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Self-Consistent Strong-Coupling Model of the Nucleon. II*

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The nonrelativistic strong-coupling theory with a recoiling source presented in an earlier paper is examined in greater detail. The self-consistent calculation is extended to include a full treatment of the bound field problem instead of the approximation of small oscillations about a constant equilibrium value. The approximations of nonrelativistic fermion kinematics and fermion eigenstates of definite angular momentum are examined carefully and found to be inconsistent. The conclusion is that the oversimplified model discussed in the first paper is not realistic. A program for modifying the model to include relativistic kinematics and angular correlations is proposed.

I. INTRODUCTION

IN a recent paper[1] we presented a first attempt at incorporating a recoiling source into the old strong-coupling model of the nucleon.[2] The basic constituents of the model were a heavy point fermion with a spin of $\frac{1}{2}$ and the neutral pseudoscalar-meson field. The only free parameters in the system were the bare mass m_0 of the fermion and the coupling constant g. The pion mass was taken to have its experimental value.

Solutions to the above model were obtained under the following set of assumptions.

(a) The fermion remains nonrelativistic both in the kinematic sense and in the sense that virtual fermion pair states can be consistently neglected.

(b) The eigenstates of the Hamiltonian can be represented as simple products of fermion and field states (the "independent-particle" assumption).

(c) The strong-coupling approximations in the form originally used by Pauli and Dancoff[2] are valid. This implies that the pion field can be constructed out of only p-wave pions and that the field strength executes only small zero-point oscillations about some large constant value.

In addition to the above three assumptions which are quite important and whose consistency is essential to the validity of our results, two additional simplifications were made to make the model more tractable and its exposition more concise. These were as follows.

(d) Isotopic spin was neglected and the assumption made that the basic result of Pauli and Dancoff (i.e., that $T=j$ gives the bound states) would continue to hold.

(e) Only spherically symmetric ($l=0$) fermion wave functions were considered. This was a technical problem rather than a limitation in principle.

It is the purpose of this paper to extend the calculations of I and examine in much greater detail the validity of the approximations made there. We have also made one improvement on the model presented in I: The self-consistency calculation has been extended to include a full solution of the field equation in the q representation. We no longer have to neglect the "rotation-vibrating coupling" as was done in I. This was shown in I to be a questionable approximation.

A first attempt has also been made at treating core states with $l>0$. We use spherical averaging techniques similar to those used in Hartree-Fock calculations but find that this method is not internally consistent. So the problem of higher core angular momentum remains unsolved.

Using the full self-consistency and spherical averaging, we obtain a spectrum of nucleon resonances which depends on just the two parameters m_0 and g. However, the values of m_0 and g needed to get reasonable values for the energies are such that assumptions (a)–(c) are badly inconsistent. We find that g and m_0 must both be rather large to get the nucleon energy down below that of its excitations, and this leads to a "kinetic" energy in the field Hamiltonian which is comparable in magnitude with the field binding energy. This makes the energy of any given state the result of the subtraction of two large numbers of about the same size, which is highly inaccurate and very sensitive to small changes in the parameters.

In summary, the purpose of this paper is to demonstrate the inconsistencies of the model introduced in I, but in so doing to expand and clarify the technical tools used for the model and to gain more physical insight into the nature of the states. These tools and insights will be valuable when we examine other formulations of the model.

This paper will be organized as follows: In Sec. II we present the expanded self-consistency calculation and include core states in which $l\neq 0$. In Sec. III we give some results of this calculation and illustrate how the spectrum is affected by changes in the parameters g and m_0. We also discuss the various contributions to the energy and how they compare with what our approximations would lead us to expect. In Sec. IV

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† Much of this work was done while the author was a visitor at the Stanford Linear Accelerator Center, Stanford, Calif.
[1] A. S. Krass, Phys. Rev. 186, 1713 (1969), hereafter referred to as I.
[2] W. Pauli and S. M. Dancoff, Phys. Rev. 62, 85 (1942). See Ref. 1 for other references.

we examine in detail our approximations and show how and why they break down.

II. FULL SELF-CONSISTENCY

As in I, we begin with the Hamiltonian

$$H=\int d^3x\,\psi^\dagger(\mathbf{x})(m_0+p^2/2m_0)\psi(\mathbf{x})$$
$$+\tfrac{1}{2}\int d^3x[\pi^2(\mathbf{x})+\phi(\mathbf{x})(-\nabla^2+\mu^2)\phi(\mathbf{x})]$$
$$-g\int d^3x\,\psi^\dagger(\mathbf{x})\boldsymbol{\sigma}\cdot\hat{r}\psi(\mathbf{x})\phi(\mathbf{x})\,, \quad (1)$$

and we begin our variational calculation with the "independent-particle" trial state vector

$$|\Psi\rangle=|\psi_{nlm_l}\rangle|\phi_{jm_j}\rangle. \quad (2)$$

These two statements involve several approximations, which we now enumerate.

(a) Nonrelativistic kinematics for the fermion.
(b) Nongradient pseudoscalar coupling.
(c) One-particle fermion states.
(d) The fermion spatial wave function and the field eigenstate (which includes the fermion spin) are uncorrelated in the sense that each is determined by averaging the other over time and space. This is in close analogy with ordinary Hartree-Fock methods for atoms and nuclei.

We will reserve comment on these approximations for Sec. V.

We now consider the set of coupled equations

$$\frac{1}{2l+1}\sum_{m_l=-l}^{l}\langle\psi_{nlm_l}|H|\psi_{nlm_l}\rangle|\phi_{jm_j}\rangle=E|\phi_{jm_j}\rangle, \quad (3a)$$

$$\frac{1}{2j+1}\sum_{m_j=-j}^{j}\langle\phi_{jm_j}|H|\phi_{jm_j}\rangle|\psi_{nlm_l}\rangle=E|\psi_{nlm_l}\rangle. \quad (3b)$$

In both of the above we have taken spherical averages to ensure that the field has a spherically symmetric source [Eq. (3a)] and that the fermion moves in a spherically symmetric potential [Eq. (3b)]. These assumptions lead to degenerate multiplets in $J=l+j$, and they can be checked for consistency by calculating the splittings in perturbation theory. (See Sec. V.)

We begin with Eq. (3a). Using Eq. (1) for H and neglecting all fermion pair states, we see that the eigenvalue problem for the field becomes

$$\left\{\tfrac{1}{2}\int d^3x[\pi^2+\phi(-\nabla^2+\mu^2)\phi]-g\boldsymbol{\sigma}\cdot\int d^3x\right.$$
$$\left.\times\left[\frac{1}{2l+1}\sum_{m_l}\psi_{nlm_l}{}^*(\mathbf{x})\psi_{nlm_l}(\mathbf{x})\right]\hat{r}\phi(\mathbf{x})\right\}|\phi_{jm_j}\rangle$$
$$=E_\phi|\phi_{jm_j}\rangle, \quad (4)$$

where we have dropped the expectation value of the first term of H.

