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Separation of Relativity Effects and Effects due to a Quadrupole Moment of the Sun in Time-Delay Measurements*

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The general-relativistic corrections and the effects due to a quadrupole moment of the Sun in measurements of the travel time of electromagnetic signals sent from Earth to artificial planets on circular and coplanar orbits are considered. It is found that a relativistic correction due to a term in the metric of second order in the mass of the Sun could not be separated from the effect due to a quadrupole moment unless additional information were available. The use of previous optical determinations of the motion of the node and perihelion of Mercury is considered and seen as adequate for a separation of the two effects to an accuracy of a few percent of the relativistic "nonlinear" effect.

INTRODUCTION

A FEW years ago, it was pointed out[1] that measurements of the travel time of radar signals bounced off the surface of other planets could be accurate enough to reveal a general-relativistic lengthening of the optical path in a gravitational field. More recently, Schiff[2] pointed out the interest of looking for an effect resulting from a term in the metric not linear in the mass of the Sun. It appears that, at this moment at least, this effect is not within the accuracy of the bounced-radar-signal technique, and also that errors in the determination of the planet radii would prevent its detection.[3] These difficulties would be alleviated in the tracking of an artificial satellite in orbit around another planet and carrying a transponder, but inaccuracies in the evaluation of classical gravitational effects could still prevent a detection of the effect.[4] The separation of this "nonlinear" relativistic correction from the classical perturbations resulting from a solar quadrupole moment is discussed here. The procedure used is the following: The planetary perturbations and the eccentricity of the orbit of the Earth are neglected. The transponder is assumed to be on an inferior circular orbit in the ecliptic plane. The relativistic predictions for the case of spherical symmetry are then compared with the classical predictions for the case of a nonzero quadrupole moment of the Sun.[5]

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[1] I. I. Shapiro, Phys. Rev. Letters **13**, 789 (1964).

[2] L. I. Schiff (unpublished); D. K. Ross and L. I. Schiff, Phys. Rev. **141**, 1215 (1966).

[3] I. I. Shapiro, Phys. Rev. **145**, 1005 (1966).

[4] As is well known, in addition to providing a far better accuracy in the distance measurements (a few meters), such a procedure would virtually eliminate radiation pressure and solar-wind effects which would cause important perturbations on the orbit of an artificial planet.

RELATIVISTIC CORRECTIONS IN SPHERICALLY SYMMETRIC FIELD

Following Eddington,[6] Robertson,[7] and Schiff,[8] the field of a spherically symmetric Sun is described by the generalized metric which is, within the frame of a curved space time, the most general expression of a spherically symmetric field. In isotropic coordinates

$$ds^2=\left[1-2\alpha\left(\frac{r_0}{r}\right)+2\beta\left(\frac{r_0}{r}\right)^2\right]c^2dt^2 - \left[1+2\gamma\left(\frac{r_0}{r}\right)\right]\left[dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right], \quad (1)$$

where r, θ, and ϕ are the polar coordinates and t is the

[5] The nonlinear term in β could also be looked for through its effect on the precession of the perihelion of ecliptical orbits. As pointed out by I. Goldberg, Phys. Rev. **149**, 1010 (1966), the separation of the relativistic effect from any possible contribution from a quadrupole moment of the Sun would be automatically achieved by use of an artificial planet at the critical inclination ($\sim 63°$) where the later effect vanishes. In such a case, however, radiation pressure and solar-wind effects could still prevent a detection of the relativistic effect.

[6] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge U. P., New York, 1957), p. 105.

[7] H. P. Robertson, *Space Age Astronomy*, edited by A. J. Deutch and W. E. Klemperer (Academic, New York, 1962), p. 228.

[8] L. I. Schiff, Proc. Natl. Acad. Sci. U. S. **46**, 871 (1960).

coordinate time. $r_0 = GM/c^2$. G is the gravitational constant and M is the mass of the Sun. α, β, and γ are numerical coefficients equal to unity in general relativity (terms of higher order than 2 in r_0/r are not retained here).

Following Shapiro, we will assume that the orbital periods and the time at inferior conjunction are known in terms of Earth proper time. (Here, time will refer to Earth proper time as indicated by Earth-based standards and periods will be in units of Earth proper time.)

