and project this onto the one-fermion subspace to obtain

$$H=\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m_0+H_\phi+ig\gamma_0\gamma_5\phi(x).$$

We can then proceed in one of two ways.

(a) Perform a Foldy-Wouthuysen transformation to determine the correct nonrelativistic limit which turns out to be (to order $1/m_0$)

$$H=m_0+\frac{p^2}{2m_0}+\frac{g^2}{2m_0}\phi^2(x)-\frac{g}{2m_0}\boldsymbol{\sigma}\cdot\nabla\phi(x)+H_\phi. \quad (33)$$

The presence of the $\phi^2(x)$ term and the gradient coupling make this a considerably more complicated problem than the one we have considered so far. We also propose to take into account the strong angular correlations between fermion and field wave functions implied by the interaction term.

(b) If we find again that nonrelativistic kinematics is untenable, the final step would be to try to solve the Dirac equation self-consistently. Whether or not this is possible is at this time an open question.

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Measuring Light-Cone Singularities*

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The scaling behavior observed in deep-inelastic electron scattering is related to the structure of the electric current commutation function in position space. We show that scaling is assured when that object has the following form, which is also consistent with Regge behavior:

$$i\langle p|[j^\mu(x),j^\nu(0)]|p\rangle=[g^{\mu\nu}\Box-\partial^\mu\partial^\nu]\left[\frac{1}{4\pi^2}\epsilon(x\cdot p)\delta(x^2)\int_0^\infty d\omega\,\frac{\cos\omega x\cdot p}{\omega^2}F_L(\omega)+\epsilon(x\cdot p)\theta(x^2)f_1(x^2,x\cdot p)\right]$$
$$+[p^\mu p^\nu\Box-p\cdot\partial(\partial^\mu p^\nu+\partial^\nu p^\mu)+g^{\mu\nu}(p\cdot\partial)^2]\left[\frac{1}{4\pi^2}\epsilon(x\cdot p)\theta(x^2)\int_0^\infty d\omega\,\frac{\sin\omega x\cdot p}{\omega x\cdot p}F_2(\omega)+\epsilon(x\cdot p)\theta(x^2)\tilde{f}_2(x^2,x\cdot p)\right].$$

In the above, $F_L=F_2-2\omega F_1$, and the F_i are the conventional scaling functions of Bjorken. The f_i are arbitrary, except that $f_2(0,x\cdot p)=0$. It is also demonstrated that when the combination $T_1+(\nu^2/q^2)T_2$ of the conventional forward Compton amplitudes, as well as T_2, are unsubtracted, a new sum rule can be derived:

$$\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle=\frac{i}{2\pi}\partial^i\delta(\mathbf{x})\int_0^\infty d\omega\frac{F_L(\omega)}{\omega^2}.$$

Finally, the consequences of the same unsubtractedness hypothesis for the electromagnetic self-mass of the target proton are discussed. The unsubtractedness hypothesis is consistent with present experimental results.

I. INTRODUCTION

In this paper we relate the remarkable regularities observed in deep-inelastic scattering[1] to the behavior of the commutator of electromagnetic currents near the light cone. That the light-cone commutator should be relevant in this connection has already been noted by several authors.[2] We show that the experimental data place stringent, but simple, restrictions on the commutator, and that the leading light-cone singularity

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[1] For a summary of the experimental data, see R. E. Taylor, SLAC Report No. SLAC-PUB-677 (unpublished). We insert here the *caveat* that the experimental data are not unimpeachable evidence for scaling. A skeptic can take refuge in the large error bars, and other uncertainties, and insist that scaling is in fact weakly broken, for example by logarithmic terms. We do not here succumb to this cautionary pessimism.

[2] B. L. Ioffe, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu 9, 163 (1969) [Soviet Phys. JETP Letters 9, 97 (1969)]; B. L. Ioffe, Phys. Letters **30B**, 123 (1969); R. Brandt, Phys. Rev. Letters **23**, 1260 (1969). These authors discussed the behavior of the light-cone commutator; see also Ref. 8. After completion of the major portion of this investigation, we learned from D. G. Boulware that he and L. S. Brown have also studied this problem. Some of their results are to be found in L. S. Brown, in *Lectures in Theoretical Physics*, edited by W. E. Brittin, B. W. Downs, and J. Downs (Interscience, New York, to be published). Other results are unpublished. L. S. Brown has derived a representation for the product of two currents, consistent with scaling, by using the spectral representation and some regularity assumptions about the behavior of the spectral functions [see Eqs. (6.45)–(6.47) of Brown's paper]. This is equivalent to our representation for the commutator, Eqs. (2.4) and (2.7) below. Brown has also discussed the connection between a q-number Schwinger term and the longitudinal electroproduction cross section in the deep-inelastic region. This discussion is equivalent to our sum rule, Eq. (2.8) below.

may be parametrized by the observed asymptotic cross sections. Also a new sum is derived which connects the asymptotic *longitudinal* cross section with a possible q-number Schwinger term in the equal-time commutator of the electromagnetic current with the charge density. Finally, we demonstrate that the divergent part of the electromagnetic self-mass of the proton, to lowest order in electromagnetism, is determined by the total electroproduction cross sections.

There are no adequate theories which exhibit the experimental high-energy phenomena. In view of this, the conclusions of this investigation do not hold in any of the usual models. Our purpose here is to translate the experimental results into theoretical constraints which must be satisfied in some future theory which will be capable of describing the electroproduction data. It is hoped that the properties revealed here will aid in the construction of this theory.

In Sec. II we state and discuss our results; the proofs are postponed to Secs. III–V. In Sec. VI, our formulas are checked against calculations in model field theories. Various technical computations are relegated to an Appendix.

II. SUMMARY AND DISCUSSION

Of interest for total inelastic scattering processes is the commutator function

$$C^{\mu\nu}(q,p)=\int d^4x\, e^{iq\cdot x}\langle p|[j^\mu(x),j^\nu(0)]|p\rangle$$
$$=\frac{1}{p\cdot q}\left(p^\mu-q^\mu\frac{q\cdot p}{q^2}\right)\left(p^\nu-q^\nu\frac{q\cdot p}{q^2}\right)\tilde{F}_2$$
$$-\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)\tilde{F}_1. \quad (2.1)$$

The state $|p\rangle$ is the target state of 4-momentum p^μ, $p^2=m^2$ (spin averaged if it is a fermion), j^μ is the electromagnetic current operator, and q is the virtual photon 4-momentum. The second equality defines a gauge-invariant tensor decomposition, and Lorentz-invariant functions $\tilde{F}_i$ have been introduced. They depend on $\nu\equiv q\cdot p$ and q^2. Frequently we shall use $\omega\equiv -q^2/2q\cdot p$ and ν as independent variables. Also the combination $\tilde{F}_2-2\omega\tilde{F}_1$ will be called $\tilde{F}_L$. In the literature $\tilde{F}_1$ is usually called W_1 and $\tilde{F}_2$ corresponds to νW_2. The total transverse and longitudinal cross sections are directly expressible in terms of the $\tilde{F}_i$.

The analysis of the light-cone singularities of the commutator of electromagnetic currents is based on the following assumptions about the functions $\tilde{F}_i$.[3]

(I) $\lim_{\nu\to\infty}\tilde{F}_i(\omega,\nu)\equiv F_i(\omega)\neq\infty$.

(II) The $F_i(\omega)$ are sufficiently regular so that they may be Fourier-transformed. The transforms can be generalized functions.

That these assumptions are experimentally verified is strongly indicated by the data: Statement (I) seems to be well established.[1,4] The validity of (II) is supported by the preliminary results that $F_2(\omega)\approx$ const; $F_L(\omega)\approx 0$.[1]

Our further results—the sum rule and the analysis of the electromagnetic self-mass—require manipulations, which are reliable only if $F_L(\omega)$ vanishes rapidly as $\omega\to 0$, or better yet if $F_L(\omega)$ vanishes identically. Of course, (I) and (II) are also required.

It should be recalled that the F_i are zero for $|\omega|>1$; ωF_1, F_2, and F_L are even in ω, and a positivity condition holds: $F_2(\omega)\geqslant 2\omega F_1(\omega)>0$. The relation to the transverse (σ_T) and longitudinal (σ_L) total cross sections is given by

$$F_1(\omega)\propto\lim_{\nu\to\infty}\nu\sigma_T(\omega,\nu)\,,$$
$$F_L(\omega)\propto\lim_{\nu\to\infty}\nu\sigma_L(\omega,\nu)\,.$$

It is convenient to introduce a tensor decomposition of $C^{\mu\nu}(q,p)$ which differs from (2.1):

$$C^{\mu\nu}(q,p)=-[q^2g^{\mu\nu}-q^\mu q^\nu]C_1$$
$$-[q^2p^\mu p^\nu-q\cdot p(q^\mu p^\nu+q^\nu p^\mu)+g^{\mu\nu}(q\cdot p)^2]C_2\,, \quad (2.2)$$

$$\tilde{F}_L(\omega,\nu)=4\omega^2\nu C_1(\omega,\nu)\,, \quad (2.3\text{a})$$

$$\tilde{F}_2(\omega,\nu)=2\omega\nu^2C_2(\omega,\nu)\,. \quad (2.3\text{b})$$

The object of present interest is

$$c^{\mu\nu}(x,p)=i\langle p|[j^\mu(x),j^\nu(0)]|p\rangle$$
$$=[g^{\mu\nu}\Box-\partial^\mu\partial^\nu]c_1(x^2,\,x\cdot p)+[p^\mu p^\nu\Box$$
$$-p\cdot\partial(\partial^\mu p^\nu+\partial^\nu p^\mu)+g^{\mu\nu}(p\cdot\partial)^2]c_2(x^2,\,x\cdot p)\,, \quad (2.4)$$

and the task is to determine the form of $c_i(x^2,\,x\cdot p)$ such that

$$C_i(\omega,\nu)=-i\int d^4x\, e^{iq\cdot x}c_i(x^2,\,x\cdot p) \quad (2.5)$$

possesses the convergent high-energy behavior in ν, at fixed ω, indicated by (I) and (2.3).