We define the source density for the field equation by

$$\rho(\mathbf{x})=\frac{1}{2l+1}\sum_{m_l}\psi_{nlm_l}{}^*(\mathbf{x})\psi_{nlm_l}(\mathbf{x})\,, \quad (5)$$

and we see immediately that it is spherically symmetric. We can also see from Eq. (4) what happens if we do not make $\rho(\mathbf{x})$ spherically symmetric. In this case the product $\psi_{lm_l}{}^*(\mathbf{x})\psi_{lm_l}(\mathbf{x})$ causes all even values of angular momentum from 0 to $2l$ to appear in $\rho(x)$. When this is multiplied by $\hat{r}$ and integrated with $\phi(x)$, portions of all odd multipoles from 1 to $2l+1$ are projected out of $\phi(x)$.

We now follow the same derivation as in I [see Eqs. (54)–(74)] and arrive at the separated Hamiltonian

$$H=\tfrac{1}{2}R|\hat{q}(\hat{q}\cdot\boldsymbol{\pi})|^2+L^2/2Tq^2+q^2/2N-g\boldsymbol{\sigma}\cdot\mathbf{q}$$
$$+\tfrac{1}{2}\int d^3x[\pi''^2(\mathbf{x})+\phi'(\mathbf{x})(-\nabla^2+\mu^2)\phi'(\mathbf{x})]$$
$$-q^{-2}\mathbf{q}(\mathbf{q}\cdot\boldsymbol{\pi})\cdot\int d^3x\,\pi'(\mathbf{x})\hat{r}\rho(\mathbf{x}). \quad (6)$$

Still following the development of I, we diagonalize the $\boldsymbol{\sigma}\cdot\mathbf{q}$ term and assume that we can measure the eigenvalue of H relative to the zero value of the "free" pion part. The last term is assumed to be small.

At this point we depart from the derivation in I, and instead of assigning the value $q_0{}^2=(gN)^2$ to the q^2 which appears in the centrifugal barrier term, we include this term in the eigenvalue problem as it stands. The "radial" part of the field equation is, therefore,

$$\left[-\tfrac{1}{2}R\left(\frac{d^2}{dq^2}+\frac{2}{q}\frac{d}{dq}\right)+\frac{(j+\frac{1}{2})^2}{2Tq^2}+\frac{q^2}{2N}-gq\right]f(q)$$
$$=E_\phi f(q). \quad (7)$$

This can also be written as follows:

$$\left[-\tfrac{1}{2}R\left(\frac{d^2}{dq^2}+\frac{2}{q}\frac{d}{dq}-\frac{\delta(\delta+1)}{q^2}\right)+\frac{q^2}{2N}-gq\right]f(q)$$
$$=E_\phi f(q)\,, \quad (8)$$

where

$$\delta(\delta+1)=(j+\tfrac{1}{2})^2/RT \quad (9)$$

forms an effective centrifugal barrier for the field amplitude equation.

Equation (8) is solved numerically after the parameters R, N, and T are calculated from the fermion source density [see Eqs. (67), (68), and (71) of I]. The resulting field wave function has the form[3]

$$\langle \mathbf{q} | U | \phi_{jm_j} \rangle = f_{n_q j}(q) \left(\frac{2j+1}{4\pi} \right)^{1/2} D_{m_j+1/2}{}^{(j)}(\alpha\beta 0) , \tag{10}$$

where n_q is the radial quantum number for the field amplitude vibrations. The other result of this calculation is, of course, the eigenvalue E_ϕ.

This completes the solution of Eq. (3a). We now use this solution in Eq. (3b) to derive the fermion Schrödinger equation. Neglecting all fermion pair states, we can reduce the fermion field equation to a single-particle Schrödinger equation by standard techniques; we get

$$\left[m_0 + \frac{p^2}{2m_0} - g \frac{1}{2j+1} \sum_{m_j} \langle \phi_{jm_j} | \boldsymbol{\sigma} \cdot \hat{r} \boldsymbol{\phi}(\mathbf{x}) | \phi_{jm_j} \rangle \right] \psi(x) = E_f \psi(x) . \tag{11}$$

This time we have dropped the expectation value of the free-pion-field Hamiltonian.

We now follow the derivation of Eqs. (79)–(83) of I except that this time the value $q_0 = gN$ is replaced by $\langle q \rangle$, where

$$\langle q \rangle = \int dq\, q^3 f_{n_q j}{}^2(q) . \tag{12}$$

So the fermion Schrödinger equation is

$$\left[-\frac{\nabla^2}{2m_0} - \frac{g\langle q \rangle}{3} \xi(\mathbf{x}) \right] \psi(\mathbf{x}) = (E_f - m_0) \psi(\mathbf{x}) . \tag{13}$$

Using the $\xi(x)$ determined from Eq. (65) of I we solve Eq. (13) numerically for the fermion wave functions. These have the form

$$\langle \mathbf{x} | \psi_{nlm_l} \rangle = R_{nl}(r) Y_{lm_l}(\Omega) , \tag{14}$$

where the $R_{nl}(r)$ are the solutions of

$$\left\{ \frac{1}{2m_0} \left[-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} \right] - \frac{g\langle q \rangle}{3} \xi(r) \right\} R_{nl}(r) = (E_f - m_0) R_{nl}(r) . \tag{15}$$

These fermion wave functions are then used to determine a new set of values of R, N, and T and, in turn, these are used to get a new solution to the q problem. This process is repeated until it converges.

The total energy of the product state is given by

$$E = \langle \Psi | H | \Psi \rangle \tag{16}$$

$$= \langle \phi,\psi | H | \phi,\psi \rangle . \tag{17}$$

[3] As in Ref. 1, we use the notation of A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton U. P., Princeton, N. J., 1957).

From Eqs. (1), (8), and (13), we know that

$$E_\phi = \langle \phi | H_\phi + H_I | \phi \rangle$$

and

$$E_f = \langle \psi | H_f + H_I | \psi \rangle ,$$

which gives

$$E_\phi + E_f = \langle \phi,\psi | H_\phi + H_f + 2H_I | \phi,\psi \rangle . \tag{18}$$

This means that

$$E = E_\phi + E_f - \langle \phi,\psi | H_I | \phi,\psi \rangle , \tag{19}$$

which is easily shown to be

$$E = E_\phi + E_f + g\langle q \rangle . \tag{20}$$

So as a result of our fully self-consistent solution, we have the complete wave function [Eqs. (10) and (14)] and the energy [Eq. (20)] of a state with any given set of fermion and field quantum numbers. We now examine some numerical results.

