A similar substitution of (C1) into (B26) gives the other desired result:

$$I_1^{+}=\tfrac{1}{3}\pi^2 m^{-2}R_1\bigg[\frac{1}{9}R_1^{-1}-\tfrac{1}{8}i\pi R_1^{-2}+\int_0^1 dz(1-z^2)^{1/2}$$
$$\times[2R_1\beta(1-\beta)]^{-2}\{R_1^{-2}+3\beta(1-\beta)(1-2\beta)\ln[(1-\beta)/\beta]+10\beta^2(1-\beta)^2\}\{\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2}$$
$$\times\sinh^{-1}[2R_1\beta(1-\beta)]-2R_1\beta(1-\beta)\}-\tfrac{1}{2}i\pi\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2}-1\}\}\bigg]. \quad \text{(C3)}$$

PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Nonassociativity of the Operators in the Crossing-Symmetric Bethe-Salpeter Equations*

ROBERT J. YAES†
Center for Particle Theory, Department of Physics, The University of Texas at Austin, Austin, Texas 78712
(Received 15 June 1970)

We discuss the properties of the crossing-symmetric Bethe-Salpeter equations which have been proposed by Taylor and by Haymaker and Blankenbecler. We consider various possible methods of solution and the possibility of application to the Veneziano amplitude. We show that the operators which appear in these equations are not mutually associative, and hence that even the linearized approximation to these equations cannot be solved by conventional techniques.

I. INTRODUCTION

IT is generally believed that the four-point function in strong-interaction theory should have the following properties: Lorentz invariance, analyticity, crossing symmetry, unitarity, and Regge asymptotic behavior. Since Lorentz invariance and analyticity are explicitly satisfied by any analytic function of the Mandelstam variables, s, t, and u, the three key properties are crossing symmetry, unitarity, and Regge behavior. Until recently, we could not obtain an amplitude having more than one of these three properties. However, we now have the simple but elegant model of Veneziano[1] which displays both crossing symmetry and Regge behavior but, alas, not unitarity.

The problem of combining crossing symmetry and unitarity is much more difficult. A set of equations for an amplitude having both these properties has been proposed by Taylor[2] and by Haymaker and Blankenbecler.[3,4] Unfortunately, being nonlinear, these equations have the disadvantage of not being soluble. All one can do is use the various iteration schemes which we shall discuss and which cannot be guaranteed to converge. In addition, since (as we shall show) the operators which appear in these equations are not mutually associative, we cannot even solve a linearized approximation to these equations by the usual techniques. In this paper we discuss the properties of these equations, the methods of obtaining iterative solutions, and the possible application to the Veneziano amplitude.

II. EQUATIONS

We consider the four-point function for the scattering of identical, spinless bosons of mass m (Fig. 1). The crossing-symmetric generalization of the Bethe-Salpeter equation proposed by Taylor[2] and by Haymaker

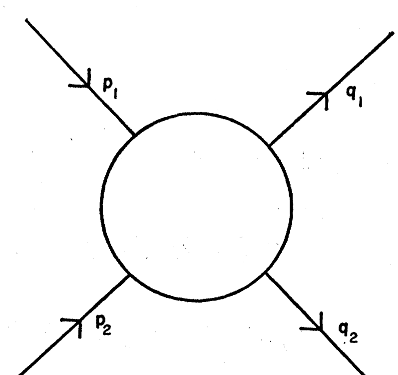

FIG. 1. Our notation for the four-point function.

* Work supported by the National Science Foundation.

† Present address: Department of Physics, Illinois Institute of Technology, Chicago, Ill. 60616.

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] J. G. Taylor, Nuovo Cimento Suppl. **1**, 988 (1963).

[3] R. W. Haymaker and R. Blankenbecler, Phys. Rev. **171**, 1581 (1968).

[4] On-shell K-matrix equations of the same form were first obtained by W. Zimmermann [Nuovo Cimento **21**, 249 (1961)]. They have been applied to various cases, including the Veneziano model, by Cordes, Ravenhall, and Schult and by Humble.

