in addition the fact that energy and momentum are locally conserved, i.e., $T_\mu{}^\nu{}_{;\nu}=0$. In a static and axial symmetric space the only nontrivial restrictions on the stresses are for $\mu=\rho$ and $\mu=z$. These restrictions are naturally called support equations.

All the potentials have now been determined and the remaining, independent, combination of field equations

$$(-g)^{1/2}(R_\rho{}^\rho+R_z{}^z)=8\pi G(-g)^{1/2}(T_0{}^0+T_\chi{}^\chi), \quad (2.10a)$$

or, equivalently,

$$(\partial^2{}_\rho+\partial^2{}_z)\lambda-2\lambda(\partial^2{}_\rho+\partial^2{}_z)\phi+2\lambda(\partial^2{}_\rho+\partial^2{}_z)\sigma +2\lambda(\nabla\phi)^2-2\nabla\lambda\cdot\nabla\phi=8\pi G(-g)^{1/2}(T_0{}^0+T_\chi{}^\chi), \quad (2.10b)$$

is satisfied as a consequence of the support equations and the above field equations. This too can be demonstrated most easily with the aid of the Bianchi identities.

III. GRAVITATIONAL FIELD OF THIN SHELLS

We have seen that in regions of empty space, there is a coordinate system (local Weyl) in which the field equations take on an extremely simple form. In this local coordinate system $\lambda=\rho$ and Eqs. (2.6) and (2.9c) are

$$\nabla^2\phi=0 \quad (3.1a)$$

and

$$D\bar{\sigma}=\rho(D\phi)^2, \quad (3.1b)$$

where $\nabla^2=(\partial^2/\partial\rho^2)+(1/\rho)(\partial/\partial\rho)+(\partial^2/\partial z^2)$ is the usual Laplacian. It might be thought that an easy way to generate solutions to Einstein's field equations is to match two empty-space solutions (i.e., Weyl solutions) across a thin shell, as in potential theory. The surface stress densities T_{ij} may have either monopole or dipole character, that is, they may be proportional to either δ functions or to normal derivatives of δ functions. T_{00} can only have monopole character since it must always be positive. We shall consider here only monopole shells since these are physically the most reasonable. Such shells will be referred to as "simple" thin shells. It is easy to show that, as in Newtonian theory, the support equations of the shell, $T_i{}^\mu{}_{;\mu}=0$, imply, for simple thin shells, that the components of the stress tensor normal to the surface vanish; that is, $T_i{}^jn_j=0$, for otherwise derivatives of δ functions would have occurred in the stresses. This means that there are three components of the stresses which must obey three support equations, so that, as in Newtonian theory, if T_{00} is specified then all the stresses are determined. However, for a disk, the support equation in the normal direction is trivially satisfied and it is possible to specify, besides T_{00}, one of the stress densities. In the next section we shall take advantage of this freedom to find exact solutions for static disks with axial symmetry.

In this section we consider the general problem of finding the gravitational field of a simple thin shell. In addition we discuss a particularly simple example which elucidates the general problem.

[9] If the region of space is not simply connected, there is an added integrability condition; see T. Morgan and H. Bondi, Proc. Roy. Soc. (to be published).

