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Scalar Form Factors just beyond the Tree Approximation*

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Using the algebra of currents and current divergences derived from the chiral $SU(3)$ symmetry-breaking model of Gell-Mann, Oakes, and Renner and a method of pole and cut dominance, we derive two-parameter effective-range formulas for scalar $(I=J=\frac{1}{2})K\pi$ and $(I=J=0)\pi\pi$ form factors. The fit to the $I=\frac{1}{2}$ s-wave $K\pi$ phase-shift data of Antich *et al.* favors their "lower" solution with $m_\kappa=1150$ MeV and supports the existence of a broad κ resonance. It is shown that the introduction of the similarly broad ϵ resonance which emerges from these considerations into the weak-interaction scheme $K_S{}^0\to\epsilon\to\pi\pi$ makes agreement with the experimental value of the $K_L{}^0$-$K_S{}^0$ mass difference difficult, if the $K_L{}^0$ mass shift is not small.

I. INTRODUCTION

In this paper the detailed derivation of two-parameter effective-range formulas for the s-wave $(I=J=0)\pi\pi$ and $(I=J=\frac{1}{2})K\pi$ form factors is presented. The version of the unitarized hard-meson approach[1,2] employed here combines the algebra of currents[3] with a method of pole and cut dominance.[4,5] Because the partial-wave amplitude which results from this generalized form-factor approximation[2] lacks a left-hand cut, our treatment may be characterized as being "just beyond the tree approximation." However, as we show below, this does not appear to limit the usefulness of two-parameter fits to the experimentally determined phase shifts in the two cases of interest.

While it was enough to use the chiral $SU(2)$ current algebra in our earlier abbreviated treatment of the scalar $\pi\pi$ form factor,[2] our present discussion of both ($\pi\pi$ and $K\pi$) s-wave systems is unified in our relying on the model of $SU(3)\times SU(3)$ symmetry breaking due to Gell-Mann, Oakes, and Renner.[6] This facilitates a fuller exploration here of the role of current-algebra constraints in our model with only right-hand-cut dynamics.

In Sec. II, we consider the s-wave $K\pi$ system in detail; Sec. III is given over to a discussion of several points relating to the s-wave $\pi\pi$ system which were omitted in our earlier paper.[2]

II. S-WAVE $K\pi$ SYSTEM AND KAPPA

Our approach to unitarization in the s-wave $(I=J=\frac{1}{2})K\pi$ channel is just a modification of the usual method of pole dominance[4,5] to include the relevant $K\pi$ cut in place of a narrow resonance pole. Thus we introduce the matrix element

$$W_{\kappa^0}=\int d^4x\, e^{-iq\cdot x}\langle K^-(p)\,|\,\theta(x_0)[\partial_\mu A_\mu{}^{(1-i2)/\sqrt{2}}(x),\\ -i\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}(0)]\,|\,0\rangle\,,\quad(1)$$

with the customary covariant normalization factors omitted. The current divergence $\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}$ is given in terms of the scalar densities u_i which transform as components of $(\mathbf{3},\mathbf{3}^*)\oplus(\mathbf{3}^*,\mathbf{3})$ in the model of Gell-Mann, Oakes, and Renner[6] by

$$\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}=-\tfrac{1}{2}i(\sqrt{\tfrac{3}{2}})cu^{(6-i7)}\,,\quad(2)$$

where[7]

$$c=2\sqrt{2}(m_\pi{}^2-m_K{}^2)/(m_\pi{}^2+2m_K{}^2)\,.\quad(3)$$

The absorptive part of $W_{\bar{\kappa}^0}$ is given formally by

$$\mathrm{Abs}W_{\kappa^0}=(1/2i)\int d^4x\, e^{-iq\cdot x}\langle K^-(p)\,|\,[\partial_\mu A_\mu{}^{(1-i2)/\sqrt{2}}(x),\\ -i\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}(0)]\,|\,0\rangle\,,\quad(4)$$

and, making use of the alternative definitions of the scalar form factor,

$$\langle K^-(p)\,|\,\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}(0)\,|\,\pi^-(q)\rangle\\ =-iF_\kappa(t)\,,\quad t=-(p-q)^2\quad(5a)$$

$$\langle K^-(p)\pi^+(q)\mathrm{out}\,|\,\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}(0)\,|\,0\rangle\\ =-\sqrt{2}\langle \bar{K}^0(p)\pi^0(q)\mathrm{out}\,|\,\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}(0)\,|\,0\rangle\\ =iF_\kappa(t)\,,\quad t=-(p+q)^2\quad(5b)$$

and the assumption of $K\pi$ cut dominance, simplifies to

$$\mathrm{Abs}W_{\kappa^0}=-\tfrac{1}{2}i\int\langle K^-(p)\,|\,\partial_\mu A_\mu{}^{(1-i2)/\sqrt{2}}(0)\\ \times\Big[\,|K^-(p')\pi^+(q')\mathrm{out}\rangle-\frac{1}{\sqrt{2}}|\bar{K}^0(p')\pi^0(q')\mathrm{out}\rangle\Big]\\ \times\frac{Q}{4\pi\sqrt{t}}\frac{d\Omega_Q}{4\pi}F_\kappa(t)\theta(t-(m_K+m_\pi)^2)\\ -i\pi\delta(q^2+m_\pi{}^2)F_\kappa(t)F_\pi m_\pi{}^2\,,\quad(6)$$

