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High-Energy Delbrück Scattering Close to the Forward Direction*

HUNG CHENG†
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

TAI TSUN WU
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

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Delbrück scattering is the elastic scattering of a photon by a static Coulomb field via electron-positron pair creation. At high energies, there are two natural scales for the momentum transfer Δ, namely, m and m^2/ω, where m is the mass of the electron, and ω is the photon energy. When Δ is much larger than the smaller scale m^2/ω, the impact factor representation holds at high energies. The impact picture is here extended to give also the high-energy behavior of the Delbrück scattering amplitude when Δ is comparable to m^2/ω. The result can be expressed in terms of generalized hypergeometric functions, which reproduces the known result in the forward direction when Δ is set equal to zero, and also joins smoothly to the impact factor representation when Δ is much larger than m^2/ω. In the present analysis, the fine-structure constant α is assumed to be small, but not $Z\alpha$. In other words, all terms of the form $\alpha(Z\alpha)^n$ in the amplitude are taken into account. It is also shown that the result for $\Delta\ll m$ is independent of the mass of the target, and hence is in particular applicable to Compton scattering by an electron.

1. INTRODUCTION

OVER a year ago, we presented a systematic discussion[1] of the high-energy behavior of all the two-body elastic scattering amplitudes in quantum electrodynamics. More precisely, the processes considered included (1) Coulomb scattering of an electron to the order Z^2e^4, (2) electron-electron scattering to the order e^4, (3) electron-positron scattering to the order e^4, (4) Delbrück scattering to the order Z^2e^6, (5) electron Compton scattering to the order e^6, and (6) photon-photon scattering to the order e^8. Here, as usual, e denotes the charge of the electron, and Ze that of the source of the Coulomb field. Higher-order effects in Ze were also given.[1] Although the original calculation is quite complicated,[2–4] substantially simpler methods to obtain the same answers were found later.[5–7]

In studying some of the processes, such as (1)–(3) above, an artifical photon mass λ is introduced to avoid infrared divergences. For other processes, such as (4)–(6) above, such a photon mass is in no way needed. We emphasize that, in our consideration of the Delbrück scattering process in Ref. 4, we carefully used massless photons all the way through. In all cases without infrared divergence, it is explicitly verified that the limiting processes of zero photon mass and of infinite energy commute. For example, consider Delbrück scattering. Let ω be the energy of the photon in the laboratory system, i.e., the coordinate system where the Coulomb field is static; then the matrix element[1,2] satisfies

$$\lim_{\lambda\to0}\lim_{\omega\to\infty}\omega^{-1}\mathfrak{M}_0^{(D)}=\lim_{\omega\to\infty}\lim_{\lambda\to0}\omega^{-1}\mathfrak{M}_0^{(D)}=\lim_{\omega\to\infty}\omega^{-1}\mathfrak{M}_0^{(D)}\big|_{\lambda=0}, \quad (1.1)$$

provided that the momentum transfer Δ is fixed at a value *different* from 0.

Closely related to this condition $\Delta\neq0$, the results on high-energy Delbrück scattering as given in Ref. 4 cannot be considered as complete, as already mentioned there. In particular, there is no obvious way of connecting those results to the previously known high-energy behavior in the exactly forward direction

$$\mathfrak{M}_0^{(D)}\sim4i\alpha^3Z^2\frac{\omega}{m^2}\frac{7}{9}\left(\ln\frac{2\omega}{m}-\frac{109}{42}-\frac{1}{2}i\pi\right), \quad (1.2)$$

as given by Racah,[8] Jost, Luttinger, and Slotnick,[9] Toll,[10] and Rohrlich and Gluckstern.[11] More generally, as previously discussed,[4] even though there is only one mass, that of the electron, in the case of Delbrück scattering, there are actually two scales for the momentum transfer Δ, namely, m and m^2/ω. Indeed, the high-energy behavior of the Delbrück scattering amplitude as given by the impact factor representation[1–5] holds for fixed nonzero Δ independent of ω, and hence does not properly take into account the second scale for Δ.

It is the purpose of the present paper to study Delbrück scattering for small momentum transfers Δ in the physically realistic case of massless photons. More precisely, we show that the present case can be dealt

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with by a suitable modification of the method[5–7] already developed.

2. QUALITATIVE CONSIDERATIONS

In our previous study[1–3] of Delbrück scattering to the order Z^2e^6, the scattering amplitude at high energies was found to be expressed in terms of the impact factor representation

$$\mathfrak{M}_0^{(D)}\sim i\omega Z^2e^2(2\pi)^{-2}\int d\mathbf{q}_\perp\frac{\mathcal{J}_{ij}^{\gamma}(\frac{1}{2}\boldsymbol{\Delta},\mathbf{q}_\perp)}{[(\mathbf{q}_\perp+\frac{1}{2}\boldsymbol{\Delta})^2][(\mathbf{q}_\perp-\frac{1}{2}\boldsymbol{\Delta})^2]},\tag{2.1}$$

provided that the momentum transfer $\boldsymbol{\Delta}$ is not zero. In (2.1), $\mathcal{J}_{ij}^{\gamma}(\frac{1}{2}\boldsymbol{\Delta},\mathbf{q})$ is the photon impact factor given by[12]

$$\mathcal{J}_{ij}^{\gamma}(\mathbf{r}_1,\mathbf{q}_\perp)=-\tfrac{1}{2}\pi^{-3}e^4\int d\mathbf{p}_\perp\int_0^1 d\beta\left\{\frac{\delta_{ij}\beta^2\mathbf{r}_1^2+2\beta(1-\beta)(p_\perp-\beta r_1)_i(p_\perp+\beta r_1)_j}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}\right.$$
$$\left.-\frac{\frac{1}{4}\delta_{ij}[\mathbf{q}_\perp+(1-2\beta)\mathbf{r}_1]^2+2\beta(1-\beta)(p_\perp+q_\perp+r_1-\beta r_1)_i(p_\perp+\beta r_1)_j}{[(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}\right\}.\tag{2.2}$$

This amplitude for Delbrück scattering differs from that for photon-photon scattering[1,2] by the appearance of only one photon impact factor instead of two. For the case of photon-photon scattering to the order e^8, the impact factor representation is actually valid[13] even for $\Delta=0$. The reason of convergence in this case with $\Delta=0$ is that, while the denominator gives a factor $(\mathbf{q}_\perp^2)^{-2}$, this singularity is canceled by two factors of $\mathbf{q}_\perp^2$ from the two photon impact factors.[14] For the case of Delbrück scattering, the presence of only one photon impact factor is insufficient to remove the singularity at $\mathbf{q}_\perp=0$. Accordingly, a factor $(\mathbf{q}_\perp^2)^{-1}$ remains, and the integral on the right-hand side of (2.1) is logarithmically divergent when $\Delta=0$.

A comparison with (1.2) shows that this logarithmic divergence is actually $\ln\omega$. More precisely, this comparison indicates that the factor $(\mathbf{q}_\perp^2)^{-1}$ should fail to hold when $|\mathbf{q}_\perp|$ is comparable to ω^{-1}. Accordingly, q_3, although of the order of ω^{-1}, cannot be neglected in the denominator, since this factor $(\mathbf{q}_\perp^2)^{-1}$ is originally $(\mathbf{q}_\perp^2+q_3^2)^{-1}$.[15] With this understanding it is possible to modify the simplified derivations of (2.1) to cover the case where Δ is of the order of magnitude m^2/ω.

3. FORMULATION OF PROBLEM

With this understanding of the importance of keeping q_3 even though it is of the order of ω^{-1}, we can modify our previous derivation of the impact factor representation so that the result also holds close to the forward direction. We have a choice of how to proceed: We can either use the momentum variables[5] or pre-Feynman perturbation method.[6,7] In the present paper, we shall follow the latter procedure. For this purpose, consider the two perturbation diagrams of Fig. 1. The longitudinal and transverse components of the various momenta are[7]

$$\begin{aligned}\mathbf{p}_1&=[\beta\omega,\mathbf{p}_\perp],\\ \mathbf{p}_2&=[(1-\beta)\omega,\ -\mathbf{p}_\perp-\mathbf{r}_1],\\ \mathbf{p}_3&=[(1-\beta)\omega,\ -\mathbf{p}_\perp+\mathbf{r}_1],\\ \mathbf{p}_4&=[\beta\omega+q_3,\ \mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1],\end{aligned}\tag{3.1}$$

and

$$\mathbf{p}_5=[(1-\beta)\omega-q_3,\ -\mathbf{p}_\perp-\mathbf{q}_\perp].$$

Since each intermediate particle is on the mass shell, the corresponding energies are given approximately by

$$\begin{aligned}E_1&\sim\beta\omega+(\mathbf{p}_\perp^2+m^2)/(2\beta\omega),\\ E_2&\sim(1-\beta)\omega+[(\mathbf{p}_\perp+\mathbf{r}_1)^2+m^2]/[2(1-\beta)\omega],\\ E_3&\sim(1-\beta)\omega+[(\mathbf{p}_\perp-\mathbf{r}_1)^2+m^2]/[2(1-\beta)\omega],\\ E_4&\sim\beta\omega+q_3+[(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1)^2+m^2]/(2\beta\omega),\end{aligned}\tag{3.2}$$

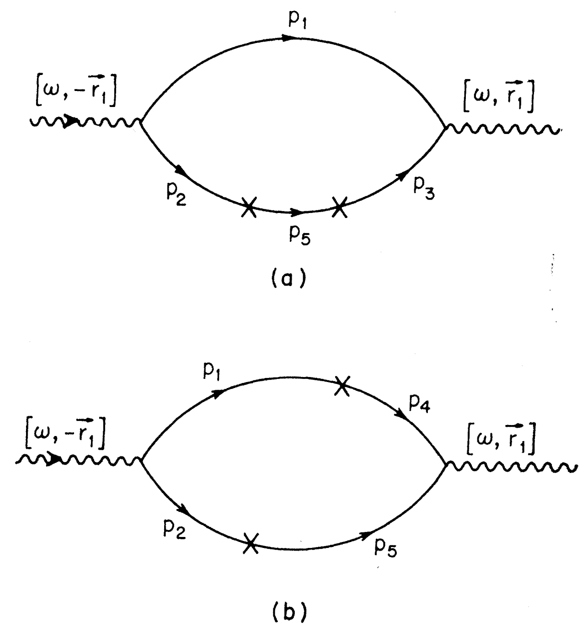

FIG. 1. Lowest-order perturbation diagrams for Delbrück scattering.