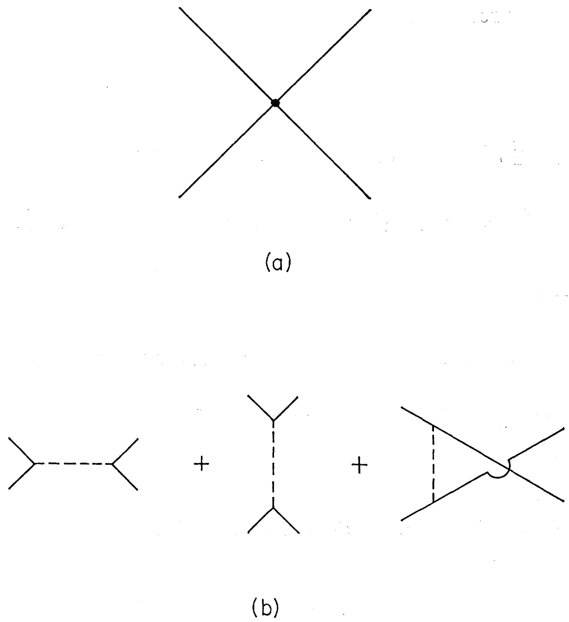

FIG. 2. Two possible choices for V: (a) a point interaction and (b) a three-pole interaction.

and Blankenbecler[3] for this amplitude can be written

$$T=K_i+K_iG_iT, \tag{1}$$

$$K_i=V+\sum_{j\neq i} K_jG_jT. \tag{2}$$

It follows immediately from Eqs. (1) and (2) that

$$T=V+\sum_i K_iG_iT, \tag{3}$$

$$T=\tfrac{1}{2}(\sum_i K_i-V). \tag{4}$$

The index i specifies the s, t, or u channel. Equation (1) is thus the familiar Bethe-Salpeter equation in the ith channel. The two-body irreducible kernel in the ith channel, K_i, satisfies Eq. (2). T is the T matrix. V is an arbitrary, two-body irreducible, crossing-symmetric input, examples of which are a point interaction $V(s,t,u)=\lambda$ [Fig. 2(a)] and a sum of three identical poles, one in each channel [Fig. 2(b)],

$$V(s,t,u)=\lambda[1/(s-M^2)+1/(t-M^2)+1/(u-M^2)].$$

All our dynamical assumptions are contained in our choice of V.

We can easily eliminate T from the equations by substituting Eq. (4) into Eq. (2), giving us an equation for K_i,

$$K_i=V-\tfrac{1}{2}\sum_{j\neq i} K_jG_jV+\tfrac{1}{2}\sum_{j\neq i,l} K_jG_jK_l. \tag{5}$$

Since T is easily determined from Eq. (4) when the K_i are known, and since K_s, K_t, and K_u are simply related by crossing symmetry, it is Eq. (5), the quadratic integral equation for the K function, that must be solved in order to solve the problem. As we shall see below, this is essentially impossible to do (except by iteration) not only because the equation is nonlinear but also because the operators in this equation is not mutually associative.

T, V, and K_i are four-point functions which can be taken as functions of the four four-momenta p_1, p_2, q_1, q_2, or as functions of the usual Mandelstam variables

$$\begin{aligned} s&=(p_1+p_2)^2=(q_1+q_2)^2,\\ t&=(p_1-q_1)^2=(p_2-q_2)^2,\\ u&=(p_1-q_2)^2=(p_2-q_1)^2 \end{aligned} \tag{6}$$

and the external mass variables p_1^2, p_2^2, p_3^2, and p_4^2. The G_i are bilateral operators which operate on two four-point functions to give us a third four-point function. Since these operators can be expressed most simply in terms of integrals over the intermediate-momentum variables, and since we need the off-shell amplitudes, in discussing the properties of these operators it will be advantageous to take the four-point functions to be functions of the momenta. On the mass shell, $p_i^2=m^2$ and the amplitudes are functions only of s, t, and u.

From Bose statistics, we immediately have

$$T(s,t,u)=T(s,u,t). \tag{7}$$

Crossing symmetry is then expressed by

$$T(s,t,u)=T(t,s,u)=T(u,t,s). \tag{8}$$

Combining Eqs. (7) and (8), we see that the T matrix is invariant under any permutation of the Mandelstam variables. The same relations, of course, hold for V. On the other hand, for the irreducible kernel, if we take

$$K_s(s,t,u)=K(s,t,u)=K(s,u,t), \tag{9}$$

we have

$$\begin{aligned} K_t(s,t,u)&=K(t,s,u)=K(t,u,s),\\ K_u(s,t,u)&=K(u,s,t)=K(u,t,s), \end{aligned} \tag{10}$$

but now $K(s,t,u)\neq K(t,s,u)\neq K(u,s,t)$ because the kernels cannot be two-body irreducible in two channels simultaneously.

