Sun. In terms of $J_{2,0}$, this is[12]

$$|J_{2,0}| \leqslant 6.8\times 10^{-10} a_E^2. \tag{9}$$

The corresponding classical contribution to the two-way travel time would be, from Eq. (9),

$$|2\delta_3 T| \leqslant 0.42(a_E/R)(1-\cos\Delta\phi)\ \mathrm{km}/c. \tag{10}$$

[12] This value corresponds to a precession of the perihelion of Mercury equal to 8 sec of arc per century, which seems a reasonable upper limit for the contribution of a quadrupole moment [I. I. Shapiro, Icarus **4**, 549 (1965)]. The contribution to the general-relativity precession of the β term of the metric is -13.3 in the same units.

This classical effect is appreciably lower than the relativistic contribution due to the term in β in the metric

$$(2\delta_2 T) = -3.4\beta(a_E/R)(1-\cos\Delta\phi)\ \mathrm{km}/c. \tag{11}$$

If detection of the first relativistic effect $\delta_1 T$ shows γ to be essentially unity, then the error on the residual on the precession of Mercury (± 8 sec of arc) must reflect on the accuracy of the determination of the coefficient β which becomes $\beta = 1\pm 0.6$. Detection of the second relativity effect $\delta_2 T$ with an accuracy limited by the error on $J_{2,0}$ [Eq. (10)] would allow a determination of β to 12%.

PHYSICAL REVIEW D VOLUME 2, NUMBER 12 15 DECEMBER 1970

Electromagnetic Wave Propagation in a Linearly Accelerating Relativistic Dielectric*

J. W. RYON AND J. L. ANDERSON
Stevens Institute of Technology, Hoboken, New Jersey 07030
(Received 17 August 1970)

A general solution is obtained for the electromagnetic waves propagating in the "vertical" direction in a linear, homogeneous, isotropic, and nondispersive dielectric medium undergoing arbitrary linear acceleration. The most significant characteristic of the solution is that the velocity of the zeros of the wave is precisely that given by the relevant velocity transformation formulas.

I. INTRODUCTION

In a previous paper[1] we have developed a generally covariant formalism for handling problems in noninertial electrodynamics. The formalism employed the naturally covariant Maxwell field equations developed by Cartan,[2] Weyl,[3] and Post,[4] the homogeneous form of which are

$$F_{[\mu\nu,\rho]} = 0, \tag{1.1}$$

$$\mathcal{G}^{\mu\nu}{}_{,\nu} = 0, \tag{1.2}$$

with $F_{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ being the antisymmetric tensor and tensor density of weight $+1$ representing **E**, **B** and **D**, **H**, respectively. The necessary connection between $F_{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ for the case of the general linear, nondispersive dielectric medium is provided by the constitutive tensor density:

$$\mathcal{G}^{\mu\nu} = \tfrac{1}{2}\chi^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{1.3}$$

The covariance of Eqs. (1.1), (1.2), and (1.3) is ensured by transformations of the form

$$F_{\mu'\nu'} = A_{\mu'}{}^{\alpha} A_{\nu'}{}^{\beta} F_{\alpha\beta}, \tag{1.4}$$

$$|A_\kappa{}^{\lambda'}|\,\mathcal{G}^{\mu'\nu'} = A_\alpha{}^{\mu'} A_\beta{}^{\nu'} \mathcal{G}^{\alpha\beta}, \tag{1.5}$$

$$|A_\kappa{}^{\lambda'}|\,\chi^{\mu'\nu'\rho'\sigma'} = A_\alpha{}^{\mu'} A_\beta{}^{\nu'} A_\tau{}^{\rho'} A_\epsilon{}^{\sigma'} \chi^{\alpha\beta\tau\epsilon}, \tag{1.6}$$

$$A_\alpha{}^{\mu'} = \partial x'^{\mu}/\partial x^{\alpha}. \tag{1.7}$$

The constitutive relation (1.3) for homogeneous, isotropic, nondispersive media was shown to take the form

$$\mathcal{G}^{\mu\nu} = (-g)^{1/2}[\mu^{-1}F^{\mu\nu} + (\epsilon-\mu^{-1}) \times (F^{\mu\sigma}u_\sigma u^\nu - F^{\nu\sigma}u_\sigma u^\mu)], \tag{1.8}$$

where ϵ and μ are, respectively, the dielectric constant and the magnetic permeability, u^μ is the local four-velocity of the medium, g is the determinant of the metric $g_{\mu\nu}$, and $F^{\mu\nu}$ is obtained from $F_{\mu\nu}$ by the usual process of raising the indices.

In the following sections the formalism will be applied to a homogeneous, isotropic, nondispersive dielectric medium undergoing arbitrary linear acceleration with respect to some inertial reference frame. In Sec. II we present and discuss the relevant kinematical and dynamical aspects of the noninertial motion. The specific form of the homogeneous field equations is obtained in Sec. III, followed in Sec. IV by the development of the general form of the solution of the field

* Based on material contained in a dissertation by J. W. Ryon submitted in partial fulfillment of the requirements for the Ph.D. degree at Stevens Institute of Technology, Hoboken, N. J.
[1] J. L. Anderson and J. W. Ryon, Phys. Rev. **181**, 1765 (1969).
[2] E. Cartan, Ann. Ecole Normale Super. Sci. **41**, 1 (1924).
[3] H. Weyl, *Space-Time-Matter* (Dover, New York, 1951), pp. 110 and 220.
[4] E. J. Post, *Formal Structure of Electromagnetics* (North-Holland, Amsterdam, 1962).

equations for waves propagating in the "vertical" direction. A thorough discussion of the wave velocities and velocity transformation formulas is given in Sec. V. Lastly, in Sec. VI we examine the specific example of hyperbolic, Born rigid motion and show that the Newtonian limit is obtained for small, constant acceleration.

II. MOTION AND COORDINATES

The theory briefly outlined in Sec. I is applicable in any coordinate system to arbitrarily moving media. In this section we discuss some of the problems involved in determining those states of motion which can be realized by physical systems. This will lead into a discussion of those properties of coordinate systems and transformations which are relevant to the application of the formalism.

The motion of the points ξ fixed in a medium is usually described by the trajectories of those points which, in an arbitrary coordinate system x^μ, are relations of the form $x^\mu = z^\mu(\tau,\xi)$. The path parameter τ is chosen to normalize the four-velocity $u^\mu = dz^\mu/d\tau$ so that one has $u^\mu g_{\mu\nu} u^\nu = 1$. Not all functions $z^\mu(\tau,\xi)$ may exhibit a normalized, timelike four-velocity; those that do are called kinematically possible trajectories (KPT's). Furthermore, not all KPT's can be realized by a given medium; those which can be realized are called dynamically possible trajectories (DPT's). Thus DPT's constitute, in general, a subset of KPT's, which, in turn, constitute a subset of the possible functions $z^\mu(\tau,\xi)$. Clearly, one needs criteria for selecting DPT's from among the functions z^μ. Since relations of the form $x^\mu = z^\mu(\tau,\xi)$ may be considered as a coordinate transformation, there is an alternative formulation of the problem. One seeks criteria for selecting from the set of all coordinate transformations those admissible transformations which can be taken to represent DPT's.

In principle, the relevant criteria must be obtained from a solution of the equations of motion of the physical system of interest using the equation of state. In general, this will be a very complicated calculation to carry out in practice, particularly since one must allow for deformations of the accelerated medium. Thus, for example, in the case of a rotating solid there must be compression and/or shear motion,[5] particularly if the system is large, to prevent the outer regions from exceeding the speed of light.