III. SPECTRA

In Fig. 1 we show the known spectrum of nucleon resonances. We show only the "well established" ones

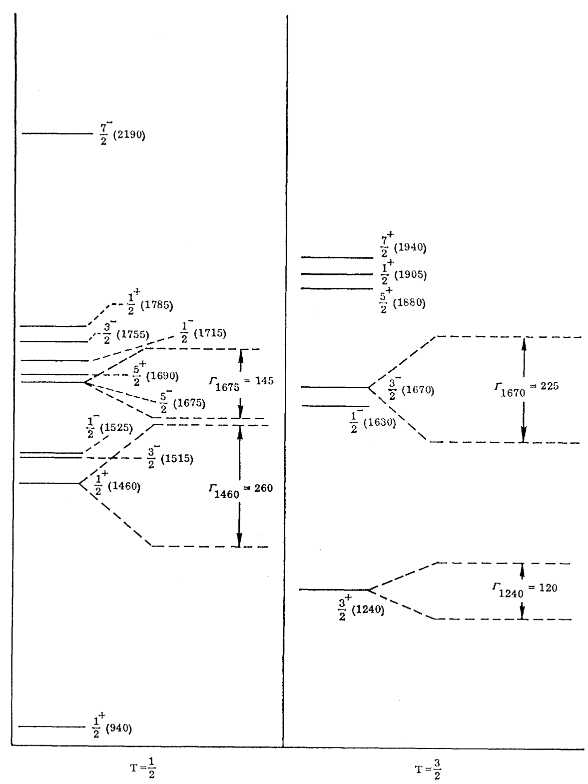

FIG. 1. Known spectrum of $T=\frac{1}{2}$ and $T=\frac{3}{2}$ nucleon resonances. The widths of several of the states are shown for comparison with the level spacings.

as of August 1969.[4] Referring to Table I, we see that the known $T=\frac{1}{2}$ spectrum can almost entirely be accounted for by the set of quantum numbers shown.[5] The only discrepancy is the presence of a $\frac{3}{2}^+$ in Table I which does not appear in the spectrum. We note that Greenberg lists a possible $\frac{3}{2}^+$ with a large width at 1900 MeV.

The $T=\frac{3}{2}$ spectrum of six states is somewhat over-described by the states listed in Table I. The spectrum is lacking two $\frac{3}{2}^+$ states and a $\frac{5}{2}^-$. Again we note that there are candidates for all three of these listed in the "possible" category by Greenberg.

Our assumption of spherical symmetry in the self-consistency calculation leads to the degeneracies indicated in Table I. It is amusing to compare these with the observed spectrum. For $T=\frac{1}{2}$, we predict a $\frac{1}{2}^-$, $\frac{3}{2}^-$ doublet, and there is a good candidate at (1525, 1515). Another $\frac{1}{2}^-$, $\frac{3}{2}^-$ doublet appears at (1715, 1755). We predict a $\frac{3}{2}^+$, $\frac{5}{2}^+$ doublet and a $\frac{5}{2}^-$, $\frac{7}{2}^-$ doublet, but the spectrum lacks one member of each.

For $T=\frac{3}{2}$, we predict a $\frac{1}{2}^-$, $\frac{3}{2}^-$, $\frac{5}{2}^-$ triplet for which the last member is missing, and our quartet of $\frac{1}{2}^+$, $\frac{3}{2}^+$, $\frac{5}{2}^+$, $\frac{7}{2}^+$ has three of its four members observed. So there seems to be some qualitative agreement, but, as we will see in Sec. V, our assumption of spherical symmetry is not at all borne out by our numerical results. We predict multiplet splittings (for the $T=\frac{3}{2}$ spectrum) which are as large as or greater than the spacings between multiplets, so we must conclude that the qualitative agreement is not significant.

TABLE I. Quantum-number assignments for the observed nucleon resonances in the strong-coupling model. The quantum numbers are defined as follows: n is the radial fermion quantum number (i.e., number of radial nodes in fermion wave function); l is the fermion orbital angular momentum (determines parity of state); j is the field angular momentum which is equal to the isospin T of the state. The last two columns give the standard spin-parity and partial wave assignments, and the degeneracies have been made explicit.

n	l	j	T	J^P	Partial wave
0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^+$	P_{11}
1	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^+$	P_{11}
2	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^+$	P_{11}
0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	S_{11}, D_{13}
1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}^-$, $\frac{3}{2}^-$	S_{11}, D_{13}
0	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}^+$, $\frac{5}{2}^+$	P_{13}, F_{15}
0	3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{2}^-$, $\frac{7}{2}^-$	D_{15}, G_{17}
0	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}^+$	P_{33}
1	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}^+$	P_{33}
0	1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}^-$, $\frac{3}{2}^-$, $\frac{5}{2}^-$	S_{31}, D_{33}, D_{35}
0	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}^+$, $\frac{3}{2}^+$, $\frac{5}{2}^+$, $\frac{7}{2}^+$	P_{31}, P_{33}, F_{35}, F_{37}
0	0	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}^+$	not applicable

[4] The data for Fig. 1 are taken from the rapporteur's review given by O. W. Greenberg, in *Proceedings of the Lund Conference on Elementary Particles, 1969*, edited by G. von Dardel (Berlingska, Lund, Sweden, 1969).

[5] We are, of course, assuming that the prediction $T=j$ where j is the *field* angular momentum, will continue to hold. This result emerged from the old strong-coupling theory and was a property of the meson field solutions. We see no reason to doubt that it will also be true in our self-consistent version.

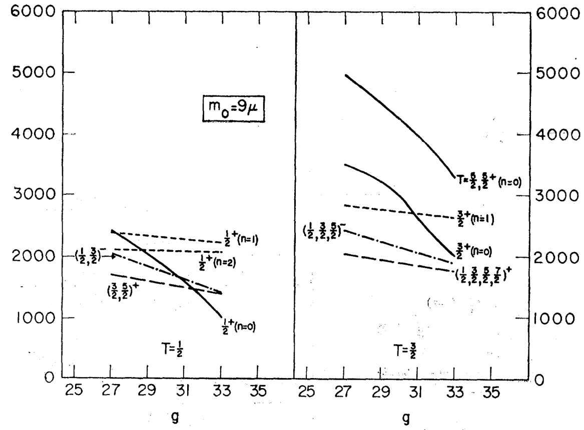

FIG. 2. Plots of the masses of some states versus the coupling constant g for fixed bare mass equal to nine pion masses. The energies are all in MeV. We have included the $T=\frac{5}{2}$ state in the $T=\frac{3}{2}$ column.

In Figs. 2 and 3, we show the behavior of some representative states as the parameters m_0 and g are changed. In Fig. 4 we show the spectrum for $g=33$ and $m_0=9$ superimposed on the observed spectrum.

In Fig. 2 we have fixed m_0 at nine pion masses and varied the coupling constant from 27 to 33. The most striking behavior is that of the ground state for each value of T. This falls dramatically as the coupling is increased. This drop is a result of the increasing concentration of the fermion wave function near the origin (see Fig. 5). This results in a highly concentrated probability density which, in its role as source for the pion field, produces a field which is very intense in the volume near the origin. This field is essentially the potential in which the fermion moves, so the result is a particle lying very low in a deep, narrow potential well.