Now taking the amplitudes to be functions of the four-momenta, we can rewrite Eq. (7) as

$$\begin{aligned} &\langle p_1,p_2|T|q_1,q_2\rangle=\langle p_1,p_2|T|q_2,q_1\rangle\\ &=\langle p_2,p_1|T|q_1,q_2\rangle=\langle p_2,p_1|T|q_2,q_1\rangle, \end{aligned} \tag{11}$$

and Eq. (8) becomes

$$\begin{aligned} \langle p_1p_2|T|q_1q_2\rangle&=\langle p_1,-q_1|T|-p_2q_2\rangle\\ &=\langle p_1,-q_2|T|q_1,-p_2\rangle. \end{aligned} \tag{12}$$

Since these equations are assumed to be valid whether or not $p_i^2=m^2$, they express crossing symmetry for the off-shell amplitudes.

Equation (11) states that the amplitudes do not change when the initial or final momenta, respectively, are permuted among themselves. If both Eqs. (11) and (12) hold, the amplitude is invariant under any permutation of the four-momenta (provided, of course, that we change the signs when we interchange an incoming momentum with an outgoing momentum). We can thus think of crossing symmetry as a kind of generalized Bose statistics. (If the particles have spin or other internal quantum numbers, there are several invariant amplitudes and the crossing relations will be more complicated.)

The irreducible kernel satisfies Eq. (9) and thus will also satisfy Eq. (11). However, from Eq. (10), we have, instead of Eq. (12),

$$\begin{aligned}\langle p_1p_2|K_t|q_1q_2\rangle&=\langle p_1,\,-q_1|K_s|-p_2,\,q_2\rangle\,,\\ \langle p_1,p_2|K_u|q_1,q_2\rangle&=\langle p_1,\,-q_2|K_s|q_1,\,-p_2\rangle.\end{aligned}\tag{13}$$

Equation (3) can now be written explicitly in terms of the momentum variables [Fig. 3(a)][5]

$$\begin{aligned}\langle p_1,p_2|T|q_1,q_2\rangle=\langle p_1,p_2|V|q_1,q_2\rangle+\sum_{k_1k_2}\langle p_1,p_2|K_s|k_1,k_2\rangle G(k_1k_2)\langle k_1,k_2|T|q_1,q_2\rangle+\sum_{k_1k_2}\langle p_1,\,-k_1|K_t|q_1,k_2\rangle G(k_1k_2)\\ \times\langle k_1,p_2|T|-k_2,\,q_2\rangle+\sum_{k_1k_2}\langle p_1,\,-k_2|K_u|k_1q_2\rangle G(k_1k_2)\langle k_1p_2|T|q_1,\,-k_2\rangle.\end{aligned}\tag{14}$$

Using Eqs. (12) and (13), we can eliminate K_t and K_u and rewrite (14) with only K_s as [Fig. 3(b)][6]

$$\begin{aligned}\langle p_1,p_2|T|q_1q_2\rangle=\langle p_1,p_2|V|q_1q_2\rangle+\sum_{k_1k_2}\langle p_1p_2|K_s|k_1k_2\rangle G(k_1k_2)\langle k_1k_2|T|q_1q_2\rangle+\sum_{k_1k_2}\langle p_1,\,-q_1|K_s|k_1k_2\rangle G(k_1k_2)\\ \times\langle k_1k_2|T|-p_2,\,q_2\rangle+\sum_{k_1k_2}\langle p_1,\,-q_2|K_s|k_1k_2\rangle G(k_1k_2)\langle k_1k_2|T|q_1,\,-p_2\rangle.\end{aligned}\tag{15}$$

The equation for the irreducible kernel [Eq. (2)] with $i=s$ becomes

$$\begin{aligned}\langle p_1,p_2|K_s|q_1,q_2\rangle=\langle p_1,p_2|V|q_1q_2\rangle+\sum_{k_1k_2}\langle p_1,\,-q_1|K_s|k_1k_2\rangle G(k_1k_2)\langle k_1,k_2|T|-p_2,\,q_2\rangle\\ +\sum_{k_1k_2}\langle p_1,\,-q_2|K_s|k_1k_2\rangle G(k_1,k_2)\langle k_1,k_2|T|q_1,\,-p_2\rangle.\end{aligned}\tag{16}$$