Because of the complexity of the equation of motion criteria, a number of substitute criteria have been advanced, usually based on the concept of a rigid body or rigid motion. The Born[6] rigid-body condition is that the body have locally constant deformation in an instantaneous, local Lorentz rest frame. Such a body has only three degrees of freedom as shown by Herglotz[7] and Nöther,[8] but for one-dimensional translation no more are required. Recently Bennett and Anderson[9] have defined a rigid body as one for which in the instantaneous, local rest frame there is no momentum flux (Landau-Lifshitz criteria) and also its Lagrangian coordinates, referred to a suitably chosen center-of-energy trajectory, are constant. They show that such a body has six degrees of freedom and thus is in closer accord with our Newtonian conception of a rigid body.

In order to surmount the various difficulties inherent in motion of extended bodies, a simple expedient will be adopted. In later sections the homogeneous electromagnetic field equations will be solved for *arbitrary* one-dimensional translation. The problem of picking out those solutions which correspond to DPT's will be left for another investigation.

The motion of the medium is most conveniently discussed from the point of view of an observer in an inertial reference frame. This procedure is rigorously valid only if one is dealing with a flat metric, but in any region of space-time where the curvature is "small" the use of a quasilocal inertial frame is a very good approximation. Let the inertial observer specify the world line of each point $\mathbf{x}'$ of the medium as a function of the proper time τ along each trajectory. The inertial observer thus obtains a family of trajectories

$$x^\mu = x^\mu(\tau,\mathbf{x}')\,, \tag{2.1}$$

which, for a medium of infinite extent, may be taken as determining a coordinate transformation between the inertial coordinates x^μ and the accelerated coordinates $x'^\mu = (\tau,\mathbf{x}')$. Note that we have put $x'^0 = \tau$.

The general form of the transformation (2.1) for one-dimensional translation along the x, x' direction is given by

$$t = t(\tau,x')\,, \tag{2.2a}$$

$$x = x(\tau,x')\,, \tag{2.2b}$$

$$y = y'\,, \tag{2.2c}$$

$$z = z'\,, \tag{2.2d}$$

so that the transformation matrix $A_{\nu'}{}^{\mu}$ has the general form

$$A_{\nu'}{}^{\mu} = \frac{\partial x^\mu}{\partial x'^\nu} = \begin{bmatrix} A_{0'}{}^{0} & A_{1'}{}^{0} & 0 & 0 \\ A_{0'}{}^{1} & A_{1'}{}^{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

The determinant of $A_{\mu'}{}^{\nu}$ is then readily obtained:

$$A = \det|A_{\mu'}{}^{\nu}| = A_{0'}{}^{0} A_{1'}{}^{1} - A_{1'}{}^{0} A_{0'}{}^{1}\,. \tag{2.4}$$

The general form of the inverse transformation is seen

[5] B. Kursunoglu, Proc. Cambridge Phil. Soc. **47**, 177 (1961).
[6] M. Born, Physik Z. **11**, 233 (1910).
[7] G. Herglotz, Ann. Phys. Lpz. **31**, 393 (1910).
[8] E. Nöther, Ann. Phys. Lpz. **31**, 919 (1910).
[9] R. Bennett and J. L. Anderson (unpublished).

by inspection of (2.3) to be

$$A_{\mu}{}^{\nu'}=\frac{\partial x'^{\nu}}{\partial x^{\mu}}=\begin{bmatrix} A_0{}^{0'} & A_1{}^{0'} & 0 & 0\\ A_0{}^{1'} & A_1{}^{1'} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$=\frac{1}{A}\begin{bmatrix} A_{1'}{}^{1} & -A_{1'}{}^{0} & 0 & 0\\ -A_{0'}{}^{1} & A_{0'}{}^{0} & 0 & 0\\ 0 & 0 & A & 0\\ 0 & 0 & 0 & A\end{bmatrix}. \quad (2.5)$$

The metric tensor is calculated from the transformation matrix and the Lorentz metric tensor $\eta_{\mu\nu}$:

$$g_{\mu\nu}=A_{\mu'}{}^{\alpha}A_{\nu'}{}^{\beta}\eta_{\alpha\beta}$$
$$=\begin{bmatrix}(A_{0'}{}^{0})^2-(A_{0'}{}^{1})^2 & A_{0'}{}^{0}A_{1'}{}^{0}-A_{0'}{}^{1}A_{1'}{}^{1} & 0 & 0\\ A_{1'}{}^{0}A_{0'}{}^{0}-A_{1'}{}^{1}A_{0'}{}^{1} & (A_{1'}{}^{0})^2-(A_{1'}{}^{1})^2 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}. \quad (2.6)$$

The determinant of the metric g, and its inverse $g^{\mu\nu}$, are easily obtained:

$$g=-A^2, \quad (2.7)$$

$$g^{\mu\nu}=\frac{1}{g}\begin{bmatrix} g_{11} & -g_{10} & 0 & 0\\ -g_{01} & g_{00} & 0 & 0\\ 0 & 0 & -g & 0\\ 0 & 0 & 0 & -g\end{bmatrix}. \quad (2.8)$$

A point fixed in the accelerating system has a world line given by

$$\mathbf{x}'=\text{const}, \quad (2.9)$$

so that the four-velocity in the noninertial frame is

$$u^{\mu'}=(g_{00})^{-1/2}(1,0,0,0). \quad (2.10)$$

Transformation of $u^{\mu'}$ to the inertial frame yields

$$u^{\mu}=(g_{00})^{-1/2}A_{0'}{}^{\mu}. \quad (2.11)$$

The local relative velocity between the two reference frames, as determined by the *inertial* observer, is then

$$v=u^1/u^0=A_{0'}{}^{1}/A_{0'}{}^{0}. \quad (2.12)$$

A point fixed in the inertial frame has a world line given by

$$\mathbf{x}=\text{const}, \quad (2.13)$$

so that the four-velocity in the inertial frame is just

$$u^{\mu}=(1,0,0,0). \quad (2.14)$$

Transformation of u^{μ} to the accelerated frame yields

$$u^{\mu'}=A_0{}^{\mu'}. \quad (2.15)$$

The local relative velocity between the two reference frames, as determined by the *accelerated* observer, is then

$$v'=u^{1'}/u^{0'}=A_0{}^{1'}/A_0{}^{0'}, \quad (2.16)$$

Equations (2.12) and (2.16) are special cases of the velocity transformation formulas appropriate to the transformation (2.2). Section V contains a discussion of these velocity transformation laws and their relevance to the present problem.

III. HOMOGENEOUS FIELD EQUATIONS

The previous sections have provided the mathematical background for the problem of electromagnetic wave propagation in linearly accelerating systems. In this section the specific form of the homogeneous field equations is obtained on the basis of the material already presented. Later sections will deal with the solution of the field equations.

With the dielectric fixed in the accelerating reference frame discussed in Sec. II, there are two types of experiments that can be carried out corresponding to measurements made by the two observers or coordinate systems. In addition, the dielectric can be at rest in the inertial frame and the noninertial observer can conduct experiments and obtain measurements. The last remaining possibility, that of medium and observer both inertial, is trivial and will not be considered. Thus the three cases of interest can be enumerated as follows:

case I, observer inertial, medium accelerated;
case II, observer accelerated, medium inertial;
case III, observer and medium coaccelerated.