Within the metric (1), the two-way time of travel of a tracking signal to a transponder on an inferior orbit can be written in the following form[3]:

$$T = T_0 \frac{(\mathcal{T}_E^{4/3} + \mathcal{T}_T^{4/3} - 2\mathcal{T}_E^{2/3}\mathcal{T}_T^{2/3}\cos\Delta\phi)^{1/2}}{(\mathcal{T}_E^{2/3} - \mathcal{T}_T^{2/3})} + 2\delta_1 T + 2\delta_2 T\,, \quad (2)$$

where $\mathcal{T}_T$ and $\mathcal{T}_E$ are the orbital periods of the transponder and of the Earth; T_0 is the travel time at inferior conjunction. The relativistic corrections are contained in the last two terms

$$\delta_1 T = \left(\frac{r_0}{c}\right)(1+\gamma)\left\{\ln\left[\frac{a_E(R + a_E - a_T\cos\Delta\phi)}{a_T(R + a_E\cos\Delta\phi - a_T)}\right] - \frac{R}{(a_E - a_T)}\ln\left(\frac{a_E}{a_T}\right)\right\}, \quad (3)$$

where a_E and a_T are the zeroth-order approximations to the orbital radii, and

$$\delta_2 T = -\left(\frac{r_0}{3c}\right)(\gamma + 2\beta)\left(\frac{a_E + a_T}{R}\right)(1 - \cos\Delta\phi)\,. \quad (4)$$

$\Delta\phi$ is the heliocentric angular distance from the Earth to the transponder, i.e.,

$$\Delta\phi = 2\pi(\tau - \tau_0)/\mathcal{T}_s\,,$$

where $\mathcal{T}_s$ is the synodic period, τ is the Earth proper time, and τ_0 is the Earth proper time at inferior conjunction. Also, R is defined by

$$R^2 = a_E^2 + a_T^2 - 2a_E a_T \cos\Delta\phi\,.$$

The first correction ($\delta_1 T$) is larger than the second one ($\delta_2 T$) ($\leqslant\sim 60$ km/c and $\leqslant 3$ km/c) and, because of its logarithmic dependence, is easier to separate from other perturbations. It is, however, independent of the nonlinear term in the metric (term in β). The second correction depends on β but, as we shall see, would not be directly separable from a classical contribution due to a solar quadrupole moment.

CLASSICAL CORRECTIONS DUE TO SOLAR QUADRUPOLE MOMENT

The gravitational field of any given body can be expressed in terms of spherical harmonics. If the center of gravity of the Sun is chosen as the origin of the coordinate system, the coefficient of the first spherical harmonic is zero. The next one is the quadrupole moment. If the harmonics of higher order than the second one are neglected, the potential is[9]

$$U = (GM/r)[1 + (J_{2,0}/2r^2)(3\cos^2\theta - 1)]\,, \quad (5)$$

where $J_{2,0}$ is an unknown numerical coefficient (if previous optical observations are not taken into account). The effect of the quadrupole moment is to alter the classical expression for the two-way travel time of the electromagnetic signals as a function of the time on Earth according to

$$T = T_0 \frac{(\mathcal{T}_E^{4/3} + \mathcal{T}_T^{4/3} - 2\mathcal{T}_E^{2/3}\mathcal{T}_T^{2/3}\cos\Delta\phi)^{1/2}}{(\mathcal{T}_E^{2/3} - \mathcal{T}_T^{2/3})} + 2\delta_3 T\,, \quad (6)$$

with

$$\delta_3 T = -\frac{J_{2,0}}{cR}\left(1 + \frac{a_E^2 + a_T^2}{2a_E a_T}\right)(1 - \cos\Delta\phi)\,, \quad (7)$$

where the periods have the same operational meaning as they have in the relativistic expressions [Eqs. (2)–(4)].

The classical correction $\delta_3 T$ due to the quadrupole moment has the same dependence on τ as the relativistic correction $\delta_2 R$ resulting from the second-order deviation from the flat metric. If these two effects are to be separated, information has to be added to the travel-time measurements.

SEPARATION OF CLASSICAL AND RELATIVISTIC EFFECTS

The classical correction term introduced in the potential (5) by the quadrupole moment is of third order in $1/r$. The β term in the metric (1) is of second order in $1/r$. This makes the former correction relatively more important on low orbits. This suggests that the optical observations on the orbit of Mercury be used to determine the possible range of $J_{2,0}$.

If the available information is limited to secular effects (which can be measured with the highest accuracy), the coefficient $J_{2,0}$ must be determined from the observed rotation of the plane of the orbit of Mercury around the axis of the quadrupole moment. This rotation is related to $J_{2,0}$ by[10]

$$\dot{\Omega} = 3\pi J_{2,0}/a^2(1-e^2)^2 \text{ rad/revolution}\,, \quad (8)$$

where a and e are the semimajor axis and the eccentricity of Mercury. As pointed out by Dicke,[11] the observed motions of the orbital plane of Mercury set an upper limit to the possible quadrupole moment of the

[9] For purpose of simplification, we neglect the fact that the axis of rotation of the Sun does not exactly coincide with the normal to the ecliptic plane. This fact would not alter the discussion significantly.

[10] D. Brouwer, Astron. J. **64**, 378 (1959).

[11] R. H. Dicke, Nature **202**, 432 (1964).

Sun. In terms of $J_{2,0}$, this is[12]

$$|J_{2,0}| \leqslant 6.8\times 10^{-10} a_E^2. \tag{9}$$

The corresponding classical contribution to the two-way travel time would be, from Eq. (9),

$$|2\delta_3 T| \leqslant 0.42(a_E/R)(1-\cos\Delta\phi)\ \mathrm{km}/c. \tag{10}$$

[12] This value corresponds to a precession of the perihelion of Mercury equal to 8 sec of arc per century, which seems a reasonable upper limit for the contribution of a quadrupole moment [I. I. Shapiro, Icarus **4**, 549 (1965)]. The contribution to the general-relativity precession of the β term of the metric is -13.3 in the same units.