We could also use Eq. (13) to rewrite Eq. (5); setting $i=s$, we have

$$\begin{aligned}\langle p_1,p_2|K_s|q_1,q_2\rangle=\langle p_1,p_2|V|q_1,q_2\rangle-\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_1|K_s|k_1,k_2\rangle G(k_1,k_2)\langle k_1,k_2|V|-p_2,\,q_2\rangle\\ -\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_2|K_s|k_1,k_2\rangle G(k_1,k_2)\langle k_1,k_2|V|q_1,\,-p_2\rangle+\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_1|K_s|k_1k_2\rangle G(k_1k_2)\langle k_1,p_2|K_s|-k_2,\,q_2\rangle\\ +\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_1|K_s|k_1k_2\rangle G(k_1k_2)\langle k_1,k_2|K_s|-p_2,\,q_2\rangle+\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_1|K_s|k_1k_2\rangle G(k_1k_2)\langle k_1,\,-q_2|K_s|-k_2,\,p_2\rangle\\ +\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_2|K_s|k_1k_2\rangle G(k_1k_2,)\langle k_1,p_2|K_s|q_1,\,-k_2\rangle+\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_2|K_s|k_1,k_2\rangle G(k_1,k_2)\\ \times\langle k_1,\,-q_1|K_s|-p_2,\,-k_2\rangle+\tfrac{1}{2}\sum_{k_1k_2}\langle p_1,\,-q_2|K_s|k_1,k_2\rangle G(k_1,k_2)\langle k_1k_2|K_s|q_1,\,-p_2\rangle.\end{aligned}\tag{17}$$

Of course, similar equations hold for K_t and K_u.

The Green's function is the product of the two single-particle propagators. If m is the physical, renormalized mass of the particles, we can take these to be the usual Feynman propagators,

$$G(k_1k_2)=(k_1^2-m^2+i\epsilon)^{-1}(k_2^2-m^2+i\epsilon)^{-1}.\tag{18}$$

The sum over intermediate states represents the usual integral over the intermediate four-momenta with the appropriate δ functions for energy-momentum conservation. With the usual normalization, the first integral term of Eq. (15) can be written

$$\begin{aligned}-(2\pi)^{-4}i\int d^4k_1d^4k_2\langle p_1p_2|K_s|k_1k_2\rangle\langle k_1k_2|T|q_1q_2\rangle\\ \times(k_1^2-m^2+i\epsilon)^{-1}(k_2^2-m^2+i\epsilon)^{-1}\\ \times\delta^4(p_1+p_2-k_1-k_2)\delta^4(k_1+k_2-q_1-q_2).\end{aligned}$$

[5] Since we must choose a consistent notation and stick to it, we take the s channel to go from right to left and the t channel from top to bottom. Bras and kets represent initial and final stages in the s channel.

[6] With an appropriate change of variables, Eqs. (1) and (15) above correspond to Eqs. (1) and (2), respectively, of Ref. 2.

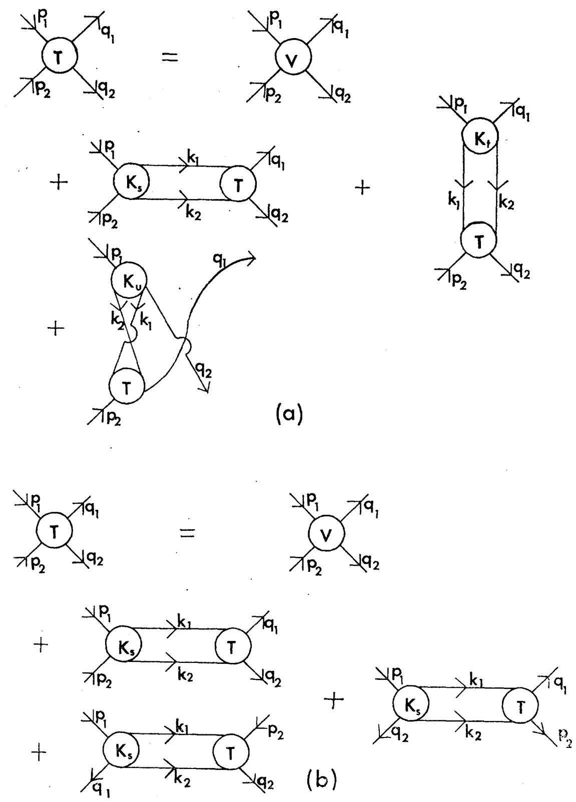

FIG. 3. Schematic diagrams (a) for Eq. (14) and (b) for Eq. (15).