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[1] R. Rockmore, Phys. Rev. Letters **24**, 541 (1970); Phys. Rev. D **2**, 593 (1970).

[2] R. Rockmore, this issue, Phys. Rev. D **2**, 2628 (1970).

[3] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[4] S. G. Brown and G. B. West, Phys. Rev. **168**, 1605 (1968).

[5] R. Rockmore, Phys. Rev. **177**, 2573 (1969).

[6] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[7] This result follows from applying low-energy considerations to the general scalar vertex of Ref. 6, $-\langle P_i(p)|u_j|P_k(p')\rangle=\alpha(t)\delta_{j0}\delta_{ik}+\beta(t)d_{ijk}$, $i,k=1,\ldots,8$, $j=0,\ldots,8$, and neglecting the dependence of α and β on t, so that

$$\alpha=-(\sqrt{\tfrac{2}{3}})\beta-(\sqrt{\tfrac{2}{3}})\frac{m_\pi{}^2}{[(\sqrt{\frac{2}{3}})+c(\sqrt{\frac{1}{3}})]}\approx 0,$$

$$\beta=-\frac{m_\pi{}^2}{[(\sqrt{\frac{2}{3}})+c(\sqrt{\frac{1}{3}})]}=\frac{m_K{}^2}{[-(\sqrt{\frac{2}{3}})+\frac{1}{2}c(\sqrt{\frac{1}{3}})]}.$$

This symmetry-breaking model seems to imply the equality of the decay constants $F_\pi=F_K$ and we shall use them interchangeably throughout.

where

$$Q(t)=\tfrac{1}{2}\{[(t-m_\pi^2-m_K^2)^2-4m_\pi^2m_K^2]/t\}^{1/2},$$

and $F_\pi=94$ MeV. As we noted earlier,[2] from rather general considerations of unitarity we expect the $l=0$ $(I=\frac{1}{2})K\pi$ partial-wave amplitude to have the form

$$\int\frac{d\Omega_Q}{4\pi}[\langle\pi^+(q')K^-(p')\text{out}|-\tfrac{1}{2}\sqrt{2}\langle\pi^0(q')\bar{K}^0(p')\text{out}|]\times\partial_\mu A_\mu^{(1+i2)/\sqrt{2}}(0)|K^-(p)\rangle=\tfrac{3}{2}[F_\kappa(t)+N(t)][\Delta_\kappa^s(t)]^{-1}F_\kappa(t),\quad(7)$$

where $N(t)$ has only the left-hand cut and the κ-field propagator $\Delta_\kappa^s(t)$ is given by

$$\Delta_\kappa^s(t)=\frac{1}{\pi}\int_{(m_K+m_\pi)^2}^\infty dt'\frac{3Q(t')|F_\kappa(t')|^2}{16\pi(\sqrt{t'})\,(t'-t)}.\quad(8)$$

Our approach, which is that of a generalized form-factor approximation, is to assume the factorization valid off-shell,

$$\begin{aligned}\int\frac{d\Omega_Q}{4\pi}&[\langle\pi^+(q')K^-(p')\text{out}|-\tfrac{1}{2}\sqrt{2}\langle\pi^0(q')\bar{K}^0(p')\text{out}|]\times\partial_\mu A_\mu^{(1+i2)/\sqrt{2}}(0)|K^-(p)\rangle\\&=-i\int\frac{d\Omega_Q}{4\pi}\frac{(p^2+m_K^2)}{F_Km_K^2}\int d^4x\,e^{ip\cdot x}\\&\quad\times[\langle\pi^+(q')K^-(p')\text{out}|-\tfrac{1}{2}\sqrt{2}\langle\pi^0(q')\bar{K}^0(p')\text{out}|]\\&\quad\times\theta(x_0)[\partial_\nu A_\nu^{(4-i5)/\sqrt{2}}(x),\partial_\mu A_\mu^{(1+i2)/\sqrt{2}}(0)]|0\rangle\\&\equiv-\frac{iF_\pi m_\pi^2f(q,p)}{(q^2+m_\pi^2)}F_\kappa(t),\end{aligned}\quad(9)$$

which still satisfies two-particle unitarity. Moreover, it is assumed that $f(q,p)$ is linear[8] in $t=-(p+q)^2$, p^2, and q^2.