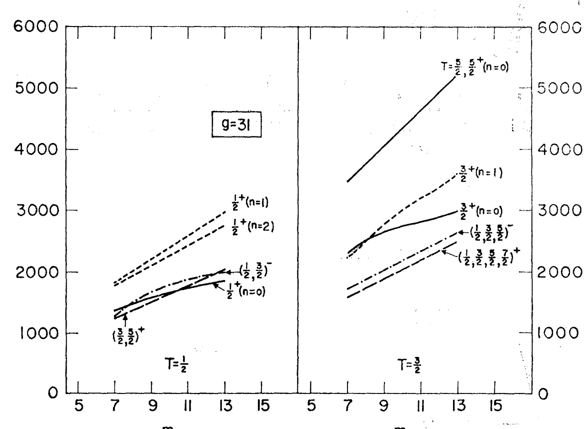

FIG. 3. Plots of the masses of some states versus the fermion bare mass m_0 (in pion mass units) for a fixed coupling constant $g=31$. The energies of states are in MeV.

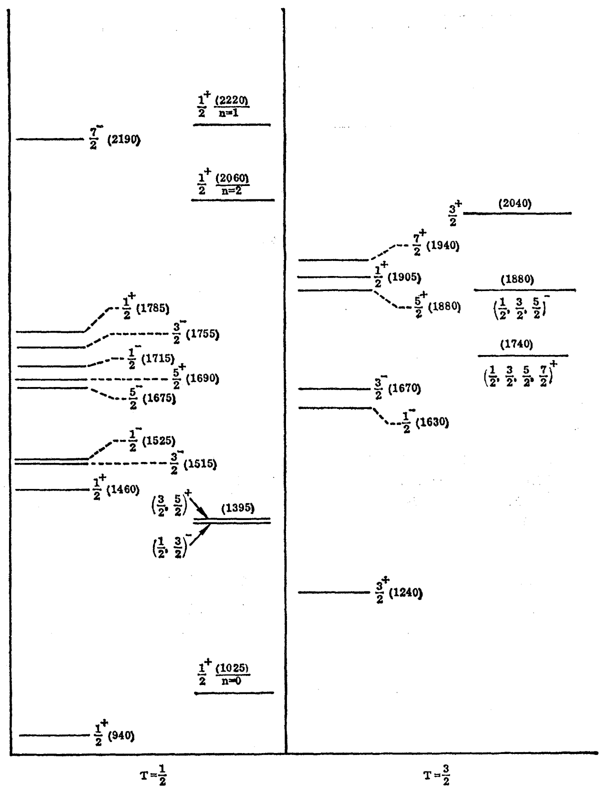

FIG. 4. The spectrum of states resulting from the choice of $g=33$, $m_0=9\,\mu$ is shown on the right in each column and the actual spectrum on the left.

For the field equation [Eq. (8)], we see that the quantity R^{-1} acts like a mass and N^{-1} like a spring constant in the oscillator equation [see Fig. 5(b)]. As the fermion wave function becomes more concentrated at the center, all three of the parameters R, N, and T increase, but R (the integral of the squared probability density of the fermion) increases most rapidly. Notice also that as T increases, the centrifugal barrier is reduced. Finally, g is being increased, so the binding term becomes more and more negative.

The result of all this is a balancing of competing effects, with the net tendency being to lower the field energy. The increase in the spring constant turns out to be a rather mild effect, so the combination of the centrifugal and binding effects is enough to overcome the decreasing mass (especially since it is the square root of the mass which is relevant) and decrease the field energy as g increases.

So we see that as g increases the total energy of the states should decrease, and it should decrease most for those states which allow the fermion to have a concentrated probability distribution. The states with $l>0$ have an additional centrifugal barrier which forces the wave function to be more spread out (see Fig. 6), but there is still some lowering of the energy as g increases. The states with $n>0$, however, seem to be very insensitive to changes in g because the combined constraints of normalization and n radial nodes always force the higher-n wave functions to spread out considerably (see Fig. 7). Finally, in Fig. 8 we show the

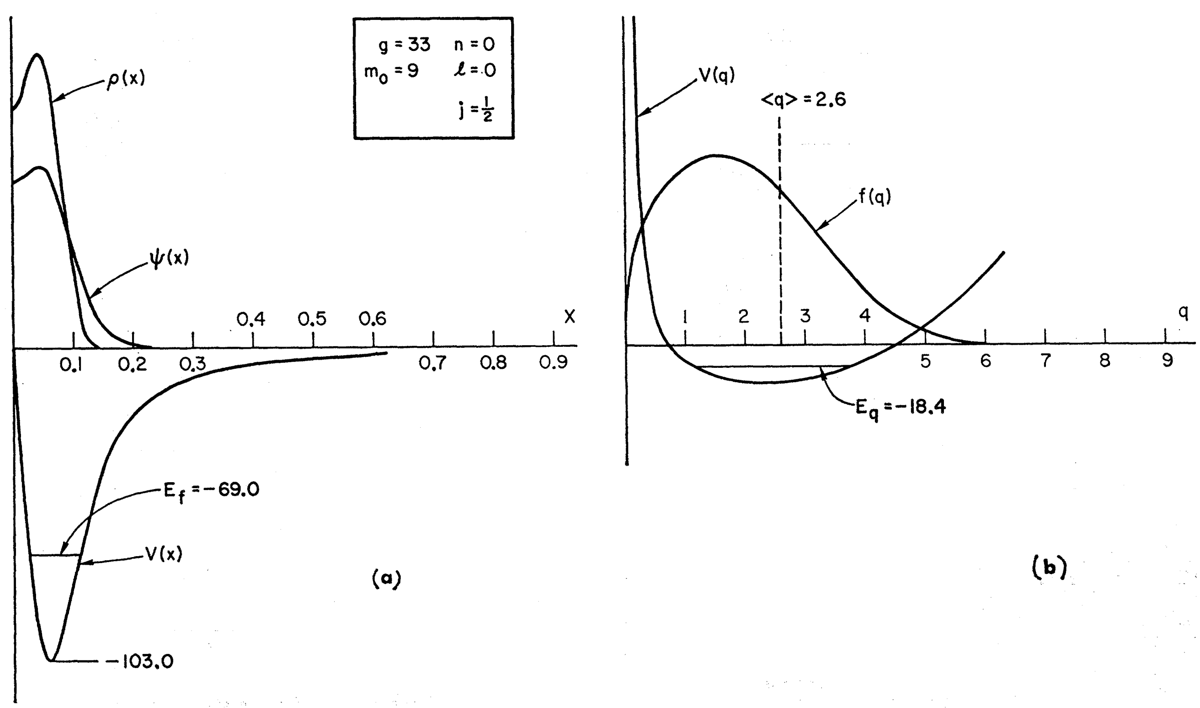

FIG. 5. (a) Plots of the final self-consistent ground-state radial wave function $\psi(r)$, probability density $\rho(r)$, and potential $V(r)$ versus r, where r is measured in pion Compton wavelengths. The energy values of $V(r)$ are measured in pion masses. (b) Plots of the effective potential for the field problem $V(q)$ and the final q-space wave function $f(q)$ for the ground-state (i.e., nucleon) problem.