III. NONASSOCIATIVITY OF OPERATORS

It is now easy to show that the operators which appear in these equations are not mutually associative. This is true particularly of the operators G_s, G_t, and G_u which appear in Eqs. (1)–(5). We shall show this specifically for the operators G_s and G_t but it is equally valid for any other pair of these operators. We consider four arbitrary four-point functions A, B, C, and D and, comparing Eqs. (3) and (14), we have

$$\langle p_1,p_2|AG_sB|k_1,k_2\rangle =\sum_{l_1l_2}\langle p_1,p_2|A|l_1l_2\rangle G(l_1l_2)\langle l_1l_2|B|k_1k_2\rangle , \quad (19a)$$

$$\langle k_1k_2|CG_tD|q_1q_2\rangle =\sum_{l_1l_2}\langle k_1,-l_1|C|q_1,l_2\rangle G(l_1l_2)\langle l_1,k_2|D|-l_2q_2\rangle . \quad (19b)$$

By substituting Eq. (19a) into Eq. (19b) and vice versa,

$$\langle p_1p_2|AG_s(BG_tC)|q_1q_2\rangle =\sum_{k_1k_2;\,l_1l_2}\langle p_1p_2|A|k_1k_2\rangle G(k_1k_2)\langle k_1,-l_1|B|q_1l_2\rangle \times G(l_1l_2)\langle l_1k_2|C|-l_2q_2\rangle \quad (20)$$

and

$$\langle p_1p_2|(AG_sB)G_tC|q_1q_2\rangle =\sum_{k_1k_2;\,l_1l_2}\langle p_1-k_1|A|l_2l_1\rangle G(l_1l_2)\langle l_1l_2|B|q_1k_2\rangle \times G(k_1k_2)\langle k_1p_2|C|-k_2q_2\rangle . \quad (21)$$

The significance of Eqs. (20) and (21) can be seen from Figs. 4(a) and 4(b), respectively. It should be obvious that $AG_s(BG_tC)$ and $(AG_sB)G_tC$ are not equivalent. In the first case, A, being the last function operated on, has two external lines while B and C have one each. In the second case, C is the last function operated on and it has two external lines while A and B have one each. If $A=B=C$, then Figs. 4(a) and 4(b) are related by $s\leftrightarrow t$ crossing. If A, B, and C are not equal, then there is no such relation between Figs. 4(a) and 4(b).

In the same way, it can also easily be seen that $(AG_sB)G_uC\neq AG_s(BG_uC)$, $(AG_tB)G_uC\neq AG_t(BG_uC)$, etc. We should emphasize the fact that the problem of nonassociativity does not occur in the single-channel case. It is only when we try to unitarize in two channels simultaneously that we run into this problem.

If the nonlinearity of the equations were the only major difficulty, we would expect that we would be able to obtain a tractable equation by making a linearized approximation to Eqs. (1)–(4) which maintains as many desirable properties of these equations as possible. The most obvious way to do this is to drop the integral term in Eq. (2) and make the approximation $K_s=K_t=K_u=V$. [Since V is crossing-symmetric, this is consistent with Eqs. (9) and (10).] Substituting this into Eq. (3), we obtain an equation of the form which we have previously discussed in the context of the quasi-potential equation,[7]

$$T=V+VG_sT+VG_tT+VG_uT. \quad (22)$$

Equation (22) is linear and explicitly crossing symmetric. It is also clear that the solution of Eq. (22) will have V as its Born approximation and will have the unitarity cuts in the right place in all three channels but not necessarily with the right discontinuity.

At first glance, we might think that it is possible, at least formally, to obtain a solution of Eq. (22) by the usual techniques. We can formally define the linear operator O by

$$AOB=AG_sB+AG_tB+AG_uB. \quad (23)$$

Then Eq. (22) can be written

$$T=V+VOT \quad (24)$$

or

$$(1-VO)T=V.$$

Then if there exists an inverse operator $(1-VO)^{-1}$ and if the operator algebra is associative, we can

[7] J. A. Campbell and R. J. Yaes, Australian J. Phys. 22, 655 (1969).

multiply on the right by $(1-VO)^{-1}$ to obtain the formal solution

$$T=(1-VO)^{-1}V. \tag{25}$$

However, it is precisely at this point that the argument breaks down. Because the operator O is defined in terms of the nonassociative operators G_s, G_t, and G_u, it does not necessarily follow that

$$(1-VO)^{-1}[(1-VO)T] =[(1-VO)^{-1}(1-VO)]T=T. \tag{26}$$

It is interesting to note that the proof of unitarity for Eq. (24) will break down for the same reason. To see this, we recall the proof of unitarity for the single-channel equation,[8]

$$T_s=V+VG_sT, \tag{27}$$

which has the solution

$$T_s=(1-VG_s)^{-1}V. \tag{28}$$

(We use the subscript s to distinguish T_s from the fully crossing-symmetric T matrix T.) Since any singularities of V in the physical region (which, in any event, could only be poles) will not contribute to the elastic cut, we shall ignore them and take V to be real in the physical region. Then the adjoint of Eq. (27) can be written