[12] See, for example, Eq. (4.12) of Ref. 5.
[13] H. Cheng and T. T. Wu, Phys. Rev. D 1, 3414 (1970).
[14] See Eq. (3) of Ref. 13.
[15] See especially Sec. 4 of Ref. 5.

and

$$E_5 \sim (1-\beta)\omega - q_3 + [(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2]/[2(1-\beta)\omega].$$

Remembering that the photon energy is

$$E_0 = (\omega^2+\mathbf{r}_1^2)^{1/2} \sim \omega + \tfrac{1}{2}\mathbf{r}_1^2/\omega\,, \tag{3.3}$$

we find that the energy denominators are

$$E_0-E_1-E_2 \sim -[(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]/[2\beta(1-\beta)\omega],$$
$$E_0-E_1-E_3 \sim -[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2]/[2\beta(1-\beta)\omega],$$
$$E_0-E_4-E_5 \sim -[(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1-\beta\mathbf{r}_1)^2+m^2]/[2\beta(1-\beta)\omega],$$

and

$$E_0-E_1-E_5 \sim q_3 - [(1-\beta)\mathbf{p}_\perp^2+\beta(\mathbf{p}_\perp+\mathbf{q}_\perp)^2 - \beta(1-\beta)\mathbf{r}_1^2+m^2]/[2\beta(1-\beta)\omega]. \tag{3.4}$$

It is important to notice that in this approximation, q_3 appears only once in these four energy denominators.

It is now straightforward to write down the Delbrück scattering amplitude at high energies as[15]

$$\mathfrak{M}_0^{(D)} \sim \omega Z^2 e^6 (2\pi^2)^{-3} \int d\mathbf{q}_\perp dq_3 d\mathbf{p}_\perp \int_0^1 d\beta [q_3^2+(\mathbf{q}_\perp-\mathbf{r}_1)^2]^{-1}[q_3^2+(\mathbf{q}_\perp+\mathbf{r}_1)^2]^{-1}$$
$$\times\left[q_3 - \frac{(1-\beta)\mathbf{p}_\perp^2+\beta(\mathbf{p}_\perp+\mathbf{q}_\perp)^2-\beta(1-\beta)\mathbf{r}_1^2+m^2}{2\beta(1-\beta)\omega} + i\epsilon\right]^{-1} A_1(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta)\,, \tag{3.5}$$

where $A_1(\mathbf{p},\mathbf{q},\mathbf{r}_1;\beta)$ is precisely the quantity in the braces of (2.2). For $\mathbf{r}_1$ not small, (2.1) follows immediately from (3.5).

In order to deal with the case of small $\mathbf{r}_1$, it is convenient to introduce

$$A_2(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta) = A_1(-\mathbf{p}_\perp-\mathbf{q}_\perp,\, \mathbf{q}_\perp,\, \mathbf{r}_1;\, 1-\beta)\,, \tag{3.6}$$

and

$$\bar{A}(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta) = \tfrac{1}{2}[A_1(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta)+A_2(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta)]. \tag{3.7}$$

It is seen from (3.5) that the three denominators explicitly given are not changed by the replacement $\mathbf{p}_\perp \to -\mathbf{p}_\perp - \mathbf{q}_\perp$ and, simultaneously $\beta \to 1-\beta$. Thus (3.5) also holds when A_1 is replaced by either A_2 or $\bar{A}$. This quantity $\bar{A}$ can be written out explicitly as

$$\bar{A} = \frac{1}{2}\left\{\frac{\delta_{ij}\beta^2\mathbf{r}_1^2+2\beta(1-\beta)(p_\perp-\beta r_1)_i(p_\perp+\beta r_1)_j}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}\right.$$
$$-\frac{\frac{1}{4}\delta_{ij}[\mathbf{q}_\perp+(1-2\beta)\mathbf{r}_1]^2+2\beta(1-\beta)(p_\perp+q_\perp+r_1-\beta r_1)_i(p_\perp+\beta r_1)_j}{[(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}$$
$$+\frac{\delta_{ij}(1-\beta)^2\mathbf{r}_1^2+2\beta(1-\beta)(p_\perp+q_\perp+r_1-\beta r_1)_i(p_\perp+q_\perp-r_1+\beta r_1)_j}{[(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1+\beta\mathbf{r}_1)^2+m^2]}$$
$$\left.-\frac{\frac{1}{4}\delta_{ij}[\mathbf{q}_\perp-(1-2\beta)\mathbf{r}_1]^2+2\beta(1-\beta)(p_\perp-\beta r_1)_i(p_\perp+q_\perp-r_1+\beta r_1)_j}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1+\beta\mathbf{r}_1)^2+m^2]}\right\}$$
$$=\tfrac{1}{8}\delta_{ij}\left\{\frac{4\beta^2\mathbf{r}_1^2}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}-\frac{[\mathbf{q}_\perp+(1-2\beta)\mathbf{r}_1]^2}{[(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2]}\right.$$
$$\left.+\frac{4(1-\beta)^2\mathbf{r}_1^2}{[(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1+\beta\mathbf{r}_1)^2+m^2]}-\frac{[\mathbf{q}_\perp-(1-2\beta)\mathbf{r}_1]^2}{[(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2][(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1+\beta\mathbf{r}_1)^2+m^2]}\right\}$$
$$+\beta(1-\beta)\left[\frac{(p_\perp-\beta r_1)_i}{(\mathbf{p}_\perp-\beta\mathbf{r}_1)^2+m^2}-\frac{(p_\perp+q_\perp+r_1-\beta r_1)_i}{(\mathbf{p}_\perp+\mathbf{q}_\perp+\mathbf{r}_1-\beta\mathbf{r}_1)^2+m^2}\right]\left[\frac{(p_\perp+\beta r_1)_j}{(\mathbf{p}_\perp+\beta\mathbf{r}_1)^2+m^2}-\frac{(p_\perp+q_\perp-r_1+\beta r_1)_j}{(\mathbf{p}_\perp+\mathbf{q}_\perp-\mathbf{r}_1+\beta\mathbf{r}_1)^2+m^2}\right]. \tag{3.8}$$

When the integration over q_3 is explicitly carried out in (3.5) with the help of the formula

$$\int_{-\infty}^{\infty} dx[(x^2+\alpha_1^2)(x^2+\alpha_2^2)(x+\alpha_3^2)]^{-1} = \frac{\pi(\alpha_1+\alpha_2+\alpha_3)}{\alpha_1\alpha_2\alpha_3(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)(\alpha_3+\alpha_1)}, \tag{3.9}$$

we get

$$\mathfrak{M}_0^{(D)}\sim -\tfrac{1}{2}i\omega Z^2e^6\pi^{-3}(2\pi)^{-2}\int d\mathbf{q}_\perp d\mathbf{p}_\perp\int_0^1 d\beta[|\mathbf{q}_\perp-\mathbf{r}_1||\mathbf{q}_\perp+\mathbf{r}_1|(|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|)]^{-1}$$

$$\times\left[|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{(1-\beta)\mathbf{p}_\perp^2+\beta(\mathbf{p}_\perp+\mathbf{q}_\perp)^2-\beta(1-\beta)\mathbf{r}_1^2+m^2}{2\beta(1-\beta)\omega}\right]\bar{A}(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta)$$

$$\times\left[|\mathbf{q}_\perp-\mathbf{r}_1|+i\frac{(1-\beta)\mathbf{p}_\perp^2+\beta(\mathbf{p}_\perp+\mathbf{q}_\perp)^2-\beta(1-\beta)\mathbf{r}_1^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}$$

$$\times\left[|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{(1-\beta)\mathbf{p}_\perp^2+\beta(\mathbf{p}_\perp+\mathbf{q}_\perp)^2-\beta(1-\beta)\mathbf{r}_1^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}. \quad (3.10)$$

In this form, we can use the approximation

$$(1-\beta)\mathbf{p}_\perp^2+\beta(\mathbf{p}_\perp+\mathbf{q}_\perp)^2-\beta(1-\beta)\mathbf{r}_1^2+m^2\sim\mathbf{p}_\perp^2+m^2, \quad (3.11)$$

and hence obtain

$$\mathfrak{M}_0^{(D)}\sim -\tfrac{1}{2}i\omega Z^2e^6\pi^{-3}(2\pi)^{-2}\int d\mathbf{q}_\perp d\mathbf{p}_\perp\int_0^1 d\beta[|\mathbf{q}_\perp-\mathbf{r}_1||\mathbf{q}_\perp+\mathbf{r}_1|(|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|)]^{-1}$$

$$\times\left[|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]\left[|\mathbf{q}_\perp-\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}$$

$$\times\left[|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}\bar{A}(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta). \quad (3.12)$$

In writing down (3.11) and hence (3.12), some of the terms with $\mathbf{q}_\perp$ and $\mathbf{r}_1$ have been deleted. These terms can be seen to be unimportant for $\mathbf{q}_\perp$ and $\mathbf{r}_1$ either of order m or of order m^2/ω. It is important to note, however, that the deletion of these terms would be incorrect if either A_1 or A_2 had been used instead of $\bar{A}$. This is the reason for introducing $\bar{A}$. Equation (3.12) is the expression that we shall study in the next two sections.

Equation (3.12) gives the high-energy behavior of the Delbrück scattering amplitude for all finite values of the momentum transfer $\mathbf{\Delta}=2\mathbf{r}_1$. This expression simplifies when $\mathbf{r}_1$ is assumed to be small. In particular, when $\mathbf{r}_1$ and $\mathbf{q}_\perp$ are both small, the $\bar{A}$ of (3.8) is approximately

$$\bar{A}\sim\tfrac{1}{4}\delta_{ij}\frac{\mathbf{r}_1^2-\mathbf{q}_\perp^2}{(\mathbf{p}_\perp^2+m^2)^2}+\beta(1-\beta)\frac{\{2[\mathbf{p}_\perp\cdot(\mathbf{q}_\perp+\mathbf{r}_1)]p_{\perp i}-(\mathbf{p}_\perp^2+m^2)(q_\perp+r_1)_i\}\{2[\mathbf{p}_\perp\cdot(\mathbf{q}_\perp-\mathbf{r}_1)]p_{\perp j}-(\mathbf{p}_\perp+m^2)(q_\perp-r_1)_j\}}{(\mathbf{p}_\perp^2+m^2)^4}. \quad (3.13)$$

Note that this expression is zero when $\mathbf{q}_\perp=\pm\mathbf{r}_1$.