The derivation now proceeds by either of two equivalent routes. (a) One considers the additional constraint imposed on the amplitude, expression (9), by the divergence condition which relates the "half-off-shell" amplitude,

$$\begin{aligned}\int\frac{d\Omega_Q}{4\pi}&\int d^4x\,e^{ip\cdot x}[\langle\pi^+(q')K^-(p')\text{out}|\\&-\tfrac{1}{2}\sqrt{2}\langle\pi^0(q')\bar{K}^0(p')\text{out}|]\theta(x_0)\\&\times[A_\nu^{(4-i5)/\sqrt{2}}(x),\partial_\mu A_\mu^{(1+i2)/\sqrt{2}}(0)]|0\rangle\\&\equiv-\frac{iF_\pi m_\pi^2}{q^2+m_\pi^2}F_\kappa(t)\left[\tilde{f}_\nu(q,p)+\frac{F_Kp_\nu f(q,p)}{p^2+m_K^2}\right],\end{aligned}\quad(10)$$

to it. This is

$$\begin{aligned}-\frac{iF_\pi m_\pi^2F_\kappa(t)}{q^2+m_\pi^2}&\left(p_\nu\tilde{f}_\nu+\frac{F_Kp^2f}{p^2+m_K^2}\right)=\frac{iF_\pi m_\pi^2F_Km_K^2fF_\kappa(t)}{(q^2+m_\pi^2)(p^2+m_K^2)}\\&+i\int\frac{d\Omega_Q}{4\pi}\int d^4x\,e^{ip\cdot x}[\langle\pi^+(q')K^-(p')\text{out}|\\&-\tfrac{1}{2}\sqrt{2}\langle\pi^0(q')\bar{K}^0(p')\text{out}|]\delta(x_0)[A_0^{(4-i5)/\sqrt{2}}(x),\\&\partial_\mu A_\mu^{(1+i2)/\sqrt{2}}(0)]|0\rangle.\end{aligned}\quad(11)$$

Since, in the Gell-Mann–Oakes–Renner model of $SU(3)\times SU(3)$ symmetry breaking,[6] one has

$$[Q_A^{(4-i5)/\sqrt{2}},\partial_\mu A_\mu^{(1+i2)/\sqrt{2}}]=(\sqrt{\tfrac{2}{3}})\frac{W_1(c)}{c}\partial_\nu V_\nu^{(6-i7)/\sqrt{2}},$$

$$Q_A^a=\int d^4x\,\delta(x_0)A_0^a(x),\quad(12)$$

Eq. (11) factorizes to

$$-p_\nu\tilde{f}_\nu=F_Kf+\frac{i(q^2+m_\pi^2)}{F_\pi m_\pi^2}(\sqrt{\tfrac{3}{2}})\frac{W_1(c)}{c},\quad(13)$$

with[6]

$$W_1(c)=\frac{(\sqrt{6})m_\pi^2}{m_\pi^2+2m_K^2}.\quad(14)$$

The polynomial constraint on $f(q,p)$, now given by

$$f=-\frac{p_\nu\tilde{f}_\nu}{F_K}+\frac{3i(q^2+m_\pi^2)}{2\sqrt{2}F_KF_\pi}\frac{1}{m_K^2-m_\pi^2},\quad(15)$$

is satisfied by taking

$$\tilde{f}_\nu=i(Ap_\nu+Bq_\nu),\quad(16)$$

with A and B momentarily free parameters. Satisfaction of Adler's self-consistency condition requires

$$-if(q,p)|_{q\to0\ (p^2=-m_K^2)}=0=\frac{Am_K^2}{F_K}+\frac{3m_\pi^2}{2\sqrt{2}F_KF_\pi}\frac{1}{m_K^2-m_\pi^2},\quad(17)$$

so that

$$-if(q,p)=\frac{3}{2\sqrt{2}F_KF_\pi}\frac{m_\pi^2}{m_\pi^2-m_K^2}+\frac{B}{2F_K}(t-m_K^2-m_\pi^2)\quad(18)$$

on the mass shell, $p^2=-m_K^2$, $q^2=-m_\pi^2$; only B remains a free parameter now. (b) Assuming the following $[t=-(p+q)^2]$:

$$-if(q,p)=\bar{A}+\bar{B}t+\bar{C}q^2+\bar{D}p^2,\quad(19)$$

one has for $q\to0$, $p^2=-m_K^2$,

$$\bar{A}+(\bar{B}-\bar{D})m_K^2=0,\quad\text{(20a)}$$

[8] This assumption is less reasonable in the unequal-mass case than in the equal-mass case (e.g., $\pi\pi$) where the s-wave projection of an amplitude linear in t, u, s, and the masses, $-q_i^2$, is still linear in t and $-q_i^2$. [For example, in $\pi\pi$ scattering, $Q^2(t)=\frac{1}{4}t-m_\pi^2$.]