Case I is the extended Fresnel-Fizeau experiment, case II is the generalized Dufour-Prunier[10] experiment, and case III is the generalized Sagnac[11]-Harras[12]-Pogany[13] experiment. For an excellent account of the original case-II and -III experiments involving rotating dielectrics, see the review article of Post.[14]

Observe that the specification of case number (I, II, or III) serves to specify both the motion of the medium and the motion of the observer. Consequently, it is superfluous to further distinguish coordinate systems by primes or other labels on tensor components. Accordingly, we shall dispense with such labels and treat all three cases together with a unified notation.

The procedure to be followed for all cases unfolds in two stages: first, the explicit evaluation of the con-

[10] A. Dufour and F. Drunier, J. Phys. (Paris) **3**, 12 (1942).
[11] G. Sagnac, Compt. Rend. **157**, 708 (1913); **157**, 1410 (1913); J. Phys. (Paris) **4**, 177 (1914).
[12] F. Harras, dissertation, Jena, 1911 (unpublished).
[13] B. Pogany, Ann. Physik **80**, 217 (1926); **85**, 244 (1928).
[14] E. J. Post, Rev. Mod. Phys. **39**, 475 (1967).

stitutive relations (1.8) in terms of the fields **E**, **B**, **D**, and **H**; second, substitution of the constitutive relations into the field equations (1.1) and (1.2) to obtain a set of equations for the vectors **E** and **B**.

Case I

Since the case-I observer is inertial, the metric is the Lorentz metric $\eta_{\mu\nu}$. Thus $F^{\mu\nu}$ is given by

$$F^{\mu\nu}=\begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}. \tag{3.1}$$

The local four-velocity of the accelerated dielectric is given by (2.11) which, by means of (2.6), can be written

$$u^\mu=\gamma(1,v,0,0)\,, \tag{3.2}$$

where

$$\gamma=(1-v^2)^{-1/2} \tag{3.3}$$

and v is given by (2.12).

The quantity $F^{\mu\sigma}u_\sigma$ is easily shown to be

$$F^{\mu\sigma}u_\sigma=\gamma(vE_1,\, E_1,\, E_2-vB_3,\, E_3-vB_2)\,. \tag{3.4}$$

Case II

The metric appropriate to the case-II accelerated observer is given by (2.6) and (2.8). The quantity $F^{\mu\nu}$ is thus

$$F^{\mu\nu}=\frac{-1}{g}\begin{bmatrix} 0 & -E_1 & (g_{11}E_2+g_{01}B_3) & (g_{11}E_3-g_{01}B_2) \\ E_1 & 0 & (-g_{00}B_3-g_{01}E_2) & (g_{00}B_2-g_{01}E_3) \\ (-g_{11}E_2-g_{01}B_3) & (g_{00}B_3+g_{01}E_2) & 0 & gB_1 \\ (-g_{11}E_3+g_{01}B_2) & (-g_{00}B_2+g_{01}E_3) & -gB_1 & 0 \end{bmatrix}. \tag{3.5}$$

The local four-velocity of the inertial medium as measured by the accelerated observer is given by (2.15) and is reproduced here:

$$u^\mu=A_0{}^{\mu'}\,. \tag{3.6}$$

The quantity $F^{\mu\sigma}u_\sigma$ is most conveniently calculated according to

$$F^{\mu\sigma}u_\sigma=g^{\mu\alpha}u^\beta F_{\alpha\beta}\,. \tag{3.7}$$

The result is

$$F^{\mu\sigma}u_\sigma=\frac{-1}{(-g)^{3/2}}\begin{bmatrix} (g_{11}A_{0'}{}^1+g_{01}A_{1'}{}^1)E_1 \\ (g_{01}A_{0'}{}^1-g_{00}A_{1'}{}^1)E_1 \\ gA_{1'}{}^1E_2+gA_{0'}{}^1B_3 \\ gA_{1'}{}^1E_3-gA_{0'}{}^1B_2 \end{bmatrix}. \tag{3.8}$$

Case III

The metric tensor appropriate to the case-III accelerated observer is the same as for case II. Consequently the quantity $F^{\mu\nu}$ is the same for case III as for case II and is given by (3.5).

The local four-velocity of the coaccelerated medium is given by (2.10) and is reproduced here:

$$u^\mu=(g_{00})^{-1/2}(1,0,0,0)\,. \tag{3.9}$$

The quantity $F^{\mu\sigma}u_\sigma$ is again calculated according to (3.7) using (3.9) for u^μ, with the result

$$F^{\mu\sigma}u_\sigma=g^{-1}g_{00}{}^{-1/2}(g_{01}E_1,\, -g_{00}E_1,\, gE_2,\, gE_3)\,. \tag{3.10}$$

Constitutive Relations

The constitutive relations for all three cases are now calculated from (1.8) by substitution of (3.1) and (3.4); (3.5) and (3.8); or (3.5) and (3.10). The results all have the form

$$D_1=\epsilon(-g)^{-1/2}E_1\,, \tag{3.11a}$$

$$D_2=\epsilon\alpha E_2-\epsilon\lambda B_3\,, \tag{3.11b}$$

$$D_3=\epsilon\alpha E_3+\epsilon\lambda B_2\,, \tag{3.11c}$$

$$H_1=\mu^{-1}(-g)^{1/2}B_1\,, \tag{3.11d}$$

$$H_2=\epsilon\beta B_2-\epsilon\lambda E_3\,, \tag{3.11e}$$

$$H_3=\epsilon\beta B_3+\epsilon\lambda E_2\,, \tag{3.11f}$$

where the quantities α, β, and λ are defined in Table I and n is the index of refraction of the dielectric,

$$n^2=1/\epsilon\mu\,. \tag{3.12}$$

At first sight the quantities α, β, and λ appearing in Table I seem to have little in common among the three cases. However, in fact, *all* the quantities in Table I share one important algebraic property in common which makes possible the general solution technique to be developed. Thus, for all three cases the quantities α, β, and λ satisfy the coefficient condition

$$\alpha\beta+\lambda^2=1/n^2=\text{const.} \tag{3.13}$$

The proof of this is straightforward but lengthy and is relegated to Appendix A.

Also the wave speed will turn out to be expressed in terms of α, β, and λ in just the right combination so that the velocity addition formulas give sensible results. Thus the particular form of α, β, and λ is crucial for the success or failure of the theory.

Homogeneous Field Equations

The homogeneous field equations for the three cases are obtained by substituting the constitutive relations (3.11) into (1.1) and (1.2). The result is

$$0=\partial_1 B_1+\partial_2 B_2+\partial_3 B_3\,, \tag{3.14a}$$

$$0=\partial_0 B_1+\partial_2 E_3-\partial_3 E_2\,, \tag{3.14b}$$

$$0=\partial_0 B_2-\partial_1 E_3+\partial_3 E_1\,, \tag{3.14c}$$

$$0=\partial_0 B_3+\partial_1 E_2-\partial_2 E_1\,, \tag{3.14d}$$

$$0=\partial_1[E_1/\surd(-g)]+\partial_2(\alpha E_2-\lambda B_3)+\partial_3(\alpha E_3+\lambda B_2)\,, \tag{3.14e}$$

$$0=\partial_0[E_1/\surd(-g)]-\partial_2(\beta B_3+\lambda E_2)+\partial_3(\beta B_2-\lambda E_3)\,, \tag{3.14f}$$

$$0=\partial_0(\alpha E_2-\lambda B_3)+\partial_1(\beta B_3+\lambda E_2)-\partial_3[B_1(-g)^{1/2}/n^2]\,, \tag{3.14g}$$

$$0=\partial_0(\alpha E_3+\lambda B_2)-\partial_1(\beta B_2-\lambda E_3)+\partial_3[B_1(-g)^{1/2}/n^2]\,. \tag{3.14h}$$