Since $c^{\mu\nu}(x,p)$, and therefore $c_i(x^2,\,x\cdot p)$, is causal

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[3] In addition to assumptions (I) and (II), our analysis relies on mathematical manipulations which may be described as *plausible* rather than *rigorous*. Thus implicitly, we are also assuming sufficient regularity and uniformity of the various expressions with which we are dealing. These assumptions are numerous and uncontrollable, and it would serve no purpose to list them here. However, we present several derivations, and this should clarify and isolate the necessary mathematical underpinnings. Nevertheless, one of our assumptions is sufficiently important to require explicit mention here: Statement (I) is assumed to hold for *all* ω, positive and negative; while experimental evidence established its validity only for $\omega>0$.

[4] The scaling of $\tilde{F}_2$ is much more firmly established than that of $\tilde{F}_1$.

(i.e., vanishes for $x^2<0$), it follows that

$$c_i(x^2, x\cdot p)=\epsilon(x\cdot p)\theta(x^2)f_i(x^2, x\cdot p) +\epsilon(x\cdot p)\delta(x^2)s_i^{(0)}(x\cdot p) +\sum_{n=1}^{N} \epsilon(x\cdot p)\delta^{(n)}(x^2)s_i^{(n)}(x\cdot p). \quad (2.6)$$

Here $\delta^{(n)}(x^2)$ is the nth derivative of $\delta(x^2)$. Only a finite number of such derivatives is present; an infinite number would destroy locality.[5,6]

A. Simple Representation

Our first result is that $F_L(\omega)$ is finite if, and only if, all derivatives of the δ function are absent from $c_1(x^2, x\cdot p)$: $s_1^{(n)}(x\cdot p)=0$, $n\geqslant 1$. Similarly, $F_2(\omega)$ is finite if, and only if, all δ functions are absent from $c_2(x^2, x\cdot p)$: $s_2^{(n)}(x\cdot p)=0$, $n\geqslant 0$. Furthermore, the leading surviving singularity near the light cone is determined[7] in terms of F_L and F_2:

$$c_1(x^2, x\cdot p)=\frac{1}{4\pi^2}\epsilon(x\cdot p)\delta(x^2)\int_0^\infty d\omega \frac{\cos\omega x\cdot p}{\omega^2}F_L(\omega) +\epsilon(x\cdot p)\theta(x^2)f_1(x^2, x\cdot p), \quad (2.7a)$$

$$c_2(x^2, x\cdot p)=\frac{1}{4\pi^2}\epsilon(x\cdot p)\theta(x^2)\int_0^\infty d\omega \frac{\sin\omega x\cdot p}{\omega x\cdot p}F_2(\omega) +\epsilon(x\cdot p)\theta(x^2)\tilde{f}_2(x^2, x\cdot p), \quad (2.7b)$$

$$\tilde{f}_2(x^2, x\cdot p)=f_2(x^2, x\cdot p)-f_2(0, x\cdot p).$$

The formulas (2.7) are derived in Sec. III. It is shown by Fourier-transforming (2.6) that finite limits are attained only when (2.7) is true. It is also demonstrated that the representation (2.7) is consistent with Regge behavior.[8]

[5] We are here ignoring the mathematically feasible situation of an infinite number of derivatives of δ functions, with coefficients sufficiently constrained so that the sum remains local.

[6] Throughout the paper we make use of the fact that expressions of the form $\epsilon(x_0)\theta(x^2)$ and $\epsilon(x_0)\delta(x^2)$ are Lorentz scalers; hence they may also be written as $\epsilon(x\cdot p)\theta(x^2)$ and $\epsilon(x\cdot p)\delta(x^2)$, since p is timelike, with positive time component.

[7] If the integrals over ω do not converge in the usual sense, they are to be interpreted as generalized functions. Such generalized functions may be ambiguous if alternative prescriptions can be given for avoiding singularities. This is further discussed at the end of Sec. III A.

[8] Note that the complete commutator function $c^{\mu\nu}(x,p)$ has singularities *more* violent than a δ function at $x^2=0$. Only the invariants $c_i(x^2,x\cdot p)$ can be characterized by the statement that they are no more singular than δ functions at the light cone. It is seen that by virtue of the double-derivative operation, which is required to pass from $c_i(x^2,x\cdot p)$ to $c^{\mu\nu}(x,p)$, the latter object is more singular than a δ function. This remains true even if $F_L=0$, in which case the invariant functions possess only step-function singularities, while the complete tensor function will involve derivatives of δ functions. Also the lightlike commutator considered by Brandt (Ref. 2) can easily be shown to involved these derivatives of $\delta(x^2)$.

B. Schwinger Term Sum Rule

The second result of this investigation is a sum rule which connects a possible q-number Schwinger term in the $[j^0,j^i]$ equal-time commutator with the asymptotic longitudinal cross section parametrized by F_L. The sum rule is

$$\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle=\frac{i}{2\pi}\partial^i\delta(\mathbf{x})\int_0^\infty d\omega\frac{F_L(\omega)}{\omega^2}. \quad (2.8)$$

Section IV is devoted to a derivation of (2.8)—several methods are presented. First we establish (2.8) directly by passing to the equal-time limit with the $0i$ components of our representation for $c^{\mu\nu}(x,p)$. Alternatively, one may use the new Bjorken high-energy limit[9] to arrive at the sum rule. These two derivations involve manipulations which may be unreliable—interchanges of limits, etc. Some light on the validity of such steps is shed by showing that (2.8) requires a no-subtraction hypothesis about the dispersive representation of certain portions of the forward Compton amplitude. The validity of this hypothesis is related to properties of F_L. The argument is presented in Sec. IV C.[10]

Whether or not the integral occurring in the sum rule converges is an experimental question. Preliminary data indicate that $F_L(\omega)$ is consistent with zero,[1] in which case convergence is obviously assured. The identical vanishing of $F_L(\omega)$ would then be a strong indication that the Schwinger term is a c number, as has been frequently conjectured. To our knowledge this is the only instance where the q-number nature of the Schwinger term is being probed experimentally.[11]

Although experimental data can be converted, through the sum rule (if it is convergent) into information about the $[j^0,j^i]$ equal-time commutator, the converse argument may be unreliable. That is, even if it is postulated that the Schwinger term is a c number, before concluding that $F_L=0$, one must first know whether the sum rule converges; specifically, it is necessary to know that $F_L(\omega)\to O(\omega^{1+\epsilon})$ as $\omega\to 0$. [There is no trouble from the infinite range in (2.8), since $F_L(\omega)=0$ for $\omega>1$.] A diverging sum rule very likely indicates that the connection between the Schwinger term and F_L is lost, rather than that the Schwinger term has infinite matrix elements; (see Sec. IV C).

Unfortunately, there is no satisfactory *a priori* argument concerning $F_L(0)$. A straightforward application

[9] J. D. Bjorken, Phys. Rev. **179**, 1547 (1969).

[10] This method is quite analogous to the techniques used by C. G. Callan, Jr. and D. J. Gross [Phys. Rev. Letters **22**, 156 (1969)] in the derivation of their sum rule.

[11] The first Weinberg sum rule [S. Weinberg, Phys. Rev. Letters **18**, 507 (1967)] is sensitive to the properties of the Schwinger term. However, its validity is not conclusive evidence for the c-number nature of this object; see D. J. Gross and R. Jackiw, Phys. Rev. **163**, 1688 (1967); D. G. Boulware and R. Jackiw, *ibid.* **186**, 1442 (1969). The sum rule (2.8) is related to a connection derived by J. M. Cornwall and R. E. Norton, *ibid.* **177**, 2584 (1969), between the Regge residue function associated with $\tilde{F}_L(\omega,\nu)$ and the Schwinger term. See also J. M. Cornwall, R. E. Norton, and D. Corrigan, Phys. Rev. Letters, **24**, 1141 (1970).

of the Regge-Pomeranchuk lore indicates $F_L(0)\neq 0$ (see Sec. III B). On the other hand, the experimental data are not inconsistent with the hypothesis that the usual Pomeranchuk pole decouples from this amplitude, or even from the entire process.

We remark here that the present rule is quite different from the one obtained by Callan and Gross.[12] In their calculation, they relate various integrals over F_2 and F_L to the *tensor* structure of

$$\int d^3x\,\langle p|[\partial^0 j^i(0,\mathbf{x}),j^j(0)]|p\rangle.$$

Our formula (2.8) makes no reference to this object.

Of course, neither of the two sum rules can be checked by explicit calculation in realistic field-theory models.[13] In Sec. VI it is shown that our result is satisfied in free-field theories. However, it is further shown that in lowest-order perturbation theory for a theory of fermions, with a vector-meson interaction, the sum rule is violated. In that theory $F_2(\omega)$ does not exist, nor are the manipulations with limits valid.

