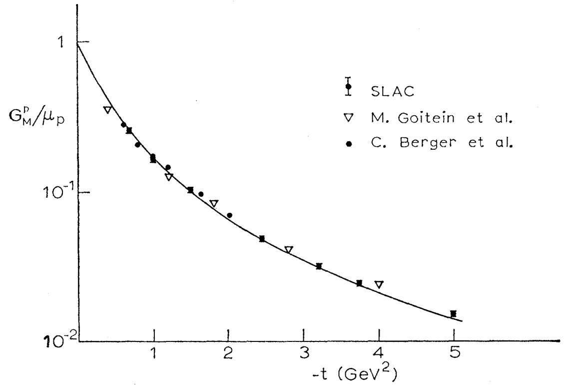

FIG. 4. Effect of nonzero $G_E^n(t)$.

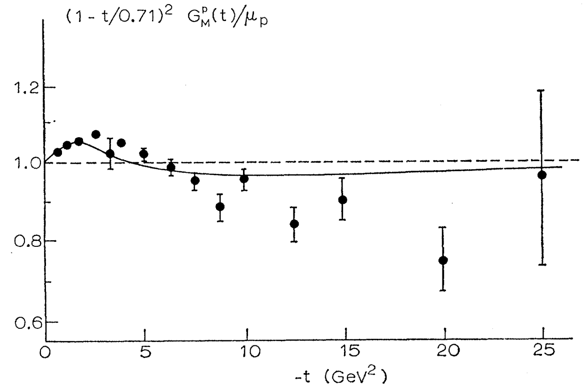

FIG. 5. $G_M^p(t)/\mu_p$ plotted relative to the empirical dipole fit. The experimental points are from Coward *et al.* (Ref. 13).

we plot $G_M^p(t)/\mu_p$ as a function of t, where $G_E^n(t)$ is taken to be zero. As expected, for small values of $|t|$ there is a small discrepancy between data points and the curve of Eq. (4). This is due to the fact that $G_E^n(t)$ is not zero for small $|t|$. To show the effect of a nonzero $G_E^n(t)$ in the calculation of $G_M^p(t)$, we will assume a functional form for the neutron electric form factor which is discussed in Ref. 12, namely,

$$G_E^n(t) = \frac{At}{1-t/4M^2} F_1^V(t)\,, \tag{10}$$

where the constant A is determined from the known slope of $G_E^n(t)$ at the point $t=0$. Figure 3 shows the corresponding neutron electric form factor plotted as a function of t. Once we have a functional form for the $G_E^n(t)$ term in formula (4), we can use it to remove the discrepancy in Fig. 2. In Fig. 4 we have shown $G_M^p(t)/\mu_p$ for small $|t|$ with a nonzero neutron electric form factor. Since a plot of $G_M^p(t)/\mu_p$ relative to the empirical dipole fit $G_M^p(t)/\mu_p = (1-t/0.71)^{-2}$ will clearly show deviation of the theoretical curve from the data, such a plot of $G_M^p(t)/\mu_p$ is given in Fig. 5, where the $G_E^n(t)$ term is also included. We also note that if we calculate a mean-square radius from the expression of $G_M^p(t)$ and take the slope of $G_E^n(t)$ from experiment, the measured value of $\langle r^2\rangle$ is obtained. It may finally be remarked that the choice $\beta=5$ which would give exact dipole behavior asymptotically gives a slightly less good fit for low t, but is not yet excluded by experiment.

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Physical-Region Constraints on Low-Energy Partial-Wave Amplitudes*

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On the basis of analyticity, crossing, and positivity of the imaginary parts of the partial-wave amplitudes, we derive inequalities on integrals involving the low partial waves of elastic $\pi^0\pi^0$ scattering in the physical region. The integrals are sensitive only to the low-energy region, and can therefore be tested once a phase-shift analysis is given. The relations can be used to discriminate between various proposed $\pi^0\pi^0$ phase shifts.

I. INTRODUCTION

ANALYTICITY, crossing, and unitarity have long been considered essential ingredients of strong-interaction physics, and much effort has been devoted to elucidating their consequences. Apart from the implications of unitarity for individual partial-wave amplitudes, most tests of these general principles (such as the verification of dispersion relations, or the Froissart bound) have involved the full amplitudes, and not merely a few partial waves. Recently, however, many different results on the partial-wave amplitudes of $\pi\pi$ scattering below threshold have been obtained. In particular, Common[1] and Yndurain[2] have found the implications of the positivity of the absorptive parts for

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[1] A. K. Common, Nuovo Cimento 63A, 863 (1969).
[2] F. J. Yndurain, Nuovo Cimento 64A, 225 (1969).

these waves; Roskies[3] has found the necessary and sufficient conditions on these waves imposed by crossing symmetry; and Martin[4] has combined crossing, analyticity, and positivity to derive inequalities on these partial waves. While these results can be useful for parametrizing the partial-wave amplitudes even above threshold, they cannot really be tested directly, since they refer to the amplitudes in an unphysical region.