IV. WAVE SOLUTIONS OF FIELD EQUATIONS

If the coefficients of the fields, namely, α, β, λ, and $(-g)^{\pm 1/2}$, in (3.14) were independent of the coordinates x^μ then the solution would be of the usual exponential form $e^{-ik_\mu x^\mu}$ with k_μ being independent of x^μ. The fact that the coefficients of the fields *do* depend upon the coordinates substantially complicates the situation. However, the coefficients *are* independent of y and z so that the dependence of the fields upon these two coordinates is still just proportional to $e^{ik_2 y+ik_3 z}$. This means that everywhere in the field equations one can replace ∂_2 and ∂_3 by ik_2 and ik_3, respectively. The t, x dependence of the fields is much more interesting and revealing and may be studied in isolation by the expedient of setting k_2 and k_3 equal to zero. In other words, we seek a solution which depends only upon t and x and which represents a wave traveling in the x direction. Such a wave is said to propagate in the vertical direction. Under these conditions the terms in ∂_2 and ∂_3 vanish and the field equations (3.14) reduce to

$$0=\partial_0 B_1=\partial_1 B_1\,, \tag{4.1a}$$

$$0=\partial_0[E_1/\surd(-g)]=\partial_1[E_1/\surd(-g)]\,, \tag{4.1b}$$

$$0=\partial_0 B_2-\partial_1 E_3\,, \tag{4.1c}$$

$$0=\partial_0(\alpha E_3+\lambda B_2)-\partial_1(\beta B_2-\lambda E_3)\,, \tag{4.1d}$$

$$0=\partial_0 B_3+\partial_1 E_2\,, \tag{4.1e}$$

$$0=\partial_0(\alpha E_2-\lambda B_3)+\partial_1(\beta B_3+\lambda E_2)\,. \tag{4.1f}$$

It is evident immediately from (4.1a) and (4.1b) that B_1 and $E_1/\surd(-g)$ are constant and consequently may be taken equal to zero. Furthermore, there is only a difference in the relative sign of the field components between the pair of equations (4.1c)–(4.1d) and the pair (4.1e)–(4.1f). Thus we concentrate on a set of equations of the form

$$0=\partial_t B+\partial_x E\,, \tag{4.2a}$$

$$0=\partial_t(\alpha E-\lambda B)+\partial_x(\beta B+\lambda E)\,, \tag{4.2b}$$

where α, β, and λ are functions of t and x given in Table I.

The first step in obtaining a solution for the fields $E(t,x)$ and $B(t,x)$ is the elimination of E by means of an auxiliary function $\eta(t,x)$ defined as follows:

$$\alpha E=-\eta B\,. \tag{4.3}$$

The field equations in terms of B and η are

$$0=\partial_t B-\partial_x(\eta B/\alpha)\,, \tag{4.4a}$$

$$0=\partial_t[(\eta+\lambda)B]+\partial_x[((\lambda\eta/\alpha)-\beta)B]\,. \tag{4.4b}$$

Define a new function θ:

$$\theta=\eta+\lambda\,, \tag{4.5}$$

so that (4.4b) becomes

$$0=\partial_t(\theta B)+\partial_x[(\lambda\theta-\lambda^2-\alpha\beta)B/\alpha]\,. \tag{4.6}$$

The term $\partial_t(\theta B)$ may be rewritten using (4.4a) and (4.5):

$$\partial_t(\theta B)=B\partial_t\theta-\alpha^{-1}\eta B\partial_x\theta+\partial_x[(\theta^2-\lambda\theta)B/\alpha]\,. \tag{4.7}$$

Substitution of this result into (4.6) yields

$$0=B\partial_t\theta-\alpha^{-1}\eta B\partial_x\theta+\partial_x[(\theta^2-\lambda^2-\alpha\beta)B/\alpha]\,. \tag{4.8a}$$

This equation along with (4.4a) in the form

$$0=\partial_t B-\partial_x[(\theta-\lambda)B/\alpha] \tag{4.8b}$$

are the equations to solve. At first glance (4.8a)–(4.8b) appear to be much more complicated than (4.2a)–(4.2b) and such would indeed be the case were it not for the coefficient condition (3.13).

Substitution of (3.13) into (4.8a) results in

$$0=B\partial_t\theta-\alpha^{-1}B\eta\partial_x\theta+\partial_x[(\theta^2-1/n^2)B/\alpha]\,. \tag{4.9}$$

Now observe that one exact solution of (4.9) is the following:

$$\theta=\pm 1/n=\text{const}\,, \tag{4.10a}$$

$$\partial_t\theta=\partial_x\theta=0\,. \tag{4.10b}$$

The remaining equation for B becomes

$$0=\partial_t B+\partial_x[(\lambda\pm n^{-1})B/\alpha]\,, \tag{4.11}$$

for which wave solutions must be found. Let B have the form

$$B(t,x)=e^{i\psi(t,x)+\phi(t,x)}\,, \tag{4.12}$$

where $\psi(t,x)$ and $\phi(t,x)$ are real. Introduction of (4.12) into (4.11) followed by separation of real and imaginary parts gives

$$0=\partial_t\psi+\alpha^{-1}(\lambda\pm n^{-1})\partial_x\psi\,, \tag{4.13a}$$

$$0=\partial_t\phi+\alpha^{-1}(\lambda\pm n^{-1})\partial_x\phi+\partial_x[\alpha^{-1}(\lambda+n^{-1})]\,. \tag{4.13b}$$

TABLE I. Constitutive coefficients for linear acceleration.

	α	β	λ
Case I	$1+(1-n^{-2})\gamma^2v^2$	$n^{-2}-(1-n^{-2})\gamma^2v^2$	$(1-n^{-2})\gamma^2v$
Case II	$(-g)^{-1/2}[(A_{1'}{}^1)^2-n^{-2}(A_{1'}{}^0)^2]$	$(-g)^{-1/2}[n^{-2}(A_{0'}{}^0)^2-(A_{0'}{}^1)^2]$	$(-g)^{-1/2}[n^{-2}A_{0'}{}^0A_{1'}{}^0-A_{1'}{}^1A_{0'}{}^1]$
Case III	$(-g)^{1/2}[(1-n^{-2})(g_{00})^{-1}+g_{11}/n^2g]$	$g_{00}/n^2\sqrt{(-g)}$	$g_{01}/n^2\sqrt{(-g)}$

Thus ψ satisfies a homogeneous equation while ϕ satisfies an inhomogeneous equation. Once these first-order partial differential equations have been solved the problem has been solved: B is obtained from (4.12) and E is obtained from (4.3).