We assume that $(-g)^{1/2}(T_\rho{}^\rho+T_z{}^z)$ is known and that it behaves as a δ function on the surface. The potential λ is then found by solving Eq. (2.6) subject to the boundary conditions that λ is continuous across the surface, its normal derivative is discontinuous by a given amount, at infinity $\lambda=\rho$, and on the axis $\lambda=0$. This is a familiar problem in two-dimensional potential theory. The last boundary condition, $\lambda=0$ on the axis, can be ensured by the use of an image surface in the unphysical ($\rho<0$) half-plane. We set $\bar{\rho}=\lambda$ and solve the Cauchy-Riemann equations, Eqs. (2.3), for $\bar{z}$. These equations imply that the tangential derivative of $\bar{z}$ equals the normal derivative of $\bar{\rho}$, etc. Thus $\bar{z}$ is discontinuous across the surface while its normal derivative is continuous. This can be viewed as a two-dimensional potential problem for $\bar{z}$ with a dipole layer source. We make the important observation that since $\bar{z}$ is discontinuous, the boundaries of the inner and outer Weyl frames do not match. This feature of local Weyl coordinates greatly complicates the problem of matching potentials continuously across the surface. It appears from Eq. (3.1a) that ϕ is a harmonic function in the two local coordinate frames so that if we assume that $(-g)^{1/2}(T_0{}^0-T_i{}^i)$ is known and behaves as a δ function on the surface, then ϕ is determined by the conditions that it is continuous across the surface, that its normal derivative in the global frame is discontinuous by the given amount, and that ϕ tends to zero at infinity. Since conformal transformation preserve angles, the normal derivatives of ϕ in the global and local frames are simply related. The normal derivative of any scalar transforms as a density of weight $\frac{1}{2}$, that is, it transforms by the square root of the Jacobian. This materially simplifies the matching procedure. However, it should be pointed out that ϕ may not be a harmonic function of $\bar{\rho}$ and $\bar{z}$ in the usual sense since the $\bar{\rho}$, $\bar{z}$ coordinate system may not be quasicylindrical. This is because conformal transformations are not necessarily globally one-to-one, nor do they ensure that $\bar{\rho}$ is non-negative.

Once the potential ϕ has been determined in the local Weyl frame, the remaining potential $\bar{\sigma}$ is found from Eq. (3.1b) by a quadrature. Then σ is known in the global frame. The interesting point is that in order to find σ we need only use the empty-space equation. The explanation for this is straightforward. First, there is no *a priori* reason, from the method we have used to calculate σ, why it should be continuous across the surface. In the general case, then, the second normal derivative of σ will be the first derivative of a δ function. The stress density $T_\chi{}^\chi$ is then determined by σ and behaves as a derivative of a δ function on the surface, as can be seen from Eq. (2.10b). This contradicts our assumption of a simple thin shell, and we must therefore require that the sources for ϕ and λ are chosen in such a manner that σ is continuous across the surface. Then, in general, the normal derivative of σ will be discontinuous, and $T_\chi{}^\chi$ will behave like a δ function on the surface. It is important to note that though first derivatives of σ occur in Eq. (2.9) and though the stress sources behave as δ functions, this does not contradict the statement that σ is continuous because, for a simple thin shell, the second derivatives of λ are δ functions which cancel the stress densities on the right-hand side of the equation. This can be seen easily by using the "simple" condition $T_i{}^j n_j=0$ with Eq. (2.6). An important point to bear in mind in this work is that if a metric satisfies the field equations off the surface, and if the discontinuities in its normal derivatives are ascribed to an energy momentum source on the surface, then the Bianchi identities and the gravitational field equations ensure that the support equations are obeyed on the surface.

In order to illustrate these procedures, consider a coordinate sphere $\rho^2+z^2=a^2$ in the global Weyl frame. We define quasispherical coordinates R, θ by

$$\rho=R\sin\theta\,,\quad z=R\cos\theta\,. \tag{3.2}$$

As the source of λ we take

$$8\pi G(-g)^{1/2}(T_\rho{}^\rho+T_z{}^z)=2Aa^{-2}\sin\theta\delta(R-a)\,. \tag{3.3}$$

This form is suggested by the facts that in flat space $(-g)^{1/2}=a^2\sin\theta$ on the sphere and that in Newtonian theory $T_\rho{}^\rho+T_z{}^z$ is a constant on a spherical shell of uniform mass density. Solving for λ and transforming to the local Weyl frames, we find that

$$\bar{\rho}=\begin{cases}\rho(1-A/a^2)\,, & R\le a\\ \rho(1-A/R^2)\,, & R\ge a\end{cases} \tag{3.4a}$$

$$\bar{z}=\begin{cases}z(1-A/a^2)\,, & R<a\\ z(1+A/R^2)\,, & R>a\end{cases} \tag{3.4b}$$

$$J=\begin{cases}(1-A/a^2)^2\,, & R<a\\ (1-A/R^2)^2+4A\sin^2\theta/R^2\,, & R>a\,.\end{cases} \tag{3.4c}$$

