Factorization of $M^J\pi \to M^{J'}\pi$ Residues within the Veneziano Model*

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(Received 25 May 1970)

The reactions $M^J\pi \to M^{J'}\pi$ are analyzed for a class of mesons, M^J. Imposing the Veneziano form in one channel, the Regge residues at poles are required to satisfy factorization conditions. These conditions are expressed in terms of the residues of the invariant amplitudes of the processes, for various hypotheses about the spacing of contributing trajectories of opposite normality. We find that factorization in one channel only is consistent with the coupling of the lowest-spin kinematically allowed particles in each helicity amplitude.

I. INTRODUCTION

SINCE the introduction of the Veneziano formula[1] there has been considerable effort expended to see to what extent it can be applied to various scattering processes as an approximation to the correct amplitudes. The first discussions were mainly limited to $\pi\pi \to \pi\pi$ and $\pi\pi \to \pi\omega$ scattering, each of which has the advantage of containing only one scattering amplitude. The simple applications here led to predictions about spacing and position of Regge trajectories and "decay" of "heavy-pions" which were in fair agreement with experiment. These successes spurred work on two-to-two scattering with higher-spin external particles. They have been considered mainly from two points of view: explicitly considering such reactions as $PV \to PV$,[2] $PN \to PN$,[3] and $N\bar{N} \to N\bar{N}$[4] and their concomitant complications of many amplitudes, and the development of 5-, 6-, etc., particle (boson) Veneziano-like formulas[5] which can be reduced at various energies to certain boson+boson → boson+boson scattering and, thus, to solve the problem at one fell swoop. An interesting exception to these is the work of Goebel, Blackmon, and Wali (GBW),[6] who were able to write Veneziano forms for $\pi\pi \to \pi S$, where S is arbitrary (integer) spin and parity, and thus showed that the kinematics for groups of reactions might be considered together.

One of the disadvantages of the explicit calculations with just low external spins is that one is never sure that the constraints arising from higher-spin reactions do not restrict the choice of acceptable functions in the lower-spin cases. In addition, the procedures for the low-external-spin solutions do not always allow one to deduce general properties of the arbitrary-spin calculation. The multiparticle Veneziano attempts have the advantage of also giving results directly on production amplitudes as well as the four-point subscattering; however, they have the disadvantage of being presently in a very preliminary stage and have not as yet been able to reproduce simple reactions like $PV \to PV$, for instance, in sufficient detail. On the applications of the Veneziano formula to spinning reactions there are, in addition to the problem of a large number of amplitudes, the difficulties of imposing factorization. There are two types of factorization constraints: factorization of helicity residues within a given scattering reaction and factorization of residues among various reactions. The GBW[6] problem, $\pi\pi \to \pi S$, is able to avoid both of these problems by being a set of reactions which have neither type of constraint. Nevertheless, these constraints are important; when factorization has been imposed a number of interesting results have been obtained. Using only $PP \to PP$, Canning[7] and Wong[8] derived the necessity of various group characteristics and trajectory spacing. Canning,[9] using factorization for selected $PP \to PP$, $PP \to PV$, and $PV \to PV$ reactions and strong assumptions about the form of the Veneziano amplitudes, obtained meson mass spectra and assignments. In many of the other Veneziano applications to meson reactions, factorization has also been used in obtaining conclusions.

In another area of application, Jacobs,[4] using the requirements of factorization in the $N\bar{N} \to N\bar{N}$ reaction, has shown the necessity of a large number of Veneziano terms in that solution.

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Grant No. EOOAR-68-0010, through the European Office of Aerospace Research.

[1] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[2] G. Canning, Nucl. Phys. **B17**, 359 (1970); E. S. Abers and V. L. Teplitz, Phys. Rev. Letters **22**, 909 (1969); A. Capella, C. A. Savoy, and A. Villani, Nuovo Cimento Letters **2**, 137 (1969); A. Capello, B. Diu, J. M. Kaplan, and D. Schiff, Nuovo Cimento **64A**, 361 (1969); E. S. Abers and V. L. Teplitz, Phys. Rev. D **1**, 624 (1970); M. A. Jacobs, presented at Boulder Conference on High Energy Physics, Boulder, Colorado, 1969 (unpublished); P. Carruthers and F. Cooper, Phys. Rev. D **1**, 1223 (1970); M. L. Whippman *ibid.* **1**, 701 (1970); P. Carruthers and E. Lasley, *ibid.* **1**, 1204 (1970).