In this paper, we find constraints on the elastic $\pi^0\pi^0$ partial-wave amplitudes in the physical region; these constraints follow from crossing, analyticity, and the positivity of the imaginary part of the elastic partial-wave amplitudes. They take the form of inequalities involving integrals over the imaginary part of a few partial waves. Moreover, the integrals are sensitive only to the low-energy region. As a result, the constraints can be tested once a low-energy phase-shift analysis is available. Conversely, one can assume the constraints to reduce the ambiguities of possible phase-shift analyses, just as dispersion relations were used to discriminate between the Fermi and Yang π-N phase shifts.[5]

After completing this work, I realized that most of the results presented here had been anticipated by Wanders.[6] This paper differs from his in emphasizing the applications to low-energy phase-shift analyses, in generalizing his sum rules away from $t=0$, and in the content of Appendix B. (A numerical error in his paper is also corrected.)

The paper is organized as follows: In Sec. II, for simplicity we develop the constraints with the unrealistic assumption that the fixed-t dispersion relations for $\pi^0\pi^0$ scattering are unsubtracted. It is shown that none of the proposed phase shifts of Malamud and Schlein[7] is consistent with the constraints. Besides indicating the anticipated necessity for a subtraction in $\pi^0\pi^0$ scattering, this also shows that the constraints are not trivial.

In Sec. III, we indicate how the analysis must be altered in the presence of subtractions. One unfortunate consequence of the alteration is that there are no longer any constraints on the imaginary part of the S wave. In this case the simplest constraint involves $l=2$ and $l=4$. Because no phase shifts have yet been proposed for these angular momenta, the constraint is not yet testable. In the narrow-resonance approximation, one obtains the result that

$$\frac{\Gamma_4}{m_4{}^9}<\frac{1}{36}\frac{\Gamma_{f_0}}{m_{f_0}{}^9}, \tag{1.1}$$

where Γ_4, m_4 are the width and mass of the lowest $l=4$ resonance with zero isospin, and Γ_{f_0}, m_{f_0} refer to the same parameters of the f_0 resonance. If the first $l=4$ resonance occurs in the S, T, U meson region around 2 BeV, the inequality restricts Γ_4 to be less than a width of the order of Γ_{f_0}.

In Sec. IV, an alternative derivation of the results is given, followed by the conclusions in Sec. V. In Appendix A we prove some required results on Legendre polynomials. In Appendix B, we prove the following theorem already mentioned in Sec. III:

Given $\text{Im}f_l(s)$, all l, $s>4\mu^2$, consistent with crossing, analyticity, and the constraints $\text{Im}f_l(s)\geqslant 0$, then, if the fixed-$t$ dispersion relations are subtracted, there exists a set of amplitudes $\text{Im}f_l'(s)$ consistent with crossing and analyticity, where

$$\begin{aligned}&\text{Im}f_l'(s)=\text{Im}f_l(s)\,,\quad l\geqslant 2\\&\text{Im}f_0'(s)\ \text{arbitrary}.\end{aligned} \tag{1.2}$$

II. NO SUBTRACTIONS

Suppose the fixed-t dispersion relation for the elastic $\pi^0\pi^0$ scattering amplitude were unsubtracted. Then we would have

$$F(s,t)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\,A(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right), \tag{2.1}$$

where $A(s',t)$ denotes the absorptive part for physical s', and we have used the $s\leftrightarrow u$ crossing properties of the amplitude. In the region $-4\mu^2<t<4\mu^2$, the absorptive part can be expanded[8] as

$$A(s',t)=\sum_l(2l+1)\,\text{Im}f_l(s')P_l\left(1+\frac{2t}{s'-4\mu^2}\right), \tag{2.2}$$

with

$$\text{Im}f_l(s')\geqslant 0\,,\quad s'\geqslant 4\mu^2. \tag{2.3}$$

While (2.1) already reflects the s, u symmetry of $F(s,t)$, we must still impose the t, u symmetry. This is most easily done by introducing the variable

$$z=1+t/2k^2=-1-u/2k^2, \tag{2.4}$$

where

$$s=4(k^2+\mu^2)\,, \tag{2.5}$$

and insisting that $F(s,t)$ be an even function of z. Eliminating t and u in favor of z by (2.4), using the expansion (2.2) in (2.1), and expanding the result as a power series in z, we impose the requirement that the

[3] R. Roskies, Nuovo Cimento **65A**, 467 (1970).