The real and imaginary parts of (4.13) are solutions of (4.11) and are

$$B_R=e^{\phi}\cos\psi\,, \tag{4.14a}$$

$$B_I=e^{\phi}\sin\psi\,, \tag{4.14b}$$

and take the form of an oscillation determined by ψ with an amplitude determined by ϕ. The question of the velocity of such a wave is complicated by the fact that different parts of the wave travel at different rates. The speed is usually determined from the condition that the change in B vanish for increments in t and x. This leads to

$$\begin{aligned}0=dt(\partial_t\phi\cos\psi-\partial_t\psi\sin\psi)&\\+dx(\partial_x\phi\cos\psi-\partial_x\psi\sin\psi)\,,&\quad\text{for}\quad B_R\end{aligned} \tag{4.15a}$$

$$\begin{aligned}0=dt(\partial_t\phi\sin\psi+\partial_t\psi\cos\psi)&\\+dx(\partial_x\phi\sin\psi+\partial_x\psi\cos\psi)\,,&\quad\text{for}\quad B_I\end{aligned} \tag{4.15b}$$

from which the velocity is obtained:

$$u_R=\left.\frac{dx}{dt}\right|_{B=\text{const}}=\frac{-\partial_t\phi\cos\psi+\partial_t\psi\sin\psi}{\partial_x\phi\cos\psi-\partial_x\psi\sin\psi}\,,\ \text{for}\ B_R \tag{4.16a}$$

$$u_I=\left.\frac{dx}{dt}\right|_{B=\text{const}}=\frac{-\partial_t\phi\sin\psi-\partial_t\psi\cos\psi}{\partial_x\phi\sin\psi+\partial_x\psi\cos\psi}\,,\ \text{for}\ B_I. \tag{4.16b}$$

However, this expression becomes infinite for the maxima and minima of B which are determined by the conditions

$$\partial B_R/\partial x=0=\partial_x\phi\cos\psi-\partial_x\psi\sin\psi\,, \tag{4.17a}$$

$$\partial B_I/\partial x=0=\partial_x\phi\sin\psi+\partial_x\psi\cos\psi\,, \tag{4.17b}$$

which are just the denominators of the expressions for u. This apparent absurdity means only that the procedure fails to provide the peak velocities and that another method is required. Despite this difficulty with the peak velocities, expression (4.16) does give good results for the velocities of the zeros of B. The zeros of B_R occur when $\cos\psi$ vanishes and those of B_I when $\sin\psi$ vanishes. Under these conditions (4.16) becomes

$$u_0=-\partial_t\psi/\partial_x\psi\,. \tag{4.18}$$

Note that this result would also be obtained if the amplitude function ϕ were a constant so that both $\partial_t\phi$ and $\partial_x\phi$ would vanish. According to (4.13b), this would require the quantity $(\lambda\pm n^{-1})/\alpha$ to be independent of x.

The equation for ψ, (4.13a), may be used to express u_0 in terms of n, α, and λ:

$$u_0=(\lambda\pm n^{-1})/\alpha\,. \tag{4.19}$$

The equations for ψ and ϕ then take the simple form

$$0=\partial_t\psi+u_0\partial_x\psi\,, \tag{4.20a}$$

$$0=\partial_t\phi+u_0\partial_x\phi+\partial_xu_0\,. \tag{4.20b}$$

The relation between E and B may be expressed very simply as

$$E=u_0B\,, \tag{4.21}$$

where we have combined (4.3), (4.5), (4.10a) and (4.19).

The inhomogeneous equation for ϕ may be transformed into an equivalent homogeneous one. To do so, the term ∂_xu_0 is first written in terms of *any* solution $\bar\psi$ of the homogeneous equation (4.20a):

$$\begin{aligned}\partial_xu_0&=-\partial_x(\partial_t\bar\psi/\partial_x\bar\psi)\\&=-(\partial_t\partial_x\bar\psi)/\partial_x\bar\psi+(\partial_t\bar\psi/\partial_x\bar\psi)(\partial_x{}^2\bar\psi/\partial_x\bar\psi)\\&=-(\partial_t+u_0\partial_x)\ln\partial_x\bar\psi\,.\end{aligned} \tag{4.22}$$

Substitution of this result into Eq. (4.20b) produces the homogeneous equation

$$0=(\partial_t+u_0\partial_x)(\phi-\ln\partial_x\bar\psi)\,. \tag{4.23}$$

Thus ψ and $\phi-\ln\partial_x\bar\psi$ satisfy the same homogeneous equation. Consequently the general solution of (4.23) for ϕ may be expressed in terms of two arbitrary solutions of the homogeneous equation:

$$\phi=\bar\psi_1+\ln\partial_x\bar\psi_2\,. \tag{4.24}$$

Note that if $\bar\psi_1$ is a solution of the homogeneous equation then so is $\ln\bar\psi_1$, so that ϕ may also be written as

$$\phi=\ln(\bar\psi_1\partial_x\bar\psi_2)\,. \tag{4.25}$$

It would thus appear as if the fields $E(t,x)$ and $B(t,x)$ depend upon three arbitrary solutions of the homogeneous equation, ψ, $\bar\psi_1$, and $\bar\psi_2$. This is illusory, as will be demonstrated. Let $\theta(t,x)$ be one particular solution of the homogeneous equation (4.20a). Then *any* other solution is given by $\psi(\theta)$, where ψ is an arbitrary function of θ. The expression for ϕ will then become

$$\phi=\ln\left(\bar{\psi}_1\frac{d\bar{\psi}_2}{d\theta}\partial_x\theta\right). \tag{4.26}$$

Observe that $d\bar{\psi}_2/d\theta$ is a solution of the homogeneous equation, which means that the product $\bar{\psi}_1(d\bar{\psi}_2/d\theta)$ is also a solution. As a consequence ϕ may be written in terms of just *one* arbitrary function $\bar{\psi}(\theta)$:

$$\bar{\psi}(\theta)=\bar{\psi}_1(d\bar{\psi}_2/d\theta), \tag{4.27}$$

$$\phi=\ln[\bar{\psi}(\theta)\partial_x\theta]. \tag{4.28}$$

The form of $B(t,x)$ as given by (4.12) is finally seen to be

$$B(t,x)=(\partial_x\theta)\bar{\psi}(\theta)e^{i\psi(\theta)}. \tag{4.29}$$

The velocity of wave zeros given by (4.18) is readily expressible in terms of θ:

$$u_0=-\partial_t\theta/\partial_x\theta. \tag{4.30}$$

The form of $E(t,x)$ may be found from the relation (4.21) between $E(t,x)$ and $B(t,x)$ and the form of $B(t,x)$ given by (4.29):

$$E(t,x)=-(\partial_t\theta)\bar{\psi}(\theta)e^{i\psi(\theta)}. \tag{4.31}$$

The results of our endeavors in this section have been to obtain a solution to the homogeneous field equations for waves traveling in the $\pm x$ direction. The solution has the form

$$B(t,x)=[\partial_x\bar{\psi}(\theta)]e^{i\psi(\theta)}, \tag{4.32a}$$

$$E(t,x)=-[\partial_t\bar{\psi}(\theta)]e^{i\psi(\theta)}, \tag{4.32b}$$

where ψ and $\bar{\psi}$ are arbitrary functions of θ which is a particular solution of the homogeneous, first-order partial differential equation

$$\partial_t\theta+u_0\partial_x\theta=0. \tag{4.33}$$

The quantity u_0 is given by

$$u_0=(\lambda\pm n^{-1})/\alpha \tag{4.34}$$

and is the velocity of the zeros of the wave.

The equation for θ, (4.33), cannot be solved until one has the specific form of u_0 as a function of t and x. This requires a knowledge of α and λ as functions of t and x which can be obtained if the transformation matrix $A_{\mu'}{}^{\nu}$ is specified. However, even without a knowledge of the specific coordinate dependence of $A_{\mu'}{}^{\nu}$ one can investigate the properties of the velocity u_0. This is done in Sec. V, where it is shown that u_0 gives precisely the same results as the appropriate velocity transformation formulas.

The solution (4.32) possesses the important characteristic of superposition: The sum of any two solutions is again a solution of the same form. Thus, for example, one has

$$(\partial_x\theta)\bar{\psi}_1e^{i\psi_1}+(\partial_x\theta)\bar{\psi}_2e^{i\psi_2}=(\partial_x\theta)\bar{\psi}_3e^{i\psi_3}, \tag{4.35}$$

where

$$(\bar{\psi}_3)^2=(\bar{\psi}_1)^2+(\bar{\psi}_2)^2+2\bar{\psi}_1\bar{\psi}_2\cos(\psi_1-\psi_2), \tag{4.36}$$

$$\psi_3=\tan^{-1}\left(\frac{\bar{\psi}_1\sin\psi_1+\bar{\psi}_2\sin\psi_2}{\bar{\psi}_1\cos\psi_1+\bar{\psi}_2\cos\psi_2}\right). \tag{4.37}$$

Clearly this relation can be extended so that the sum of any number of solutions of the form (4.32) will again be a solution of the same form.