[3] Shan-Yuan Chu and Bipin Desai, Phys. Rev. **188**, 2215 (1969); S. Fenster and K. C. Wali, Phys. Rev. D **1**, 1409 (1970); J. Namyslowski and M. Sawicki (unpublished); R. H. Graham and J. W. Moffat (unpublished); M. L. Blackmon and K. C. Wali (unpublished); R. F. Amann, Phys. Rev. D **2**, 561 (1970); J. Maharana and R. Ramachandran (unpublished); M. H. Vaughn and D. Y. Wong (private communication).

[4] M. A. Jacobs, Phys. Rev. **184**, 1574 (1969); R. H. Mutter (unpublished); B. K. Pal (unpublished).

[5] For a summary of these papers, see C. Lovelace, Review paper at the Irvine Conference on Regge Poles, 1969 (unpublished).

[6] C. J. Goebel, M. L. Blackmon, and K. C. Wali, Phys. Rev. **182**, 1485 (1969).

[7] G. P. Canning, Nucl. Phys. **B14**, 437 (1969).

[8] D. Y. Wong, Phys. Rev. **183**, 1412 (1969); see also H. Harari, Phys. Rev. Letters **20**, 1395 (1968); J. L. Rosner, *ibid.* **21**, 950 (1968); and C. B. Chiu and J. Finkelstein, Phys. Letters **27B**, 510 (1968).

[9] G. P. Canning, Nucl. Phys. **B17**, 359 (1970).

It is with this knowledge of the importance of the factorization constraints within the Veneziano model that we have decided to study the requirement of simultaneous factorization of all s-channel residues in reactions of the form $M^J\pi \to M^{J'}\pi$, for an infinite class of particles, M^J. This is the first class of reactions which have both normality contributions and in which both types of factorization constraints can be imposed. Hence, they are the first general class of reactions from which we can discover if the form of residues predicted by the Veneziano representation is compatible with factorization and if the solutions of lower-spin reactions are either changed or invalidated by the inclusion of higher spin reactions. In Sec. II we reevaluate the $M^J\pi \to \pi\pi$ problem from a point of view which is generalizable to the other reactions of interest. The kinematics of the $M^J\pi \to M^{J'}\pi$ reactions are discussed in Sec. III. We note that, to our knowledge, the invariant amplitudes of even these simple reactions have never been given before.[10] We therefore present a method, applicable within our one-channel approach, of selecting an acceptable set of invariant amplitudes. The Regge residues of $M^J\pi \to M^{J'}\pi$, $M^J\pi \to \pi\pi$, $M^J\pi \to \delta\pi$, $\pi\pi \to \pi\pi$, and $\delta\pi \to \delta\pi$, where δ is $J^{PIG}=0^{+1-}$ [possibly the $\delta(962)$ or $\pi_N(1016)$], are then required to factorize. In Sec. IV we discuss the applications and conclusions from the form of the resulting equations.

II. $M^J\pi \to \pi\pi$ AND $M^J\pi \to \delta\pi$ REACTIONS

The Veneziano formula is the representation of the invariant amplitudes for a four-point scattering process as a linear combination of Euler beta functions, $B_p{}^{m,n} = \Gamma(m-\alpha_1)\Gamma(n-\alpha_2)/\Gamma(m+n-p-\alpha_1-\alpha_2)$, with $m,n \geq 0$, $\min(m,n) \geq p \geq 0$, and $\alpha_{1,2}$ linear trajectories in two of the s,t,u channels. The linear identity $B_p{}^{m,n} = B_p{}^{m,n-1} - B_p{}^{m+1,n-1} + pB_{p+1}{}^{m,n-1}$ means that the general triple sum of $B_p{}^{m,n}$'s can be reduced to a sum on only two indices, for instance,[11]

$$A = \sum_{m,r} [a_{m,r} B_{m-r}{}^{m,n} + b_{m,r}(B_{m+1-r}{}^{m,m+1} - B_{m+1-r}{}^{m+1,m})], \quad m \geq r \geq 0, \quad b_{m,0}=0.$$