[4] A. Martin, Nuovo Cimento **58A**, 303 (1968); **63A**, 167 (1969); G. Auberson *et al.*, CERN Report No. TH-1032, 1969 (unpublished); or Ref. 7, p. 715.

[5] W. C. Davidon and M. L. Goldberger, Phys. Rev. **104**, 1119 (1956).

[6] G. Wanders, Nuovo Cimento **63A**, 108 (1969).

[7] E. Malamud and P. Schlein, in Proceedings of the Argonne Conference on $\pi\pi$ and $K\pi$ Interactions, 1969, p. 93 (unpublished).

[8] A. Martin, Nuovo Cimento **42A**, 930 (1966).

coefficients of z^{2n+1} vanish. This gives

$$\frac{1}{\pi}\sum_{l=0}^{\infty}(2l+1)\int_{4\mu^2}^{\infty}ds'\,\mathrm{Im}f_l(s')\times\left[\frac{P_l^{(2n+1)}(1-4k^2/(s'-4\mu^2))}{(2n+1)!}\left(\frac{4k^2}{s'-4\mu^2}\right)^{2n+1}\frac{1}{s'-s}-\sum_{r=0}^{2n+1}\frac{P_l^{(r)}(1-4k^2/(s'-4\mu^2))}{r!}\left(\frac{-2(s'+2k^2)}{s'-4\mu^2}\right)^r\times\left(\frac{2k^2}{s'+2k^2}\right)^{2n+1}\frac{1}{s'+2k^2}\right]=0,\quad(2.6)$$

where $P_l^{(r)}(z)$ denotes $d^rP_l(z)/dz^r$. The expansions are valid provided

$$-4\mu^2<t<4\mu^2,\quad(2.7)$$

i.e.,

$$-4\mu^2<2k^2(z-1)<4\mu^2,\quad(2.8)$$

and so, near $z=0$, we must have

$$-4\mu^2<-2k^2<4\mu^2,\quad(2.9)$$

i.e.,

$$-4\mu^2<s<8\mu^2.\quad(2.10)$$

In order not to have any singularity in the physical s' region, we shall restrict s to satisfy

$$-4\mu^2<s<4\mu^2.\quad(2.11)$$

We thus have a series of sum rules, labeled by a discrete index n and a continuous index k^2. But they involve all partial waves. Now using (2.3) we show that they can be rewritten as inequalities involving only low partial waves.

Consider $n=0$, for example. We obtain

$$\int_{4\mu^2}^{\infty}ds'\,\mathrm{Im}f_0(s')g_0(s',s)=\sum_{\substack{l=2\\ l\text{ even}}}^{\infty}\int_{4\mu^2}^{\infty}ds'\,\mathrm{Im}f_l(s')g_l(s',s),\quad(2.12)$$

with

$$g_l(s',s)=(2l+1)\left[\frac{-P_l((s'-s)/(s'-4\mu^2))}{(s'+2k^2)^2}+\frac{2P_l'((s'-s)/(s'-4\mu^2))}{s'-4\mu^2}\times\left(\frac{1}{s'+2k^2}+\frac{1}{s'-s}\right)\right],\quad l\geqslant 2\quad(2.13)$$

$$=1/(s'+2k^2)^2,\quad l=0.\quad(2.14)$$

In Appendix A, we show that

$$g_l(s',s)>0\quad(2.15)$$

for

$$s'\geqslant 4\mu^2,\quad l\geqslant 4,\quad -4\mu^2[(8/3)^{1/2}-1]\leqslant s\leqslant 4\mu^2.$$

Using the positivity of $\mathrm{Im}f_l(s')$, we obtain

$$\int_{4\mu^2}^{\infty}ds'\,\mathrm{Im}f_0(s')g_0(s',s)\geqslant\int_{4\mu^2}^{\infty}ds'\,\mathrm{Im}f_2(s')g_2(s',s),\quad(2.16)$$

that is,

$$\int_{4\mu^2}^{\infty}ds'\frac{\mathrm{Im}f_0(s')}{(s'+2k^2)^2}\geqslant 5\int_{4\mu^2}^{\infty}ds'\frac{\mathrm{Im}f_2(s')}{(s'-4\mu^2)^2(s'+2k^2)^2}\times[48k^4+24k^2(s'-4\mu^2)+11s'^2-16s'\mu^2-16\mu^4]\quad(2.17)$$

for

$$-\mu^2(8/3)^{1/2}\leqslant k^2\leqslant 0.$$

This is a typical example of the constraints one can obtain in this formalism.