The present representation for the Schwinger term, (2.8), bears a striking resemblance to the analogous relation for the vacuum-expectation value of that object, expressed in terms of the total electroannihilation cross section, $\sigma(q^2)$[14]:

$$\langle 0|[j^0(0,\mathbf{x}),j^j(0)]|0\rangle = i\partial^i\delta(\mathbf{x})\frac{1}{32\pi^2\alpha^2}\int_0^\infty dq^2 q^2\sigma(q^2). \quad (2.9)$$

C. Self-Mass Divergence

The existence of $F_2(\omega)$, the vanishing of $F_L(\omega)$, the no-subtraction hypothesis about a portion of the forward Compton amplitude, and the Cottingham formula[15] for the electromagnetic self-mass can be combined into a calculation of the logarithmic divergence of that object. The answer is[16]

$$\delta m \propto \left(L+\int_0^2 d\omega\, F_2(\omega)\right)\ln\infty + (\text{finite terms}). \quad (2.10)$$

In the above, δm is the self-mass of the target, to lowest order in electromagnetic interactions. L is related to a subtraction constant in a fixed q^2 dispersion relation for the Compton amplitude; hence it is not directly related to the measured cross sections. In Sec. V, we show that if the sum rule of Sec. II B is correct, then the divergent part of the electromagnetic self-mass can be computed in terms of measurable (in principle) deep-inelastic cross sections. If $F_L(\omega)$ does not vanish, then the self-mass is quadratically divergent. If $F_L(\omega)$ is identically zero, there is probably still a logarithmic divergence; the logarithmically divergent term has the form

$$\delta m_{\text{div}} = \int^{-\infty}\frac{dq^2}{q^2}\left(M-\frac{m^2}{\pi}\int_0^\infty d\omega\, F_2(\omega)\right). \quad (2.11)$$

Here M is a non-negative quantity, determined by the nonscaling corrections to the cross sections.

III. DERIVATION OF REPRESENTATION

We present the derivation of the position-space representation [Eq. (2.7)] for the commutator function. Our method is a direct exercise in Fourier analysis. At the end of this section, it is demonstrated that the representation is consistent with Regge phenomenology.

A. Fourier-Transform Analysis

The general form for $c_i(x^2, x\cdot p)$ which follows from causality [Eq. (2.6)] is repeated here for reference:

$$c_i(x^2, x\cdot p) = \epsilon(x\cdot p)\theta(x^2)f_i(x^2, x\cdot p) + \epsilon(x\cdot p)\delta(x^2)s_i^{(0)}(x\cdot p) + \sum_{n=1}^{N}\epsilon(x\cdot p)\delta^{(n)}(x^2)s_i^{(n)}(x\cdot p). \quad (3.1)$$

In addition to the derivatives of δ functions, which we have exhibited explicitly in (3.1), there may also be contributions to $c_i(x^2, x\cdot p)$ which behave as "fractional derivatives" of δ functions. By this we mean generalized

[12] C. G. Callan, Jr. and D. J. Gross, Phys. Rev. Letters **22**, 156 (1969).

[13] In theories where the electromagnetic current is bilinear in Fermi fields which interact by meson exchange, F_1 and F_2 do not exist beyond the Born approximation, while F_L does not exist beyond first-order perturbation theory; see R. Jackiw and G. Preparata, Phys. Rev. Letters **22**, 975 (1969); **22**, 1162(E), (1969); Phys. Rev. **185**, 1748 (1969); S. L. Adler and Wu-Ki Tung, Phys. Rev. Letters **22**, 978 (1969); Phys. Rev. D **1**, 2846 (1970). Only in the physically unrealistic, superrenormalizable scalar meson theory with a cubic meson interaction do the F_i exist, at least in lowest-order perturbation theory. However, all is not well, since the F_i possess a singularity of the form $(\omega-1)^{-1}$: D. J. Gross and R. Jackiw (unpublished).

[14] See, e.g., J. D. Bjorken, Phys. Rev. **148**, 1467 (1966). Formal sum rules based on this commutator can be derived; J. D. Bjorken, *ibid.* **148**, 1467 (1966); V. Gribov *et al.*, Phys. Letters **24B**, 557 (1967); J. Dooher, Phys. Rev. Letters **19**, 600 (1967); R. Jackiw and G. Preparata, Ref. 13.

[15] W. N. Cottingham, Ann. Phys. (N. Y.) **25**, 424 (1963).

[16] Heinz Pagels, Phys. Rev. **185**, 1990 (1969); G. B. West (unpublished). It should be remarked here that this expression for the logarithmic divergence of δm does not coincide with the one of J. D. Bjorken, *ibid.* **148**, 1467 (1966); i.e., it is not simply related to $\langle p|[\partial^0 j^i(0,\mathbf{x}),j_i(0)]|p\rangle$. This difference is traceable to Bjorken's use of an unsubtracted dispersion relation for T_1, while we permit a subtraction. For more general treatments of the connection between commutators and self-mass divergences, see J. M. Cornwall and R. E. Norton, *ibid.* **173**, 1637 (1968); D. G. Boulware and S. Deser, *ibid.* **175**, 1912 (1968). Our formula (2.10) is consistent with these investigations.

functions of the form

$$[(x^2-i\epsilon x\cdot p)^{-\alpha}-(x^2+i\epsilon x\cdot p)^{-\alpha}]s_i^{(\alpha)}(x\cdot p)\,,$$

where α is an arbitrary, positive noninteger quantity. By arguments completely analogous to those presented below, which show that the derivatives of δ functions violate scaling, and hence must be absent, one can rule out $\alpha>1$ in c_1 and $\alpha>0$ in c_2.

The following properties are true: (1) The functions $f_i(x^2, x\cdot p)$ must be integrable at $x^2=0$. The reason for this that the Fourier transform of $\epsilon(x\cdot p)\theta(x^2)f_i(x^2, x\cdot p)$ is observable; therefore, it must exist (away from the elastic point $-q^2=\pm 2\nu$). Since $x^2=0$ is an endpoint for the Fourier integration, the integral must converge there, and we conclude that $f_i(x^2, x\cdot p)=O((x^2)^{-1+\epsilon})$ near $x^2=0$. (2) The mass spectrum imposes the condition that the Fourier transforms of $f_i(x^2, x\cdot p)$ and $s_i^{(n)}(x\cdot p)$ in the one-dimensional variable $x\cdot p$ have finite support.[17] (3) From their definition, $f_i(x^2, x\cdot p)$ and $s_i^{(n)}(x\cdot p)$ are seen to be even in $x\cdot p$.

We now Fourier-transform the expression (3.1) with respect to x. Consider first the terms involving the δ function and derivatives thereof. A typical expression is

$$I_i^{(n)}\equiv i\int d^4x\, e^{iq\cdot x}\epsilon(x\cdot p)\delta^{(n)}(x^2)s_i^{(n)}(x\cdot p)$$
$$=\int_{-\infty}^{\infty}d\alpha\,\epsilon(\alpha)s_i^{(n)}(\alpha)i\int d^4x\, e^{iq\cdot x}\times\delta(x\cdot p-\alpha)\delta^{(n)}(x^2)\,. \quad (3.2a)$$

The four-dimensional x integral may be evaluated, in the p rest frame, with the result

$$\int d^4x\, e^{iq\cdot x}\delta(x\cdot p-\alpha)\delta^{(n)}(x^2)=-\frac{\pi m^{2n-1}}{2^n\alpha^{2n}}\frac{1}{|\mathbf{q}|}\times\left(-|\mathbf{q}|\frac{\partial}{\partial|\mathbf{q}|}\right)^n\left(\exp\frac{i}{m}(q_0\alpha-|\mathbf{q}\alpha|)-\exp\frac{i}{m}(q_0\alpha+|\mathbf{q}\alpha|)\right). \quad (3.2b)$$

Insertion of (3.2b) into (3.2a) yields the following expression, once the symmetry properties $s_i^{(n)}(\alpha)$ are taken into account:

$$I_i^{(n)}=-\frac{\pi m^{2n-1}}{2^n}\frac{1}{|\mathbf{q}|}\left(-|\mathbf{q}|\frac{\partial}{\partial|\mathbf{q}|}\right)^n\times\int_{-\infty}^{\infty}d\alpha\frac{s_i^{(n)}(\alpha)}{\alpha^{2n}}\left(\exp\frac{i\alpha}{m}(q_0-|\mathbf{q}|)-\exp\frac{i\alpha}{m}(q_0+|\mathbf{q}|)\right). \quad (3.2c)$$

[17] This is most easily seen from the spectral representation for the commutator function, see L. S. Brown, Ref. 2.