The existence of both $a_{m,r}$ and $b_{m,r}$ coefficients in this sum allows sufficient freedom for the residues of A at a pole in either of its two channels to be written as $P(\alpha)\nu^{\alpha-n}/\alpha!$, where α is the value of the trajectory at the pole, ν is $(t-u)$ in the s channel or the appropriate difference of Mandelstam variables in the other channels, and $P(\alpha)$ is a polynomial in α (of finite degree for a finite number of a's and b's). The polynomials $P(\alpha)$ can be fixed separately, along with n, in both channels with certain restrictions. With no symmetry requirements, restricting the amplitude to behave asymptotically as $\nu_i{}^{\alpha-n}$, $n>0$ in one channel eliminates the poles at $\alpha=0$ in both channels and the poles in α up to $\alpha=n-1$ in the i $(=s,\ t,$ or $u)$ channel. This requires $P(\alpha)$ to include $\alpha(\alpha-1)\cdots(\alpha-n+1)$ in the i channel and α in the other channel. If the amplitude is required to be symmetric or antisymmetric, the poles in α up to $\alpha=n-1$ are eliminated in both channels and the polynomials in both channels include factors of $\alpha, \alpha-1, \ldots, (\alpha-n+1)$. Nevertheless, it is still possible (and convenient) to discuss many of the problems of the two channels separately, and to treat these correlations at a later stage. When isospin is included, the absence of $I=2$ poles in $1\otimes1 \to 1\otimes1$, or $I=\frac{3}{2}$ poles in $1\otimes\frac{1}{2} \to 1\otimes\frac{1}{2}$ isospin scattering (mesons) produces restrictions on the $I=0, \frac{1}{2}$, or 1 channels. In a model in which the pole structure of the Veneziano formula is exact, this leads to certain exchange degeneracies.[12] If duality or the Veneziano model is relaxed so that only the leading few poles (parents and a finite number of daughter levels) are believed, it is possible to include ephemeral, daughterlike poles in exotic channels and allow the $I=0, \frac{1}{2}, 1$ isospins to be almost independent.[13] For simplicity, therefore, we shall discuss the factorization restrictions on the Regge residues in one channel only, and one isospin in this channel.

In this section we discuss $M^J\pi \to \pi\pi$ and $M^J\pi \to \delta\pi$ reactions. We call this channel the s channel and label the resonances that occur in it with $I=1$ names. Thus, if we choose the particles M^J to have the quantum numbers $P=(-)^{J+1}$, $I=1$, $G=-(\pi, A_1,$ etc.), the ρ trajectory (ρ, A_2, etc.) is the dominant trajectory in the reaction $M^J\pi \to \pi\pi$. In the reaction $M^J\pi \to \delta\pi$, the dominant trajectory is the B [$B(1220)$, etc.]. These two trajectories differ by approximately 1. Within the Veneziano model, it is consistent to assume them to differ by exactly 1.[14] Thus, a single trajectory function α is used throughout to give both the ρ and B trajectories.

We proceed as follows: We construct the helicity states of M^J; we contract these helicity states with a set of invariant operators; we find the linear combinations of the invariant amplitudes which have simple and independent asymptotic form; writing the Veneziano representation for these amplitudes, we obtain a form for the Regge residues of the processes. These residues will be required to satisfy factorization conditions in the processes $M^J\pi \to M^{J'}\pi$ discussed in Sec. III.

The kinematics for the reactions are shown in Fig. 1, with $q_1{}^2 = q_2{}^2 = \mu^2$, $p_a{}^2 = a^2$, $p_b{}^2 = b^2$. We allow the mass of particle b (π or δ) to be different from μ. We construct the helicity tensor of particle a out of J spin-1 polarization vectors $\epsilon_\lambda{}^\mu$, where $\lambda = +, 0, -$ and μ is the Lorentz

[10] Invariant amplitudes for general processes have been discussed by A. C. Hearn, Nuovo Cimento **21**, 333 (1961), but the discussion there is incomplete and leads to too many amplitudes for the processes.

[11] M. A. Jacobs (Ref. 2); R. E. Kreps and M. S. Milgran, Phys. Rev. D **1**, 2271 (1970); M. J. Whippman, *ibid.* 701 (1970).

[12] See C. Lovelace (Ref. 5) for references on this point; also J. Mandula, J. Weyers, and G. Zweig, Phys. Rev. Letters **23**, 266 (1969); G. P. Canning, this issue, Phys. Rev. D **2**, 2426 (1970).

[13] For examples of this, see D. Y. Wong, Phys. Rev. **181**, 1900 (1969); and M. A. Jacobs (Ref. 2).