We have tested the constraint at $k^2=0$ for the proposed S-wave phase shifts of Malamud and Schlein[7] which range from 420 MeV to 1 BeV in $\sqrt{s}$. We assumed for simplicity that both sides of the equation were dominated by their $I=0$ terms, because the $I=2$ terms are small in any model. Because the weight functions fall like $1/s'^2$ for large s', the higher-energy behavior of $\mathrm{Im}f_0(s')$ and $\mathrm{Im}f_2(s')$ is unimportant. One can put an upper bound on the error involved in cutting off the S-wave integration above 1 BeV by saturating the unitary bound

$$\mathrm{Im}f_0(s')\leqslant\left(\frac{4s'}{s'-4\mu^2}\right)^{1/2}\quad(2.18)$$

in this region. In the energy region below 420 MeV, we assumed the fit

$$\delta=ak,\quad(2.19)$$

with a chosen to reproduce the phase shift at 420 MeV. We saturated $\mathrm{Im}f_2(s')$ by the f_0 resonance, and it did not matter very much whether we used the narrow-resonance approximation or a Breit-Wigner distribution with the correct threshold, i.e.,

$$f_2(s')=-\left(\frac{4s'}{s'-4\mu^2}\right)^{1/2}\left(\frac{k'}{k_0}\right)^5\times\frac{m_{f_0}\Gamma_{f_0}}{s-s_0+im_{f_0}\Gamma_{f_0}(k'/k_0)^5},\quad(2.20)$$

where

$$s'=4k'^2+4\mu^2,\quad s_0=(m_{f_0})^2=4k_0^2+4\mu^2.\quad(2.21)$$

For the "up-up" solution, the left-hand side of Eq. (2.17) was $(0.5\pm0.1)/4\mu^2$, whereas the right-hand side exceeded $1.25/4\mu^2$. Thus the sum rule is unambiguously violated. The situation is similar for the "up-down" solution and even worse for the "down-up" solution.

The failure of this sum rule is not surprising. It arises because the no subtraction assumption is unwarranted. It does, however, demonstrate that these sum rules are nontrivial, and that with suitable modifications in the presence of subtractions they may be useful in discriminating between different proposed phase shifts. It is also interesting that one can establish the necessity for a subtraction of the fixed-t $\pi^0\pi^0$ dispersion relation, by looking at the low-energy S and D waves only. Wanders[6] also noticed that the low-energy data implied the necessity for subtractions.

III. SUBTRACTIONS

In the presence of subtractions, the $\pi^0\pi^0$ elastic amplitude can still be written in terms of its absorptive part up to an arbitrary constant. The correct expression is[9]

$$F(s,t)=C+\frac{1}{\pi}\int_{4\mu^2}^{\infty}ds'\,A(s',t)\left(\frac{1}{s'-s}+\frac{1}{s'-u}-\frac{1}{s'-t}-\frac{1}{s'-4\mu^2+2t}\right)+\frac{2}{\pi}\int_{4\mu^2}^{\infty}ds'\int_0^t dt'\,A(s',t')\left[\frac{1}{(s'-t')^2}-\frac{1}{(s'-4\mu^2+2t')^2}\right], \quad (3.1)$$

where C is an arbitrary real constant. The s-u symmetry is manifest, and we must again impose the t-u symmetry. As in (2.4), we introduce the variable z and insist that $F(s,t)$ be an even function of z. Following the same technique as in Sec. II, we find the following sum rules:

$$(4k^2)^{2n+1}\sum_l (2l+1)\int_{4\mu^2}^{\infty}ds'\,\mathrm{Im}f_l(s')\left(\sum_{r=1}^{2n}\frac{P_l^{(r)}(1-4k^2/(s'-4\mu^2))}{(r-1)!(s'-4\mu^2)^r}\frac{(-1)^{r+1}}{(s'-s)^{2n+2-r}}+2\sum_{r=1}^{2n}\frac{P_l^{(r)}(1-4k^2/(s'-4\mu^2))}{(r-1)!(s'-4\mu^2)^r}\frac{[2n+2-r+(-1)^r(r-2n)]}{(2s'-4\mu^2+s)^{2n+2-r}}\right)=0. \quad (3.2)$$

Since the sums start with $r=1$, we see that the term with $n=0$ vanishes identically. We also see that the S wave never contributes in the sum rule since $P_0^{(r)}(x)=0$ for $r>0$. This is a consequence of the result that in the presence of subtractions, crossing and analyticity do not restrict the imaginary part of the S waves if the imaginary part of the higher partial waves are specified. This is proved in detail in Appendix B. The simplest nontrivial sum rule in the case of subtractions is for $n=1$. The result is (canceling an over-all factor $3s-1$),

$$\sum_l (2l+1)\int_{4\mu^2}^{\infty}ds'\frac{\mathrm{Im}f_l(s')}{(s'-4\mu^2)(s'-s)^2(2s'-4\mu^2+s)^2}\left[\frac{s+4\mu^2-4s'}{s'-4\mu^2}P_l^{(2)}\left(\frac{s'-s}{s'-4\mu^2}\right)+\frac{(4\mu^2)^2+3s^2+12s'^2-6s'(s+4\mu^2)}{(s'-s)(2s'-4\mu^2+s)}P_l^{(1)}\left(\frac{s'-s}{s'-4\mu^2}\right)\right]=0. \quad (3.3)$$