Next a Fourier representation for $s_i^{(n)}(\alpha)$ is introduced:

$$s_i^{(n)}(\alpha)=\int_{-\infty}^{\infty}d\beta\, e^{i\alpha\beta}\tilde{s}_i^{(n)}(\beta)\,. \quad (3.3)$$

$\tilde{s}_i^{(n)}(\beta)$ has finite support. Our expression for $I_i^{(n)}$ now takes the form

$$I_i^{(n)}=-\frac{\pi m^{2n-1}}{2^n}\int_{-\infty}^{\infty}d\beta\,\tilde{s}_i^{(n)}(\beta)\frac{1}{|\mathbf{q}|}\left(-|\mathbf{q}|\frac{\partial}{\partial|\mathbf{q}|}\right)^n\times\int_{-\infty}^{\infty}\frac{d\alpha}{\alpha^{2n}}\left[\exp\frac{i\alpha}{m}(q_0-|\mathbf{q}|+\beta m)-\exp\frac{i\alpha}{m}(q_0+|\mathbf{q}|+\beta m)\right]. \quad (3.4)$$

When $n\geqslant 1$, the α integral becomes[18]

$$\frac{1}{|\mathbf{q}|}\left(|\mathbf{q}|\frac{\partial}{\partial|\mathbf{q}|}\right)^n\frac{i\pi}{(2n-1)!m^{2n-1}}\times[(q_0-|\mathbf{q}|+\beta m)^{2n-1}\epsilon(q_0-|\mathbf{q}|+\beta m)-(q_0+|\mathbf{q}|+\beta m)^{2n-1}\epsilon(q_0+|\mathbf{q}|+\beta m)]. \quad (3.5)$$

We determine the asymptotic form of (3.5) as $\nu=mq_0$ gets large, with ω fixed, i.e., with $|\mathbf{q}|\simeq q_0+m\omega$. It is easy to see that (3.5) and hence $I_i^{(n)}(\omega,\nu)$, becomes in this limit proportional to ν^{2n-2}. According to (2.3), $I_1^{(n)}(\omega,\nu)$ $(I_2^{(n)}(\omega,\nu))$ must have a finite limit when multiplied by ν (ν^2). This is impossible for $n\geqslant 1$, and it must be concluded that all derivatives of δ functions are absent from $c_i(x^2, x\cdot p)$.

For $n=0$, (3.4) is simply

$$I_i^{(0)}=\frac{-2\pi^2}{m|\mathbf{q}|}\left[\tilde{s}_i^{(0)}\left(\frac{q_0-|\mathbf{q}|}{m}\right)-\tilde{s}_i^{(0)}\left(\frac{-q_0-|\mathbf{q}|}{m}\right)\right], \quad (3.6a)$$

and in the desired region this approaches

$$I_i^{(0)}\to-(2\pi^2/\nu)[\tilde{s}_i^{(0)}(\omega)-\tilde{s}_i^{(0)}(-2\nu/m^2)]. \quad (3.6b)$$

For suficiently large ν, $\tilde{s}_i^0(-2\nu/m^2)$ vanishes because of the mass spectrum limitations on its support. Therefore, from (2.3) it follows that no δ function contributes to $c_2(x^2, x\cdot p)$, while

$$8\pi^2\omega^2\tilde{s}_1^{(0)}(\omega)=F_L(\omega)\,, \quad (3.7a)$$

$$s_1^{(0)}(x\cdot p)=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega x\cdot p}}{\omega^2}F_L(\omega)=\frac{1}{4\pi^2}\int_0^{\infty}d\omega\frac{\cos\omega x\cdot p}{\omega^2}F_L(\omega)\,. \quad (3.7b)$$

[18] M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge U. P., Cambridge, England, 1959), p. 43.

Next, the step-function contributions to (3.1) are Fourier-transformed:

$$I_i' \equiv i \int d^4x\, e^{iq\cdot x} \epsilon(x\cdot p)\theta(x^2) f_i(x^2, x\cdot p)$$

$$= \int_{-\infty}^{\infty} d\alpha\, \epsilon(\alpha) i \int d^4x\, e^{iq\cdot x}\theta(x^2) f_i(x^2,\alpha)\delta(x\cdot p-\alpha). \quad (3.8a)$$

The four-dimensional x integral is partially evaluated in the rest frame of p by performing the x_0 and the angular three-dimensional integrations. Use of the symmetry properties of $f_i(x^2,\alpha)$ in α, as well as the asymptotic formula for $|\mathbf{q}|$ in terms of q_0, yields an expression valid when $\nu \to \infty$:

$$I_i'(\omega,\nu) \to -\frac{2\pi}{\nu} \int_0^{\infty} d\alpha \int_0^{\alpha^2/m^2} dx\, f_i\left(\frac{\alpha^2}{m^2} - x, \alpha\right)$$

$$\times \left[\cos\left(\frac{\nu}{m^2}\alpha - \frac{\nu}{m}\sqrt{x} - m\omega\sqrt{x}\right) - \cos\left(\frac{\nu}{m^2}\alpha + \frac{\nu}{m}\sqrt{x} + m\omega\sqrt{x}\right)\right]. \quad (3.8b)$$

As ν gets large, the contribution from the last cosine term is negligible because of rapid oscillations of the argument, while the first cosine term emphasizes the region $\sqrt{x} \simeq \alpha/m - (m\omega/\nu)\alpha$. Therefore, for large ν, (3.8b) is equal to

$$I_i'(\omega,\nu) \to -\frac{2\pi}{\nu} \int_0^{\infty} d\alpha\, f\left(\frac{2\omega\alpha^2}{\nu}, \alpha\right) \times \int_0^{\alpha^2/m^2} dx \cos\left(\frac{\nu}{m^2}\alpha - \frac{\nu}{m}\sqrt{x} - m\omega\sqrt{x}\right)$$

$$\to \frac{2\pi i}{\nu^2} \int_{-\infty}^{\infty} d\alpha\, \alpha e^{i\omega\alpha} f_i\left(\frac{2\omega\alpha^2}{\nu}, \alpha\right). \quad (3.8c)$$

From (2.3) we see that the contribution to F_L of this term is

$$\frac{8\pi i}{\nu} \int_{-\infty}^{\infty} d\alpha\, \alpha e^{i\omega\alpha} f_1\left(\frac{2\omega\alpha^2}{\nu}, \alpha\right). \quad (3.9)$$

Since $f_1(x^2, x\cdot p)$ can be no more singular than $(x^2)^{-1+\epsilon}$ for small x^2, (3.9) increases with ν no faster than $\nu^{-\epsilon}$. Thus the step-function terms, make no contribution to F_L. For F_2, (3.8c) gives

$$F_2(\omega) = \lim_{\nu\to\infty}\left[-4i\pi\omega \int_{-\infty}^{\infty} d\alpha\, \alpha e^{i\omega\alpha} f_2\left(\frac{2\omega\alpha^2}{\nu}, \alpha\right)\right]. \quad (3.10a)$$

Since by hypothesis this limit *does* exist, $f_2(x^2, x\cdot p)$ must be regular at $x^2=0$, and we obtain

$$F_2(\omega) = -4\pi i\omega \int_{-\infty}^{\infty} d\alpha\, \alpha e^{i\omega\alpha} f_2(0,\alpha), \quad (3.10b)$$

$$f_2(0, x\cdot p) = \frac{i}{8\pi^2} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega x\cdot p}}{\omega x\cdot p} F_2(\omega)$$

$$= \frac{1}{4\pi^2} \int_0^{\infty} d\omega \frac{\sin\omega x\cdot p}{\omega x\cdot p} F_2(\omega). \quad (3.11)$$

This completes the derivation of the position-space representation (2.7). However, we must expose here a tacit assumption which has been made. Throughout the derivation it has been demanded that $\nu C_1(\omega,\nu)$ and $\nu^2 C_2(\omega,\nu)$ approach finite limits since, according to (2.3), these limits are equal to the finite quantities

$$\lim_{\nu\to\infty} 4\nu C_1(\omega,\nu) = F_L(\omega)/\omega^2, \quad (3.12a)$$

$$\lim_{\nu\to\infty} 2\nu^2 C_2(\omega,\nu) = F_2(\omega)/\omega. \quad (3.12b)$$

Unfortunately, the division by powers of ω, employed in passing from (2.3) to (3.12), may be ambiguous in the context of generalized functions. Arbitrary multiples of a δ function and of its derivative may be present on the right-hand side of (3.12). Similarly, another example of such ambiguities arises in the *mathematically* correct solution to (3.7a):

$$\tilde{s}_1^{(0)}(\omega) = \frac{1}{8\pi^2} \frac{F_L(\omega)}{\omega^2} + n_1\delta(\omega) + n_2\delta^{(1)}(\omega),$$

$$s_1^{(0)}(x\cdot p) = \frac{1}{8\pi^2} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} e^{i\omega x\cdot p} F_L(\omega) + n_1 - in_2 x\cdot p. \quad (3.13)$$

In the above, n_1 and n_2 are arbitrary parameters. The term proportional to n_2 may be eliminated by invoking crossing symmetry. A related problem is the definition of the Fourier transforms over ω which occur in our representation, in the case that the integral does not converge in the usual sense. Clearly the integrations must then be interpreted as defining generalized functions. However, such generalized functions can be ambiguous, since there may exist several alternative methods for handling the singularity. Thus, for example, if $F_L(0)$ is nonvanishing, we interpret $F_L(0)/\omega^2$ as a principle value, and there may be an additional arbitrary term proportional to $\delta(\omega)$, for the reasons given above.

We have not determined the correct and unambiguous prescription for handling these terms over which we have no control; they have simply been set to zero throughout. We suspect that they are related to possible subtraction constants in a dispersive representation of the forward Compton amplitude (see Sec. VI C).

We remark that the same form for the commutator function (2.7) may be also derived with the help of the conventional spectral representation which is satisfied by that amplitude: Jost, Lehmann, and Dyson[19] or Deser, Gilbert, and Sudarshan.[20] We do not present such a derivation here, since it makes use of techniques already developed by Brown,[2] who applied them to a study of the product (rather than commutator) of two currents.

B. Regge Limit

We demonstrate that the representation (2.7) is consistent with the conventional ideas about Regge behavior, i.e., fixed q^2 and large ν behavior.[21] The calculation is performed for $\tilde{F}_2$; completely analogous considerations apply to $\tilde{F}_L$ and to $\tilde{F}_1$.