4. SPECIAL CASE OF FORWARD SCATTERING

Before launching into the calculation for Δ of the order ω^{-1}, we first consider the special case $\Delta=0$ to see how the known result (1.2) can be derived from the present considerations. For this special case, it follows from (3.8) and (3.12) that

$$\mathfrak{M}_0^{(D)}|_{\Delta=0}\sim -\tfrac{1}{4}i\omega Z^2e^6\pi^{-3}(2\pi)^{-2}\int d\mathbf{q}_\perp d\mathbf{p}_\perp\int_0^1 d\beta|\mathbf{q}_\perp|^{-3}\left[2|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]\left[|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-2}$$

$$\times\left\{-\tfrac{1}{4}\delta_{ij}\frac{\mathbf{q}_\perp^2}{[\mathbf{p}_\perp^2+m^2][(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2]}+\beta(1-\beta)\left[\frac{p_{\perp i}}{\mathbf{p}_\perp^2+m^2}-\frac{(p_\perp+q_\perp)_i}{(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2}\right]\left[\frac{p_{\perp j}}{\mathbf{p}_\perp^2+m^2}-\frac{(p_\perp+q_\perp)_j}{(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2}\right]\right\}. \quad (4.1)$$

We carry out the integration in (4.1) by dividing the region of integration into two pieces:

$$\mathfrak{M}_0^{(D)}|_{\Delta=0}\sim -\tfrac{1}{4}i\omega Z^2e^6\pi^{-3}(2\pi)^{-2}(I_{10}+I_{20}), \quad (4.2)$$

where I_{10} and I_{20} are given by the same integral as that of (4.1) except that the $\mathbf{q}_\perp$ integration is restricted, respectively, to $|\mathbf{q}_\perp|<\delta$ and $|\mathbf{q}_\perp|>\delta$. Here the quantity δ is chosen to satisfy

$$m\gg\delta\gg m^2/\omega. \quad (4.3)$$

For these two regions of integration, different approximations can be carried out. Thus

$$I_{10}=\int_{|q_\perp|<\delta} d\mathbf{q}_\perp d\mathbf{p}_\perp \int_0^1 d\beta |\mathbf{q}_\perp|^{-3}\left[2|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]\left[|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-2}$$
$$\times\left\{-\tfrac{1}{4}\delta_{ij}\frac{\mathbf{q}_\perp^2}{(\mathbf{p}_\perp^2+m^2)^2}+\beta(1-\beta)\frac{[2(\mathbf{p}_\perp\cdot\mathbf{q}_\perp)p_{\perp i}-(\mathbf{p}_\perp^2+m^2)q_{\perp i}][2(\mathbf{p}_\perp\cdot\mathbf{q}_\perp)p_{\perp j}-(\mathbf{p}_\perp^2+m^2)q_{\perp j}]}{(\mathbf{p}_\perp^2+m^2)^4}\right\} \quad (4.4)$$

and

$$I_{20}=2\int_{|q_\perp|>\delta} d\mathbf{q}_\perp d\mathbf{p}_\perp \int_0^1 d\beta (\mathbf{q}_\perp^2)^{-2}\left\{-\tfrac{1}{4}\delta_{ij}\frac{\mathbf{q}_\perp^2}{[\mathbf{p}_\perp^2+m^2][(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2]}\right.$$
$$\left.+\beta(1-\beta)\left[\frac{p_{\perp i}}{\mathbf{p}_\perp^2+m^2}-\frac{(p_\perp+q_\perp)_i}{(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2}\right]\left[\frac{p_{\perp j}}{\mathbf{p}_\perp^2+m^2}-\frac{(p_\perp+q_\perp)_j}{(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2}\right]\right\}. \quad (4.5)$$

We proceed to carry out these integrations for a δ satisfying (4.3). As a first step, because of the rotational invariance in the exactly forward direction, both I_{10} and I_{20} must be proportional to δ_{ij}:

$$I_{10}=\delta_{ij}\int_{|q_\perp|<\delta} d\mathbf{q}_\perp d\mathbf{p}_\perp \int_0^1 d\beta |\mathbf{q}_\perp|^{-3}\left[2|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]\left[|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-2}$$
$$\times\left[-\frac{1}{4}\frac{\mathbf{q}_\perp^2}{(\mathbf{p}_\perp^2+m^2)^2}+\tfrac{1}{2}\beta(1-\beta)\frac{(\mathbf{p}_\perp^2+m^2)^2\mathbf{q}_\perp^2-4m^2(\mathbf{p}_\perp\cdot\mathbf{q}_\perp)^2}{(\mathbf{p}_\perp^2+m^2)^4}\right], \quad (4.6)$$

and

$$I_{20}=2\delta_{ij}\int_{|q_\perp|>\delta} d\mathbf{q}_\perp d\mathbf{p}_\perp (\mathbf{q}^2)^{-2}\left[-\frac{1}{4}\frac{\mathbf{q}_\perp^2}{(\mathbf{p}_\perp^2+m^2)[(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2]}+\frac{1}{6}\frac{\mathbf{p}_\perp^2}{(\mathbf{p}_\perp^2+m^2)^2}-\frac{1}{6}\frac{\mathbf{p}_\perp\cdot(\mathbf{p}_\perp+\mathbf{q}_\perp)}{(\mathbf{p}_\perp^2+m^2)[(\mathbf{p}_\perp+\mathbf{q}_\perp)^2+m^2]}\right]. \quad (4.7)$$

Consider I_{10} first. Let

$$y=|\mathbf{q}_\perp|; \quad (4.8)$$

then averaging over the direction of $\mathbf{q}_\perp$ yields

$$I_{10}=2\pi\delta_{ij}\int d\mathbf{p}_\perp \int_0^1 d\beta \int_0^\delta dy \left[2y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-2}$$
$$\times\left[-\frac{1}{4}\frac{1}{(\mathbf{p}_\perp^2+m^2)^2}+\tfrac{1}{2}\beta(1-\beta)\frac{(\mathbf{p}_\perp^2+m^2)^2-2m^2\mathbf{p}_\perp^2}{(\mathbf{p}_\perp^2+m^2)^4}\right]. \quad (4.9)$$

For $\delta\gg m^2/\omega$, the y integration can be carried out simply:

$$I_{10}\sim 2\pi\delta_{ij}\int d\mathbf{p}_\perp \int_0^1 d\beta\left[2\ln\frac{2\beta(1-\beta)\omega\delta}{\mathbf{p}_\perp^2+m^2}-1-\pi i\right]\left[-\frac{1}{4}\frac{1}{(\mathbf{p}_\perp^2+m^2)^2}+\tfrac{1}{2}\beta(1-\beta)\frac{(\mathbf{p}_\perp^2+m^2)^2-2m^2\mathbf{p}_\perp^2}{(\mathbf{p}_\perp^2+m^2)^4}\right], \quad (4.10)$$

which gives immediately

$$I_{10}\sim -2\pi^2 m^{-2}\delta_{ij}\{(7/36)[2\ln(2\omega\delta/m^2)-1-\pi i]-43/36\}. \quad (4.11)$$

Attention is now turned to the I_{20} of (4.7). After introducing a Feynman parameter x, the $\mathbf{p}_\perp$ integration can be carried out[16]:

$$I_{20}=\tfrac{1}{3}\pi\delta_{ij}\int_0^1 dx\int_{|q_\perp|>\delta} d\mathbf{q}_\perp(\mathbf{q}_\perp^2)^{-1}\times(-1-x+x^2)[x(1-x)\mathbf{q}_\perp^2+m^2]^{-1}. \quad (4.12)$$

For $\delta\ll m$, integrations over $\mathbf{q}_\perp$ and then x give

$$I_{20}\sim\tfrac{1}{3}m^{-2}\pi^2\delta_{ij}\int_0^1 dx(-1-x+x^2)\ln\{m^2/[x(1-x)\delta^2]\}$$
$$=\tfrac{1}{3}m^{-2}\pi^2\delta_{ij}[-(7/3)\ln(m/\delta)-41/18]. \quad (4.13)$$

When (4.11) is added to (4.13), the sum is

$$I_{10}+I_{20}\sim -\pi^2 m^{-2}\delta_{ij}\{(7/9)[\ln(2\omega/m)-\tfrac{1}{2}i\pi]-109/54\}, \quad (4.14)$$

where as expected δ does not appear. Finally, the substitution of (4.14) into (4.2) gives the known result (1.2).

[16] Compare Eq. (3.4) of Ref. 2.

5. DELBRÜCK SCATTERING WITH SMALL MOMENTUM TRANSFER

We are now ready to study Delbrück scattering near the forward direction, i.e., Delbrück scattering with a momentum transfer of the order of m^2/ω. More specifically, we proceed to generalize the considerations of Sec. 4 to this case of small but nonzero momentum transfer.