As in Sec. II, we show in Appendix A that the coefficient of $\mathrm{Im}f_l(s')$ is negative for all $l\geqslant 4$ and $4\mu^2(1-2/\sqrt{3})<s<4\mu^2$, so that we obtain the inequality

$$\int_{4\mu^2}^{\infty}ds'\frac{\mathrm{Im}f_2(s')}{(s'-4\mu^2)^2(2s'-4\mu^2+s)^3}\geqslant\frac{3}{4}\int_{4\mu^2}^{\infty}ds'\frac{\mathrm{Im}f_4(s')}{(s'-4\mu^2)^4}\frac{1}{(2s'-4\mu^2+s)^3}\times[48s'^2-54s'(4\mu^2)+42ss'-21s^2+13(4\mu^2)^2]. \quad (3.4)$$

We see that the convergence of the integrals is very rapid, and, consequently, such a relation can be well tested once a low-energy phase shift is given. However, such $\pi\pi$ phase shifts have not yet been proposed.

In the narrow-resonance approximation, saturating the $l=2$ term with the f_0 meson, and the $l=4$ term with an $I=0$ resonance of mass m_4 and width Γ_4, we find (for masses $\gg 4\mu^2$)

$$\Gamma_{f_0}/(m_{f_0})^9\geqslant 36\Gamma_4/(m_4)^9. \quad (3.5)$$

[It is reasonable to include only one resonance in each partial wave, since for large s' the coefficient of $\mathrm{Im}f_l(s')$ falls like $1/s'^5$.] If $m_4\sim 2$ BeV, the region of the S, T, U mesons, we have the result[10]

$$\Gamma_{f_0}>\Gamma_4.$$

[9] A. Martin, Nuovo Cimento **47A**, 265 (1967).

[10] There is a numerical error in Ref. 6, which claims $\Gamma_4<60$ MeV, whereas that calculation should give $\Gamma_4<600$ MeV. For Wanders's choice $m_4=1.97$ BeV, $m_{f_0}=1.26$ BeV, and $\Gamma_{f_0}=145$ MeV, we would obtain $\Gamma_4<230$ MeV. The difference arises because Wanders gives only a rough estimate of the coefficient of $\mathrm{Im}f_4$ in (3.4).

Of course, given a particular model, one can strengthen this inequality. For example, in the Lovelace-Veneziano amplitude for $\pi\pi$ scattering with positive widths,[11] in the limit of small μ^2 we find

$$\left[\frac{\Gamma_{f_0}}{\Gamma_4}\left(\frac{m_4}{m_{f_0}}\right)^9\right]_{\text{Lovelace-Veneziano}} \geqslant 36\times\frac{98}{27}. \quad (3.6)$$

Going back to (3.3), we see that there is an infinite set of sum rules labeled by a discrete parameter n ($n=1, 2, 3\cdots$) and a continuous parameter s ($-4\mu^2 < s < 4\mu^2$). We can write these as

$$\sum_{l=2}^{\infty}(2l+1)\int_{4\mu^2}^{\infty} ds'\, \mathrm{Im} f_l(s') h_l^{(n)}(s',s)=0. \quad (3.7)$$

Given any functions $\rho^{(n)}(s)$, we can therefore write

$$\sum_n \int_{-4\mu^2}^{4\mu^2} ds\, \rho^{(n)}(s) \sum_l (2l+1)\int_{4\mu^2}^{\infty} ds' \times \mathrm{Im} f_l(s') h_l^{(n)}(s',s)=0 \quad (3.8)$$

or

$$\sum_{l=2}^{\infty}(2l+1)\int_{4\mu^2}^{\infty} ds'\, \mathrm{Im} f_l(s') H_l(s')=0, \quad (3.9)$$

with

$$H_l(s')=\sum_{n=1}^{\infty}\int_{-4\mu^2}^{4\mu^2} ds\, h_l^{(n)}(s',s)\rho^{(n)}(s). \quad (3.10)$$

It would be very interesting if one could choose $\rho^{(n)}(s)$ so that $H_l(s')\geqslant 0$ for $l\geqslant 4$, $s'\geqslant 4\mu^2$ and $H_2(s')$ is positive somewhere between $4\mu^2$ and ∞. We could then write

$$\int_{4\mu^2}^{\infty} ds'\, \mathrm{Im} f_2(s') H_2(s')<0, \quad (3.11)$$

and this would be a nontrivial constraint on the D wave alone from crossing and positivity. Such a relation would for example, give bounds on the locations of D-wave resonances in a narrow-resonance approximation. Unfortunately, I have not been able to obtain such a relation nor have I been able to prove that such a relation is impossible, although I suspect that the latter is correct.