We see that $\bar{\rho}$ is continuous while $\bar{z}$ is discontinuous across the surface, and that the normal derivative of $\bar{\rho}$ is discontinuous while the normal derivative of $\bar{z}$ is continuous. Let us first consider the case where $|A|<a^2$. The inside surface of the sphere is then mapped into a sphere of radius $a(1-A/a^2)$ while the outside surface is mapped into a prolate ($A>0$) or oblate $A<0$ spheroid with semiaxes $a(1+A/a^2)$ and $a(1-A/a^2)$. The coordinate values of the points on the two surfaces coincide only on a circle in the $z=0$ plane. One can see from Eqs. (3.4) that for $|A|<a^2$ the mappings are one-to-one and that $\bar{\rho}$ remains positive. If $A>a^2$ then $\bar{\rho}$ is negative throughout the entire inner region and part of the outer region. In this case the local Weyl coordinate systems are difficult to interpret. If $A<-a^2$ then the mapping $z\rightarrow\bar{z}$ is not one-to-one in the outer region. There are points in the new frame which have identical $\bar{\rho}$, $\bar{z}$ coordinate values, but which correspond to distinct points in the original frame. On physical grounds one would require that $A>0$ since the stresses must support the sphere against its self-attraction. We shall actually show that, since the magnitude of the energy density must be greater than the stress density, we must take $0 \leqslant A \leqslant a^2$ in the example below.

We shall now assume that $0 \leqslant A \leqslant a^2$. Since the outer surface is a prolate spheroid, it is convenient to introduce in the outer region prolate spheroidal coordinates defined by

$$\bar{\rho}^2 = (\xi^2 - d^2)(1 - \eta^2), \tag{3.5a}$$

$$\bar{z} = \xi\eta, \tag{3.5b}$$

where $d^2 = 4A$, so that the outer surface is given by $\xi = a(1 + A/a^2)$. It is important to note that ξ and η are related simply to R and θ of the global frame by

$$\xi = R(1 + A/R^2), \tag{3.6a}$$

$$\eta = \cos\theta. \tag{3.6b}$$

In terms of these coordinates the Jacobian of the outer transformation has the simple form

$$J = (\xi^2 - d^2\eta^2)/R^2. \tag{3.7}$$

Since ϕ is a harmonic function in the two local frames, we can write

$$\phi = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta), \quad R \leq a \tag{3.8a}$$

$$\phi = \sum_{l=0}^{\infty} B_l P_l(\eta) Q_l(\xi/d), \quad R \geq a \tag{3.8b}$$

where P_l and Q_l are Legendre functions of the first and second kinds, respectively. Since $\eta = \cos\theta$, we see that ϕ is continuous only if $A_l a^l = B_l Q_l(a + A/a)$. The simplest choice of the coefficients A_l and B_l is to take both the sphere and the ellipsoid to be equipotentials in which case

$$\phi = \frac{MG}{d} \ln\left(\frac{2a - d}{2a + d}\right), \quad R \leq a \tag{3.9a}$$

and

$$\phi = \frac{MG}{2d} \ln\left(\frac{\xi - d}{\xi + d}\right), \quad R \geq a \tag{3.9b}$$

where M is the mass, since at infinity $\phi \to -MG/R$. Calculating the normal derivative of ϕ in the global frame one finds for the source

$$-4\pi G(-g)^{1/2}(T_0{}^0 - T_i{}^i) = (MG \sin\theta/a)\delta(R - a). \tag{3.10}$$

We find $\bar{\sigma}$ in the local frames to be

$$\bar{\sigma} = 0, \quad R < a \tag{3.11a}$$

$$\bar{\sigma} = \frac{M^2G^2}{2d^2} \ln\left(\frac{\xi^2 - d^2}{\xi^2 - \eta^2 d^2}\right), \quad R > a \tag{3.11b}$$

so that in the global frame

$$\sigma = \ln(1 - A/a^2), \quad R < a \tag{3.12a}$$

and

$$\sigma = \frac{M^2G^2}{2d^2} \ln\left(\frac{\xi^2 - d^2}{\xi^2 - d^2\eta^2}\right) + \tfrac{1}{2} \ln\left(\frac{\xi^2 - d^2\eta^2}{R^2}\right), R > a. \tag{3.12b}$$