for $p\to 0$, $q^2=-m_\pi^2$,

$$\bar{A}+(\bar{B}-\bar{C})m_\pi^2=0\,, \tag{20b}$$

and for $q,\,p\to 0$,

$$-F_\pi F_K\bar{A}=(\sqrt{\tfrac{3}{2}})W_1(c)/c\,, \tag{20c}$$

with f given on-shell by

$$-if(q,p)=\bar{B}(t-m_K^2-m_\pi^2)+\frac{3}{2\sqrt{2}F_KF_\pi}\frac{m_\pi^2}{m_\pi^2-m_K^2}. \tag{21}$$

The alternative free parameter $\bar{B}$ is related to the old B by

$$\bar{B}=B/2F_K\,. \tag{22}$$

Since we have

$$\int\frac{d\Omega_Q}{4\pi}[\langle\pi^+(q')K^-(p')\text{out}|-\tfrac{1}{2}\sqrt{2}\langle\pi^0(q')\bar{K}^0(p')\text{out}|]\times\partial_\mu A_\mu{}^{(1+i2)/\sqrt{2}}(0)|K^-(p)\rangle=\frac{F_\pi m_\pi^2}{q^2+m_\pi^2}F_\kappa(t)\times\left[\frac{B}{2F_K}(t-m_K^2+q^2)-\frac{3q^2}{2\sqrt{2}F_KF_\pi(m_\pi^2-m_K^2)}\right], \tag{23}$$

the absorptive part of W_{κ^0} may be written compactly as

$$\text{Abs}W_{\kappa^0}=-i\frac{3Q(t)}{16\pi(\sqrt{t})}|F_\kappa(t)|^2\frac{F_\pi m_\pi^2}{q^2+m_\pi^2}\left[\frac{B}{3F_K}(t-m_K^2+q^2)-\frac{q^2}{\sqrt{2}F_KF_\pi(m_\pi^2-m_K^2)}\right]\theta(t-(m_K+m_\pi)^2)-i\pi\delta(q^2+m_\pi^2)F_\kappa(t)F_\pi m_\pi^2. \tag{24}$$

We reconstruct the function W_{κ^0} from its absorptive part (with respect to t) up to an unknown pole in q^2 with assumed constant residue $-i\gamma$,

$$W_{\kappa^0}(q^2,t)=-\frac{im_\pi^2}{q^2+m_\pi^2}\frac{1}{\pi}\int_{(m_K+m_\pi)^2}^\infty dt'\,\frac{3Q(t')|F_\kappa(t')|^2}{16\pi(\sqrt{t'})(t'-t)}\times\left[\tfrac{1}{3}B(t'-m_K^2+q^2)-\frac{q^2}{\sqrt{2}F_K(m_\pi^2-m_K^2)}\right]+\frac{i\gamma}{q^2+m_\pi^2}; \tag{25}$$

next comparing with absorptive part of W_{κ^0} with respect to q^2 as given by Eq. (24), we find

$$F_\pi m_\pi^2F_\kappa(t)=-\frac{m_\pi^2}{\pi}\int_{(m_K+m_\pi)^2}^\infty dt'\,\frac{3Q(t')|F_\kappa(t')|^2}{16\pi(\sqrt{t'})(t'-t)}\times\left[\tfrac{1}{3}B(t'-m_K^2-m_\pi^2)+\frac{m_\pi^2}{\sqrt{2}F_K(m_\pi^2-m_K^2)}\right]+\gamma. \tag{26}$$

The algebra of current divergences[6] imposes the constraint

$$\lim_{q\to 0,\,t\to m_K^2}W_{\kappa^0}(q^2,t)=i\langle K^-(p)|\times[Q_A{}^{(1-i2)/\sqrt{2}},\partial_\lambda V_\lambda{}^{(6-i7)/\sqrt{2}}]|0\rangle=-\tfrac{1}{2}i(\sqrt{\tfrac{3}{2}})F_Km_K^2\frac{c}{W_4(c)}, \tag{27}$$

so that

$$W_{\kappa^0}(0,m_K^2)=-\frac{i}{\pi}\int_{(m_K+m_\pi)^2}^\infty dt'\,\frac{3Q(t')|F_\kappa(t')|^2}{16\pi\sqrt{t'}}\times\tfrac{1}{3}B+\frac{i\gamma}{m_\pi^2}=\tfrac{1}{2}i\sqrt{2}F_K(m_K^2-m_\pi^2). \tag{28}$$