IV. ALTERNATIVE FORMULATION

In Sec. III, we found a family of sum rules involving the absorptive parts of the partial-wave amplitudes in the physical region. These were labeled by an integer n and a continuous label s. One could, of course, take moments of those sum rules with respect to s to convert them into a family of sum rules labeled by two integers. For example, Wanders's results[6] come from taking successive derivatives at $s=4\mu^2$. In this section we show another simple way of achieving this, and of finding how many of these sum rules are independent.

We must impose that $F(s,t)$ defined by (3.1) is symmetric in s and t. It also follows that $\partial^2F/\partial s\partial t$ is symmetric in s and t. No information is lost by imposing the symmetry on $\partial^2F/\partial s\partial t$ rather than on F, since the symmetry of F follows from that of $\partial^2F/\partial s\partial t$ by (3.1). From (3.1) and (2.2) we have

$$\frac{\partial^2F}{\partial s\partial t}=\frac{2}{\pi}\sum_l (2l+1)\int_{4\mu^2}^{\infty} ds'\, \mathrm{Im} f_l(s') \times\left\{P_l'\left(1+\frac{2t}{s'-4\mu^2}\right)\left[\frac{1}{(s'-s)^2}-\frac{1}{(s'+s+t-4\mu^2)^2}\right]\frac{1}{s'-4\mu^2} +P_l\left(1+\frac{2t}{s'-4\mu^2}\right)\frac{1}{(s'+s+t-4\mu^2)^3}\right\}. \quad (4.1)$$

Now introducing the variables

$$x=s-\tfrac{4}{3}\mu^2, \quad (4.2)$$

$$y=t-\tfrac{4}{3}\mu^2, \quad (4.3)$$

$$z=\frac{s'-\frac{4}{3}\mu^2}{s'-4\mu^2}, \quad (4.4)$$

and expanding everything as a power series in x and y, we obtain

$$\frac{\partial^2F}{\partial s\partial t}=\frac{2}{\pi}\sum_l (2l+1)\int_{4\mu^2}^{\infty} ds'\, \mathrm{Im} f_l(s') \sum_{r=0}^{\infty}\sum_{q=0}^{\infty}\left[\frac{P_l^{(r+1)}(z)}{r!}\left(\frac{2y}{s'-4\mu^2}\right)^r \times\frac{1}{s'-4\mu^2}\frac{q[x^{q-1}+(-1)^q(x+y)^{q-1}]}{(s'-\frac{4}{3}\mu^2)^{q+1}}+\frac{P_l^{(r)}(z)}{r!}\left(\frac{2y}{s'-4\mu^2}\right)^r\frac{1}{2}\frac{q(q-1)(x+y)^{q-2}(-1)^q}{(s'-\frac{4}{3}\mu^2)^{q+1}}\right]. \quad (4.5)$$

This must be symmetric in x and y, which we can obtain by equating the coefficients of x^py^{n-p} and of $x^{n-p}y^p$ for each n and p. The resulting sum rules are

$$0=\frac{2}{\pi}\sum_l (2l+1)\int_{4\mu^2}^{\infty} ds'\frac{\mathrm{Im} f_l(s')}{(s'-\frac{4}{3}\mu^2)^{n+3}}\sum_{m=1}^{n-p+1}\frac{(2z)^m}{m!}P_l^{(m)}(z)C_{np}(m), \quad (4.6)$$

where

$$C_{np}(m)=(n-m+2)\left[m\delta_{m,p+1}-m\delta_{m,n-p+1}+(n-m+1)!(-1)^{n-m}\left(\frac{p+1}{(n-p)!(p-m+1)!}-\frac{n-p+1}{p!(n-p-m+1)!}\right)\right] \quad (4.7)$$

[11] C. Lovelace, Phys. Letters **28B**, 264 (1968); F. Wagner, Nuovo Cimento **63A**, 393 (1969).

and

$$n=2, 3, 4, \ldots; \quad p=0, 1, 2\cdots[\tfrac{1}{2}n-\tfrac{1}{2}],$$

and z is given by (4.4). Here $[m]$ denotes the largest integer $\leqslant m$.

The simple form of these sum rules is a consequence of the choice of variables x and y, which are a convenient set of variables because $s=t=u=\frac{4}{3}\mu^2$ is the symmetry point of the amplitude.