According to the usual lore, $\tilde{F}_2$ should behave as $\nu^{\alpha-1}$ in the Regge limit. Assuming $\alpha=1$, due to Pomeranchuk exchange, one is lead to the conclusion that $\tilde{F}_2$ is independent of ν for large ν. This may be combined with the deep-inelastic limit, with the result that $\tilde{F}_2$ is also independent of q^2 in the Regge region, i.e.,

$$\lim_{\nu\to\infty;\ q^2\ \text{fixed}} \tilde{F}_2(\omega,\nu)=\tilde{F}_2(0,\infty)=F_2(0). \qquad (3.14)$$

To see that (2.7) is consistent with (3.14), one merely needs to Fourier-transform $c_2(x^2, x\cdot p)$. A formula like (3.8b) is again obtained, except that the independent variables are now q^2 and ν. Explicitly, we find

$$I_2'(q^2,\nu) \underset{\nu\to\infty}{\longrightarrow} -\frac{2\pi}{\nu}\int_0^\infty d\alpha \int_0^{\alpha^2/m^2} dx\, f_2\!\left(\frac{\alpha^2}{m^2}-x,\alpha\right) \times\left[\cos\!\left(\frac{\nu}{m^2}\alpha-\frac{\nu}{m}\sqrt{x}+\frac{mq^2}{2\nu}\sqrt{x}\right) -\cos\!\left(\frac{\nu}{m^2}\alpha+\frac{\nu}{m}\sqrt{x}-\frac{mq^2}{2\nu}\sqrt{x}\right)\right]. \qquad (3.15a)$$

Destructive interference eliminates the contribution of the second cosine term, while the argument of the first emphasizes $\sqrt{x}\simeq\alpha/m$. Therefore,

$$I_2'(q^2,\nu) \underset{\nu\to\infty}{\longrightarrow} -\frac{2\pi}{\nu}\int_0^\infty d\alpha\, f_2(0,\alpha) \times\int_0^{\alpha^2/m^2} dx\,\cos\!\left(\frac{\nu}{m^2}\alpha-\frac{\nu}{m}\sqrt{x}+\frac{mq^2}{2\nu}\sqrt{x}\right) \to\frac{2\pi i}{\nu^2}\int_{-\infty}^\infty d\alpha\,\alpha e^{-iq^2\alpha/2\nu} f_2(0,\alpha). \qquad (3.15b)$$

[19] R. Jost and H. Lehmann, Nuovo Cimento 5, 1598 (1957); F. J. Dyson, Phys. Rev. 111, 1717 (1958).

[20] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. 115, 731 (1959).

[21] There are many discussions which combine the Regge limit with the deep-inelastic limit; see D. H. I. Abarbanel, M. Goldberger, and S. Treiman, Phys. Rev. Letters 22, 500 (1969); H. Harari, *ibid.* 22, 1078 (1969); R. Brandt, *ibid.* 22, 1149 (1969).

According to (3.10b), this is equal to

$$I_2'(q^2,\nu) \underset{\nu\to\infty}{\longrightarrow} -(1/q^2\nu)F_2(-q^2/2\nu). \qquad (3.15c)$$

Finally, from (2.3b), we get

$$\tilde{F}_2(q^2,\nu) \underset{\nu\to\infty}{\longrightarrow} F_2(0), \qquad (3.15d)$$

which verifies (3.14). An analogous calculation for $\tilde{F}_L(q^2,\nu)$ shows that

$$\tilde{F}_L(q^2,\nu) \underset{\nu\to\infty}{\longrightarrow} F_L(0). \qquad (3.16)$$

It should be remarked that at a time when $F_2(\omega)$ seemed constant at small ω, the above Regge argument indicated that the Pomeranchuk *did* couple in the relevant channel, with a strength given by the constant value of $F_2(\omega)$. However, further experimental study has indicated a *decreasing* behavior for $F_2(\omega)$ at small ω,[1] and at the present time one has no way of determining $F_2(0)$ from the data. Indeed a possibility exists that $F_2(0)=0$. Also the preliminary experimental results are consistent with $F_L(\omega)=F_L(0)=0$. Thus the Pomeranchuk pole may be decoupling from the amplitude F_L. Furthermore, if it does couple to F_2, the intercept may be less than 1.

IV. DERIVATION OF SUM RULE

Several derivations of the sum rule (2.8) are given. The first proceeds directly from (2.7), evaluated at equal times. Next, the deep-inelastic limit of Bjorken[9] is also shown to yield our formula. This section is then concluded with a demonstration that a no-substraction hypothesis, about the dispersive representation of a portion of the forward Compton amplitude, is necessary for the validity of the sum rule.

A. First Method

From (2.4) it follows that

$$i\langle p|[j^0(x),j^i(0)]p\rangle=-\partial^0\partial^i c_1(x^2, x\cdot p) +[p_0p^i\square-p\cdot\partial(\partial^0 p^i+\partial^i p_0)]c_2(x^2, x\cdot p). \qquad (4.1)$$

Since the c_i are odd in x_0, only the odd part of the differential operators survives at equal times:

$$\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle=\lim_{x_0\to 0}[i\partial^i\partial^0 c_1(x^2, x\cdot p) +i(p^ip^j+g^{ij}p_0^2)\partial_j\partial^0 c_2(x^2, x\cdot p)]. \qquad (4.2)$$

An expression of the form $\theta(x^2)\epsilon(x\cdot p)f(x^2, x\cdot p)$ has the property that it vanishes upon differentiation by x_0, when x_0 is set to zero. Therefore, according to (2.7), we are left with

$$\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle$$
$$=\frac{i}{4\pi^2}\lim_{x_0\to0}\partial^i\partial^0\left[\epsilon(x\cdot p)\delta(x^2)\int_0^\infty d\omega\frac{\cos\omega x\cdot p}{\omega^2}F_L(\omega)\right]$$
$$=\frac{i}{2\pi}\lim_{x_0\to0}\partial^i\partial^0\left[d(x|0)\int_0^\infty d\omega\frac{\cos\omega x\cdot p}{\omega^2}F_L(\omega)\right]$$
$$=\frac{i}{2\pi}\partial^i\left[\delta(\mathbf{x})\int_0^\infty d\omega\frac{\cos\omega\mathbf{x}\cdot\mathbf{p}}{\omega^2}F_L(\omega)\right]. \quad (4.3)$$

In the last equation above, use has been made of the fact that the causal zero-mass function, $d(x|0)$, vanishes at $x_0=0$, while its time derivative is a three-dimensional δ function at that point. When the integral

$$\int_0^\infty d\omega\frac{\cos\omega\mathbf{x}\cdot\mathbf{p}}{\omega^2}F_L(\omega)$$

is convergent, in the usual sense, (4.3) verifies (2.8).

B. Second Method

Our second method for deriving (2.8) makes use of the new Bjorken high-energy limit.[9] We take the $0i$ components of (2.1), form the limit $p_0\to\infty$ at fixed $\mathbf{q}$, in the frame $\mathbf{p}=\mathbf{q}|\mathbf{p}|/|\mathbf{q}|$, subject to the constraints $p^2=m^2$, i.e., $p_0\simeq|\mathbf{p}|$, and ω constant, i.e., $q_0\simeq-2p_0\omega-|\mathbf{q}|$. It then follows that

$$2p_0C^{0i}(q,p)\to\tfrac{1}{2}q^iF_L(\omega)/\omega^2. \quad (4.4)$$

Also $C^{0i}(q,p)$ may be written in this limit as

$$C^{0i}(q,p)\to\int d^4x\,e^{-2ip_0\omega x_0}e^{-i|\mathbf{q}|x_0}e^{-i\mathbf{q}\cdot\mathbf{x}}\times\langle p|[j^0(x),j^i(0)]|p\rangle. \quad (4.5)$$

An integration over ω of (4.5) produces a δ function in $2p_0x_0$, which then evaluates the x_0 integral at equal times:

$$\lim_{p_0\to\infty}\int_{-\infty}^{\infty}d\omega\,2p_0C^{0i}(q,p)$$
$$=\tfrac{1}{2}q^i\int_{-\infty}^{\infty}d\omega\frac{F_L(\omega)}{\omega^2}$$
$$=2\pi\int d^3x\,e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle. \quad (4.6)$$

This is the desired result. A cautionary reminder must be inserted concerning the validity of an interchange of limit and integral.

It is easy to show that the Callan-Gross[12] sum rule may be derived by the same techniques presented here, except that one considers the ij components of $C^{\mu\nu}$, and integrates ωC^{ij} over ω.