One final point should be mentioned. The solution obtained here is for waves propagating in a direction parallel to the acceleration. The solution for waves propagating perpendicular to the direction of the acceleration has yet to be found. The problem of perpendicular or "horizontal" propagation is complicated by the fact that the solution will depend upon x as well as y and t or z and t. This is because of the x dependence of the coefficients α, β, and λ in the field equations (3.14).

V. VELOCITY OF WAVE ZEROS

The velocity u_0 of the zeros of the wave which was found in Sec. IV can be investigated without an explicit solution of (4.33). We begin with a discussion of the velocity transformation law between the ordinary local three-velocities u^r and $u^{s'}$ of a physical system at the point x^μ or x'^μ as measured in the inertial and accelerated frames, respectively. The velocities are defined in terms of coordinate differentials

$$u^r=dx^r/dt, \tag{5.1a}$$

$$u^{r'}=dx^{r'}/dx'^0, \tag{5.1b}$$

so that use of the transformation matrix $A_{\mu'}{}^{\nu}$ leads to the following calculation:

$$u^r=\frac{A_{\mu'}{}^r dx'^\mu}{A_{\nu'}{}^0 dx'^\nu}=\frac{A_{0'}{}^r+A_{s'}{}^r u^{s'}}{A_{0'}{}^0+A_{s'}{}^0 u^{s'}}. \tag{5.2}$$

The ratio $A_{0'}{}^r/A_{0'}{}^0$ is the relative velocity of the two coordinate systems at the point x^μ:

$$v^r=\frac{\partial x^r/\partial x'^0}{\partial t/\partial x'^0}. \tag{5.3}$$

The velocity transformation formula thus takes the form

$$u^r=\frac{v^r+u^{s'}A_{s'}{}^r/A_{0'}{}^0}{1+u^{s'}A_{s'}{}^0/A_{0'}{}^0}. \tag{5.4}$$

An identical calculation yields the inverse transformation law

$$u^{r'}=\frac{v^{r'}+u^s A_s{}^{r'}/A_0{}^{0'}}{1+u^s A_s{}^{0'}/A_0{}^{0'}},\tag{5.5}$$

with

$$v^{r'}=\frac{\partial x'^r/\partial t}{\partial x'^0/\partial t}.\tag{5.6}$$

Note that $v^{r'}$ in (5.5) is the relative velocity of the two frames as measured in the accelerated frame and is not necessarily equal to $-v^r$ measured in the interial frame.

In our cases we are interested only in velocities having a single nonvanishing component u^1 or $u^{1'}$. Thus, with superfluous superscripts suppressed, the velocity transformations are

$$u=\frac{v+u'A_{1'}{}^1/A_{0'}{}^0}{1+u'A_{1'}{}^0/A_{0'}{}^0},\tag{5.7a}$$

$$u'=\frac{v'+uA_1{}^{1'}/A_0{}^{0'}}{1+uA_1{}^{0'}/A_0{}^{0'}},\tag{5.7b}$$

where the relative velocities are given by

$$v=A_{0'}{}^1/A_{0'}{}^0,\tag{5.8a}$$

$$v'=A_0{}^{1'}/A_0{}^{0'},\tag{5.8b}$$

and are identical to (2.12) and (2.16). In the inertial system, light signals in vacuum are propagated along the x axis with velocities given by

$$u=c=\pm1.\tag{5.9}$$

The corresponding velocities in the accelerated system are obtained from (5.7b) and (5.9):

$$c'=u'=\frac{v'\pm A_1{}^{1'}/A_0{}^{0'}}{1\pm A_1{}^{0'}/A_0{}^{0'}}.\tag{5.10}$$

Substitution of (5.8b) into (5.10) yields

$$c'=\frac{A_0{}^{1'}\pm A_1{}^{1'}}{A_0{}^{0'}\pm A_1{}^{0'}}.\tag{5.11}$$

Clearly, c' will not in general equal ±1. This is not a violation of relativity if it can be shown that c' as given by (5.11) is the quantity which causes the metric quadratic form $g_{\mu\nu}u^{\mu'}u^{\nu'}$ to vanish. We shall see that this is indeed the case.

The vanishing of the quadratic form $g_{\mu\nu}u^{\mu'}u^{\nu'}$ gives the null light cone:

$$0=g_{\mu\nu}u^{\mu'}u^{\nu'}=(u^{0'})^2[g_{00}+2g_{01}c'+g_{11}(c')^2].\tag{5.12}$$

The equation is readily solved for c':

$$c'=[-g_{01}\mp\sqrt{(-g)}]/g_{11}.\tag{5.13}$$

The substitution of

$$g_{01}=A_{0'}{}^0A_{1'}{}^0-A_{0'}{}^1A_{1'}{}^1,\tag{5.14}$$

$$g_{11}=(A_{1'}{}^0)^2-(A_{1'}{}^1)^2,\tag{5.15}$$

$$\sqrt{(-g)}=A\tag{5.16}$$

into (5.13) yields

$$c'=\frac{-A_{0'}{}^0\pm A_{0'}{}^1}{A_{1'}{}^0\mp A_{1'}{}^1}.\tag{5.17}$$

Finally we substitute the components of $A_\mu{}^{\nu'}$ from (2.5) for those of $A_{\mu'}{}^\nu$ in (5.17) to obtain

$$c'=\frac{A_0{}^{1'}\pm A_1{}^{1'}}{A_0{}^{0'}\pm A_1{}^{0'}},\tag{5.18}$$

which is identical with (5.11). Thus the coordinate velocity u' may have a value outside of the limits ±1 yet still be within the light cone. There is a simple test by which one may determine if any given function $u'(x'^0,x'^1)$ is in the light cone. Substitute the function into (5.7a) and see if the resulting values of u are within the light cone. Alternately, if the function $u'(x'^0,x'^1)$ can be expressed in the form (5.7b) with u being within the light cone, then u' is a physically realizable velocity, at least in principle.

With the preceding background in velocity transformation formulas in mind, we now turn to a consideration of u_0. The function u_0 may be calculated from the expressions for α and λ given in Table I. The result for case I is

$$u_0=\frac{(1-n^{-2})\gamma^2v\pm n^{-1}}{1+(1-n^{-2})\gamma^2v^2}=\frac{v\pm n^{-1}}{1\pm v/n}.\tag{5.19}$$

The result is immediately recognizable as the velocity addition law for inertial observers. Thus $\pm1/n$ is taken to be the speed of the wave zeros with respect to the instantaneous, local inertial rest frame of the medium. Such a result is hardly surprising and quite gratifying. Indeed, in the limit of zero acceleration such a result must be obtained or the theory would be a failure. One may wonder why there are no additional terms in (5.19) reflecting the fact that the medium is accelerated. The answer is that such terms would have to vanish with the acceleration which would require the terms to be proportional to powers of gradients of the velocity. But such gradients were explicitly excluded from the constitutive relations as given by (1.8). Thus the result can only have its inertial form (5.19).