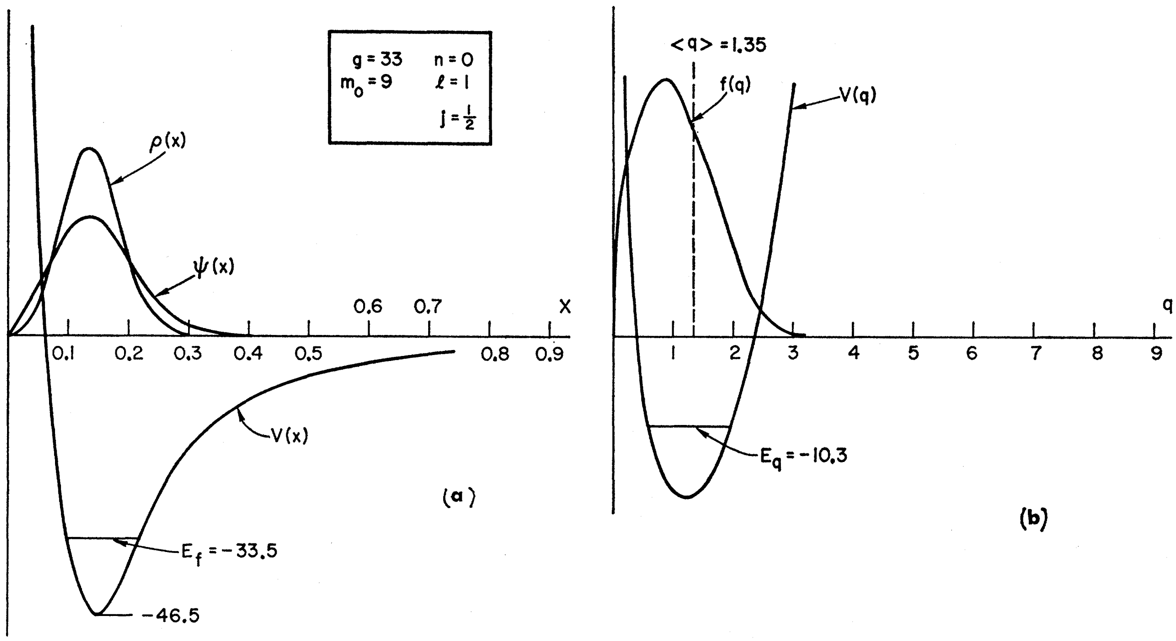

FIG. 6. (a) Plots of $\psi(r)$, $\rho(r)$, and $V(r)$ for the state with $T=\frac{1}{2}$ and $J^P=(\frac{1}{2}^-,\frac{3}{2}^-)$. (b) Plots of $V(q)$ and $f(q)$ for the same state.

results for the $\Delta_{3/2,3/2}$ resonance. The extra centrifugal barrier in the field Hamiltonian causes the field energy to increase, and because the effective mass in the field equations is so small, the effect is too large, and the Δ lies much too high in energy.

The set of solutions shown in Fig. 4 comes as close as we can reasonably get to the actual spectrum. It is quite apparent that there are still serious discrepancies between the calculated and observed spectra for this model. In order to get the ground state to lie below the

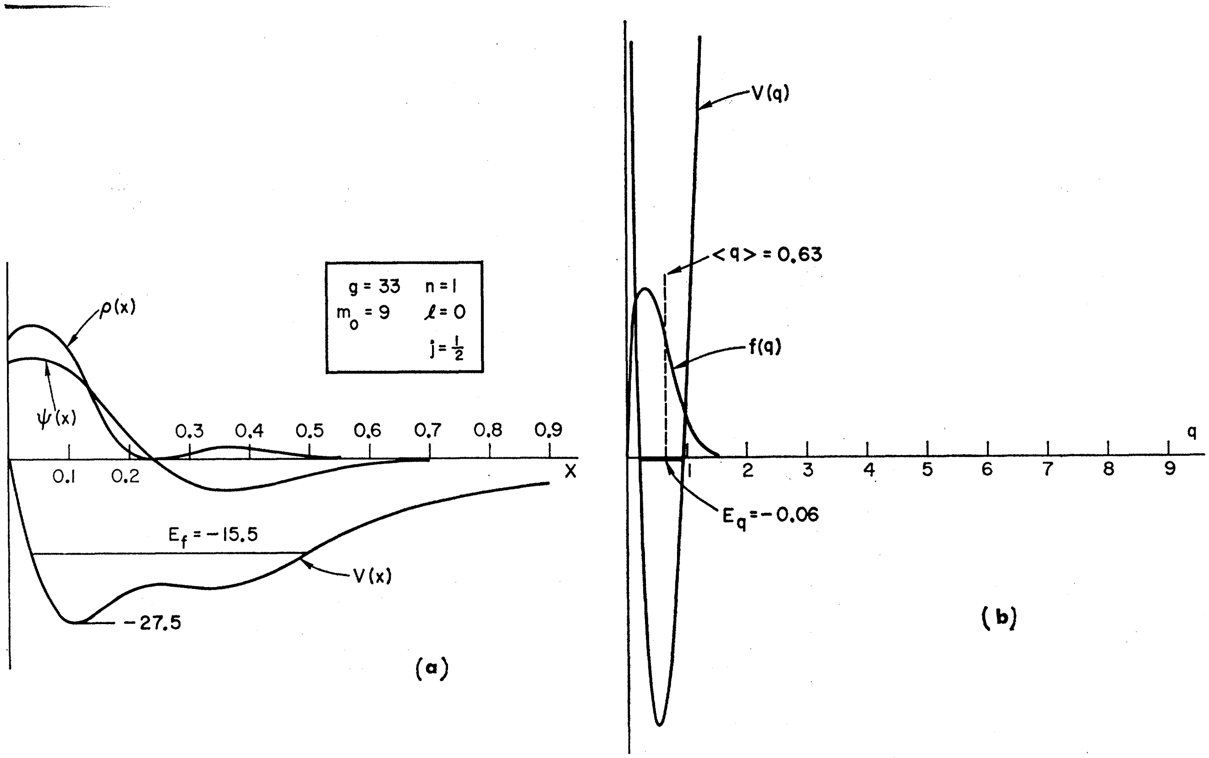

FIG. 7. (a) Plots of $\psi(r)$, $\rho(r)$, and $V(r)$ for the state with $T=\frac{1}{2}$ and $J^P=\frac{1}{2}^+$ which has energy equal to 2220 MeV in Fig. 4. This is the "Roper" resonance in the strong-coupling model. (b) Plots of $V(q)$ and $f(q)$ for the same state.