C. Relation to Compton Amplitude

In order to gain some understanding about the conditions under which the sum rule (2.8) is valid, that is, about the conditions which permit the manipulations employed in the previous derivations, we now show that (2.8) follows if there are no subtractions in the dispersive representation of a certain portion of the forward Compton amplitude. The amplitude in question is

$$T^{\mu\nu}(q,p)=i\int d^4x\,e^{iqx}\langle p|T^*j^\mu(x)j^\nu(0)|p\rangle$$
$$=-(g^{\mu\nu}-q^\mu q^\nu/q^2)T_1(q^2,\nu)$$
$$+(p^\mu-q^\mu q\cdot p/q^2)(p^\nu-q^\nu q\cdot p/q^2)T_2(q^2,\nu). \quad (4.7)$$

The T_i satisfy fixed-q^2 dispersion relations in ν:

$$T_1(q^2,\nu)=T_1(q^2)+\frac{\nu^2}{2\pi}\int_{(-q^2/2)^2}^\infty d\nu'^2\frac{W_1(q^2,\nu')}{\nu'^2(\nu'^2-\nu^2)}, \quad (4.8)$$

$$T_2(q^2,\nu)=\frac{1}{2\pi}\int_{(-q^2/2)^2}^\infty d\nu'^2\frac{W_2(q^2,\nu')}{(\nu'^2-\nu^2)}. \quad (4.9)$$

In the above, the W_i are related simply to our $\tilde{F}_i$: $W_1=\tilde{F}_1$, $\nu W_2=\tilde{F}_2$. We have allowed for one subtraction in (4.8) while no subtractions are present in (4.9). This reflects the usual Regge ideas: $W_1\to\nu^\alpha$, $W_2\to\nu^{\alpha-2}$, $\alpha=1$. If one forms the combination

$$T_L(q^2,\nu)\equiv T_1(q^2,\nu)+(\nu^2/q^2)T_2(q^2,\nu), \quad (4.10)$$

one may verify that the absorptive part of $T_L(q^2,\nu)$ is essentially $\tilde{F}_L$. Therefore, to the extent that one may extrapolate the present electroproduction data to the conclusion that the Pomeranchuk pole decouples from this amplitude, it is plausible to assume that $T_L(q^2,\nu)$ is unsubtracted.[22–24]

When the fixed-q^2 unsubtracted dispersion relation for T_L is written in terms of the variable $\omega'=-q^2/2\nu'$, one arrives at the formula

$$T_L(q^2,\nu)=\frac{\omega^2}{2\pi}\int_0^\infty\frac{d\omega'}{\omega'^2}\frac{\tilde{F}_L(\omega',q^2)}{\omega'^2-\omega^2}. \quad (4.11)$$

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[22] Of course a subtraction in a dispersion relation is not necessitated only by divergences in the dispersive integral. A term without an absorptive part may be present in the scattering amplitude—we must assume that this does not occur for T_L. Furthermore, it is clear that subdominant Regge behavior must be consistent with our unsubtractedness hypothesis.

[23] Such an assumption *necessitates* a subtraction in $T_1(q^2,\nu)$. For if $T_L(q^2,\nu)$ and $T_1(q^2,\nu)$ are unsubtracted, it follows from (4.11) that $\nu^2T_2(q^2,\nu)$ is unsubtracted. But according to (4.10b), $T_2(q^2,\nu)$ is also unsubtracted. Hence a superconvergence relation holds; $\int d\nu^2W_2(q^2,\nu)=0$. This violates the positivity of W_2.

[24] In a different context, some of the consequences of a no-subtraction hypothesis for *combinations* of T_1 with ν^2T_2 have been explored by D. J. Gross and Heinz Pagels, Phys. Rev. **172**, 1391 (1968).

In terms of these same variables, we have for T_2

$$T_2(q^2,\nu)=\frac{2\omega^2}{\pi q^2}\int_0^\infty d\omega'\frac{\tilde{F}_2(\omega',q^2)}{\omega'^2-\omega^2}. \quad (4.12)$$

In the above the independent arguments of $\tilde{F}_i$ are ω' and q^2, rather than ω' and ν.

Consider now the limit of T_L and T_2 as $q_0\to i\infty$. Since $\omega^2=(q^2/2\nu)^2$ approaches ∞ in this limit, we find

$$\lim_{q_0\to i\infty} T_2(q^2,\nu)=\lim_{q_0\to i\infty}\frac{-2}{\pi q^2}\int_0^\infty d\omega' F_2(\omega')=0, \quad (4.13a)$$

$$\begin{aligned}\lim_{q_0\to i\infty} T_L(q^2,\nu)&=-\frac{1}{2\pi}\int_0^\infty \frac{d\omega'}{\omega'^2}F_L(\omega')\\ &=\lim_{q_0\to i\infty}\left(T_1(q^2,\nu)+\frac{\nu^2}{q^2}T_2(q^2,\nu)\right)\\ &=\lim_{q_0\to i\infty} T_1(q^2,\nu). \quad (4.13b)\end{aligned}$$

On the other hand, the $0i$ component of $T^{\mu\nu}$ is just the T product of j^0 and j^i (the seagull necessitated by a possible Schwinger term is present only in the ij component of $T^{\mu\nu}$)[25]; hence the large q_0 limit may be evaluated by the help of the Bjorken-Johnson-Low formula[26]

$$\begin{aligned}\lim_{q_0\to i\infty} q_0 T^{0i}(q,p)&=-\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle\\ &=\lim_{q_0\to i\infty} q^i T_1(q^2,\nu)\\ &=-\frac{q^i}{2\pi}\int_0^{-\infty}\frac{d\omega'}{\omega'^2}F_L(\omega'). \quad (4.13c)\end{aligned}$$

This completes the verification of the sum rule.

It is now also clear that if a subtraction were required on the dispersion relation for $T_L(q^2,\nu)$, one could not arrive at the desired sum rule. In that case (4.11) would be replaced by

$$T_L(q^2,\nu)=T_L(q^2,0)+\frac{1}{2\pi}\int_0^\infty d\omega'\frac{\tilde{F}_L(\omega',q^2)}{\omega'^2-\omega^2}. \quad (4.14a)$$

The analog of (4.13b) now becomes

$$\begin{aligned}\lim_{q_0\to i\infty} T_L(q^2,\nu)&=T_L(-\infty,0)\\ &=\lim_{q_0\to i\infty} T_1(q^2,\nu). \quad (4.14b)\end{aligned}$$

An application of Bjorken-Johnson-Low[26] theorem yields, in the present instance, the uninformative result

$$\begin{aligned}\lim_{q_0\to i\infty} q_0 T^{0i}(q,p)&=-\int d^3x\, e^{-i\mathbf{q}\cdot\mathbf{x}}\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle\\ &=\lim_{q_0\to i\infty} q^i T_1(q^2,\nu)\\ &=q^i T_L(-\infty,0). \quad (4.14c)\end{aligned}$$

It is to be recalled that the derivation by Callan and Gross[12] of their sum rule employed the techniques presented in this subsection. They worked with the i,j components of (4.7) and used the representations (4.8) and (4.9). Their derivation is *invalid* in the presence of Schwinger terms, since in that case $T^{ij}(q,p)$ does not coincide with the T product to which the Bjorken-Johnson-Low[26] theorem may be applied—additional seagulls are necessarily present. However, it is easy to modify their derivation to account for this seagull, and their conclusion remains unchanged.[27]

V. SELF-MASS DIVERGENCE

In this section we show how the unsubtractedness of T_L leads to the conclusion that the self-mass of the proton, to lowest order in electromagnetism, is expressible in terms of measurable quantities.

The Cottingham formula[15] for the self-mass may be expressed in the form

$$\delta m\propto\int_0^{-\infty}\frac{dq^2}{q^2}\int_{-1}^1 dz(1-z^2)^{1/2}q^2T(q^2,i\nu). \quad (5.1)$$

In the above T is $g^{\mu\nu}T_{\mu\nu}$ and $z\equiv\nu/m\surd(-q^2)$. In presenting (5.1), we have made use of a Wick rotation ($q\to iq$). The conventional dispersive representation for T_1 and T_2, (4.10), expressed in terms of the variables ω and q^2, together with the scaling properties of $W_i(q^2,\nu)$ permits one to perform the z integrations in the asymptotic region $q^2\to-\infty$. The result is that (5.1) is divergent. The infinite portion of this quality is proportional to

$$\delta m_{\rm div}\propto\int^{-\infty}\frac{dq^2}{q^2}\left[q^2T_1(q^2)+\frac{m^2}{2\pi}\int_0^\infty d\omega(F_2(\omega)+2\omega F_1(\omega))\right], \quad (5.2)$$

where $T_1(q^2)$ is the subtraction constant in the dispersion relation for $T_1(q^2,\nu)$, i.e., $T_1(q^2,0)\equiv T_1(q^2)$.

[25] D. J. Gross and R. Jackiw, Nucl. Phys. **B14**, 269 (1969).

[26] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966); K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37-38**, 74 (1966).

[27] We remark here that if the experimental data lead to the conclusion that a no-subtraction hypothesis for T_L is *not* tenable, then it may still be possible to derive a sum rule for the one-particle matrix elements of the Schwinger term. In order to do this, one would examine the feasibility of a no-subtraction hypothesis for combinations of T_1 and ν^2T_2 which differ from T_L.