From the integral equation for the scalar form factor $F_\kappa(t)$, we have

$$\frac{1}{2i}[F_\kappa(t+i\epsilon)-F_\kappa(t-i\epsilon)]=-\frac{1}{F_\pi}\frac{3Q(t)}{16\pi\sqrt{t}}|F_\kappa(t)|^2\times\left[\tfrac{1}{3}B(t-m_K^2-m_\pi^2)+\frac{m_\pi^2}{\sqrt{2}F_K(m_\pi^2-m_K^2)}\right]\times\theta(t-(m_K+m_\pi)^2); \tag{29}$$

thus,[1]

$$\frac{1}{\sqrt{2}}F_K(m_K^2-m_\pi^2)=\frac{BF_\pi}{6\pi i}\times\int_{\circlearrowleft}dt'\,\frac{F_\kappa(t')}{\tfrac{1}{3}B(t'-m_K^2-m_\pi^2)+m_\pi^2[\sqrt{2}F_K(m_\pi^2-m_K^2)]^{-1}}+\frac{\gamma}{m_\pi^2}=F_\pi F_\kappa\left(m_K^2+m_\pi^2-\frac{3m_\pi^2}{\sqrt{2}BF_K(m_\pi^2-m_K^2)}\right)+\frac{\gamma}{m_\pi^2}. \tag{30}$$

To simplify matters, we may set the unknown residue γ equal to zero at this point. Then the boundary condition

$$F_\kappa\left(m_K^2+m_\pi^2-\frac{3m_\pi^2}{\sqrt{2}BF_K(m_\pi^2-m_K^2)}\right)=\tfrac{1}{2}\sqrt{2}(m_K^2-m_\pi^2) \tag{31}$$

is to be imposed on the unsubtracted integral equation

$$F_\kappa(t)=-\frac{1}{\pi F_\pi}\int_{(m_K+m_\pi)^2}^\infty dt'\,\frac{3Q(t')|F_\kappa(t')|^2}{16\pi(\sqrt{t'})(t'-t)}\times\left[\tfrac{1}{3}B(t'-m_K^2-m_\pi^2)+\frac{m_\pi^2}{\sqrt{2}F_K(m_\pi^2-m_K^2)}\right], \tag{32a}$$

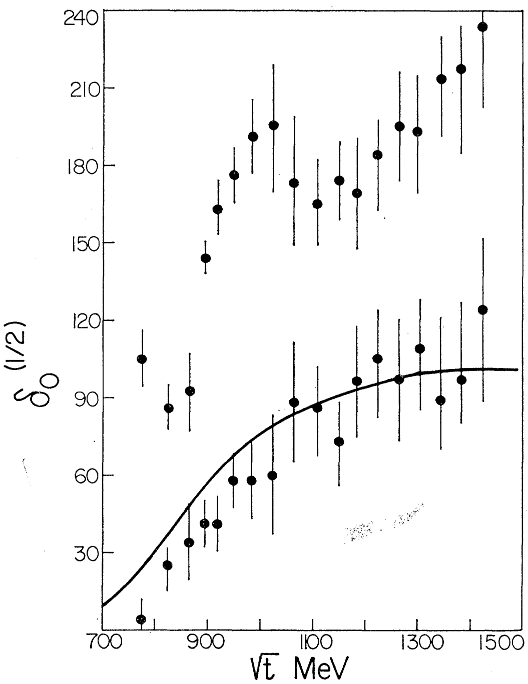

FIG. 1. Plot of the s-wave, $I=\frac{1}{2}$ $K\pi$ phase shift, $\delta_0^{(1/2)}$, derived from the two-parameter expression (39). The fit is determined by taking $m_\kappa=1150$ MeV as in Antich *et al.* (Ref. 11) and $B\simeq-\sqrt{2}/F_\pi m_K^2$ for which $a_0^{(1/2)}\simeq 0.11 m_\pi^{-1}$. The doubling of data points taken from Ref. 11 reflects the determination of phase shifts from experimental data modulo π. Our fit prefers the lower solution.

or, alternatively, merely eliminating γ, we have to deal with the subtracted integral equation[9]

$$F_\kappa(t)=\frac{1}{\sqrt{2}}(m_K^2-m_\pi^2)$$
$$-\frac{1}{\pi F_\pi}\left[\tfrac{1}{3}B(t-m_K^2-m_\pi^2)-\frac{m_\pi^2}{\sqrt{2}F_K(m_\pi^2-m_K^2)}\right]$$
$$\times\int_{(m_K+m_\pi)^2}^{\infty}dt'\,\frac{3Q(t')\,|F_\kappa(t')|^2}{16\pi(\sqrt{t'})(t'-t)}. \quad (32b)$$