This classical effect is appreciably lower than the relativistic contribution due to the term in β in the metric

$$(2\delta_2 T) = -3.4\beta(a_E/R)(1-\cos\Delta\phi)\ \mathrm{km}/c. \tag{11}$$

If detection of the first relativistic effect $\delta_1 T$ shows γ to be essentially unity, then the error on the residual on the precession of Mercury (± 8 sec of arc) must reflect on the accuracy of the determination of the coefficient β which becomes $\beta = 1\pm 0.6$. Detection of the second relativity effect $\delta_2 T$ with an accuracy limited by the error on $J_{2,0}$ [Eq. (10)] would allow a determination of β to 12%.

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Electromagnetic Wave Propagation in a Linearly Accelerating Relativistic Dielectric*

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A general solution is obtained for the electromagnetic waves propagating in the "vertical" direction in a linear, homogeneous, isotropic, and nondispersive dielectric medium undergoing arbitrary linear acceleration. The most significant characteristic of the solution is that the velocity of the zeros of the wave is precisely that given by the relevant velocity transformation formulas.

I. INTRODUCTION

In a previous paper[1] we have developed a generally covariant formalism for handling problems in noninertial electrodynamics. The formalism employed the naturally covariant Maxwell field equations developed by Cartan,[2] Weyl,[3] and Post,[4] the homogeneous form of which are

$$F_{[\mu\nu,\rho]} = 0, \tag{1.1}$$

$$\mathcal{G}^{\mu\nu}{}_{,\nu} = 0, \tag{1.2}$$

with $F_{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ being the antisymmetric tensor and tensor density of weight $+1$ representing **E**, **B** and **D**, **H**, respectively. The necessary connection between $F_{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ for the case of the general linear, nondispersive dielectric medium is provided by the constitutive tensor density:

$$\mathcal{G}^{\mu\nu} = \tfrac{1}{2}\chi^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{1.3}$$

The covariance of Eqs. (1.1), (1.2), and (1.3) is ensured by transformations of the form

$$F_{\mu'\nu'} = A_{\mu'}{}^{\alpha} A_{\nu'}{}^{\beta} F_{\alpha\beta}, \tag{1.4}$$

$$|A_\kappa{}^{\lambda'}|\,\mathcal{G}^{\mu'\nu'} = A_\alpha{}^{\mu'} A_\beta{}^{\nu'} \mathcal{G}^{\alpha\beta}, \tag{1.5}$$

$$|A_\kappa{}^{\lambda'}|\,\chi^{\mu'\nu'\rho'\sigma'} = A_\alpha{}^{\mu'} A_\beta{}^{\nu'} A_\tau{}^{\rho'} A_\epsilon{}^{\sigma'} \chi^{\alpha\beta\tau\epsilon}, \tag{1.6}$$

$$A_\alpha{}^{\mu'} = \partial x'^{\mu}/\partial x^{\alpha}. \tag{1.7}$$

The constitutive relation (1.3) for homogeneous, isotropic, nondispersive media was shown to take the form

$$\mathcal{G}^{\mu\nu} = (-g)^{1/2}[\mu^{-1}F^{\mu\nu} + (\epsilon-\mu^{-1}) \times (F^{\mu\sigma}u_\sigma u^\nu - F^{\nu\sigma}u_\sigma u^\mu)], \tag{1.8}$$

where ϵ and μ are, respectively, the dielectric constant and the magnetic permeability, u^μ is the local four-velocity of the medium, g is the determinant of the metric $g_{\mu\nu}$, and $F^{\mu\nu}$ is obtained from $F_{\mu\nu}$ by the usual process of raising the indices.

In the following sections the formalism will be applied to a homogeneous, isotropic, nondispersive dielectric medium undergoing arbitrary linear acceleration with respect to some inertial reference frame. In Sec. II we present and discuss the relevant kinematical and dynamical aspects of the noninertial motion. The specific form of the homogeneous field equations is obtained in Sec. III, followed in Sec. IV by the development of the general form of the solution of the field

* Based on material contained in a dissertation by J. W. Ryon submitted in partial fulfillment of the requirements for the Ph.D. degree at Stevens Institute of Technology, Hoboken, N. J.

[1] J. L. Anderson and J. W. Ryon, Phys. Rev. **181**, 1765 (1969).

[2] E. Cartan, Ann. Ecole Normale Super. Sci. **41**, 1 (1924).

[3] H. Weyl, *Space-Time-Matter* (Dover, New York, 1951), pp. 110 and 220.

[4] E. J. Post, *Formal Structure of Electromagnetics* (North-Holland, Amsterdam, 1962).