The situation is much the same as that for accelerated clocks, the rate of which is presumed to depend only upon velocity and not at all upon acceleration or higher derivatives of velocity. For sufficiently high accelerations the presumption is clearly wrong: clocks dropped to the floor from a great height will break and cease to function. In a similar way one can expect the hypothesis of acceleration-independent constitutive relations to break down for sufficiently large accelerations. The fact that u_0 is exact and correct to all orders of v makes the empirical determination of the breakdown point simply a matter of comparing the experimental results to (5.19). Such a program, while simple in principle, may be difficult in practice.

The emergence of the velocity addition law is compelling evidence that the formalism is correct. The result is quite sensitive to the precise form of the constitutive relations and any change in the quantities α, β, and λ could alter the result appreciably. It must be admitted, however, that no proof has been offered to show the uniqueness of the result to the present formalism. The possibility exists that other constitutive relations may also yield the velocity addition law.

The velocity u_0 for case II is calculated as follows:

$$u_0=\frac{n^{-2}A_{0'}{}^0A_{1'}{}^0-A_{1'}{}^1A_{0'}{}^1\pm n^{-1}(A_{0'}{}^0A_{1'}{}^1-A_{1'}{}^0A_{0'}{}^1)}{(A_{1'}{}^1)^2-n^{-2}(A_{1'}{}^0)^2}$$

$$=\frac{-A_{0'}{}^1\pm A_{0'}{}^0/n}{A_{1'}{}^1\mp A_{1'}{}^0/n}$$

$$=\frac{A_0{}^{1'}\pm A_1{}^{1'}/n}{A_0{}^{0'}\pm A_1{}^{0'}/n}. \qquad (5.20)$$

The result is identical to the velocity transformation formula (5.7b) provided that $\pm 1/n$ is identified as the wave velocity in the inertial medium. This result is quite satisfying.

Finally for case III the velocity u_0 is calculated as follows:

$$u_0=\frac{g_{01}/n^2\sqrt{(-g)}\pm 1/n}{\sqrt{(-g)}[(1-n^{-2})(g_{00})^{-1}+g_{11}/n^2g]}$$

$$=\frac{\pm g_{00}/n\sqrt{(-g)}}{1\mp g_{01}/n\sqrt{(-g)}}. \qquad (5.21)$$

This result purports to be the speed of the wave zeros in an accelerated medium as measured by the accelerated observer. In view of the fact that the *comoving inertial* observer would measure the speed as $\pm 1/n$, the complicated expression (5.21) requires some explanation. If (5.21) is in fact correct, it should be possible to obtain the result from the velocity transformation formula (5.7b) and the wave-zero velocity measured in the inertial frame (5.19). Such is indeed the case as the following calculation shows:

$$u_0=\frac{-A_{0'}{}^1+uA_{0'}{}^0}{A_{1'}{}^1-uA_{1'}{}^0}$$

$$=\frac{-A_{0'}{}^1(1\pm v/n)+A_{0'}{}^0(v\pm n^{-1})}{A_{1'}{}^1(1\pm v/n)-A_{1'}{}^0(v\pm n^{-1})}. \qquad (5.22)$$

Substitution of (5.8a) for v leads to

$$u_0=\frac{\pm n^{-1}[(A_{0'}{}^0)^2-(A_{0'}{}^1)^2]}{A\pm n^{-1}[A_{1'}{}^1A_{0'}{}^1-A_{0'}{}^0A_{1'}{}^1]}$$

$$=\frac{\pm g_{00}/n\sqrt{(-g)}}{1\mp g_{01}/n\sqrt{(-g)}}, \qquad (5.23)$$

which is the same as (5.21).

The foregoing results are most important, in fact, essential for the success of the formalism. We have found that the expressions for wave velocity u_0, obtained on the basis of the constitutive relations (1.8) and the solution developed in Sec. IV, are precisely identical to the velocity transformation formulas, obtained on the basis of the transformation matrix and the principle that the speed of light in an instantaneous, local inertial frame is $1/n$. It should be emphasized that these results hold for arbitrary linear translation of the accelerated reference frame. One must also recall that the results are expected to be modified by sufficiently high accelerations (gravitational fields).

VI. CASE III FOR HYPERBOLIC, BORN RIGID MOTION

A solution of the first-order partial differential equation (4.33) for θ requires the explicit dependence of u_0 upon coordinates to be known. Such knowledge can be obtained only if the coordinate dependence of the transformation matrix and metric tensor are known. But the transformation matrix can be determined only from the motion of the noninertial reference frame. Thus the solution of (4.33) for θ involves all the difficulties which were discussed, and bypassed, in Sec. II.

Here, as an illustration of the kind of calculation required to find θ, and hence B and E, we shall just choose a hypothetical accelerated motion that intuition judges to be a reasonable candidate for actual accelerated motion. Thus we take the following as the world lines of the points $\mathbf{x}'$ of the noninertial system referred to the inertial frame:

$$t=a^{-1}(x')\sinh[a(x')\tau], \qquad (6.1a)$$

$$x=x_0+a^{-1}(x')\cosh[a(x')\tau], \qquad (6.1b)$$

$$y=y', \qquad (6.1c)$$

$$z=z'. \qquad (6.1d)$$

The motion represented by (6.1) is a displacement of the points $\mathbf{x}'$ in the x direction, each point $\mathbf{x}'$ undergoing the well-known hyperbolic motion with a proper acceleration $a(x')$ which is an arbitrary function of x' and independent of the proper time τ. The velocity of each point $\mathbf{x}'$, in the inertial frame, is given by

$$v=\frac{\partial x/\partial\tau}{\partial t/\partial\tau}=\frac{\sinh[a(x')\tau]}{\cosh[a(x')\tau]}=\frac{t}{x-x_0}. \tag{6.2}$$

The locus of points $\mathbf{x}'$ having constant velocity v is thus given by the *linear* relationship

$$t=v(x-x_0). \tag{6.3}$$

This implies that a succession of instantaneous Lorentz frames can be found in which *all* points $\mathbf{x}'$ are momentarily at rest. Thus, with respect to these inertial frames, the accelerated frame can be said to move as a rigid body. This is one of the simplest, nontrivial examples of Born rigid motion. For a thorough discussion of this and other cases consult, for example, the dissertation of Ryon.[15]

Relation (6.1b) evaluated at $\tau=0$ gives the deformation of the x' mesh system compared to that of x:

$$x=x_0+1/a(x'). \tag{6.4}$$

In terms of ratios of infinitesimal displacements this gives

$$A=\left.\frac{dx}{dx'}\right|_{\tau=0}=\frac{d}{dx'}\left(\frac{1}{a(x')}\right). \tag{6.5}$$

Thus the deformation is determined by the proper acceleration of each point $\mathbf{x}'$.

The transformation matrix $A_{\mu'}{}^{\nu}$ obtained by differentiation of (6.1) is

$$A_{\mu'}{}^{\nu}=\begin{bmatrix}\cosh a\tau & (\sinh a\tau-a\tau\cosh a\tau)A & 0 & 0\\ \sinh a\tau & (\cosh a\tau-a\tau\sinh a\tau)A & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{6.6}$$

The determinant of $A_{\mu'}{}^{\nu}$ is

$$\det|A_{\mu'}{}^{\nu}|=A=[dx/dx']_{\tau=0}, \tag{6.7}$$

which explains the choice of symbol in (6.5).

The metric tensor is easily shown to be

$$g_{\mu\nu}=\begin{bmatrix}1 & -Aa\tau & 0 & 0\\ -Aa\tau & A^2(a^2\tau^2-1) & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}, \tag{6.8}$$

and its determinant is

$$g=-A^2. \tag{6.9}$$

[15] J. W. Ryon, III, Ph.D. thesis, Stevens Institute of Technology, 1970 (unpublished).