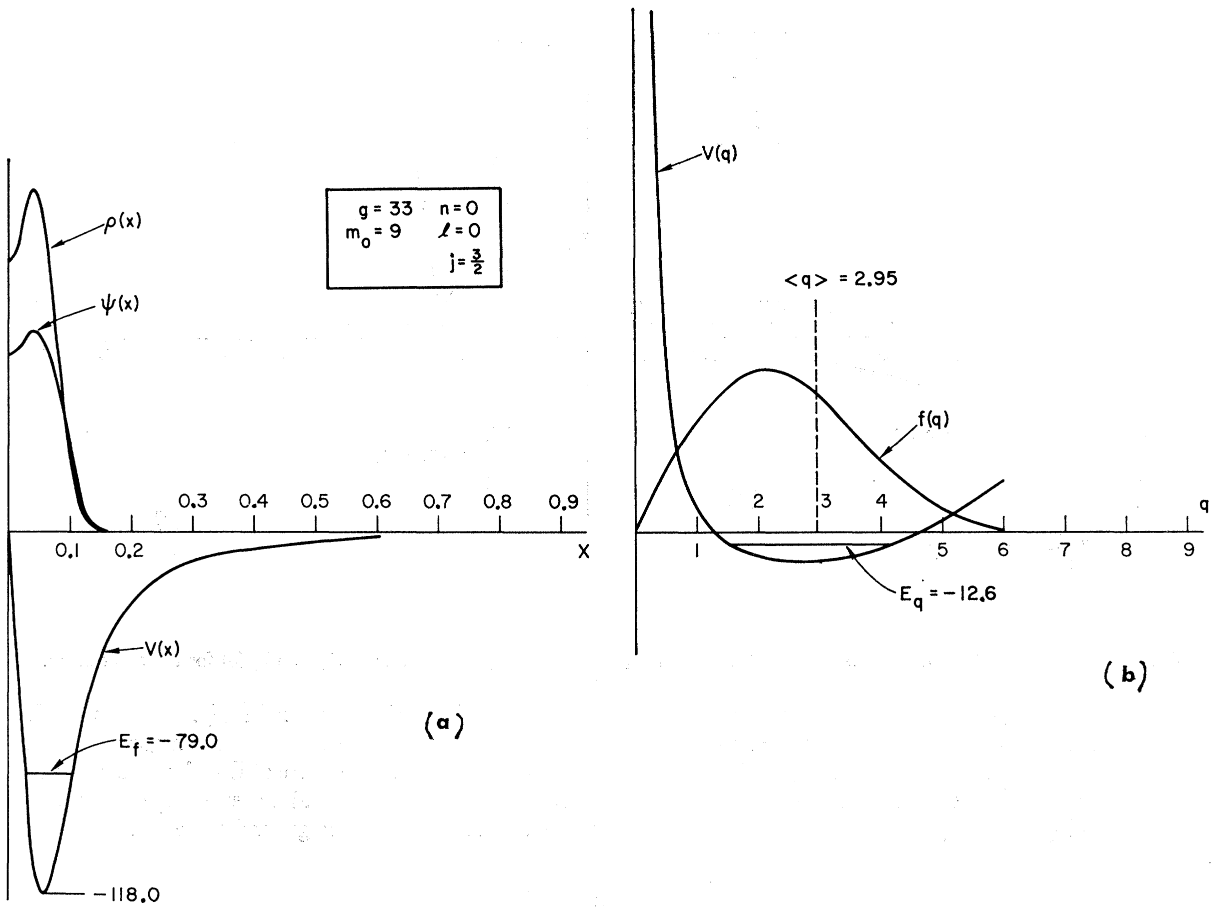

FIG. 8. (a) Plots of $\psi(r)$, $\rho(r)$, and $V(r)$ for the state with $T=\frac{3}{2}$ and $J^p=\frac{3}{2}^+$, the Δ_{33} resonance. (b) Plots of $V(q)$ and $f(q)$ for the same state.

excited states, g must be quite large. But no matter how large we make g, we cannot get the first radial excitation (i.e., the Roper 1460) to lie below the rotational excitations of the core fermion.

There is one possibility for fixing this, which depends on the fact that the $n=1$ and $n=2$ radial excitations are close together (indeed the $n=2$ state lies below the $n=1$). Since our states are not orthogonal to each other (in effect they are solutions to different Hamiltonians), there will be mixing between levels with the same angular wave functions, and they will repel each other. This refinement is not considered in this paper since the other difficulties in the model make it somewhat irrelevant.

There is no reason to exclude "radial" excitations in the q variable. Referring to Figs. 5(b), 6(b), etc., we see that this would imply finding an $f(q)$ with one radial node. We have looked for and found such solutions, but because of the very tight binding (in particular, the large value of R which corresponds to a small mass in the oscillator equation), this radially excited state has an extremely large energy. These states are therefore far above the region which is depicted in Figs. 1 and 4 and not of great interest as yet. There are also reasons to expect that these states will be very broad.[1]

There are more details of the spectra to be examined, and these are treated in Sec. IV, in which we discuss the validity of our approximations.

IV. CHECKS OF APPROXIMATIONS

In this section we examine our results of Sec. III and discuss the degree to which they are consistent with the approximations we have made. We will find some rather disturbing inconsistencies which will lead us to the conclusion that the model presented in this paper is inadequate in its present form. We conclude by proposing a possible alternative.

A. Nonrelativistic Kinematics

Before we do any quantitative calculations, we can see very quickly that we are in trouble on this assumption. Referring to Fig. 5(a), which gives the ground-state wave function, and recalling that the bare mass of the fermion in this case is nine pion masses, we note that the binding energy is nearly eight times as large as the mass. The average height above the bottom of the well is about 17 pion masses, meaning that the average kinetic energy is roughly twice the mass.

Another relevant observation is the radius of the probability density. In the ground state, the fermion is confined almost completely within a region of radius 0.1 pion Compton wavelength. But this is almost exactly the Compton wavelength of the fermion itself, so we expect that the formation of virtual pairs will not be negligible.

The approximation of nonrelativistic kinematics has been checked quantitatively. We have calculated the expectation value of the third and fourth terms in the expansion

$$E=(p^2+m^2)^{1/2}=m+\frac{p^2}{2m}-\frac{1}{8}\frac{p^4}{m^3}+\frac{1}{16}\frac{p^6}{m^6}-\cdots \tag{21}$$

and compared their sum to the bare mass and the kinetic energy of the fermion. The results for four of our states are given in Table II. One look at this table is enough to convince us that our assumption of nonrelativistic kinematics is patently ridiculous. Even in the best case, the Roper excitation, the fact that the relativistic correction is possitive shows that the fourth term of (21) is larger than the third term.

There is very little more to be said about this problem. We must either find a formulation of the model in which the binding need not be so strong or go to a Dirac equation for the fermion.

B. Spherical Averaging

We have been working with states which are products of fermion states with a given l and field states with a given j. Since the solution of each separate eigenvalue problem is obtained by spherically averaging over the coordinates of the other system, the net result is an energy which depends only on the values of l and j and not on their vector sum. We now proceed to test this assumption by returning to the basic Hamiltonian of Eq. (1) and calculating the "fine structure."

Even at the outset we should be apprehensive. We have seen that the expectation value of the interaction term in Eq. (1) is very large in our results. Indeed, it is the essence of a strong-coupling approximation that this term shall dominate the energy. But all the contribution to the fine-structure splitting comes from this term, so it seems almost *a priori* inconsistent to enforce spherical symmetry for the basic equations.

We consider the matrix element of

$$H_I=-g\int d^3x\,\psi^\dagger(x)\boldsymbol{\sigma}\cdot\hat{r}\psi(x)\phi(x) \tag{22}$$

in a state with given values of l, j, and J. We evaluate this matrix element in the "body-fixed frame"[6] so that

[6] C. J. Goebel, in *Non-Compact Groups in Particle Physics*, edited by Y. Chow (Benjamin, New York, 1966).