For this purpose we return to (3.12) and write, analogous to (4.2),

$$\mathfrak{M}_0^{(D)} \sim -\tfrac{1}{4} i\omega Z^2 e^6 \pi^{-3}(2\pi)^{-2}(I_1+I_2)\,, \tag{5.1}$$

where I_1 and I_2, similar to I_{10} and I_{20}, are the contributions from the regions $|\mathbf{q}_\perp|<\delta$ and $|\mathbf{q}_\perp|>\delta$, respectively. Since

$$|\mathbf{r}_1| \ll m \tag{5.2}$$

and (4.3) holds, we have

$$I_2 = I_{20}\,, \tag{5.3}$$

which is explicitly given by (4.13). It is therefore sufficient to concentrate on I_1, which is the integral, from (3.12) and (3.13),

$$I_1 = 2\int_{|\mathbf{q}_\perp|<\delta} d\mathbf{q}_\perp d\mathbf{p}_\perp \int_0^1 d\beta \big[|\mathbf{q}_\perp-\mathbf{r}_1||\mathbf{q}_\perp+\mathbf{r}_1|(|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|)\big]^{-1}\left[|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]$$
$$\times\left[|\mathbf{q}_\perp-\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}\left[|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}\left[\frac{1}{4}\delta_{ij}\frac{\mathbf{r}_1^2-\mathbf{q}_\perp^2}{(\mathbf{p}_\perp^2+m^2)^2}\right.$$
$$\left.+\beta(1-\beta)\frac{\{2[\mathbf{p}_\perp\cdot(\mathbf{q}_\perp+\mathbf{r}_1)]p_{\perp i}-(\mathbf{p}_\perp^2+m^2)(q_\perp+r_1)_i\}\{2[\mathbf{p}_\perp\cdot(\mathbf{q}_\perp-\mathbf{r}_1)]p_{\perp j}-(\mathbf{p}_\perp^2+m^2)(q_\perp-r_1)_j\}}{(\mathbf{p}_\perp^2+m^2)^4}\right]. \tag{5.4}$$

The first step in the reduction of this complicated integral is to use

$$\big[|\mathbf{q}_\perp-\mathbf{r}_1||\mathbf{q}_\perp+\mathbf{r}_1|(|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|)\big]^{-1}\left[|\mathbf{q}_\perp-\mathbf{r}_1|+|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]$$
$$\times\left[|\mathbf{q}_\perp-\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}\left[|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}$$
$$=\big[(\mathbf{q}_\perp+\mathbf{r}_1)^2-(\mathbf{q}_\perp-\mathbf{r}_1)^2\big]^{-1}\left\{|\mathbf{q}_\perp-\mathbf{r}_1|^{-1}\left[|\mathbf{q}_\perp-\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}-|\mathbf{q}_\perp+\mathbf{r}_1|^{-1}\left[|\mathbf{q}_\perp+\mathbf{r}_1|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}\right\}. \tag{5.5}$$

It is therefore desirable to change to the variables $\mathbf{q}_\perp\pm\mathbf{r}_1$. With due care in the changes of variables, we get

$$I_1 = I_{11}+I_{12}\,, \tag{5.6}$$

where

$$I_{11} = -\pi\int d\mathbf{p}_\perp\int_0^1 d\beta\{-\tfrac{1}{2}\delta_{ij}(\mathbf{p}_\perp^2+m^2)^{-2}+\beta(1-\beta)(\mathbf{p}_\perp^2+m^2)^{-4}[(\mathbf{p}_\perp^2+m^2)^2\delta_{ij}-4m^2p_{\perp i}p_{\perp j}]\} \tag{5.7}$$

and

$$I_{12} = 4\int_{|\mathbf{q}_\perp|<\delta} d\mathbf{q}_\perp d\mathbf{p}_\perp\int_0^1 d\beta|\mathbf{q}_\perp|^{-1}\left[|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}\big[(\mathbf{q}_\perp+2\mathbf{r}_1)^2-\mathbf{q}_\perp^2\big]^{-1}$$
$$\times\left[\frac{1}{4}\delta_{ij}\frac{\mathbf{r}_1^2-(\mathbf{q}_\perp+\mathbf{r}_1)^2}{(\mathbf{p}_\perp^2+m^2)^2}+\beta(1-\beta)\frac{\{2[\mathbf{p}_\perp\cdot(\mathbf{q}_\perp+2\mathbf{r}_1)]p_{\perp i}-(\mathbf{p}_\perp^2+m^2)(q_\perp+2r_1)_i\}[2(\mathbf{p}_\perp\cdot\mathbf{q}_\perp)p_{\perp j}-(\mathbf{p}_\perp^2+m^2)q_{\perp j}]}{(\mathbf{p}_\perp^2+m^2)^4}\right]. \tag{5.8}$$

In (5.8), principal values must be taken along $|\mathbf{q}_\perp+2\mathbf{r}_1|=|\mathbf{q}_\perp|$. Since

$$I_{11} = 7\pi^2 m^{-2}\delta_{ij}/18\,, \tag{5.9}$$

we concentrate on I_{12} of (5.8).

The integration over the angular variable of $\mathbf{p}_\perp$ is the easiest one to carry out:

$$I_{12} = \int_{|\mathbf{q}_\perp|<\delta} d\mathbf{q}_\perp d\mathbf{p}_\perp\int_0^1 d\beta|\mathbf{q}_\perp|^{-1}\left[|\mathbf{q}_\perp|+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}\big[\mathbf{r}_1\cdot\mathbf{q}_\perp+\mathbf{r}_1^2\big]^{-1}$$
$$\times\left[-\tfrac{1}{4}\delta_{ij}\frac{\mathbf{q}_\perp\cdot(\mathbf{q}_\perp+2\mathbf{r}_1)}{(\mathbf{p}_\perp^2+m^2)^4}+\beta(1-\beta)\frac{\tfrac{1}{2}(\mathbf{p}_\perp^2)^2[\mathbf{q}_\perp\cdot(\mathbf{q}_\perp+2\mathbf{r}_1)]\delta_{ij}+m^4(q_\perp+2r_1)_iq_{\perp j}}{(\mathbf{p}_\perp^2+m^2)^4}\right]. \tag{5.10}$$

In the last numerator, we have omitted a term $q_{\perp i}r_{1j}-r_{1i}q_{\perp j}$, because such a term cannot contribute after integration. Let us use the variable $y=|\mathbf{q}_\perp|$ of (4.8) together with an angular variable defined by

$$\mathbf{q}_\perp\cdot\mathbf{r}_1=yr_1\cos\theta,$$

when $r_1=|\mathbf{r}_1|$; then

$$I_{12\parallel}=\int d\mathbf{p}_\perp\int_0^\delta y\,dy\int_{-\pi}^{\pi}d\theta\int_0^1 d\beta\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}r_1^{-1}(y\cos\theta+r_1)^{-1}$$
$$\times\left[-\frac{1}{4}\frac{y+2r_1\cos\theta}{(\mathbf{p}_\perp^2+m^2)^2}+\beta(1-\beta)\frac{\frac{1}{2}(\mathbf{p}_\perp^2)^2(y+2r_1\cos\theta)+m^4(y\cos\theta+2r_1)\cos\theta}{(\mathbf{p}_\perp^2+m^2)^4}\right]\quad(5.11)$$

and

$$I_{12\perp}=\int d\mathbf{p}_\perp\int_0^\delta y\,dy\int_{-\pi}^{\pi}d\theta\int_0^1 d\beta\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}r_1^{-1}(y\cos\theta+r_1)^{-1}$$
$$\times\left[-\frac{1}{4}\frac{y+2r_1\cos\theta}{(\mathbf{p}_\perp^2+m^2)^2}+\beta(1-\beta)\frac{\frac{1}{2}(\mathbf{p}_\perp^2)^2(y+2r_1\cos\theta)+m^4y\sin^2\theta}{(\mathbf{p}_\perp^2+m^2)^4}\right],\quad(5.12)$$

where $I_{12\parallel}$ and $I_{12\perp}$ are the values of I_{12} when the photon is linearly polarized in the scattering plane and perpendicular to the scattering plane, respectively. It is somewhat simpler to deal with the sum and difference of (5.11) and (5.12). Define

$$I^\pm=\tfrac{1}{2}(I_{12\perp}\pm I_{12\parallel});\quad(5.13)$$

then

$$I^+=\frac{1}{2}\int d\mathbf{p}_\perp\int_0^\delta y\,dy\int_{-\pi}^{\pi}d\theta\int_0^1 d\beta\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}$$
$$\times r_1^{-1}\frac{y+2r_1\cos\theta}{y\cos\theta+r_1}\{-\tfrac{1}{2}(\mathbf{p}_\perp^2+m^2)^{-2}+\beta(1-\beta)$$
$$\times[(\mathbf{p}_\perp^2)^2+m^4](\mathbf{p}_\perp^2+m^2)^{-4}\},\quad(5.14)$$

and

$$I^-=\frac{1}{2}\int d\mathbf{p}_\perp\int_0^\delta y^2dy\int_0^{\pi}d\theta\int_0^1 d\beta\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}$$
$$\times r_1^{-1}\beta(1-\beta)(\mathbf{p}_\perp^2+m^2)^{-4}m^4(y\cos\theta+r_1)^{-1}.\quad(5.15)$$

Note that I^- is known to be zero for $r_1\to 0$.

The next step is to integrate over θ. By using principal values, we have

$$\int_{-\pi}^{\pi}d\theta(y\cos\theta+r_1)^{-1}=\begin{cases}2\pi(r_1^2-y^2)^{-1/2} & \text{for } y<r_1\\ 0 & \text{for } y>r_1.\end{cases}\quad(5.16)$$

Accordingly,

$$I^-=\pi m^4\int d\mathbf{p}_\perp\int_0^{r_1}y^2dy(r_1^2-y^2)^{-1/2}$$
$$\times\int_0^1 d\beta\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}$$
$$\times r_1^{-1}\beta(1-\beta)(\mathbf{p}_\perp^2+m^2)^{-4},\quad(5.17)$$

while

$$I^+=I_1^++I_2^+,\quad(5.18)$$

where

$$I_1^+=-\pi\int d\mathbf{p}_\perp\int_0^{r_1}dy(2r_1^2-y^2)(r_1^2-y^2)^{-1/2}$$
$$\times\int_0^1 d\beta\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}r_1^{-1}\{-\tfrac{1}{2}(\mathbf{p}_\perp^2+m^2)^{-2}$$
$$+\beta(1-\beta)[(\mathbf{p}_\perp^2)^2+m^4](\mathbf{p}_\perp^2+m^2)^{-4}\},\quad(5.19)$$

and

$$I_2^+=2\pi\int d\mathbf{p}_\perp\int_0^\delta dy\int_0^1 d\beta\left[y+i\frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)\omega}\right]^{-1}$$
$$\times\{-\tfrac{1}{2}(\mathbf{p}_\perp^2+m^2)+\beta(1-\beta)[(\mathbf{p}_\perp^2)^2+m^2](\mathbf{p}_\perp^2+m^2)^{-4}\}$$
$$\sim-2\pi^2m^{-2}\left[\frac{7}{36}\left(2\ln\frac{2\omega\delta}{m^2}-\pi i\right)-\frac{43}{36}\right].\quad(5.20)$$

Note that I^- and I_1^+ are both independent of δ. A comparison of (5.20) and (5.9) with (4.11) shows that

$$I_{10}\sim I_{11}+I_2^+.\quad(5.21)$$

We therefore obtain the result that

$$\mathfrak{M}_0^{(D)}-\mathfrak{M}_0^{(D)}|_{\Delta=0}\sim-\tfrac{1}{4}i\omega Z^2e^6\pi^{-3}(2\pi)^{-2}\times(I_1^+\pm I^-),\quad(5.22)$$

where I_1^+ and I^- are given by (5.19) and (5.17), and the upper (lower) sign should be used when the photon is linearly polarized perpendicular to the scattering plane (in the scattering plane). Equation (5.22) is the desired result.