It is worthwhile noting that the boundary condition [Eq. (31)] is imposed at the zero [which we might expect from soft-meson considerations[10] to be located slightly below the physical threshold $(m_K+m_\pi)^2$],

$$t=m_K^2+m_\pi^2-\frac{3m_\pi^2}{\sqrt{2}BF_K(m_\pi^2-m_K^2)},$$

of the s-wave $K\pi$ amplitude rather than at $t=0$ as in the discussion of Gell-Mann, Oakes, and Renner.[6] Since the κ ($m_\kappa\simeq1150$ MeV) is a relatively low-lying enhancement, in the sense that $t_{\rm th}{}^{K\pi}/m_\kappa{}^2\simeq\frac{1}{4}$, as compared with the ϵ ($m_\epsilon\simeq900$ MeV), for which $t_{\rm th}{}^{\pi\pi}/m_\epsilon{}^2\simeq\frac{1}{9}$ [$t_{\rm th}{}^{ij}=(m_i+m_j)^2$, the threshold energy squared], we do not ignore this distinction here as in our earlier discussion[2] of the $(I=J=0)\pi\pi$ channel.

A simple method for obtaining an effective-range solution for $F_\kappa(t)$ has recently been given by the author.[1] $F_\kappa(t)$ has the form

$$F_\kappa(t)=[\alpha+\beta t+\psi(t)h(t)-\psi(0)h(0)]^{-1}, \quad (33)$$

where

$$h(t)=1-t\int_{(m_K+m_\pi)^2}^{\infty}\frac{dt'Q(t')}{(\sqrt{t'})t'(t'-t)}$$
$$=\frac{2Q(t)}{\sqrt{t}}\ln\left(\frac{(\sqrt{t})+2Q}{m_K+m_\pi}\right)-\frac{i\pi Q(t)}{\sqrt{t}},$$
$$t\geq(m_K+m_\pi)^2 \quad (34)$$

and

$$\psi(t)=-\frac{3}{16\pi^2F_\pi}\left[\tfrac{1}{3}B(t-m_K^2-m_\pi^2)+\frac{m_\pi^2}{\sqrt{2}F_K(m_\pi^2-m_K^2)}\right], \quad (35)$$

from unitarity considerations. From the boundary condition [Eq. (31)], we find

$$\alpha=\frac{\sqrt{2}}{m_K^2-m_\pi^2}+\psi(0)$$
$$-\beta\left[m_K^2+m_\pi^2-\frac{3m_\pi^2}{\sqrt{2}BF_K(m_\pi^2-m_K^2)}\right], \quad (36)$$

so that

$$F_\kappa(t)=\left[\frac{\sqrt{2}}{m_K^2-m_\pi^2}-\frac{16\pi^2F_\pi}{B}\beta\psi(t)+\psi(t)h(t)\right]^{-1} \quad (37)$$

is the anticipated two-parameter effective-range formula for $F_\kappa(t)$. To compare with the $K\pi$ scattering phase shifts $\delta_0^{(1/2)}$ obtained by Antich *et al.*[11] in a study of the reaction $K^+p\to K^+\pi^-\Delta^{++}$, we require that $\delta_0^{(1/2)}(m_R)=\pi/2$ for $m_R=m_\kappa=1150$ MeV (the value given in Ref. 11) *and* $\psi(m_K^2-\frac{1}{2}m_\pi^2)=0$, at the location of the soft-meson s-wave amplitude zero, $t\simeq m_K^2-\frac{1}{2}m_\pi^2$. In this case, $B\simeq-\sqrt{2}/F_Km_K^2$, and from

$$t_0^{(1/2)}=(\sqrt{t}/Q)e^{i\delta_0^{(1/2)}}\sin\delta_0^{(1/2)}$$
$$=\pi\psi(t)F_\kappa(t), \quad (38)$$

we find $a_0^{(1/2)}=t_0^{(1/2)}(t_{\rm th}{}^{K\pi})/(m_Km_\pi)\simeq0.11m_\pi^{-1}$, approximately half the soft-meson result.[10] The fit de-

[9] Equations (26) and (28) imply $\gamma=0$ and this is reflected in Eq. (32b).

[10] R. W. Griffith, Phys. Rev. **176**, 1705 (1968).

[11] P. Antich, A. Callahan, R. Carson, B. Cox, D. Denegri, L. Ettlinger, D. Feiock, D. Gillespie, G. Goodman, G. Luste, R. Mercer, A. Pevsner, R. Sekulin, and R. Zdanis, in Proceedings of the Conference on $\pi\pi$ and $K\pi$ Interactions, edited by F. Loeffler and E. Malamud, Argonne National Laboratory, 1969, p. 508 (unpublished).

rived from

$$\frac{Q}{\sqrt{t}}\cot\delta_0^{(1/2)} = \frac{\sqrt{2}/(m_K^2-m_\pi^2)-(16\pi^2/B)F_K\beta\psi(t)+\psi(t)\,\mathrm{Re}h(t)}{\pi\psi(t)}, \tag{39}$$

has been plotted in Fig. 1 along with the data points of Ref. 11. Note that the experimental phase shifts are determined modulo π in a Chew-Low extrapolation using the one-pion-exchange model. Our fit plainly favors the lower solution which "resonates" at 1150 MeV. (However, this "resonance" has a "width" of several hundred MeV.)