TABLE II. Columns 1–3 give the quantum numbers of the states in question, and columns 4–6 list, respectively, the fermion bare mass m_0, the fermion kinetic energy [i.e., the expectation value of the second term of Eq. (21)], and the relativistic correction to the fermion energy [the third and fourth terms of Eq. (21)]. The energies are measured in pion masses.

n	l	j	m_0	KE	$\Delta E_{\rm rel}$	J^p
0	0	$\frac{1}{2}$	9	17.4	308	$\frac{1}{2}^+(N)$
0	0	$\frac{3}{2}$	9	19.3	435	$\frac{3}{2}^+(\Delta)$
1	0	$\frac{1}{2}$	9	5.1	1.4	$\frac{1}{2}^+$
0	1	$\frac{1}{2}$	9	11.4	21.6	$\frac{1}{2}^-, \frac{3}{2}^-$

the transformation U [see Eqs. (24)–(27) of I] must be applied to H_I first. As we have seen in I, the effect of this transformation is to change $\boldsymbol{\sigma}$ to $\hat{q}$ and reduce the two-component spinor equation to a one-component equation. Our matrix element is now

$$-g\langle\Psi_{nljJM}|\int d^3x\,\psi^\dagger(x)\hat{q}\cdot\hat{r}\psi(x)\phi(x)|\Psi_{nljJM}\rangle, \tag{23}$$

where[3]

$$\langle\mathbf{q},\mathbf{r}|\Psi_{nljJM}\rangle=R_{nl}(r)f_j(q)\sum_{m,m'}\langle lmjm'|ljJM\rangle \times Y_{lm}(\Omega)(-1)^{m'-1/2}C_{m'+1/2}{}^{(j)}(\Gamma).$$

The next step is to assume that the expectation value of $\phi(x)$ in the state in question is given by $\mathbf{q}\cdot\hat{r}\xi(x)$. With this assumption, the matrix element (23) breaks up into three distinct factors:

$$\langle\Psi_{nljJ}|H_I|\Psi_{nljJ}\rangle=-g\left[\int r^2dr\,R_{nl}{}^2(r)\xi(r)\right] \times\left[\int dq\,q^3f_j{}^2(q)\right]\times\langle\Omega_{ljJ}|(\hat{q}\cdot\hat{r})^2|\Omega_{ljJ}\rangle, \tag{24}$$

where we have suppressed the obviously irrelevant index M. The first factor in (24) is just

$$\int r^2dr\,\rho(r)\xi(r)=3, \tag{25}$$

which can be seen by using Eqs. (5) and (14) of this paper and Eq. (62) of I. The second factor of (24) is just $\langle q\rangle$ [Eq. (12)], so we are left with

$$\langle\Psi_{nljJ}|H_I|\Psi_{nljJ}\rangle=-3g\langle q\rangle\langle\Omega_{ljJ}|(\hat{q}\cdot\hat{r})^2|\Omega_{ljJ}\rangle. \tag{26}$$

The problem has now reduced to the evaluation of the angular matrix element, and this is done using standard Wigner-Eckhart theorem techniques.[3] We write

$$(\hat{q}\cdot\hat{r})^2=\tfrac{1}{3}+Q_{ij}R_{ij}, \tag{27}$$

where

$$Q_{ij}=\hat{q}_i\hat{q}_j-\tfrac{1}{3}\delta_{ij}, \tag{28}$$

and R_{ij} is the analogous tensor made from $\hat{r}$. Equation

(26) becomes

$$\langle\Psi_{nljJ}|H_I|\Psi_{nljJ}\rangle=-g\langle q\rangle-3g\langle q\rangle\langle\Omega_{ljJ}|Q_{ij}R_{ij}|\Omega_{ljJ}\rangle$$
$$=-g\langle q\rangle-g\langle q\rangle\gamma_{ljJ}\,, \tag{29}$$

where we recognize the first term as our spherically averaged interaction energy and the second term as the "perturbation" which will break the degeneracy. The result is

$$\gamma_{ljJ}=\frac{1}{24\sqrt{2}}\left[\frac{2j+1}{j(j+1)}\right]^2 \times\frac{3X(X-1)-4j(j+1)l(l+1)}{(2l-1)(2l+3)}, \tag{30}$$

where

$$X=j(j+1)+l(l+1)-J(J+1)\,. \tag{31}$$

It is easy to verify that when either $l=0$ or $j=\frac{1}{2}$, $\gamma_{ljJ}=0$. So we predict zero splitting for all the $T=\frac{1}{2}$ core excitations. As we have seen in Sec. III, this is not a bad prediction for the two complete doublets.

The only splittings occur in the $T=\frac{3}{2}$ spectrum (and of course also when $T>\frac{3}{2}$). The γ_{ljJ}'s for these multiplets can be expressed in the form

$$\gamma=\tfrac{1}{2}\sqrt{2}(16/15)^2\alpha\,, \tag{32}$$

and the α's are given in Table III. We note that all of the above γ's are reasonably small compared to 1, so that the splitting is only a small fraction of the binding energy. However, as we can see from Fig. 9, where the splittings have been incorporated into the $T=\frac{3}{2}$ spectrum, they are not small compared to the separations between multiplets. This is because $g\langle q\rangle=50.0$ for the $l=1$ state and 28.9 for the $l=2$ state. In fact it is only luck which keeps the lowest-lying levels from having negative energy. If the coupling were only a little stronger, this would occur.

We conclude that our procedure of calculating the degenerate energy levels by spherically averaging the potentials and then calculating the splittings in perturbation theory is not valid. The fine structure must somehow be incorporated into the zeroth-order solution. At this time we have no answer to this problem.

We conclude that since each of the approximations made in I has been shown to be invalid, the model must be rejected. However, the model studied in these two papers has been only the simplest possible generalization of the static model. In future work we hope to explore other versions of the model in which the problems of relativistic kinematics and angular correlations between the fermion and field wave functions are treated in a more realistic way.