6. PROPERTIES OF SCATTERING AMPLITUDE

It is clear that both I_1^+ and I^- are functions of the variable

$$R_1=r_1\omega/m^2,\quad(6.1)$$

not of r_1 and ω separately. In terms of R_1,

$$I^- = \pi m^4 \int d\mathbf{p}_\perp \int_0^1 y^2 dy (1-y^2)^{-1/2} \times \int_0^1 d\beta \left[y + i \frac{\mathbf{p}_\perp^2 + m^2}{2\beta(1-\beta)m^2 R_1} \right]^{-1} \times \beta(1-\beta)(\mathbf{p}_\perp^2+m^2)^{-4}, \quad (6.2)$$

and

$$I_1^+ = -\pi \int d\mathbf{p}_\perp \int_0^1 dy(2-y^2)(1-y^2)^{-1/2} \times \int_0^1 d\beta \left[y + i \frac{\mathbf{p}_\perp^2+m^2}{2\beta(1-\beta)m^2R_1} \right]^{-1} \{ -\tfrac{1}{2}(\mathbf{p}_\perp^2+m^2)^{-2} + \beta(1-\beta)[(\mathbf{p}_\perp^2)^2+m^4](\mathbf{p}_\perp^2+m^2)^{-4} \}. \quad (6.3)$$

Clearly, as $R_1 \to 0$, both I^- and I_1^+ approach zero.

Next consider the limit $R_1 \to \infty$. In this limit

$$I^- \sim \pi^2 m^{-2}/18, \quad (6.4)$$

while the more complicated behavior of I_1^+ is given by

$$I_1^+ \sim -\pi \int d\mathbf{p}_\perp \int_0^1 d\beta \left[2 \ln \frac{4\beta(1-\beta)m^2R_1}{\mathbf{p}_\perp^2+m^2} - 1 - i\pi \right] \times \{ -\tfrac{1}{2}(\mathbf{p}_\perp^2+m^2)^{-2} + \beta(1-\beta) \times [(\mathbf{p}_\perp^2)^2+m^4](\mathbf{p}_\perp^2+m^2)^{-4} \} = 2\pi^2 m^{-2} \left[\frac{7}{36}(2\ln 4R_1 - 1 - \pi i) - \frac{43}{36} \right]. \quad (6.5)$$

The substitution of (6.4) and (6.5) into (5.22) yields, together with (1.2),

$$\mathfrak{M}_{01}{}^{(D)} \sim 4i\alpha^3 Z^2 \omega m^{-2}(7/9)[\ln(m/\Delta)+19/21]$$

and (6.6)

$$\mathfrak{M}_{011}{}^{(D)} \sim 4i\alpha^3 Z^2 \omega m^{-2}(7/9)[\ln(m/\Delta)+22/21]$$

for $m \gg \Delta \gg m^2/\omega$. These expressions are in complete agreement with those previously obtained[17] from the impact factor representation. Therefore (5.22) indeed joins correctly the impact factor result (2.21) to the known expression (1.2) for the exactly forward direction.

There are many equivalent ways of writing (6.2) and (6.3). We shall mention only a few. Let

$$x = (\mathbf{p}_\perp^2+m^2)/m^2; \quad (6.7)$$

then

$$I^- = \pi^2 m^{-2} \int_1^\infty dx \int_0^1 y^2 dy (1-y^2)^{-1/2} \times \int_0^1 d\beta \{ y + ix/[2\beta(1-\beta)R_1] \}^{-1} x^{-4} \beta(1-\beta) \quad (6.8)$$

and

$$I_1^+ = -\pi^2 m^{-2} \int_1^\infty dx \int_0^1 dy (2-y^2)(1-y^2)^{-1/2} \times \int_0^1 d\beta \{ y + ix/[2\beta(1-\beta)R_1] \}^{-1} \times [-\tfrac{1}{2}x^{-2} + x^{-4}\beta(1-\beta)(x^2-2x+2)]. \quad (6.9)$$

Let us assume for the time being that

$$R_1 < 2. \quad (6.10)$$

Then

$$\{ y + ix/[2\beta(1-\beta)R_1] \}^{-1} = -\sum_{n=1}^\infty (2iR_1)^n x^{-n} [\beta(1-\beta)]^n y^{n-1}. \quad (6.11)$$

The substitution of (6.11) into (6.8) gives

$$I^- = -\tfrac{1}{4}\pi^{5/2} m^{-2} \sum_{n=1}^\infty (2iR_1)^n \frac{\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2}n+\frac{5}{2})} \frac{[\Gamma(n+2)]^2}{\Gamma(2n+4)}. \quad (6.12)$$

When the Legendre duplication formula for gamma functions is used in (6.12), I^- can be alternatively expressed as

$$I^- = -(\sqrt{2}/64)\pi^3 m^{-2} \sum_{n=1}^\infty (\tfrac{1}{2}iR_1)^n \times \frac{[\Gamma(\frac{1}{2}n+1)]^2 \Gamma(\frac{1}{2}n+\frac{3}{2})}{\Gamma(\frac{1}{2}n+\frac{5}{2})\Gamma(\frac{1}{2}n+5/4)\Gamma(\frac{1}{2}n+7/4)}. \quad (6.13)$$

Thus the real and imaginary parts of I^- are, respectively,

$$\mathrm{Re}I^- = (\sqrt{2}/256)\pi^3 m^{-2} R_1^2 \sum_{n=0}^\infty (-\tfrac{1}{4}R_1^2)^n \times \frac{[\Gamma(n+2)]^2 \Gamma(n+\frac{5}{2})}{\Gamma(n+\frac{7}{2})\Gamma(n+9/4)\Gamma(n+11/4)} \quad (6.14)$$

and

$$\mathrm{Im}I^- = -(\sqrt{2}/128)\pi^3 m^{-2} R_1 \sum_{n=0}^\infty (-\tfrac{1}{4}R_1^2)^n \times \frac{[\Gamma(n+\frac{3}{2})]^2 \Gamma(n+2)}{\Gamma(n+3)\Gamma(n+7/4)\Gamma(n+9/4)}. \quad (6.15)$$

These expressions can be written in terms of generalized hypergeometric functions as follows:

$$\mathrm{Re}I^- = (2/525)\pi^2 m^{-2} R_1^2 \times {}_4F_3 \left[\begin{matrix} 1, & 2, & 2, & 5/2; & -\tfrac{1}{4}R_1^2 \\ 7/2, & 9/4, & 11/4 & & \end{matrix} \right] \quad (6.16)$$

[17] See Eqs. (4.7) and (4.8) of Ref. 4.

and[18]

$$\mathrm{Im}I^- = -(1/240)\pi^3 m^{-3} R_1 \times {}_4F_3\left[\begin{matrix}1, & 3/2, & 3/2, & 2; & -\frac{1}{4}R_1{}^2 \\ 3, & 7/4, & 9/4 & & \end{matrix}\right]. \quad (6.17)$$

Equations (6.16) and (6.17) hold even without the restriction (6.10). Similar results can be written down for $I_1{}^+$.

We conclude with the following remark on the Mellin transforms for I^- and $I_1{}^+$. Generalized hypergeometric functions all have simple Mellin transforms. In particular, as seen most easily through the Meijer G function,[19] the Mellin transform is a quotient of products of gamma functions. This provides an alternative way of deriving asymptotic behaviors (6.4) and (6.5). This alternative method has the great advantage of giving as many terms as one wants for the behavior of I^- and $I_1{}^+$ when R_1 is large. The details of some of the calculation along this line are to be found in Appendix A.

In Appendices B and C we study $I_1{}^+$ and I^- in a somewhat different way. The integral representation for I^- and $I_1{}^+$ as given by (6.8) and (6.9) are first reduced to double integrals in Appendix B. Then one of the two integrations in the double integral is carried out in Appendix C so that both I^- and $I_1{}^+$ are expressed as single integrals.

7. HIGHER-ORDER CORRECTIONS

There is one important difference between Delbrück scattering on the one hand and Compton scattering and photon-photon scattering on the other, namely, there are two parameters in the case of Delbrück scattering. The perturbation expansion for the Delbrück scattering amplitude is a double series in the two variables α and $Z\alpha$, where the lowest-order term obtained in the three preceding sections is of the order $\alpha(Z\alpha)^2$. Physically α is quite small, but $Z\alpha$ need not be; in fact, $Z\alpha$ for uranium is about $\frac{2}{3}$. It is therefore useful to sum over all terms of the form $\alpha(Z\alpha)^n$ for Delbrück scattering. For Compton scattering and photon-photon scattering, no corresponding summation is physically meaningful. It is for this reason that, for the case of fixed $\Delta\neq 0$, higher-order terms for Delbrück scattering only are treated in the original paper.[1] We extend here that consideration to small or zero momentum transfer.