[14] Of many sources assuming this, see, e.g., D. Y. Wong (Ref. 8).

index. Our conventions for ϵ_λ are that ϵ_0 has a positive component in the z direction (the direction of the 3-momentum of particle a in the s-channel c.m. system, $s > s_{\text{threshold}}$) and $\epsilon_\pm = (\sqrt{\tfrac{1}{2}})(\pm\hat{e}_x + i\hat{e}_y)$. The scattering occurs in the xz plane.

A state $|J,\lambda\rangle$ with spin J, helicity λ, can be constructed from J states $|1,\lambda\rangle$ by applying the helicity lowering operator, J^-_{op}, $(J-\lambda)$ times to the state $|1,+;1\rangle|1,+;2\rangle\cdots|1,+;J\rangle$ or the raising operator, J^+_{op}, $(J+\lambda)$ times to $|1,-;1\rangle|1,-;2\rangle\cdots|1,-;J\rangle$. The result is

$$|J,\lambda\rangle = \mathfrak{N}_\lambda{}^J \sum_{\{\lambda_i\}} 2^{-c}|1,\lambda_1\rangle|1,\lambda_2\rangle\cdots|1,\lambda_J\rangle\,,$$
$$\sum_{i=1}^{J} \lambda_i = \lambda\,, \tag{1}$$

where $\mathfrak{N}_\lambda{}^J$ is a normalization constant and c is the number of occurrences of -1 in $\{\lambda_i\}$. The $\mathfrak{N}_\lambda{}^J$ satisfy $\mathfrak{N}_\lambda{}^J 2^\lambda = \mathfrak{N}_{-\lambda}{}^J$.

The product of the normalities, $n = P(-)^J$, of the external particles in $M_a{}^J\pi_1 \to \pi_b\pi_2$ is $+1$ and hence there are an even number of $\epsilon^{\mu\nu\rho\sigma}$'s occurring in the invariant operators. A complete set of these tensor operators with no $\epsilon^{\mu\nu\rho\sigma}$'s is well known to be

$$\mathcal{O}_i{}^J = \frac{i!(J-i)!}{J!} \sum_{\text{permutations}} (\mathcal{P}_1\cdots\mathcal{P}_{J-i}\mathcal{Q}_{J-i+1}\cdots\mathcal{Q}_J)\,, \quad i=0,\ldots,J, \tag{2}$$

where $\mathcal{P} = p_a - p_b$ and $\mathcal{Q} = -q_1 + q_2$. The helicity amplitudes[15] are then given by

$$f_\lambda = \sum_{i=0}^{J} \langle \mathcal{O}_i{}^J | J,\lambda\rangle I_i\,,$$

for which I_i are kinematic-structure free. The Lorentz contractions symbolized by the bra and ket vectors are accomplished in the following stages. The state

$$|1,+\rangle\cdots|1,+\rangle|1,0\rangle\cdots|1,0\rangle|1,-\rangle\cdots|1,-\rangle$$

with a occurrences of $+1$, b of 0, c of -1, in the prescribed order, is contracted against

$$\sum_{i=0}^{J} I_i \langle \mathcal{O}_i{}^J|$$

with the help of Table I. Using the symmetry in Lorentz indices, the sum over the set of direct-product states having the same Clebsch-Gordan coefficients is taken. Then the sum over these sets is taken. The calculation is tedious but straightforward, and the result is

$$f_\lambda = \mathfrak{N}_\lambda{}^J \sum_{a,b,c,i,k} \frac{(-)^{c+k}2^{-c}i!(J-k)!}{a!(b-k)!c!k!(i-k)!} \times (U_b)^{J-b}(V_b)^{b-k}(X_a)^k I_i\,. \tag{3}$$

The multiple sum is a bit deceptive since we must also require

$$a+b+c=J\,,\quad a-c=\lambda\,,$$

and hence is actually a triple sum over, say, c, i, and k. The range of summation is simply determined by the factorials present. The k summation has appeared because we have incorporated the V portion of the $\langle\mathcal{Q}|1,0\rangle$ with the V portion of $\langle\mathcal{P}|1,0\rangle$. The form of Eq. (3) allows the introduction of a linear combination of I's which will be seen to have independent asymptotic behavior. These amplitudes are defined as

$$I_i = \sum_m \frac{(-)^{i-m}m!}{(m-i)!i!} A_m\,, \tag{4a}$$

$$A_m = \sum_i \frac{i!}{(i-m)!m!} I_i\,. \tag{4b}$$

Substituting Eq. (4a) into (3), we can remove the half-angle factor $(1-z^2)^{\lambda/2}$ and obtain the parity-conserving