This result is well known.[16] The novel feature in the present development is the use of the unsubtracted dispersion relation for T_L. From (4.10) and (4.11) it follows that

$$T_1(q^2,\nu)=\frac{\omega^2}{2\pi}\int_0^\infty \frac{d\omega'}{\omega'^2}\frac{\tilde{F}_L(\omega',q^2)}{\omega'^2-\omega^2}-\frac{\nu^2}{q^2}T_2(q^2,\nu)\,. \quad (5.3)$$

As $\nu\to 0$, $\omega\to\infty$; therefore

$$T_1(q^2)=-\frac{1}{2\pi}\int_0^\infty \frac{d\omega'}{\omega'^2}\tilde{F}_L(\omega',q^2)\,. \quad (5.4)$$

Note that as $q^2\to-\infty$,

$$T_1(q^2)\to-\frac{1}{2\pi}\int_0^\infty \frac{d\omega'}{\omega'^2}F_L(\omega')\,;$$

hence in order to avoid quadratically diverging self-masses, there must be no q-number Schwinger term.[28]

To proceed, we note that (5.4) implies

$$T_1(q^2)=-\frac{1}{2\pi}\int_0^\infty \frac{d\omega'}{\omega'^2}\left(\tilde{F}_L(\omega',q^2)-\frac{4m^2\omega'^2}{q^2}\tilde{F}_2(\omega',q^2)\right)-\frac{2m^2}{q^2\pi}\int_0^\infty d\omega'\tilde{F}_2(\omega',q^2)$$

$$=-M(q^2)-\frac{2m^2}{q^2\pi}\int_0^\infty d\omega'\tilde{F}_2(\omega',q^2)\,. \quad (5.5)$$

Here $M(q^2)\geq 0$ since

$$\tilde{F}_L(\omega',q^2)-\frac{4m^2\omega'^2}{q^2}\tilde{F}_2(\omega',q^2)\geq 0\,.$$

Hence

$$q^2T_1(q^2)\xrightarrow[-q^2\to\infty]{} M-\frac{2m^2}{\pi}\int_0^\infty d\omega' F_2(\omega')\,, \quad (5.6)$$

where

$$M=\lim_{-q^2\to\infty}-q^2M(q^2)\geq 0\,.$$

Inserting this formula into (5.2) and using $F_2=2\omega F_1$ gives

$$\delta m_{\rm div}=\int^{-\infty}\frac{dq^2}{q^2}\left(M-\frac{m^2}{\pi}\int_0^\infty d\omega'\frac{F_2(\omega')}{\omega'}\right). \quad (5.7)$$

Since the coefficient of the logarithmic divergence is the difference of two positive quantities, no definite statement can be made concerning the logarithmically divergent part of δm, though a cancellation seems unlikely. It should be noted that M is in principle measurable, since it is determined by the nonscaling corrections to $\tilde{F}_L(\omega,\nu)$ at large ν.

We remark here also that the validity of the unsubtracted dispersion relation for T_L requires that

$$\lim_{\nu\to\infty}\nu\frac{\sigma_L}{\sigma_T}\bigg|_{q^2\text{ fixed}}=0\,, \quad (5.8)$$

where we have used the experimental fact that $\sigma_T\to$ const as $\nu\to\infty$. If relation (5.8) can be extended into the deep inelastic region, namely,

$$\lim_{\nu\to\infty}\nu\frac{\sigma_L}{\sigma_T}\bigg|_{\omega\text{ fixed}}=0\,, \quad (5.9)$$

then it is easy to show that $M=0$.

[28] The connection between quadratically divergent self-masses and q-number Schwinger terms is well known; see the papers cited in Ref. 16.

VI. MODEL CALCULATIONS

We present here model calculations relevant to our results. First the free-field calculation both for charged bosons and fermions is seen to verify the present conclusions. Then it is demonstrated that in a theory of fermions interacting through a neutral vector meson, the first-order perturbative calculation violates the sum rule. The reasons for the violation are discussed.

A. Free, Charged Bosons

The forward Compton amplitude $T^{\mu\nu}$ for free, charged bosons is given by the diagrams of Fig. 1. We find, in the notation (4.7) and (4.10),

$$T_1(q^2,\nu)=-2\,, \quad (6.1a)$$

$$T_2(q^2,\nu)=-8q^2/(q^4-4\nu^2)\,, \quad (6.1b)$$

$$T_L(q^2,\nu)=-2q^4/(q^4-4\nu^2)\,. \quad (6.1c)$$

Note that although T_1 requires a subtraction, T_2 and T_L do not. In the notation (2.2) and (2.3), the commutator function $C^{\mu\nu}$ leads to the following invariants:

$$C_1(\omega,\nu)=2\pi\epsilon(p_0+q_0)\delta(q^2+2q\cdot p)-2\pi\epsilon(p_0-q_0)\delta(q^2-2q\cdot p)=(\pi/\nu)[\delta(1-\omega)+\delta(1+\omega)]\,, \quad (6.2a)$$

$$C_2(\omega,\nu)=(-8\pi/q^2)\epsilon(p_0+q_0)\delta(q^2+2q\cdot p)+(8\pi/q^2)\epsilon(p_0-q_0)\delta(q^2-2q\cdot p)=(2\pi/\nu^2)[\delta(1-\omega)-\delta(1+\omega)]\,, \quad (6.2b)$$

$$F_L(\omega)=F_2(\omega)=4\pi[\delta(1-\omega)+\delta(1+\omega)]\,. \quad (6.3)$$

The equality of F_L and F_2 is a verification of the Callan-Gross[12] sum rule for this theory.

It now follows that

$$c_1(x^2,\,x\cdot p)=-2IM\int\frac{d^4q}{(2\pi)^3}e^{-iq\cdot x}\epsilon(p_0+q_0)\times\delta(q^2+2q\cdot p)\,, \quad (6.4a)$$

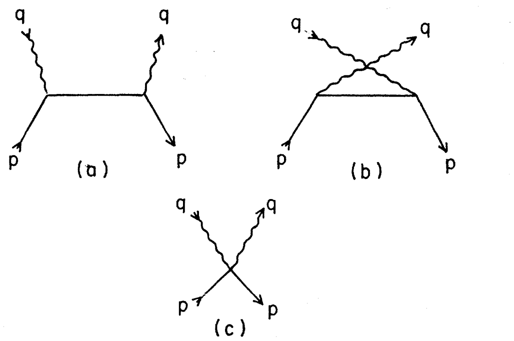

FIG. 1. Forward Compton amplitude in Born approximation. (a) and (b) pole terms; (c) seagull term.

$$c_2(x^2, x\cdot p) = -4IM\int \frac{d^4q}{(2\pi)^3} e^{-iq\cdot x}\epsilon(p_0+q_0) \times\delta(q^2+2q\cdot p)\,. \quad (6.4b)$$

The integrals are evaluated in the Appendix. The result is

$$c_1(x^2, x\cdot p) = (1/\pi)\epsilon(x\cdot p)\delta(x^2)\cos x\cdot p + \epsilon(x\cdot p)\theta(x^2) f_1(x^2, x\cdot p)\,, \quad (6.5a)$$

$$c_2(x^2, x\cdot p) = (1/\pi)\epsilon(x\cdot p)\theta(x^2)(\sin x\cdot p)/x\cdot p + \epsilon(x\cdot p)\theta(x^2)\tilde{f}_2(x^2, x\cdot p)\,, \quad (6.5b)$$

$$\tilde{f}_2(0, x\cdot p) = 0\,.$$

On the other hand, inserting (6.3) directly into (2.7) also yields (6.5), thus verifying the position-space representation.

To check the sum rule, we note that from canonical commutators it follows that

$$\langle p|[j^0(0,x), j^i(0)]|p\rangle = 2i\partial^i\delta(\mathbf{x})\langle p|\varphi^*(0)\varphi(0)|p\rangle = 2i\partial^i\delta(\mathbf{x})\,. \quad (6.6)$$

This same expression is arrived at by a direct evaluation of the sum rule (2.8) with $F_L(\omega)$ as given by (6.3).

B. Free, Charged Fermions

The spin-averaged forward, Compton amplitude for free, charged fermions is again given by the diagrams of Fig. 1, except that there is no seagull. The conclusion of a straightforward calculation is

$$T_1(q^2,\nu) = 8\nu^2/(q^4-4\nu^2)\,, \quad (6.7a)$$

$$T_2(q^2,\nu) = -8q^2/(q^4-4\nu^2)\,, \quad (6.7b)$$

$$T_L(q^2,\nu) = 0\,. \quad (6.7c)$$

Again T_1 needs a subtraction, while T_2 and T_L do not. The commutator function is described by the invariants

$$C_1(\omega,\nu) = 0\,, \quad (6.8a)$$

$$C_2(\omega,\nu) = (-8\pi/q^2)\epsilon(p_0+q_0)\delta(q^2+2q\cdot p) + (8\pi/q^2)\epsilon(p_0-q_0)\delta(q^2-2q\cdot p) = (2\pi/\nu^2)[\delta(1-\omega)-\delta(1+\omega)]\,, \quad (6.8b)$$

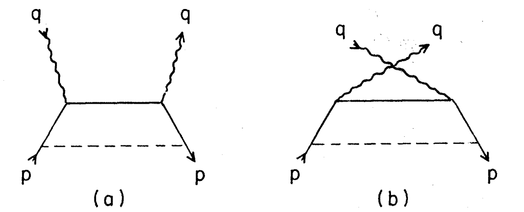

FIG. 2. Forward Compton amplitude with first-order "strong"-interaction correction (dashed line). It is understood that vertex and self-energy insertions need to be included.

$$F_L(\omega) = 0\,, \quad (6.9a)$$

$$F_2(\omega) = 4\pi[\delta(1-\omega)+\delta(1+\omega)]\,. \quad (6.9b)$$

The vanishing of F_L is a consequence of the Callan-Gross[12] sum rule for this model.

Since $C_2(\omega,\nu)$ of (6.8b) and $F_2(\omega)$ of (6.9b) coincide with the corresponding functions calculated for the boson case, we conclude that the representation (2.7) is valid here also. The sum rule is trivially valid. There is no q-number Schwinger term in the free-field theory, and F_L does vanish.