As given in Ref. 1, the inclusion of all the higher-order terms in $Z\alpha$ with the restriction to lowest order in α changes the impact factor representation only slightly. Explicitly, (2.1) is modified to[1,2]

$$\mathfrak{M}^{(D)} \sim i\omega Z^2 e^2 (2\pi)^{-2} \times \int d\mathbf{q}_\perp \frac{\mathcal{g}_{ij}{}^\gamma(\frac{1}{2}\boldsymbol{\Delta},\mathbf{q}_\perp)}{[(\mathbf{q}_\perp+\frac{1}{2}\boldsymbol{\Delta})^2]^{1-iZ\alpha}[(\mathbf{q}_\perp)-\frac{1}{2}\boldsymbol{\Delta}^2]^{1+iZ_\alpha}}. \quad (7.1)$$

Let us study the difference (7.1) and (2.1), which is the correction due to higher-order terms in $Z\alpha$,

$$\delta\mathfrak{M}^{(D)} = \mathfrak{M}^{(D)} - \mathfrak{M}_0{}^{(D)} \sim i\omega Z^2 e^2 (2\pi)^{-2} \int d\mathbf{q}_\perp \mathcal{g}_{ij}{}^\gamma(\tfrac{1}{2}\boldsymbol{\Delta},\mathbf{q}_\perp) \times \{[(\mathbf{q}_\perp+\tfrac{1}{2}\boldsymbol{\Delta})^2]^{-1+iZ\alpha}[(\mathbf{q}_\perp-\tfrac{1}{2}\boldsymbol{\Delta})^2]^{-1-iZ\alpha} - [(\mathbf{q}_\perp+\tfrac{1}{2}\boldsymbol{\Delta})^2]^{-1}[(\mathbf{q}_\perp-\tfrac{1}{2}\boldsymbol{\Delta})^2]^{-1}\}. \quad (7.2)$$

Note that the integrand is zero if $\boldsymbol{\Delta}$ is formally set equal to zero. We shall calculate the limit of $\delta\mathfrak{M}^{(D)}$ as $\Delta\to 0$.

When Δ is small, the contribution to $\delta\mathfrak{M}^{(D)}$ comes entirely from the region where $\mathbf{q}_\perp$ is also small. Since, by definition of A_1 and (3.7),

$$\mathcal{g}_{ij}{}^\gamma(\mathbf{r}_1,\mathbf{q}_\perp) = -\tfrac{1}{2}\pi^{-3}e^4 \int d\mathbf{p}_\perp \int_0^1 d\beta\, A_1(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta) = -\tfrac{1}{2}\pi^{-3}e^4 \int d\mathbf{p}_\perp \int_0^1 d\beta\, \bar{A}(\mathbf{p}_\perp,\mathbf{q}_\perp,\mathbf{r}_1;\beta), \quad (7.3)$$

we can use the approximation (3.13) for the present purpose. Accordingly,

$$\lim_{\Delta\to 0} \delta\mathfrak{M}^{(D)} = -\tfrac{1}{2}i\omega Z^2 e^6 \pi^{-3}(2\pi)^{-2} \int d\mathbf{q}_\perp d\mathbf{p}_\perp \int_0^1 d\beta \times \left[\frac{1}{4}\delta_{ij}\frac{\mathbf{r}_1{}^2-\mathbf{q}_\perp{}^2}{(\mathbf{p}^2+m^2)^2} + \beta(1-\beta)\frac{\{2[\mathbf{p}_\perp\cdot(\mathbf{q}_\perp+\mathbf{r}_1)]p_{\perp i}-(\mathbf{p}_\perp{}^2+m^2)(q_\perp+r_1)_i\}\{2[\mathbf{p}_\perp\cdot(\mathbf{q}_\perp-\mathbf{r}_1)]p_{\perp j}-(\mathbf{p}_\perp{}^2+m^2)(q_\perp-r_1)_j\}}{(\mathbf{p}_\perp{}^2+m^2)^4}\right] \times \{[(\mathbf{q}_\perp+\mathbf{r}_1)^2]^{-1+iZ\alpha}[(\mathbf{q}_\perp-\mathbf{r}_1)^2]^{-1-iZ\alpha} - [(\mathbf{q}_\perp+\mathbf{r}_1)^2]^{-1}[(\mathbf{q}_\perp-\mathbf{r}_1)^2]^{-1}\}. \quad (7.4)$$

In this form, the $\mathbf{p}_\perp$ integration and the β integration can be easily carried out to give

$$\lim_{\Delta\to 0} \delta\mathfrak{M}^{(D)} = -\tfrac{1}{2}i\omega Z^2 e^6 \pi^{-2} m^{-2}(2\pi)^{-2} \int d\mathbf{q}_\perp [(2/9)\delta_{ij}(\mathbf{r}_1{}^2-\mathbf{q}_\perp{}^2) - (1/18)(r_{1i}r_{1j}-q_{\perp i}q_{\perp j})] \times \{[(\mathbf{q}_\perp+\mathbf{r}_1)^2]^{-1+iZ\alpha}[(\mathbf{q}_\perp-\mathbf{r}_1)^2]^{-1-iZ\alpha} - [(\mathbf{q}_\perp+\mathbf{r}_1)^2]^{-1}[(\mathbf{q}_\perp-\mathbf{r}_1)^2]^{-1}\}. \quad (7.5)$$

[18] This ImI^- is actually an elliptic integral. See Appendix C.

[19] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, Chaps. IV and V.

In order to carry out the $\mathbf{q}_\perp$ integration, we use a generalization of the Feynman representation

$$[(\mathbf{q}_\perp+\mathbf{r}_1)^2]^{-1+iZ\alpha}[(\mathbf{q}_\perp-\mathbf{r}_1)^2]^{-1-iZ\alpha}=(\pi Z\alpha)^{-1}\sinh(\pi Z\alpha)\int_0^1 dx\, x^{-iZ\alpha}(1-x)^{iZ\alpha}[\mathbf{q}_\perp^2-2(1-2x)\mathbf{r}_1\cdot\mathbf{q}_\perp+\mathbf{r}_1^2]^{-2}. \quad (7.6)$$

The substitution of (7.6) into (7.5) gives, after some rearrangement to avoid divergence,

$$\lim_{\Delta\to 0}\delta\mathfrak{M}^{(D)}=-\tfrac{1}{2}i\omega Z^2e^6\pi^{-2}m^{-2}(2\pi)^{-2}\Bigg[(\pi Z\alpha)^{-1}\sinh(\pi Z\alpha)\int_0^1 dx\, x^{-iZ\alpha}(1-x)^{iZ\alpha}$$
$$\times\int d\mathbf{q}_\perp\{[\mathbf{q}_\perp^2+4x(1-x)\mathbf{r}_1^2]^{-2}-[\mathbf{q}_\perp^2+\mathbf{r}_1^2]^{-2}\}\{(2/9)\delta_{ij}[4x(1-x)\mathbf{r}_1^2-\mathbf{q}_\perp^2]$$
$$-(1/18)[4x(1-x)r_{1i}r_{1j}-q_{\perp i}q_{\perp j}]\}-(\text{the value of this term at }\alpha=0)\Bigg]$$
$$=-\tfrac{1}{2}\delta_{ij}i\omega Z^2e^6\pi^{-2}m^{-2}(2\pi)^{-2}\Bigg[(\pi Z\alpha)^{-1}\sinh(\pi Z\alpha)\int_0^1 dx\, x^{-iZ\alpha}(1-x)^{iZ\alpha}\{-(7\pi/36)\ln[4x(1-x)]\}$$
$$-(\text{the value of this term at }\alpha=0)\Bigg]$$
$$=i\omega\delta_{ij}Z^2\alpha^3m^{-2}(14/9)[\psi(1+iZ\alpha)+\psi(1-iZ\alpha)+2\gamma], \quad (7.7)$$

where ψ is the logarithmic derivative of the gamma function, and γ is Euler's constant, numerically 0.57722. This is the desired result.

Since the limiting value exists as $\Delta\to 0$, (7.7) is a good approximation for all small momentum transfers. Thus the structure of the $Z\alpha$ corrections to Delbrück scattering is rather similar to the case of photon-photon scattering close to the forward direction,[13] and is much simpler than the lowest-order contribution.

8. RESULT FOR DELBRÜCK SCATTERING

We summarize here the present result on high-energy Delbrück scattering close to the forward direction. To all orders in $Z\alpha$ but only to the lowest order in α, the matrix element is approximately

$$\mathfrak{M}^{(D)}\sim 4i\alpha^3Z^2(\omega/m^2)\Bigg[(7/9)[\ln(2\omega/m)-(109/42)-\tfrac{1}{2}i\pi+\gamma+\mathrm{Re}\,\psi(1+iZ\alpha)]+4\int_1^\infty dx\int_0^1 dy(1-y^2)^{-1/2}$$
$$\times\int_0^1 d\beta\{y+ixm^2/[\beta(1-\beta)\Delta\omega]\}^{-1}\{(2-y^2)[-\tfrac{1}{2}x^{-2}+x^{-4}\beta(1-\beta)(x^2-2x+2)]\mp y^2\beta(1-\beta)x^{-4}\}\Bigg], \quad (8.1)$$

where the upper (lower) sign should be used when the photon is linearly polarized perpendicular to the scattering plane (in the scattering plane). Alternative forms for the integrals in (8.1) are given in Sec. 5, and the appendices. This result (8.1) holds when

$$\Delta\ll m. \quad (8.2)$$

The special case of $\Delta=0$ has been studied by Davies, Bethe, and Maximon.[20]

9. COMPTON SCATTERING

In terms of the impact factor representation,[1,2] the Compton scattering amplitude at high energies is given by

$$\mathfrak{M}_0^{(C)}\sim\tfrac{1}{2}ie^2m^{-1}\delta_{12}s(2\pi)^{-2}\times\int d\mathbf{q}_\perp\frac{\mathcal{G}_{ij}^{\gamma}(\tfrac{1}{2}\mathbf{\Delta},\mathbf{q}_\perp)}{[(\mathbf{q}_\perp+\tfrac{1}{2}\mathbf{\Delta})^2][(\mathbf{q}_\perp-\tfrac{1}{2}\mathbf{\Delta})^2]} \quad (9.1)$$

to the order e^4 when the fixed momentum transfer $\mathbf{\Delta}$ is not zero. Here, in the c.m. system, δ_{12} means that the helicity of the electron is not changed,[1,2] and s is, as usual, the square of the c.m. energy. The remarkable similarity between (9.1) and (2.1) has previously been noted.