TABLE I. Lorentz contractions used in the text.

$$\langle\mathcal{P}|a_\pm{}^1\rangle = \pm U_b \qquad \langle\mathcal{Q}|a_\pm{}^1\rangle = \pm U_b$$
$$\langle\mathcal{P}|a_0{}^1\rangle = V_b \qquad \langle\mathcal{Q}|a_0{}^1\rangle = V_b - X_a$$
$$\langle\mathcal{P}|b_\pm{}^1\rangle = \mp U_a \qquad \langle\mathcal{Q}|b_\pm{}^1\rangle = \mp U_a$$
$$\langle\mathcal{P}|b_0{}^1\rangle = V_a \qquad \langle\mathcal{Q}|b_0{}^1\rangle = V_a - X_b$$

$$\langle a_\pm{}^1|b_\pm{}^1\rangle = -\tfrac{1}{2}(1+z)$$
$$\langle a_\pm{}^1|b_\mp{}^1\rangle = -\tfrac{1}{2}(1-z)$$
$$\langle a_\pm{}^1|b_0{}^1\rangle = \mp Y_b$$
$$\langle a_0{}^1|b_\pm{}^1\rangle = \pm Y_a$$
$$\langle a_0{}^1|b_0{}^1\rangle = [\phi_a\psi_a\phi_b\psi_b - z(s+a^2-\mu^2)(s+b^2-\mu^2)]/4abs$$
$$\langle\mathcal{A}|a_\pm{}^1\rangle = \phi_a\psi_a\phi_b\psi_b(1-z^2)^{1/2}/s^{1/2}$$
$$\langle\mathcal{A}|a_0{}^1\rangle = 0$$

$$U_m = -\phi_m\psi_m(1-z^2)^{1/2}/2\sqrt{2}s^{1/2}$$
$$V_m = [-\phi_m\psi_m(s+N^2-\mu^2)z + \phi_N\psi_N(s+m^2-\mu^2)]/4Ns$$
$$X_m = \phi_m\psi_m/m$$
$$Y_m = (s+m^2-\mu^2)(1-z^2)^{1/2}/2\sqrt{2}ms^{1/2}$$

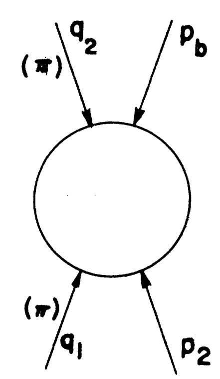

FIG. 1. Kinematics of the reactions.

[15] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

helicity amplitudes of Gell-Mann *et al.*,[16]

$$\mathfrak{F}_\lambda{}^+ = 2\mathfrak{N}_\lambda{}^J \sum_{a,b,c,k} \frac{(-)^c 2^{-c}(J-k)!}{a!(b-k)!c!}(1-z^2)^c \times (\bar{U}_b)^{J-b}(V_b)^{b-k}(X_a)^k A_k\,, \quad (5)$$

where $\bar{U}_b$ is U_b with the factor $(1-z^2)^{1/2}$ removed. The notation $\mathfrak{F}^{\pm}$ means that a trajectory with normality $\pm$ dominates the amplitude for $z\rightarrow\infty$. In addition, Gell-Mann *et al.*[16] showed that a particle with spin α contributes to $\mathfrak{F}_\lambda{}^{\pm}$ as $z^{\alpha-|\lambda|}$ $(z\rightarrow\infty)$ if it has the dominant normality. From (5) we easily see that sequentially determining $\mathfrak{F}_\lambda{}^+$, $\lambda=J,J-1,\ldots,0$ to be as $z^{\alpha-\lambda}$ with independent coefficients requires $A_k\rightarrow z^{\alpha-J+k}$ also independently. We parametrize the A_k leading behavior as

$$(\text{residue of } A_k \text{ at } \alpha=\text{integer}) = \frac{A_k{}^J(\alpha)(t-u)^{\alpha-J+k}}{(\alpha-J+k)!}, \quad (6)$$

where $A_k{}^J(\alpha)$ are finite polynomials in α. Also, we define

$$\mathfrak{F}_\lambda{}^+ \underset{z\rightarrow\infty}{\longrightarrow} \mathcal{C}(\alpha,\lambda,0)\beta_\lambda{}^{J0}\beta_0{}^{00}z^{\alpha-\lambda}. \quad (7)$$