$$T_s^\dagger=V+T_s^\dagger G_s^\dagger V=V+VG_s^\dagger T_s^\dagger. \tag{29}$$

If we subtract Eq. (29) from Eq. (28) and define ΔT_s and ΔG_s by

$$\Delta T_s=T_s-T_s^\dagger, \quad \Delta G_s=G_s-G_s^\dagger, \tag{30}$$

we have

$$\Delta T_s=VG_sT_s-VG_s^\dagger T_s^\dagger=VG_s\Delta T_s-V\Delta G_sT^\dagger \tag{31}$$

or

$$(1-VG_s)\Delta T_s=V\Delta G_sT_s^\dagger; \tag{32}$$

hence, multiplying by $(1-VG_s)^{-1}$ and using Eq. (28),

$$\Delta T_s=T_s\Delta G_sT_s^\dagger. \tag{33}$$

This is just the elastic unitarity relation

$$\langle p_1p_2|\Delta T_s|q_1q_2\rangle=(2\pi)^{-2}i\int d^4k_1d^4k_2\langle p_1p_2|T_s|k_1k_2\rangle \times\theta(k_1^0)\delta(k_1^2-m^2)\theta(k_2^0)\delta(k_2^2-m^2)\langle k_1k_2|T_s^\dagger|q_1q_2\rangle \times\delta^4(p_1+p_2-k_1-k_2)\delta^4(k_1+k_2-q_1-q_2). \tag{34}$$

Equation (34) is nonvanishing only when all the θ functions and δ functions are satisfied, so we have

$$s=(p_1+p_2)^2=(k_1+k_2)^2=k_1^2+k_2^2+2(k_1^0k_2^0-\mathbf{k}_1\cdot\mathbf{k}_2) =2m^2+2[(k_1^2+m^2)^{1/2}(k_2^2+m^2)^{1/2}-\mathbf{k}_1\cdot\mathbf{k}_2]>4m^2. \tag{35}$$

Since the external lines are on the mass shell, it is easy to see that we must also satisfy the inequalities $-(s-4m^2)<t<0$, $(-s-4m^2)<u<0$. These three inequalities define the s-channel physical region.

[8] See, e.g., V. A. Alessandrini and R. L. Omnes, Phys. Rev. **136**, B472 (1964).

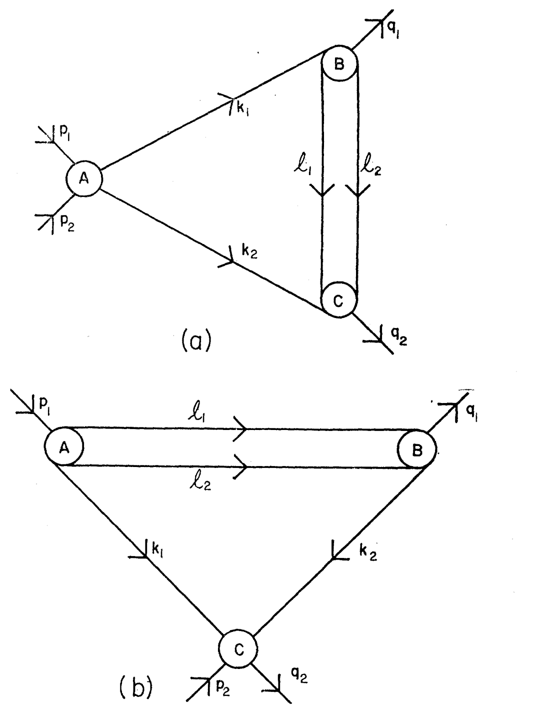

FIG. 4. Schematic representations (a) for $AG_s(BG_tC)$ and (b) for $(AG_sB)G_tC$.

Terms like $T_t\Delta G_tT_t^\dagger$ should not vanish only in the t-channel physical region and terms like $T_u\Delta G_uT_u^\dagger$ vanish everywhere but in the u-channel physical regions. Since the physical regions cannot overlap, if T has the elastic unitarity cuts in all three channels, we would have

$$\Delta T=T\Delta G_sT^\dagger+T\Delta G_tT^\dagger+T\Delta G_uT^\dagger =T\Delta OT. \tag{36}$$

However, Eq. (36) does not follow from Eq. (24) for the same reason that Eq. (25) does not. We should point out, however, that the solution of the original nonlinear set of equations [Eqs. (1)–(4)] will satisfy Eq. (36) below the inealstic thresholds. Above the inelastic thresholds, we will get additional contributions from the multiparticle cuts in the K_i. It seems as though it is impossible to satisfy both *elastic* unitarity and crossing symmetry simultaneously everywhere.