For a given n, not all the relations corresponding to a given value of p are independent. This can be traced to the original symmetry of $F(s,t)$ under s, u interchanges. To count the number of independent constraints for a given n, define

$$G(x,y)=F(s,t). \tag{4.8}$$

A term of given n corresponds to extracting from $\partial^2 G/\partial x\partial y$ the homogeneous polynomial of degree n in x and y. This arises from the term of degree $n+2$ in $G(x,y)$, which can be written as

$$G_{n+2}(x,y)=\sum_{m=0}^{n+2}\alpha_m x^m y^{n+2-m}, \tag{4.9}$$

with $n+3$ independent coefficients α_m. But the s, u symmetry of $F(s,t)$ implies that

$$G_{n+2}(x,y)=G_{n+2}(-x-y, y), \tag{4.10}$$

so that $G_{n+2}(x,y)$ has really only $[\frac{1}{2}n]+2$ independent coefficients and $\partial^2 G_{n+2}/\partial x\partial y$ has only $[\frac{1}{2}n]+1$. After symmetrization of $\partial^2 G_{n+2}/\partial x\partial y$ in x and y, G_{n+2} will be totally symmetric under interchange of x, y, $-(x+y)$. There are then[12] $[\frac{1}{2}n]+1-[\frac{1}{3}n+\frac{1}{3}]$ independent coefficients in $G_{n+2}(x,y)$ and also in $\partial^2/\partial x\partial y G_{n+2}(x,y)$. Thus the number of independent coefficients has been reduced from $[\frac{1}{2}n]+1$ to $[\frac{1}{2}n]+1-[\frac{1}{3}n+\frac{1}{3}]$, which means that there are $[\frac{1}{3}n+\frac{1}{3}]$ additional constraints. Thus as p varies from 0 to $[\frac{1}{2}n-\frac{1}{2}]$ only $[\frac{1}{3}n+\frac{1}{3}]$ of the sum rules (4.6) for a given n are independent.

V. CONCLUSIONS

On the basis of crossing, positivity of the absorptive parts of the partial-wave amplitudes, and analyticity, we have derived inequalities involving only the imaginary parts of the low partial wave of $\pi^0\pi^0$ scattering at low energies. These may be used to discriminate between proposed low-energy phase shifts in $\pi^0\pi^0$ scattering. In the realistic case where the fixed-t dispersion relation is subtracted, we have shown that no constraints can be derived on the absorptive part of the S wave. The ambiguity in this wave can be reduced if one uses the nonlinear form of unitarity $\mathrm{Im} f_l(s)\geqslant\rho(s)|f_l(s)|^2$, but this appears very difficult.

One can apply our techniques to other processes besides $\pi^0\pi^0$. Since the signs of the imaginary parts of the partial waves play a crucial role in deriving the inequalities, the techniques are applicable only to processes which are elastic in each channel, i.e.,

$$A+A\rightarrow A+A \quad \text{or} \quad A+\bar{A}\rightarrow A+\bar{A}.$$

$\pi\pi$ elastic scattering with isospin can be treated very much along the lines of the present paper. Many features in a narrow-resonance approximation have already been discussed by Wanders.[6] Elastic nucleon-nucleon scattering would be most interesting to study. The chief problem is the existence of the unphysical cut in the partial waves of $N\bar{N}\rightarrow N\bar{N}$. But the sign of the imaginary parts of the partial wave in this region is known.[13] Whether one can then salvage any useful results remains to be seen.

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APPENDIX A

(a) We wish to show that

$$\frac{-P_l(1-4k^2/(s'-4\mu^2))}{(s'+2k^2)^2}+\frac{2P_l'(1-4k^2/(s'-4\mu^2))}{s'-4\mu^2}\times\left(\frac{1}{s'+2k^2}+\frac{1}{s'-s}\right)\geqslant 0 \tag{A1}$$

for

$$s'\geqslant 4\mu^2,\quad 4\mu^2[1-(8/3)^{1/2}]\leqslant s\leqslant 4\mu^2,\quad l\geqslant 4.$$

Defining the variables

$$v=(s'-s)/(s'-4\mu^2), \tag{A2}$$

$$w=s'/4\mu^2, \tag{A3}$$

we must show that

$$\left(\frac{3w-1}{w-1}\right)^2-v^2\geqslant 2v\frac{P_l(v)}{P_l'(v)} \tag{A4}$$

if

$$1\leqslant v\leqslant\frac{w+(8/3)^{1/2}-1}{w-1},\quad w\geqslant 1. \tag{A5}$$

The left-hand side of (A4) is a decreasing function of w for fixed v, so it is sufficient to verify the relation at the largest value of w consistent with (A5). The right-hand side of the equation is a decreasing function of l,

[12] Techniques for calculating the number of independent coefficients can be found in R. Roskies, J. Math. Phys. 11, 482 (1970).

[13] S. MacDowell, Phys. Rev. D (to be published); G. Mahoux, Saclay Report, 1969 (unpublished).

so that it suffices to choose $l=4$. The result is then immediate.