C. Interacting, Charged Fermions

In the theory of charged fermions interacting with a massive vector particle, the forward Compton amplitude, to lowest order in these interactions, is given by Fig. 2. To this order it is known that the Schwinger term is a c number,[29] i.e.,

$$\lim_{q_0\to i\infty} q_0 T^{0i}(q,p) = 0\,, \quad (6.10)$$

but that $F_L(\omega)$ does not vanish[13]:

$$F_L(\omega) \propto \theta(1-\omega^2)\omega^2\,, \quad (6.11)$$

and that $F_2(\omega)$ does not exist[13]:

$$\tilde{F}_2(\omega,\nu) \underset{\nu\to\infty}{\longrightarrow} \infty\,. \quad (6.12)$$

Clearly our representation is not satisfied—hypotheses (I) and (II) are not valid. Also the sum rule is violated since

$$\int_0^\infty d\omega \frac{F_L(\omega)}{\omega^2} \neq 0\,.$$

Note that the violation occurs even though everything is finite.

The failure of the sum rule is rather subtle. The essential steps for its derivations are (4.11)–(4.13), i.e.,

$$\lim_{q_0\to i\infty} T_2(q^2,\nu) = 0\,,$$

$$\lim_{q_0\to i\infty} T_L(q^2,\nu) = -\frac{1}{2\pi}\int_0^\infty \frac{d\omega}{\omega^2} F_L(\omega)\,.$$

[29] The relevant calculation has been performed by Adler and Tung, Ref. 13.

From the explicitly calculated fact that $\lim_{q_0\to i\infty} q_0 T^{0i} = 0$,[29] it follows that $\lim_{q_0\to i\infty} T_i(q^2,\nu)=0$, $i=1, 2$. Therefore the culprit is $T_L(q^2,\nu)$, while $T_2(q^2,\nu)$ *does* go to zero in this limit, even though our method of proving this result is invalid in the present example, since F_2 does not exist. The failure of the sum rule occurs because $T_L(q^2,\nu)$ does not satisfy an unsubtracted dispersion relation.[30] Evidently in our second derivation the interchange of limit and integral is not allowed in the present instance. We have not been able to explicate the precise reasons which prevent the first derivation from being valid. This point is being studied further. Its resolution must await the explicit calculation of the position-space representation for this model. We suspect that the position-space representation will have additional terms arising from a $\delta(\omega)$ contribution to $F_L(\omega)/\omega^2$, as a consequence of the subtraction in the dispersion relation for $T_L(q^2,\nu)$.[30] An intriguing question is whether or not the present violation is related to the violation of the Callan-Gross sum rule, which also occurs in this example.[13] In any case, it should be remembered that this model has none of the experimentally desirable features. Therefore it may not be a serious counterexample to our results.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge informative conversations with Professor D. G. Boulware, Professor H. Pagels, and Professor G. Preparata. Professor Boulware stressed to us that care must be exercised in interpreting the sum rule. One of us (G.B.W.) wishes to thank Professor Sidney Drell and his colleagues at SLAC for their kind hospitality during the summer of 1969, where part of this work was done.

APPENDIX

Here we evaluate the integrals given in (6.4). The expression for (6.4a) is easy:

$$\begin{aligned}
c_1(x^2, x\cdot p) &= -2IM\int \frac{d^4q}{(2\pi)^3} e^{-iq\cdot x}\epsilon(p_0+q_0)\delta([q+p]^2-m^2) \\
&= -2IMe^{ip\cdot x}\int \frac{d^4q}{(2\pi)^3} e^{-iq\cdot x}\epsilon(q_0)\delta(q^2-m^2) \\
&= 2IMie^{ix\cdot p}d(x|m^2) \\
&= \frac{1}{\pi}\epsilon(x\cdot p)\delta(x^2)\cos x\cdot p - \frac{m^2}{4\pi}\epsilon(x\cdot p)\theta(x^2)\frac{2J_1(\sqrt{(m^2x^2)})}{\sqrt{(m^2x^2)}} \\
&= (1/\pi)\epsilon(x\cdot p)\delta(x^2)\cos x\cdot p + \epsilon(x\cdot p)\theta(x^2) f_1(x^2, x\cdot p)\,.
\end{aligned} \tag{A1}$$

This verifies (6.5a). We have introduced the causal function $d(x|m^2)$ with mass m.

The second integral is more recondite:

$$\begin{aligned}
c_2(x^2, x\cdot p) &= -4IM\int \frac{d^4q}{(2\pi)^3} e^{-iq\cdot x}\epsilon(p_0+q_0)\frac{\delta((q+p)^2-m^2)}{q\cdot p} \\
&= -4IMe^{ix\cdot p}\int \frac{d^4q}{(2\pi)^3} e^{-iq\cdot x}\frac{\epsilon(q_0)\delta(q^2-m^2)}{q\cdot p-m^2}\,.
\end{aligned} \tag{A2}$$

It is easy to verify that the integrand does not have a singularity at $q\cdot p=m^2$. Hence we may, for convenience, treat that point by the principal-value convention. By the convolution theorem, (A2) is recast into

$$c_2(x^2, x\cdot p) = 4REe^{ix\cdot p}\int d^4y\, d(y|m^2)\int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)}\frac{1}{q\cdot p-m^2}\,. \tag{A3}$$

The q integral may be evaluated in the p rest frame:

$$\begin{aligned}
\int \frac{d^4q}{(2\pi)^4}\frac{e^{-iq\cdot(x-y)}}{m(q_0-m)} &= \delta(\mathbf{x}-\mathbf{y})\frac{e^{im(y_0-x_0)}}{m}\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}e^{iq_0(y_0-x_0)} \\
&= \delta(\mathbf{x}-\mathbf{y})(e^{im(y_0-x_0)}/m)\tfrac{1}{2}i\epsilon(y_0-x_0)\,.
\end{aligned} \tag{A4}$$

[30] Explicit calculation verifying this statement has been performed by Dr. Anthony Zee (private communication).

Therefore, we arrive at the following expression:

$$c_2(x^2, x\cdot p) = \frac{2}{m} IM \int_{-\infty}^{\infty} dy_0 e^{imy_0} d(y_0, \mathbf{x}|m^2)\epsilon(x_0-y_0). \tag{A5}$$

The free-field commutator function is decomposed,

$$d(x|m^2) = (1/2\pi)\epsilon(x_0)\delta(x^2) + 2\pi\epsilon(x_0)\theta(x^2)\mathcal{J}_1(\lambda^2,x^2), \tag{A6}$$

and the contribution from each of the two terms in (A6) is evaluated separately. First we consider

$$\begin{aligned}
\frac{2}{m} IM \int_{-\infty}^{\infty} dy_0 e^{imy_0}\epsilon(x_0-y_0)\frac{\epsilon(y_0)}{2\pi}\delta(y_0{}^2-\mathbf{x}^2) &= IM\frac{1}{2\pi m|\mathbf{x}|}[e^{im|\mathbf{x}|}\epsilon(x_0-|\mathbf{x}|) - e^{-im|\mathbf{x}|}\epsilon(x_0+|\mathbf{x}|)] \\
&= \frac{\sin m|\mathbf{x}|}{2\pi m|\mathbf{x}|}[\epsilon(x_0-|\mathbf{x}|)+\epsilon(x_0+|\mathbf{x}|)] \\
&= \frac{\sin m|\mathbf{x}|}{\pi m|\mathbf{x}|}\epsilon(x_0)\theta(x^2) = \frac{1}{\pi}\epsilon(x_0)\theta(x^2)\frac{\sin([(x\cdot p)^2-m^2x^2]^{1/2})}{[(x\cdot p)^2-m^2x^2]^{1/2}}.
\end{aligned} \tag{A7a}$$

The last equation in (A7a) is the frame-independent version of the previous expression, for timelike p. Next, we have

$$\begin{aligned}
&\frac{4\pi}{m} IM \int_{-\infty}^{\infty} dy_0 e^{imy_0}\epsilon(x_0-y_0)\epsilon(y_0)\theta(y_0{}^2-\mathbf{x}^2)\mathcal{J}_1(m^2, y_0{}^2-\mathbf{x}^2) \\
&\quad = \frac{2\pi}{m}\int_0^{\infty} dy_0 \sin my_0[\epsilon(x_0-y_0)+\epsilon(x_0+y_0)]\theta(y_0{}^2-\mathbf{x}^2)\mathcal{J}_1(m^2, y_0{}^2-\mathbf{x}^2) \\
&\quad = \frac{2\pi}{m}\epsilon(x_0)\theta(x^2)\int_0^{\infty} dy_0 \sin my_0\theta(x_0{}^2-y_0{}^2)\theta(y_0{}^2-\mathbf{x}^2)\mathcal{J}_1(m^2, y_0{}^2-\mathbf{x}^2) \\
&\quad = \frac{2\pi}{m}\epsilon(x_0)\theta(x^2)\int_{|\mathbf{x}|}^{|x_0|} dy_0 \sin my_0\, \mathcal{J}_1(m^2, y_0{}^2-\mathbf{x}^2).
\end{aligned} \tag{A7b}$$

Combining (A7a) with (A7b), we conclude that

$$\begin{aligned}
&c_2(x^2, x\cdot p) = (1/\pi)\epsilon(x\cdot p)\theta(x^2)(\sin x\cdot p)/x\cdot p + \epsilon(x\cdot p)\theta(x^2)\tilde{f}_2(x^2, x\cdot p), \\
&\tilde{f}_2(0, x\cdot p) = 0.
\end{aligned} \tag{A8}$$

This verifies (6.5b).