IV. METHODS OF SOLUTION

One method of approach is the variational principle proposed by Haymaker and Blankenbecler, which is, however, rather complicated. In addition, in order to apply it, not only must one know the kernel K_i and the functional form for T but one must also have the four-body T matrix T_4. To our knowledge this variational principle has not yet been applied to any concrete cases

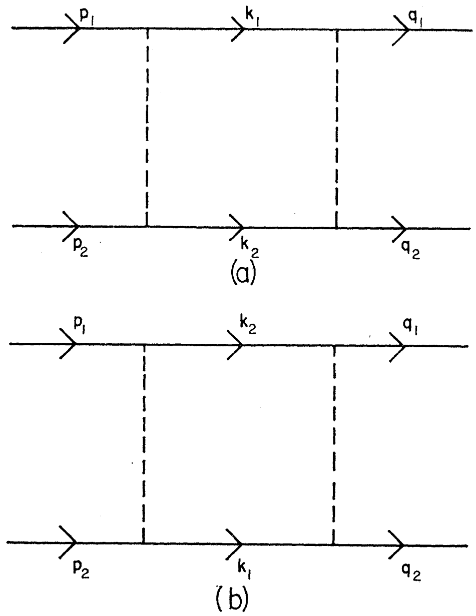

FIG. 5. Two diagrams which appear when Eqs. (15) and (16) are iterated once with V given by Fig. 2(b).

either by the above-mentioned authors or by anyone else.

A more obvious approach is straightforward iteration in powers of V. Exact crossing symmetry is maintained and unitarity is satisfied to the order to which we iterate. This just gives us the Feynnan diagrams to a given order which we could obtain easily enough without reference to our equations. However, the equations serve as a convenient mnemonic device for making certain that we have included all relevant diagrams.

As an example, we consider the case where V is given by Fig. 2(b), Since V is a sum of three terms and since there are three terms in the equation, a single iteration will give us 27 diagrams to second order in V. Diagrams where the internal lines are interchanged both appear. For example, both the graphs in Figs. 5(a) and 5(b) appear in the sum. Hence, only the bubble graphs appear only once. We thus have three bubble graphs [Fig. 6(a)], one in each channel, 12 vertex-correction graphs, two each of Figs. 6(b) and 6(c), in each channel, and 12 box and annihilation graphs of the type of Figs. 6(d) and 6(e).

Another approach is to maintain unitarity exactly and crossing symmetry only to a given order in V. This merely involves iterating the equations for K_i to a given order in V and then solving Eq. (1) with this K_i. Thus, for example, to first order, $K_s = V$ and Eq. (1) becomes

$$T = V + VG_sT. \tag{37}$$

(If V had only the t-channel pole, this would be the Bethe-Salpeter equation in the ladder approximation.)

To second order, the kernel is

$$K_s = V + VG_tV + VG_uV, \tag{38}$$

etc. This procedure of adding more terms to the irreducible kernel can also be accomplished without reference to these equations. It suffers from the disadvantage that at each stage we must solve a Bethe-Salpeter equation with a quite complicated kernel.

If for some reason Eq. (37) is a good approximation, we can treat the higher-order terms in the kernel as perturbations and we can use the perturbation theory which we have previously developed[9] for the Faddeev equations. We define

$$\begin{aligned} K_s &= K_0 + \delta K_s, \quad K_0 = V \\ T_0 &= V + VG_sT_0, \\ T &= K_s + K_sG_sT = T_0 + \delta T. \end{aligned} \tag{39}$$

Then to first order in δK_s,

$$\delta T = (1 + T_0G_s)\delta K_s(1 + G_sT_0), \tag{40}$$

and so on. If there are bound-state poles, we can also obtain the shift in binding energies and form factors by this method.

V. DISCUSSION

Two important questions that remain are whether we can simplify the equations by replacing the product of Feynman propagators [Eq. (18)] by a quasipotential propagator[10] and whether we can apply the equations

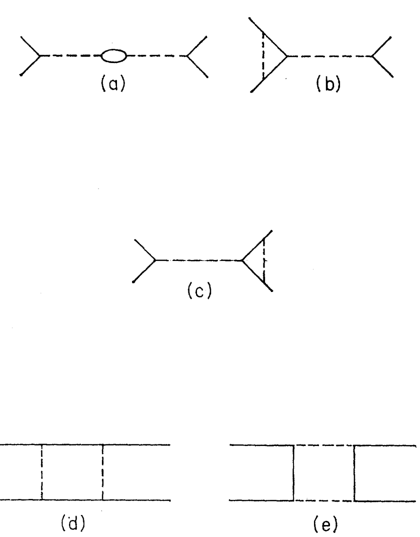

FIG. 6. Types of diagrams which appear when Eqs. (15) and (16) are iterated once with V given by Fig. 2(b).