The similarity between the high-energy behavior of the amplitudes for Compton and Delbrück scattering is actually even more striking. For this purpose, consider (9.1) instead in the laboratory system, where the elec-

[20] H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. 93, 788 (1954).

tron is initially at rest. Let ω be the energy of the incident photon, then

$$s=(\omega+m)^2-\omega^2\sim 2m\omega. \tag{9.2}$$

Suppose that (9.1) is expressed in terms of ω instead of s; then[21]

$$\mathfrak{M}_0^{(C)}\sim i\omega e^2\delta_{12}(2\pi)^{-2} \times\int d\mathbf{q}_\perp\frac{\mathcal{g}_{ij}{}^{\gamma}(\frac{1}{2}\mathbf{\Delta},\mathbf{q})}{[(\mathbf{q}_\perp+\frac{1}{2}\mathbf{\Delta})^2][(\mathbf{q}_\perp-\frac{1}{2}\mathbf{\Delta})^2]}. \tag{9.3}$$

Except for δ_{12}, this is exactly (2.1) with $Z=1$. Consequently, the result (5.22) for Delbrück scattering also holds for electron Compton scattering

$$\mathfrak{M}_0^{(C)}-\mathfrak{M}_0^{(C)}|_{\Delta=0}\sim -\tfrac{1}{4}i\omega e^6\pi^{-3} \times\delta_{12}(2\pi)^{-2}(I_1^+\pm I^-), \tag{9.4}$$

with

$$\mathfrak{M}_0^{(C)}|_{\Delta=0}\sim 4i\alpha^3(\omega/m^2)(7/9) \times[\ln(2\omega/m)-109/42-\tfrac{1}{2}i\pi]\delta_{12}. \tag{9.5}$$

In (9.4), the integrals I_1^+ and I^- are given by (5.19) and (5.17), and the upper and lower signs should be used in the same way as (5.22). The result (9.4) holds when $\Delta\ll m$, and implies that the target mass is irrelevant.

The result for high-energy Compton scattering is rather confused in the literature. Equation (9.5) implies, in particular, through the optical theorem, that the total cross section for

$$\gamma+e^-\to e^++2e^-$$

is approximately

$$\alpha r_0^2[(28/9)\ln(2\omega/m)-218/27] \tag{9.6}$$

in the extreme relativistic limit ($\omega\gg m$), where r_0 is the classical radius of the electron. Moreover (9.6) holds independently of the mass of the target particle (of charge e). This result contradicts that of Joseph,[22] who obtained 100/9 instead of 218/27, but agrees with Suh and Bethe.[23]

Together with earlier work on the impact factor representation[1] and on photon-photon scattering close to the forward direction, the present results complete the description of the high-energy behaviors for *all* two-body elastic scattering amplitudes in quantum electrodynamics when the momentum transfer is not large. The application of all the considerations of the present paper to scalar electrodynamics is completely straightforward.

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APPENDIX A

In this appendix, we study the Mellin transform of I^- and I_1^+. Define

$$\bar{I}_1{}^+(\zeta)=m^2\pi^{-2}\int_0^\infty I_1{}^+R_1{}^{-1-\zeta}dR_1 \tag{A1}$$

and

$$\bar{I}^-(\zeta)=m^2\pi^{-2}\int_0^\infty I^-R_1{}^{-1-\zeta}dR_1; \tag{A2}$$

then it follows from (6.8) and (6.9) that[24]

$$\begin{aligned}\bar{I}_1{}^+(\zeta)&=-\pi(\csc\pi\zeta)e^{-i\pi\zeta/2}\int_1^\infty dx\int_0^1 dy(2-y^2)(1-y^2)^{-1/2}\int_0^1 d\beta\\&\qquad\times y^{-1+\zeta}[\tfrac{1}{2}\beta^{-1}(1-\beta)^{-1}x]^{-\zeta}[-\tfrac{1}{2}x^{-2}+x^{-4}\beta(1-\beta)(x^2-2x+2)]\\&=\tfrac{1}{8}\pi^{3/2}(\csc\pi\zeta)e^{-i\pi\zeta/2}2^{\zeta}(14+20\zeta+9\zeta^2+\zeta^3)\frac{\Gamma(\frac{1}{2}\zeta)}{\Gamma(\frac{1}{2}\zeta+\frac{5}{2})}\frac{[\Gamma(1+\zeta)]^2}{\Gamma(4+2\zeta)},\end{aligned} \tag{A3}$$

and

$$\begin{aligned}\bar{I}^-(\zeta)&=\pi(\csc\pi\zeta)e^{-i\pi\zeta/2}\int_1^\infty dx\int_0^1 dy\, y^2(1-y^2)^{-1/2}\int_0^1 d\beta y^{-1+\zeta}[\tfrac{1}{2}\beta^{-1}(1-\beta)^{-1}x]^{-\zeta}x^{-4}\beta(1-\beta)\\&=\tfrac{1}{4}\pi^{3/2}(\csc\pi\zeta)e^{-i\pi\zeta/2}2^{\zeta}\frac{\Gamma(\frac{1}{2}\zeta+1)}{\Gamma(\frac{1}{2}\zeta+\frac{5}{2})}\frac{[\Gamma(2+\zeta)]^2}{\Gamma(4+2\zeta)}.\end{aligned} \tag{A4}$$

[21] It should be noted that, in general, for Δ of the order of m, δ_{12} is somewhat complicated. The spin state of the Compton scattered electron is, at high energies, determined completely by that of the electron before scattering. The relation, however, is particularly simple only in the c.m. system or, more generally, in a system where the electron and the photon both move rapidly in opposite directions. When Δ is much smaller than m, the recoil electron moves very slowly, and δ_{12} has the meaning of not changing the spin state of the electron even in the laboratory system.

[22] J. Joseph, Ph.D. thesis, State University of Iowa, 1955 (unpublished). See also J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Reading, Mass., 1955), p. 250.

[23] K. S. Suh and H. A. Bethe, Phys. Rev. **115**, 672 (1959).

[24] See Eqs. (3.7) and (3.8) of Ref. 4.

It is seen that the Mellin transforms are of quite simple forms.

We list the simplest of the behaviors of $\bar{I}_1^+(\zeta)$ and $\bar{I}^-(\zeta)$ near their singularities.

(i) When ζ is near zero,

$$\bar{I}_1^+(\zeta)=\tfrac{1}{8}(\sqrt{\pi})\zeta^{-1}(1-\tfrac{1}{2}i\pi\zeta)(1+\zeta\ln 2)(14+20\zeta)\times 2\zeta^{-1}[\Gamma(\tfrac{5}{2})]^{-1}\tfrac{1}{6}\{1+\tfrac{1}{2}\zeta[5\psi(1)-\psi(\tfrac{5}{2})-4\psi(4)]\}+O(1)$$
$$=(7/9)\zeta^{-2}[1+\zeta(-\tfrac{1}{2}i\pi+2\ln 2-25/7)]+O(1)\,, \tag{A5}$$

and

$$\bar{I}^-(\zeta)=\zeta^{-1}/18+O(1)\,. \tag{A6}$$

(ii) When ζ is near 1,

$$\bar{I}_1^+(\zeta)=-i\pi(1-\zeta)^{-1}(11/2)/5!+O(1)\,, \tag{A7}$$

and

$$\bar{I}^-(\zeta)=-\tfrac{1}{8}i\pi(1-\zeta)^{-1}/5!+O(1)\,. \tag{A8}$$

The asymptotic behaviors (6.4) and (6.5) follow immediately from (A6) and (A5), respectively. Higher-order terms for these asymptotic expansion can be obtained directly from the behaviors of $\bar{I}_1^+(\zeta)$ and $\bar{I}^-(\zeta)$ near $\zeta=-1,\ -2,\ -3$, etc. The leading terms for small R_1 are

$$I_1^+=-11i\pi^3m^{-2}R_1/240+O(R_1^2) \tag{A9}$$

and

$$I^-=-i\pi^3m^{-2}R_1/240+O(R_1^2)\,. \tag{A10}$$

It is seen that (A10) is consistent with (6.15). Higher-order terms in (A9) can be obtained either from (6.9) directly or from the behavior of $\bar{I}^+(\zeta)$ near $\zeta=2,\ 3,\ 4$, etc. Unlike their behavior for large R_1, the series expansion for I_1^+ and I^- are convergent power series.

APPENDIX B

In this appendix, we reduce the triple integrals (6.8) and (6.9) to double integrals. The reduction is quite elegant for (6.8) but rather messy for (6.9).

a. Reduction of I^-

Beginning with (6.8), we change the variable x by

$$x=y/z; \tag{B1}$$

then the y integration can be carried out because

$$\int_z^1 y^{-2}dy(1-y^2)^{-1/2}=z^{-1}(1-z^2)^{1/2}. \tag{B2}$$

We therefore get immediately the desired result

$$I^-=2\pi^2m^{-2}R_1\int_0^1 d\beta\int_0^1 z^2dz(1-z^2)^{1/2}\beta^2(1-\beta)^2\times[i+2R_1\beta(1-\beta)z]^{-1}. \tag{B3}$$

b. Reduction of I_1^+

We believe that the case of I_1^+ is intrinsically much more complicated. It is necessary to split the right-hand side of (6.9) into a number of separate terms and treat them differently. The following procedure may not be the best one. Define

$$I_{11}^+=\int_1^\infty dx\int_0^1 dy(2-y^2)(1-y^2)^{-1/2}\int_0^1 d\beta\times\{y+ix/[2\beta(1-\beta)R_1]\}^{-1}x^{-2} \tag{B4}$$

and, for $j=2,\ 3,\ 4$,

$$I_{1j}^+=\int_1^\infty dx\int_0^1 dy(2-y^2)(1-y^2)^{-1/2}\int_0^1 d\beta\times\{y+ix/[2\beta(1-\beta)R_1]\}^{-1}\beta(1-\beta)x^{-j}; \tag{B5}$$

then

$$I_1^+=\pi^2m^{-2}(\tfrac{1}{2}I_{11}^+-I_{12}^++2I_{13}^+-2I_{14}^+)\,. \tag{B6}$$

The change of variable (B1) is to be used for the two cases I_{13}^+ and I_{14}^+. The necessary integrals analogous to (B2) are, respectively,

$$\int_z^1 dy\,y^{-3}(2-y^2)(1-y^2)^{-1/2}=z^{-2}(1-z^2)^{1/2}, \tag{B7}$$

and

$$\int_z^1 dy\,y^{-4}(2-y^2)(1-y^2)^{-1/2}=\tfrac{1}{3}z^{-3}(2+z^2)(1-z^2)^{1/2}. \tag{B8}$$