Note that inside the sphere, space is flat. We see that in order for σ to be continuous we must take $d^2 = M^2G^2$ or $A = M^2G^2/4$. Indeed in Newtonian theory, the stress density of a uniform sphere is given by Eq. (3.3) with this value of A. Now, calculating the discontinuity in the normal derivative of σ across the surface, we find that

$$8\pi G(-g)^{1/2} T_\chi{}^\chi = (2A/a^2) \sin\theta \delta(R - a). \tag{3.13}$$

We can now calculate the ratio of the energy density to the trace of the stress densities. We find that

$$T_0{}^0/(T_\rho{}^\rho + T_z{}^z + T_\chi{}^\chi) = 1 - (MGa/2A). \tag{3.14}$$

From general considerations, one must require that this ratio be less than -1. We shall refer to this as the stress condition. In our example it places the limit $|A| < \tfrac{1}{4}a^2$. If we allow A to approach a^2 then the stress densities become infinitely larger than the energy density, and the sphere in the global frame is mapped into a rod of length $2MG$ in the exterior local frame. The inside of the sphere is mapped onto the origin.

If we make the identification $\bar{R} = \xi + MG$, $\cos\theta = \eta$, we find that

$$\phi = \tfrac{1}{2} \ln(1 - 2MG/\bar{R}).$$

Indeed, the exterior solution for the sphere found above is just the Schwarzschild solution, where $\bar{R}$ is the usual curvature coordinate defined by the property of intrinsic spheres that their area is given by $4\pi\bar{R}^2$. Note that $\bar{R}$ is related to R by

$$\bar{R} = R\left(1 + \frac{MG}{2R}\right)^2. \tag{3.15}$$

This means that the global, isothermal coordinates used in this example are in fact isotropic coordinates, and that the solution derived above is the Schwarzschild solution in isotropic coordinates. The critical radius, at which the stresses become infinite, is the usual Schwarzschild radius, which in isotropic coordinates is $R = \tfrac{1}{2}MG$ and which in curvature coordinates is $\bar{R} = 2MG$.

A well-known problem of curvature coordinates is that several of the field equations contain only first derivatives and that if one solves for the field of a thin shell in this coordinate system one finds that ϕ is not continuous across the shell. The reason for this can be seen easily. Since the normal derivatives of the potentials are discontinuous across the shell, the rate at which spheres change their area is also discontinuous there. This means that though the mapping from isotropic coordinates to curvature coordinates is continuous, the derivatives of the mapping are discontinuous, and, hence, the transformed metric is discontinuous across the shell.

The Curzon metric, which is only defined in the exterior part of the global Weyl frame, is given by $\lambda=\rho$, $\phi=-MG/R$, and $=(-M^2G^2\rho^2/2R^4)$, where $R^2=\rho^2+z^2$. If we match this metric to an interior flat-space solution across a sphere of radius a, we find that (i) since λ is continuous across the surface $(-g)^{1/2}(T_\rho{}^\rho+T_z{}^z)=0$; (ii) since σ is discontinuous, $(-g)^{1/2}T_\rho{}^z$ and $(-g)^{1/2}(T_\rho{}^\rho-T_z{}^z)$ behave as δ functions on the surface; and (iii) $(-g)^{1/2}T_\chi{}^\chi$ behaves as the derivative of a δ function, which in turn implies that $(-g)^{1/2}T_0{}^0$ behaves as the derivative of a δ function. Inasmuch as $T_0{}^0$ must always be negative, such behavior must be ruled out on physical grounds. Consequently, a physically reasonable source of the Curzon metric cannot so easily be found.