[9] R. J. Yaes, Phys. Rev. **170**, 1236 (1968); Nucl. Phys. **A131**, 623 (1969).

[10] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963); R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966); V. G. Kadyshevsky, Nucl. Phys. **B6**, 125 (1968); F. Gross, Phys. Rev. **186**, 1448 (1969).

to the Veneziano amplitude (i.e., set V=Veneziano) and obtain meaningful results. After expending much time and effort on these questions,[11] we have come to the conclusion that the answer to both is probably no.

The quasipotential kernel has spurious left-hand cuts.[12] Hence, if we inserted the quasipotential propagators in Eq. (3), the t- and u-channel terms would have spurious right-hand cuts in the s-channel physical region. We have been unable to find a method of avoiding this difficulty. If we try to remedy the situation by multiplying the propagators by the appropriate θ functions to remove the spurious cuts, we would destroy the analyticity of the amplitude.

It would seem as though the first iteration scheme would be applicable to the Veneziano amplitude. The Veneziano amplitude has an infinite number of poles in each of the three channels and hence can be considered to be an infinite sum of graphs of the type in Fig. 2(b). Thus, if we take V equal to the Veneziano amplitude and apply the iteration scheme, we would obtain an infinite set of graphs of the type seen in Fig. 6, as well as higher-order graphs. These will have the appropriate cut structure.

The major difficulty in applying this method to the Veneziano model is the fact that we must know V with all four lines off the mass shell. If we just use the on-shell Veneziano amplitude, the integrals will be very badly divergent since the integrands will behave like Γ functions of the integration variables. To correct this, it is necessary to assume that V goes to zero at least as fast as an inverse Γ function in the off-shell variables. Since the off-shell continuation otherwise is arbitrary, it is questionable whether numerical results obtained by this method would have any significance.

We have thus seen that the problem of obtaining amplitudes which are simultaneously unitary and crossing symmetric is even more difficult than we first imagined. Not only are the equations which we must use nonlinear, but the operators which appear in these equations are not mutually associative. Hence, even the linearized approximation to these equations cannot be solved by conventional techniques and we must fall back on the usual perturbative methods of field theory. Thus, since we have no satisfactory method of approaching the problem when the input consists of a point interaction or a sum of three poles, we certainly cannot deal with the case where the input is the Veneziano amplitude which is an infinite sum of poles.[13] It would thus seem that the solution of the problem of obtaining a unitarized Veneziano amplitude (that is, a unitary, crossing-symmetric, Regge-behaved, dual amplitude) will not be obtained until we are able to obtain amplitudes satisfying both unitarity and crossing symmetry with simple inputs.

ACKNOWLEDGMENTS

The author is deeply indebted to Professor John A. Campbell for his constant advice and encouragement. The author is also grateful to Professor Antonio Pagnamenta, Professor Richard A. Brandt, and Professor S. N. Biswas for valuable conversations.

[11] J. A. Campbell and R. J. Yaes (unpublished). Preliminary numerical calculations using the equations and off-shell continuation of Ref. 7 indicate that the change in the amplitude at threshold due to unitarization will be negligible. The equation for the $\pi^0\pi^0$ amplitude using the parameters of C. Lovelace [Phys. Letters **28B**, 264 (1968)] was iterated once. It was found that at threshold, on the mass shell, $(VG_sV+VG_tV+VG_uV)/V<0.002$, independent of the choice of the parameter c in the off-shell continuations. Because of the conceptual difficulties inherent in this model, some of which are described above, further numerical calculations were not carried out.

[12] M. K. Polivanov and S. S. Khoruzhi, Zh. Eksperim. i Teor. Fiz. **46**, 339 (1963) [Soviet Phys. JETP **19**, 232 (1964)].

[13] In setting V equal to the Veneziano amplitude in Eqs. (1)–(4), we would be taking the position that all of the poles in the Veneziano amplitude are "elementary" particles. It is also possible that we would have to put in only a finite number of poles and the other resonance poles would appear as composite states; that is, the Born series would diverge at the resonant energies. In order to see which is actually the case, we would have to have a method of obtaining solutions of the equations. This is precisely what we do not have.