TABLE III. The quantity α in Eq. (32) is given for each of the J values possible for a given set of l, j.

l, j	$J=\frac{1}{2}$	$J=\frac{3}{2}$	$J=\frac{5}{2}$	$J=\frac{7}{2}$
1, $\frac{3}{2}$	$\frac{1}{4}$	$-\frac{1}{5}$	1/20	$\cdots$
2, $\frac{3}{2}$	$\frac{1}{4}$	0	$-5/28$	14

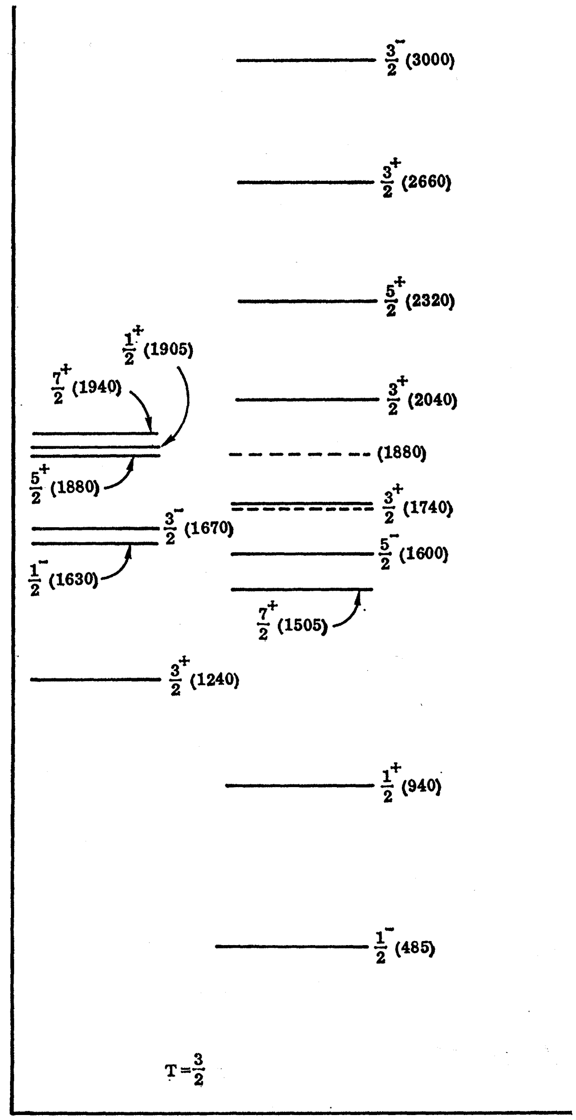

FIG. 9. Spectrum of $T=\frac{3}{2}$ resonances. The observed states are on the left, and on the right is the calculated spectrum in which the fine-structure splitting has been taken into account. We note that the vertical scale of this figure is expanded by a factor of 2 from that of Figs. 1 and 4.

Our proposal is to start with the relativistic Hamiltonian

$$H=\int d^3x\,\psi^\dagger(x)(\boldsymbol{\alpha}\cdot\mathbf{p}+\gamma_0 m_0)\psi(x) +\int d^3x(\pi^2+|\nabla\phi|^2+\mu^2\phi^2) +ig\int d^3x\,\psi^\dagger(x)\gamma_0\gamma_5\psi(x)\phi(x)$$

and project this onto the one-fermion subspace to obtain

$$H=\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m_0+H_\phi+ig\gamma_0\gamma_5\phi(x).$$

We can then proceed in one of two ways.

(a) Perform a Foldy-Wouthuysen transformation to determine the correct nonrelativistic limit which turns out to be (to order $1/m_0$)

$$H=m_0+\frac{p^2}{2m_0}+\frac{g^2}{2m_0}\phi^2(x)-\frac{g}{2m_0}\boldsymbol{\sigma}\cdot\nabla\phi(x)+H_\phi. \quad (33)$$

The presence of the $\phi^2(x)$ term and the gradient coupling make this a considerably more complicated problem than the one we have considered so far. We also propose to take into account the strong angular correlations between fermion and field wave functions implied by the interaction term.

(b) If we find again that nonrelativistic kinematics is untenable, the final step would be to try to solve the Dirac equation self-consistently. Whether or not this is possible is at this time an open question.

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Measuring Light-Cone Singularities*

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The scaling behavior observed in deep-inelastic electron scattering is related to the structure of the electric current commutation function in position space. We show that scaling is assured when that object has the following form, which is also consistent with Regge behavior:

$$i\langle p|[j^\mu(x),j^\nu(0)]|p\rangle=[g^{\mu\nu}\Box-\partial^\mu\partial^\nu]\left[\frac{1}{4\pi^2}\epsilon(x\cdot p)\delta(x^2)\int_0^\infty d\omega\,\frac{\cos\omega x\cdot p}{\omega^2}F_L(\omega)+\epsilon(x\cdot p)\theta(x^2)f_1(x^2,x\cdot p)\right]$$
$$+[p^\mu p^\nu\Box-p\cdot\partial(\partial^\mu p^\nu+\partial^\nu p^\mu)+g^{\mu\nu}(p\cdot\partial)^2]\left[\frac{1}{4\pi^2}\epsilon(x\cdot p)\theta(x^2)\int_0^\infty d\omega\,\frac{\sin\omega x\cdot p}{\omega x\cdot p}F_2(\omega)+\epsilon(x\cdot p)\theta(x^2)\tilde{f}_2(x^2,x\cdot p)\right].$$

In the above, $F_L=F_2-2\omega F_1$, and the F_i are the conventional scaling functions of Bjorken. The f_i are arbitrary, except that $f_2(0,x\cdot p)=0$. It is also demonstrated that when the combination $T_1+(\nu^2/q^2)T_2$ of the conventional forward Compton amplitudes, as well as T_2, are unsubtracted, a new sum rule can be derived:

$$\langle p|[j^0(0,\mathbf{x}),j^i(0)]|p\rangle=\frac{i}{2\pi}\partial^i\delta(\mathbf{x})\int_0^\infty d\omega\frac{F_L(\omega)}{\omega^2}.$$

Finally, the consequences of the same unsubtractedness hypothesis for the electromagnetic self-mass of the target proton are discussed. The unsubtractedness hypothesis is consistent with present experimental results.

I. INTRODUCTION

In this paper we relate the remarkable regularities observed in deep-inelastic scattering[1] to the behavior of the commutator of electromagnetic currents near the light cone. That the light-cone commutator should be relevant in this connection has already been noted by several authors.[2] We show that the experimental data place stringent, but simple, restrictions on the commutator, and that the leading light-cone singularity

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[1] For a summary of the experimental data, see R. E. Taylor, SLAC Report No. SLAC-PUB-677 (unpublished). We insert here the *caveat* that the experimental data are not unimpeachable evidence for scaling. A skeptic can take refuge in the large error bars, and other uncertainties, and insist that scaling is in fact weakly broken, for example by logarithmic terms. We do not here succumb to this cautionary pessimism.

[2] B. L. Ioffe, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu 9, 163 (1969) [Soviet Phys. JETP Letters 9, 97 (1969)]; B. L. Ioffe, Phys. Letters **30B**, 123 (1969); R. Brandt, Phys. Rev. Letters **23**, 1260 (1969). These authors discussed the behavior of the light-cone commutator; see also Ref. 8. After completion of the major portion of this investigation, we learned from D. G. Boulware that he and L. S. Brown have also studied this problem. Some of their results are to be found in L. S. Brown, in *Lectures in Theoretical Physics*, edited by W. E. Brittin, B. W. Downs, and J. Downs (Interscience, New York, to be published). Other results are unpublished. L. S. Brown has derived a representation for the product of two currents, consistent with scaling, by using the spectral representation and some regularity assumptions about the behavior of the spectral functions [see Eqs. (6.45)–(6.47) of Brown's paper]. This is equivalent to our representation for the commutator, Eqs. (2.4) and (2.7) below. Brown has also discussed the connection between a q-number Schwinger term and the longitudinal electroproduction cross section in the deep-inelastic region. This discussion is equivalent to our sum rule, Eq. (2.8) below.