(b) We wish to show that

$$\frac{s+4\mu^2-4s'}{s'-4\mu^2}P_l^{(2)}\left(\frac{s'-s}{s'-4\mu^2}\right)+\frac{(4\mu^2)^2+3s^2+12s'^2-6s'(s+4\mu^2)}{(s'-s)(2s'-4\mu^2+s)}P_l^{(1)}\left(\frac{s'-s}{s'-4\mu^2}\right)\geqslant 0 \quad \text{(A6)}$$

for

$$s'\geqslant 4\mu^2,\quad 4\mu^2(1-2/\sqrt{3})\leqslant s\leqslant 4\mu^2,\quad l\geqslant 4.$$

With the variables v and w, it is equivalent to showing that

$$v\frac{P_l^{(2)}(v)}{P_l^{(1)}(v)}+3-\frac{4}{1-v^2[(w-1)/(3w-1)]^2}\geqslant 0 \quad \text{(A7)}$$

if

$$1\leqslant v\leqslant \frac{w+2/\sqrt{3}-1}{w-1},\quad w\geqslant 1. \quad \text{(A8)}$$

The left-hand side of (A7) is again a decreasing function of w and an increasing function of l. It therefore suffices to verify the relation for $l=4$ and w the largest value consistent with (A8). The result then follows immediately.

APPENDIX B

Given partial-wave amplitudes $f_l(s)$ for elastic $\pi^0\pi^0$ scattering consistent with crossing, analyticity, and the constraints

$$\left(\frac{4s}{s-4\mu^2}\right)^{1/2}\geqslant \operatorname{Im}f_l(s)\geqslant 0,\quad s\geqslant 4\mu^2 \quad \text{(B1)}$$

we shall show that if the fixed-t dispersion relation for the amplitude is subtracted, we can find new amplitudes $f_l'(s)$ satisfying the same properties but for which

$$\operatorname{Im}f_l'(s)=\operatorname{Im}f_l(s),\quad l\geqslant 2,\ s\geqslant 4\mu^2 \quad \text{(B2)}$$

and $\operatorname{Im}f_0'(s)$ is consistent with (B1) but otherwise arbitrary.

Proof: Let

$$f_l'(s)=f_l(s)+\frac{4}{\pi(s-4\mu^2)}\times\int_{4\mu^2}^{\infty}ds'\,g(s')Q_l\left(1+\frac{2s'}{s-4\mu^2}\right),\quad l\geqslant 2 \quad \text{(B3)}$$

$$f_0'(s)=f_0(s)+\frac{4}{\pi}\int_{4\mu^2}^{\infty}ds'\,g(s')\times\left[\frac{1}{4(s'-s)}+\frac{1}{s-4\mu^2}Q_0\left(1+\frac{2s'}{s-4\mu^2}\right)-\frac{1}{4s'}-\frac{1}{4\mu^2}Q_0\left(\frac{2s'}{4\mu^2}-1\right)\right], \quad \text{(B4)}$$

with $g(s)$ real and otherwise arbitrary as long as the integrals converge. Then

$$\operatorname{Im}f_l'(s)=\operatorname{Im}f_l(s),\quad s\geqslant 4\mu^2,\ l\geqslant 2 \quad \text{(B5)}$$

$$\operatorname{Im}f_0'(s)=\operatorname{Im}f_0(s)+g(s),\quad s\geqslant 4\mu^2. \quad \text{(B6)}$$

If $\operatorname{Im}f_0$, $\operatorname{Im}f_0'$ both satisfy (B1), then

$$|g(s)|<2\left(\frac{4s}{s-4\mu^2}\right)^{1/2}, \quad \text{(B7)}$$

so that all the integrals do converge. Using the relation

$$\sum_{l\text{ even}}(2l+1)P_l(z)Q_l(z')=\frac{1}{2}\left(\frac{1}{z'-z}+\frac{1}{z'+z}\right), \quad \text{(B8)}$$

we can perform the partial-wave sum

$$F'(s,t)=\sum(2l+1)f_l'(s)P_l(z_s) \quad \text{(B9)}$$

to obtain

$$F'(s,t)=F(s,t)+\frac{1}{\pi}\int ds'\,g(s')\times\left(\frac{1}{s'-s}+\frac{1}{s'-t}+\frac{1}{s'-u}-\frac{3}{s'}\right)+\frac{1}{\pi}\int ds'\,g(s')\left[\frac{2}{s'}-\frac{1}{\mu^2}Q_0\left(\frac{2s'}{4\mu^2}-1\right)\right]. \quad \text{(B10)}$$

Thus $F'(s,t)$ is crossing symmetric if $F(s,t)$ is, and its fixed-t dispersion relation requires no more subtractions than $F(s,t)$ does, provided $F(s,t)$ needs at least one.