Therefore

$$I_{13}^+=2R_1\int_0^1 d\beta\int_0^1 dz(1-z^2)^{1/2}\beta^2(1-\beta)^2\times[i+2R_1\beta(1-\beta)z]^{-1} \tag{B9}$$

and

$$I_{14}^+=\tfrac{2}{3}R_1\int_0^1 d\beta\int_0^1 dz(2+z^2)(1-z^2)^{1/2}\beta^2(1-\beta)^2\times[i+2R_1\beta(1-\beta)z]^{-1}. \tag{B10}$$

It is interesting to note that

$$I^-+2I_{13}^+-3I_{14}^+=0\,. \tag{B11}$$

For I_{11}^+ and I_{12}^+, this change of variable (B1) fails to work because the corresponding y integral is transcendental. This makes it impossible to carry out the z integration as desired in Appendix C. To avoid this problem, we write I_{11}^+ and I_{12}^+ as the sums

$$I_{11}^+=I_{15}^++I_{16}^+ \tag{B12}$$

and

$$I_{12}^+=I_{17}^++I_{18}^+\,, \tag{B13}$$

where

$$I_{15}^{+}=\int_1^\infty dx\int_0^1 dy(1-y^2)^{-1/2}\int_0^1 d\beta \times\{y+ix/[2\beta(1-\beta)R_1]\}^{-1}x^{-2}, \quad \text{(B14)}$$

$$I_{16}^{+}=\int_1^\infty dx\int_0^1 dy(1-y^2)^{1/2}\int_0^1 d\beta \times\{y+ix/[2\beta(1-\beta)R_1]\}^{-1}x^{-2}, \quad \text{(B15)}$$

$$I_{17}^{+}=\int_1^\infty dx\int_0^1 dy(1-y^2)^{-1/2}\int_0^1 d\beta \times\{y+ix/[2\beta(1-\beta)R_1]\}^{-1}\beta(1-\beta)x^{-2}, \quad \text{(B16)}$$

and

$$I_{18}^{+}=\int_1^\infty dx\int_0^1 dy(1-y^2)^{1/2}\int_0^1 d\beta \times\{y+ix/[2\beta(1-\beta)R_1]\}^{-1}\beta(1-\beta)x^{-2}. \quad \text{(B17)}$$

The two terms I_{15}^+ and I_{17}^+ can still be treated in the same way as before. By (B1) and (B2) we get

$$I_{15}^{+}=2R_1\int_0^1 d\beta\int_0^1 dz(1-z^2)^{1/2}\beta(1-\beta) \times[i+2R_1\beta(1-\beta)z]^{-1} \quad \text{(B18)}$$

and

$$I_{17}^{+}=I_{13}^{+}. \quad \text{(B19)}$$

For I_{16}^+ and I_{18}^+, we use the change of variable

$$x=\beta(1-\beta)x' \quad \text{(B20)}$$

instead of (B1). This makes it possible to integrate over β:

$$I_{16}^{+}=2\int_4^\infty dx'\,x'^{-2}\int_0^\infty dy(1-y^2)^{1/2} \times(y+\tfrac{1}{2}ix'/R_1)^{-1}\ln\frac{x'^{1/2}+(x'-4)^{1/2}}{x'^{1/2}-(x'-4)^{1/2}} \quad \text{(B21)}$$

and

$$I_{18}^{+}=\int_4^\infty dx'\,x'^{-2}\int_0^\infty dy(1-y^2)^{1/2}(y+\tfrac{1}{2}ix'/R_1)^{-1} \times[(x'-4)/x']^{1/2}. \quad \text{(B22)}$$

We are unable to avoid the appearance of a logarithmic factor in (B21), and there is some evidence that it is impossible to avoid such a factor.

It remains to put (B21) and (B22) in a form more similar to those of the other integrals. For this purpose, let

$$x'=[\beta(1-\beta)]^{-1}, \quad \text{(B23)}$$

and call the dummy variable y of integration z; then

$$I_{16}^{+}=2R_1\int_0^1 d\beta\int_0^1 dz(1-z^2)^{1/2}\beta(1-\beta)(1-2\beta) \times[i+2R_1\beta(1-\beta)z]^{-1}\ln[(1-\beta)/\beta] \quad \text{(B24)}$$

and

$$I_{18}^{+}=R_1\int_0^1 d\beta\int_0^1 dz(1-z^2)^{1/2}\beta(1-\beta)(1-2\beta)^2 \times[i+2R_1\beta(1-\beta)z]^{-1}. \quad \text{(B25)}$$

Finally, the substitution of (B13)–(B19), (B9), and (B10) into (B6) gives

$$I_1^{+}=\tfrac{1}{3}\pi^2m^{-2}R_1\int_0^1 d\beta\int_0^1 dz(1-z^2)^{1/2}\beta(1-\beta) \times[i+2R_1\beta(1-\beta)z]^{-1}\{3(1-2\beta)\ln[(1-\beta)/\beta] +2\beta(1-\beta)(5-2z^2)\}. \quad \text{(B26)}$$

As a check, we note that (A9) follows readily from (B26).

APPENDIX C

It is the purpose of this appendix to carry out the z integration in (B3) and (B26). The basic integral is

$$\int_0^1 dz(1-z^2)^{1/2}[i+2R_1\beta(1-\beta)z]^{-1} =[2R_1\beta(1-\beta)]^{-2}\{\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2} \times\sinh^{-1}[2R_1\beta(1-\beta)]-2R_1\beta(1-\beta)\} -\tfrac{1}{2}\pi i\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2}-1\}\}. \quad \text{(C1)}$$

Note that the imaginary part is much simpler than the real part. The substitution of (C1) into (B4) gives

$$I^{-}=\tfrac{1}{2}\pi^2m^{-2}R_1^{-1}\Big[\frac{1}{9}R_1-\tfrac{1}{8}i\pi\int_0^1 d\beta[2R_1\beta(1-\beta)]^{-2} \times\{\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2} \times\sinh^{-1}[2R_1\beta(1-\beta)]-2R_1\beta(1-\beta)\} -\tfrac{1}{2}i\pi\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2}-1\}\}\Big]. \quad \text{(C2)}$$

Note that the first term of (C2) is just that of (6.4). The imaginary part of the right-hand side of (C2) is expressible in terms of an elliptic integral of the third kind.

A similar substitution of (C1) into (B26) gives the other desired result:

$$I_1^+=\tfrac{1}{3}\pi^2 m^{-2}R_1\Bigg[\frac{1}{9}R_1^{-1}-\tfrac{1}{8}i\pi R_1^{-2}+\int_0^1 dz(1-z^2)^{1/2}$$
$$\times[2R_1\beta(1-\beta)]^{-2}\{R_1^{-2}+3\beta(1-\beta)(1-2\beta)\ln[(1-\beta)/\beta]+10\beta^2(1-\beta)^2\}\{\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2}$$
$$\times\sinh^{-1}[2R_1\beta(1-\beta)]-2R_1\beta(1-\beta)\}-\tfrac{1}{2}i\pi\{[1+4R_1^2\beta^2(1-\beta)^2]^{1/2}-1\}\}\Bigg]. \quad \text{(C3)}$$

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Nonassociativity of the Operators in the Crossing-Symmetric Bethe-Salpeter Equations*

ROBERT J. YAES†
Center for Particle Theory, Department of Physics, The University of Texas at Austin, Austin, Texas 78712
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We discuss the properties of the crossing-symmetric Bethe-Salpeter equations which have been proposed by Taylor and by Haymaker and Blankenbecler. We consider various possible methods of solution and the possibility of application to the Veneziano amplitude. We show that the operators which appear in these equations are not mutually associative, and hence that even the linearized approximation to these equations cannot be solved by conventional techniques.

I. INTRODUCTION

It is generally believed that the four-point function in strong-interaction theory should have the following properties: Lorentz invariance, analyticity, crossing symmetry, unitarity, and Regge asymptotic behavior. Since Lorentz invariance and analyticity are explicitly satisfied by any analytic function of the Mandelstam variables, s, t, and u, the three key properties are crossing symmetry, unitarity, and Regge behavior. Until recently, we could not obtain an amplitude having more than one of these three properties. However, we now have the simple but elegant model of Veneziano[1] which displays both crossing symmetry and Regge behavior but, alas, not unitarity.

The problem of combining crossing symmetry and unitarity is much more difficult. A set of equations for an amplitude having both these properties has been proposed by Taylor[2] and by Haymaker and Blankenbecler.[3,4] Unfortunately, being nonlinear, these equations have the disadvantage of not being soluble. All one can do is use the various iteration schemes which we shall discuss and which cannot be guaranteed to converge. In addition, since (as we shall show) the operators which appear in these equations are not mutually associative, we cannot even solve a linearized approximation to these equations by the usual techniques. In this paper we discuss the properties of these equations, the methods of obtaining iterative solutions, and the possible application to the Veneziano amplitude.

II. EQUATIONS

We consider the four-point function for the scattering of identical, spinless bosons of mass m (Fig. 1). The crossing-symmetric generalization of the Bethe-Salpeter equation proposed by Taylor[2] and by Haymaker

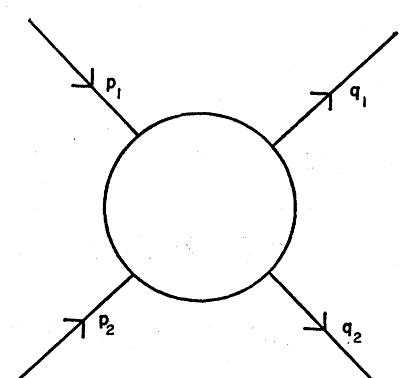

FIG. 1. Our notation for the four-point function.

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† Present address: Department of Physics, Illinois Institute of Technology, Chicago, Ill. 60616.

[1] G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2] J. G. Taylor, Nuovo Cimento Suppl. 1, 988 (1963).

[3] R. W. Haymaker and R. Blankenbecler, Phys. Rev. 171, 1581 (1968).

[4] On-shell K-matrix equations of the same form were first obtained by W. Zimmermann [Nuovo Cimento 21, 249 (1961)]. They have been applied to various cases, including the Veneziano model, by Cordes, Ravenhall, and Schult and by Humble.