The factor $\mathcal{C}$ is the coefficient of $z^{\alpha-\lambda}$ in $e_{\lambda\mu}{}^{\alpha+}$ of Gell-Mann *et al.*, and we need note only that

$$\mathcal{C}(\alpha,\lambda,0)\mathcal{C}(\alpha,0,\mu) = \mathcal{C}(\alpha,\lambda,\mu)\mathcal{C}(\alpha,0,0)\,.$$

The Regge residues have already been written as factorized $\pi\pi\rho^\alpha$ and $\pi M^J\rho^\alpha$ three-point couplings. Equations (5)–(7) determine $\beta_\lambda{}^{J0}\beta_0{}^{00}$ straightforwardly as

$$\beta_\lambda{}^{J0}\beta_0{}^{00} = \mathfrak{N}_\lambda{}^J\mathcal{C}(\alpha,\lambda,0)^{-1} \times \sum_{c,k}\frac{2(J-k)!}{(\lambda+c)!(J-\lambda-2c-k)!c!(\alpha-J+k)!} \times \left(\frac{-(s+a^2-\mu^2)}{4a\phi_a\psi_a}\right)^J \left(\frac{\sqrt{2}as^{1/2}}{s+a^2-\mu^2}\right)^\lambda \left(\frac{a^2s}{(s+a^2-\mu^2)^2}\right)^c \times \left(\frac{4\phi_a{}^2\psi_a{}^2}{s+a^2-\mu^2}\right)^k \left(\frac{\phi_a\psi_a\phi_b\psi_b}{s}\right)^\alpha A_k{}^J, \quad (8)$$

$$\phi_a = [s-(a+\mu)^2]^{1/2}, \quad \psi_a = [s-(a-\mu)^2]^{1/2}.$$

Sums over a and b have been carried out in evaluating Eq. (8). We see that the implied factorization is possible since the J,λ dependence involves only a,μ (not b,μ) masses.

In principle, daughter residues could also be evaluated; however, they are more complicated and are not discussed in this paper.

[16] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

For $\pi_a\pi_1\rightarrow\pi_b\pi_a$, we have

$$(\beta_0{}^{00})^2 = \mathcal{C}(\alpha,0,0)^{-1}(\phi_a\psi_a\phi_b\psi_b/s)^\alpha H^+\,, \quad (8')$$

where $H^+=A_0{}^0$.

In addition, the scattering $\delta_a\pi_1\rightarrow\delta_b\pi_2$ has

$$(\bar{\beta}_0{}^{00})^2 = \mathcal{C}(\alpha,0,0)^{-1}(\phi_a\psi_a\phi_b\psi_b/s)^\alpha H^-\,, \quad (8'')$$

where α is here interpreted as the negative-normality B-particle trajectory and the $\delta\pi B^\alpha$ vertex has been called $\bar{\beta}$ instead of β. The masses a,b should also be reinterpreted accordingly.

The above procedures can also be applied to $M^J\pi\rightarrow\delta\pi$. In these reactions the B trajectory dominates. The product of the normalities of the external particles is -1 and an odd number of $\epsilon^{\mu\nu\rho\sigma}$'s are required in each of the invariant tensor operators. A well-known complete set of these operators is

$$\mathcal{Q}^{\mu_1}(\mathcal{O}_i{}^{J-1})^{\mu_2\cdots\mu_J}, \quad i=0,\ldots,J-1, \qquad \mathcal{Q}^\mu = 4\sqrt{2}i\epsilon^{\mu\nu\rho\sigma}Q_{1\nu}P_{b\rho}Q_{2\sigma}\,, \quad (9)$$

where the operator $\mathcal{O}_i{}^{J-1}$ is as defined in Eq. (2) and the symmetrization is to be taken over only the Lorentz indices shown. We note (Table I) that the axial vector $\mathcal{Q}^\mu$ cannot couple to $\epsilon_0{}^\mu$ and gives the same sign contribution to $\epsilon_\pm{}^\mu$, whereas both polar vectors contribute oppositely to $\epsilon_\pm{}^\mu$. The net effect of these sign differences is to guarantee that there will be an over-all factor λ in f_λ in the equation corresponding to Eq. (3). The combinatorial analysis is such that, besides the λ, the only other changes in (3) are $(J-k)\rightarrow(J-k-1)$ and a factor $\phi_a\psi_a\phi_b\psi_b(1-z^2)^{1/2}$ from the $\mathcal{Q}$ contraction. Since the i summation has not changed, substitution (4) is again appropriate, although we opt to call the amplitudes C_m to avoid confusion with our previously defined A_m. In terms of the residues $C_k{}^J$ [Eq. (6)], we find