The case we shall examine is case III, dielectric and observer both accelerated, which corresponds to the generalized Sagnac experiment. For this we must calculate u_0 from (5.21). This can be done by inspection with (6.8) and (6.9):

$$u_0=\pm[(nA)(1\pm a\tau/n)]^{-1}. \tag{6.10}$$

Substitution of this expression for u_0 into Eq. (4.33) for θ yields

$$(1\pm a\tau/n)\partial_\tau\theta\pm(1/nA)\partial_{x'}\theta=0. \tag{6.11}$$

The first step in obtaining a solution of (6.11) is to change variables from x' to ξ defined as

$$\xi=1/a(x') \tag{6.12}$$

so that

$$\partial_{x'}=A\,\partial_\xi. \tag{6.13}$$

Then we note the useful fact

$$\partial_\xi a=\partial_\xi(1/\xi)=-a^2. \tag{6.14}$$

Finally Eq. (6.11) may be transformed to a convenient form, first by multiplying by $a(x')$ and then by using (6.13) and (6.14). The result is

$$[\partial_\xi(\ln a\pm a\tau/n)]\partial_\tau\theta-[\partial_\tau(\ln a\pm a\tau/n)]\partial_\xi\theta=0. \tag{6.15}$$

Clearly, a particular solution of (6.15) is given by

$$\theta=-\ln a(x')\mp a(x')\tau/n. \tag{6.16}$$

The quantities $\partial_{x'}\theta$ and $\partial_\tau\theta$ are seen to be

$$\partial_{x'}\theta=a(x')A(1+a\tau/n), \tag{6.17a}$$

$$\partial_\tau\theta=\mp a(x')/n. \tag{6.17b}$$

The fields E and B are thus determined. From (4.32) we have upon substitution of (6.17)

$$B(\tau,x')=aA(1\pm a\tau/n)\bar{\psi}(\theta)e^{i\psi(\theta)}, \tag{6.18a}$$

$$E(\tau,x')=\pm(a/n)\bar{\psi}(\theta)e^{i\psi(\theta)}, \tag{6.18b}$$

with θ given by (6.16).

Let us specialize the fields (6.18) to the simplest wave-like solution possible. To do this set

$$A=1, \tag{6.19}$$

$$\bar{\psi}(\theta)=1. \tag{6.20}$$

Reference to (6.5) discloses that (6.19) is equivalent to

$$1/a(x')=x'+x_0', \tag{6.21}$$

where x_0' is a constant.

Next require that $e^{i\psi}$ vary like $e^{\mp i\omega\tau}$ at $x'=0$:

$$\psi(\theta(\tau,0))=\mp\omega\tau. \tag{6.22}$$

Now solve

$$\theta(\tau,0)=\theta_0=\ln(x_0')\mp\tau/nx_0' \tag{6.23}$$

for τ and substitute in (6.22):

$$\psi(\theta_0)=\omega nx_0'(\theta_0-\ln x_0'). \tag{6.24}$$

Put θ for θ_0 in (6.24) to obtain

$$\psi(\theta)=\omega n x_0' \ln(1+x'/x_0') \mp \omega\tau/(1+x'/x_0'). \quad (6.25)$$

This is the result we want. Observe that for x_0' large and positive, so that the following inequality holds:

$$|x'/x_0'| \ll 1, \quad (6.26)$$

the expression for ψ obtained in (6.25) can be written as

$$\psi(\theta)=\omega n x' \mp \omega\tau. \quad (6.27)$$

This will be recognized as giving the familiar plane-wave solution for the fields:

$$B(\tau,x')=a(1+a\tau/n)e^{i(kx'\mp\omega\tau)}, \quad (6.28a)$$

$$E(\tau,x')=\pm(a/n)e^{i(kx'\mp\omega\tau)}, \quad (6.28b)$$

with the dispersion law

$$k=\omega n. \quad (6.29)$$

Note that the condition (6.26) is equivalent, by virtue of (6.21), to a small, constant acceleration. Thus, once again the Newtonian limit is recovered:

$$B(\tau,x')=a_0 e^{i(kx'\mp\omega\tau)}, \quad (6.30a)$$

$$E(\tau,x')=\pm(a_0/n)e^{i(kx'\mp\omega\tau)}. \quad (6.30b)$$

VII. SUMMARY

The electromagnetic field equations for a linear, homogeneous, isotropic, nondispersive dielectric medium undergoing arbitrary (noninertial) linear displacement were obtained from the constitutive relations advanced by Anderson and Ryon. A general solution of these field equations was then developed for the waves propagating in the vertical direction, parallel to the direction of the acceleration. It was found that the expressions for the velocities of the zeros of the (quasi-sinusoidal) waves are exactly those given by the velocity transformation formulas obtained directly from the coordinate transformation and the principle that the speed of light be $1/n$ in the local, instantaneous rest frame.

The problem of picking out the physically realizable motions from the set of all coordinate transformations is a separate issue and has not been determined here. However, a reasonable candidate for accelerated motion was taken to be the case of hyperbolic, Born rigid motion. For this particular situation the electromagnetic fields for the vertical waves were worked out in detail for the case of the dielectric medium and the observer both coaccelerated. In the limit of small, constant acceleration the wave was shown to reduce to Newtonian form.

The range of possible applications of the present theory is potentially enormous. It would be most desirable if solutions for dispersive media could be found. It is not unreasonable to suppose that the combination of acceleration (or gravitational fields) and dispersion would produce some interesting new effects. The most likely practical applications appear to be interferometer, laser, or microwave devices, all of which deal with electromagnetic beams. The analysis of such devices would involve finding the wave solutions which satisfy the boundary conditions appropriate to the particular device.

APPENDIX

We with to prove the coefficient condition

$$\alpha\beta+\lambda^2=1/n^2. \quad (A1)$$

Substitute from Table I into (A1) successively for cases I, II and III.

Case I

$$\begin{aligned}\alpha\beta+\lambda^2&=[1+(1-n^{-2})\gamma^2v^2][n^{-2}-(1-n^{-2})\gamma^2v^2]\\&\qquad+(1-n^{-2})^2\gamma^4v^2\\&=n^{-2}-(1-n^{-2})^2\gamma^2v^2+(1-n^{-2})^2(1-v^2)\gamma^4v^2\\&=1/n^2. \end{aligned}\quad (A2)$$

Case II

$$\begin{aligned}\alpha\beta+\lambda^2&=-g^{-1}\{[(A_{1'}{}^1)^2-n^{-2}(A_{1'}{}^0)^2][n^{-2}(A_{0'}{}^0)^2\\&\qquad-(A_{0'}{}^1)^2]+[n^{-2}A_{0'}{}^0A_{1'}{}^0-A_{1'}{}^1A_{0'}{}^1]^2\}\\&=-g^{-1}n^{-2}[(A_{0'}{}^0A_{1'}{}^1)^2-2A_{0'}{}^0A_{1'}{}^1A_{1'}{}^0A_{0'}{}^1\\&\qquad+(A_{1'}{}^0A_{0'}{}^1)^2]\\&=-A^2/n^2g=1/n^2.\end{aligned}\quad (A3)$$

Case III

$$\begin{aligned}\alpha\beta+\lambda^2&=n^{-2}(1-n^{-2})+n^{-4}g^{-1}g_{00}g_{11}-n^{-4}g^{-1}(g_{01})^2\\&=n^{-2}(1-n^{-2})+n^{-4}=1/n^2.\end{aligned}\quad (A4)$$