III. S-WAVE $\pi\pi$ SYSTEM AND EPSILON

If we introduce the (isoscalar) sigma field,[6]

$$\sigma = -[W_1(c)/\sqrt{3}](u_8+\sqrt{2}u_0), \tag{40}$$

and the matrix element

$$W_\epsilon = \int d^4x\, e^{-iq\cdot x}\langle\pi^+(p)|\theta(x_0) \times[\partial_\mu A_\mu^{(1+i2)/\sqrt{2}}(x),\sigma(0)]|0\rangle, \tag{41}$$

with

$$\langle\pi^+(p)|\sigma(0)|\pi^+(q)\rangle = -F(t),\quad t=-(p-q)^2 \tag{42a}$$

and

$$\langle\pi^+(p)\pi^-(q)\mathrm{out}|\sigma(0)|0\rangle = F(t),\quad t=-(p+q)^2 \tag{42b}$$

then the derivation in this case goes through in much the same fashion as before. Since this has been adequately sketched in our earlier paper,[2] we remark that the boundary condition

$$F(0) = -m_\pi^2, \tag{43}$$

to be imposed on the integral equation

$$F(t) = \frac{1}{\pi F_\pi}\int_{4m_\pi^2}^\infty dt' \frac{3Q(t')}{16\pi\sqrt{t'}}\left[-\frac{1}{F_\pi}+\tfrac{1}{2}B(2m_\pi^2-t')\right] \times\frac{|F(t')|^2}{t'-t}, \tag{44}$$

is simply a *good* approximation to the correct one,

$$F(2m_\pi^2-2/BF_\pi) = -m_\pi^2, \tag{45}$$

in this case.[12]

[12] It was noted in our earlier communication that the method of J. J. Brehm, E. Golowich, and S. C. Prasad [Phys. Rev. Letters 25, 67 (1970)] does not appear to handle properly the problem of subtractions so that their (one-parameter) fit is erroneously overconstrained. We must emphasize that our two-parameter fit (Ref. 2) for t_{00} precisely reduces to the unitarized soft-pion result of L. S. Brown and R. L. Goble [Phys. Rev. Letters 20, 346 (1968)] in the nonresonant limit ($\beta=0$) if the remaining free parameter B is fitted by requiring an s-wave amplitude zero at the corresponding soft-pion point, $t=\frac{1}{2}m_\pi^2$.

We observe that the generalized form-factor approximation allows one to express the scalar propagator $\Delta^s(t)$ and its inverse $[\Delta^s(t)]^{-1}$ in compact form. Thus,

$$\begin{aligned}\Delta^s(t) &= \frac{1}{\pi}\int_{4m_\pi^2}^\infty \frac{dt'3Q(t')|F(t')|^2}{16\pi(\sqrt{t'})(t'-t)} \\ &= -\frac{1}{2\pi i}\left(\frac{2F_\pi}{B}\right)\int_{\circlearrowleft} dt'\frac{F(t')}{(t'-t)(t'-2m_\pi^2+2/BF_\pi)} \\ &= -\frac{2F_\pi}{B}\left[\frac{F(t)-F(2m_\pi^2-2/BF_\pi)}{t-2m_\pi^2+2/BF_\pi}\right],\end{aligned} \tag{46}$$

with $F(t)$ precisely given by

$$F(t) = \left\{-\frac{1}{m_\pi^2}+\beta\left(t-2m_\pi^2+\frac{2}{BF_\pi}\right) -\frac{3}{16\pi^2F_\pi}h(t)\left[\tfrac{1}{2}B(t-2m_\pi^2)+\frac{1}{F_\pi}\right]\right\}^{-1}. \tag{47}$$

If it is assumed[13] that $K_S{}^0$ weak interactions are mediated by the ϵ, which implies an effective weak interaction of the form[13]

$$\mathcal{H}_w = G_F\lambda_\epsilon K_S{}^0\sigma, \tag{48}$$

then one finds, following the scheme $K_S{}^0\to\epsilon\to\pi\pi$,

$$\begin{aligned}\Gamma(K_S{}^0\to\pi\pi) &= \hbar/\tau_S \\ &= \frac{G_F{}^2\lambda_\epsilon{}^2}{m_K}\frac{3Q(m_K{}^2)}{16\pi m_K}|F(m_K{}^2)|^2 \\ &= \frac{G_F{}^2\lambda_\epsilon{}^2}{m_K}\,\mathrm{Im}\Delta^s(m_K{}^2),\end{aligned} \tag{49}$$

with a $K_S{}^0$ mass shift δ_S (second order in G_F) given by

$$\begin{aligned}\delta_S &= \mathrm{Re}\left\{i\int d^4x\langle K_S{}^0(p)|T(\mathcal{H}_w(x),\mathcal{H}_w(0))|K_S{}^0(p)\rangle\right\} \\ &= -\frac{G_F{}^2\lambda_\epsilon{}^2}{2m_K}\,\mathrm{Re}\Delta^s(m_K{}^2).\end{aligned} \tag{50}$$