IV. GRAVITATIONAL FIELDS OF DISKS

For an axially symmetric, simple, thin disk there is an important simplification due to the plane symmetry of the disk and the fact that there is no interior local Weyl frame. Assume that $\bar{\rho}$ and $\bar{z}$ have been solved for and that they are quasicylindrical. In general, the image of the disk under this transformation is a surface with an unphysical interior, since if the normal derivative of λ is discontinuous, then $\bar{z}$ must be discontinuous. However, ϕ, σ, and $|\bar{z}|$ are all automatically continuous because of the plane symmetry of the problem. In addition this symmetry implies that, in the global frame,

$$\phi_{,z}=\pm\tfrac{1}{2}4\pi G\int_{-\epsilon}^{+\epsilon}\frac{(-g)^{1/2}}{\lambda}(T_0{}^0-T_i{}^i)dz \qquad (4.1)$$

on the lower (+) and upper (−) surfaces of the disk. In the local Weyl frame, the normal derivative of ϕ is $(J)^{1/2}\phi_{,z}$. This means that if $(-g)^{1/2}(T_0{}^0-T_i{}^i)$ is known, then ϕ obeys a Neumann boundary-value problem, that is, ϕ is a harmonic function whose normal derivative is given on the surface. That σ is automatically continuous across the disk is a consequence of the fact that there is only one support equation to be satisfied, i.e., $T_\rho{}^\mu{}_{;\mu}=0$. On a simple disk $T_0{}^0$ and $T_\rho{}^\rho$ can be chosen independently, and the remaining stress $T_\chi{}^\chi$ is then calculated from σ using Eq. (2.10). This method of determining $T_\chi{}^\chi$ automatically ensures that the support equation is obeyed.

In summary, if $\bar{\zeta}$ is an analytic function of ρ and z everywhere except on the disk, and if it generates a conformal transformation which maps the quasicylindrical coordinates ρ, z into quasicylindrical coordinates $\bar{\rho}$, $\bar{z}$, and if ϕ is a harmonic function of $\bar{\rho}$ and $\bar{z}$, then $\bar{\zeta}$ and ϕ generate a solution of the field equations for a disk and one merely has to check whether they correspond to physically realistic sources.

As an example, consider a disk of radius d in the global frame with

$$8\pi G(-g)^{1/2}T_\rho{}^\rho=(2A\rho/d)(d^2-\rho^2)^{-1/2}\delta(z)\,. \qquad (4.2)$$

One finds that, writing $\zeta=\rho+iz$,

$$\bar{\zeta}=\zeta-(A/d)[\zeta-(\zeta^2-d^2)^{1/2}] \qquad (4.3)$$

and

$$J=|\partial\bar{\zeta}/\partial\zeta|\,. \qquad (4.4)$$

The new coordinates are quasicylindrical only if $0\leqslant A\leqslant 1$, in which case the disk is mapped in an oblate $(0<A<\frac{1}{2})$ or prolate $(\frac{1}{2}<A<1)$ spheroid with semi-axes of length $(1-A)d$ and Ad. If

$$-4\pi G\lambda^{-1}(-g)^{1/2}(T_0{}^0-T_i{}^i)=S(\rho)\delta(z)\,,$$

then the normal derivative of ϕ on the upper surface, in the local frame, is given by $\frac{1}{2}J^{1/2}S(\rho)$. The Neumann problem can be solved straightforwardly since the Laplacian separates in spheroidal coordinates. However, it does not appear possible to choose an energy density for this model which satisfies the stress condition at the edge of the disk.

In a subsequent paper, specific disk models will be discussed in detail. It is of particular interest to obtain solutions for collisionless dust disks since if one has a distribution function which gives rise to a self-consistent solution to the collisionless Boltzmann equation,[10] one is assured of having a physically reasonable equation of state.

ACKNOWLEDGMENT

One of the authors (T.M.) would like to thank Professor F. Pirani for his hospitality at Kings College, where part of this work was performed.

[10] J. Synge, Trans. Roy. Soc. Canada III, **28**, 127 (1934), has given a clear exposition of the theory of statistical distributions of particles in a Riemannian background space. A. G. Walker, Proc. Edinburgh Math. Soc. **4**, 238 (1934–36) has applied Synge's work to the particular case where there are no collisions. He showed that the solution of the collisionless Boltzmann equation is equivalent to obtaining first integrals of the geodesic equations. More recently E. D. Fackerell, Astrophys. J. **153**, 643 (1968), has obtained self-consistent solutions to the collisionless Boltzmann equation for spherically symmetric systems, and J. Ipser and K. Thorne [see, for example, Astrophys. J. **154**, 251 (1968)] have developed criteria for the stability of such solutions.