Thus,

$$\begin{aligned}\delta_S &= -\frac{\hbar}{2\tau_S}\frac{\mathrm{Re}\Delta^s(m_K{}^2)}{\mathrm{Im}\Delta^s(m_K{}^2)} \\ &= \frac{\hbar}{2\tau_S}\frac{\mathrm{Re}[F(m_K{}^2)]^{-1}}{\mathrm{Im}[F(m_K{}^2)]^{-1}} \\ &= -\frac{\hbar}{2\tau_S}\cot\delta_{00}(m_K{}^2).\end{aligned} \tag{51}$$

[13] R. Arnowitt, P. Nath, P. Pond, and M. H. Friedman, Northeastern University report, 1969 (unpublished).

Note that since $\delta_{00}(m_K^2) \simeq \pi/4$ in our model, the value of δ_S which results is *twice* as large as that calculated by Arnowitt *et al.*[13] Hence in this model the calculated $K_L^0 - K_S^0$ mass difference $\delta_L - \delta_S$ can no longer be said to be "in excellent agreement with experiment."[13]

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Collinear Dispersion Relations and the K_{e4} Decay Rate

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We study the K_{e4} decay form factors using a method introduced by Fubini and Furlan. The amplitudes which we extrapolate from the soft-pion limits to the physical point differ slightly from the previously used ones. Our choice is motivated by the collinear parametrization. These amplitudes are simply related to the K_{e4} form factors and to the K-π scattering amplitudes, which appear on equal footing. The calculated decay rate $\Gamma = (2.3 \pm 0.3) \times 10^3$ sec^{-1} lies within the experimental error.

I. INTRODUCTION

THE form factors for the decay K_{e4} were first calculated by Callan and Treiman[1] from current algebra. These authors contract over one of the pions of the final state at a time and obtain values for the form factors in the two soft-pion limits. Their results for the form factor F_3, however, differ considerably, depending on which of the momenta of the two pions is put equal to zero. Weinberg[2] later explained the rapid variation of F_3 by taking a nearby K pole explicitly into account.

In all these calculations the form factors F_1 and F_2, on which the decay rate $\Gamma_{K_{e4}^+}$ only depends, were taken to be constant. The results of Refs. 1 and 2 give for the K_{e4}^+ decay rate

$$\Gamma_{K_{e4}^+} = (1.6 \pm 0.2) \times 10^3 \text{ sec}^{-1},$$

whereas experimentally,[3]

$$\Gamma_{K_{e4}^+} = (2.9 \pm 0.6) \times 10^3 \text{ sec}^{-1}.$$

It seems to us, however, that the discrepancy between theory and experiment could be accounted for by the variation of the form factors between the soft-pion limit and the physical point.

In this paper we apply an extrapolation method of Fubini and Furlan[4] which makes use of the collinear parametrization in the rest frame of the particles to study the appropriate matrix elements. The method of Ref. 4 has several advantages:

(a) The ambiguity that different choices of the amplitudes to be extrapolated may lead to different results on the mass shell is resolved, the physical amplitudes at threshold and the ones related to them by crossing being directly related to the soft-pion limits through dispersion relations.

(b) The Low representation of the amplitudes determines their asymptotic behavior, thus giving information about the possibility of writing dispersion relations and the number of subtractions needed.

(c) Anomalous thresholds are absent in the physical sheet, where the dispersion relations are written.

(d) Since we are working in the rest frame, we can make use of strong parity and angular momentum selection rules to calculate the corrections to the soft-pion limits.

The form factors F_1, F_2, and F_3, and the K-π scattering amplitudes at the threshold will appear naturally on the same footing in sum rules.

We obtain for the K_{e4}^+ decay rate

$$\Gamma_{K_{e4}^+} = (2.3 \pm 0.3) \times 10^3 \text{ sec}^{-1}.$$

II. COLLINEAR PARAMETRIZATION AND K_{e4} FORM FACTORS

The K_{e4}^+ form factors are defined in the following way:

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[1] C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 153 (1966).

[2] S. Weinberg, Phys. Rev. Letters 17, 336 (1966).

[3] R. P. Ely *et al.*, LRL Report No. UCRL 18626, 1968 (unpublished).

[4] S. Fubini and G. Furlan, Ann. Phys. (N. Y.) 48, 322 (1968).