$$\gamma_\lambda{}^{J0}\beta_0{}^{00} = (-2\sqrt{2}\lambda\phi_a\psi_a)\mathfrak{N}_\lambda{}^J\mathcal{C}(\alpha,\lambda,0)^{-1} \times \sum_{c,k}\frac{2(J-k-1)!}{(\lambda+c)!(J-\lambda-2c-k)!c!(\alpha-J-k)!} \times \left(\frac{-(s+a^2-\mu^2)}{4a\phi_a\psi_a}\right)^J \left(\frac{\sqrt{2}as^{1/2}}{s+a^2-\mu^2}\right)^\lambda \left(\frac{a^2s}{(s+a^2-\mu^2)^2}\right)^c \times \left(\frac{4\phi_a{}^2\psi_a{}^2}{s+a^2-\mu^2}\right)^k \left(\frac{\phi_a\psi_a\phi_b\psi_b}{s}\right)^\alpha C_k{}^J, \quad (10)$$

where $\bar{\beta}_0{}^{00}$ is again the coupling of $\delta\pi$ to the particle on the B trajectory of spin α.

III. $M^J\pi\rightarrow M^{J'}\pi$ REACTIONS

In this section we consider the reaction $M^J\pi\rightarrow M^{J'}\pi$. The first part of the section is devoted to the kinematic difficulties for these reactions. In Sec. II, the discussion was facilitated by the fact that there was only one spinning particle. For these reactions, simple counting arguments suffice to obtain the correct number of invariant operators which are known to give the correct analytic structure. For $A_1\pi \to A_1'\pi$, the operators $\mathcal{P}\mathcal{P}$, $\mathcal{P}\mathcal{Q}$, $\mathcal{Q}\mathcal{P}$, $\mathcal{Q}\mathcal{Q}$, and $g^{\mu\nu}$ are known to be an acceptable set. However, the generalization of these operators to our more complicated problem provides far too many more operators than there are independent amplitudes. Most of the methods commonly available for checking sets of test invariant operators[17] are sufficiently complicated to preclude their use here. However, the method used in Sec. II for evaluating the kinematics can be applied to our too-large set. It is then obvious that an acceptable subset of these operators must allow, *a priori*, the simultaneous independence of all couplings not related by parity. Requiring this, we are able to determine invariant amplitudes for the processes. The only other complication of a second spinning particle is the fact that both normalities of particles contribute. We have pointed out that, for $M^J=\pi$, A_1, etc., the leading trajectory is ρ with B, approximately one unit below, also contributing. If we had chosen $M^J=\omega$, A_2, etc., the same trajectories would contribute; however, the trajectories fit into the kinematics differently depending on whether the normality of the trajectory equals or is opposite to the product of normalities of the incoming (and outgoing) particles. Hence the second choice of particles M^J effectively interchanges ρ and B. In order to consider both cases, we shall retain the notation of our original set of particles, but allow $\alpha_\rho=\alpha_B+1$ and $\alpha_\rho=\alpha_B-1$ separately. For completeness we also consider parity doubling, $\alpha_\rho=\alpha_B$.

The analysis proceeds much the same as in Sec. II. A set of operators is defined (too many) which are Lorentz contracted against $\epsilon(a)_\lambda{}^{\{\mu\}}\epsilon(b)_{\lambda'}{}^{\{\nu\}*}$ to give helicity amplitudes. The parity-dominated combinations are taken; these amplitudes are then used to select out the correct set of operators which give acceptable invariant amplitudes. The hypotheses of parity doubling or nondoubling are imposed and factorization requirements are solved for.

The phase conventions for $\epsilon(b)_\lambda$ are that $\epsilon(b)_0$ has a positive component in the z' direction, the 3-space momentum direction of b when $s>s_{\text{threshold}}$, and $\epsilon(b)_\pm=(\sqrt{\tfrac{1}{2}})(\pm\hat{a}_{x'}+i\hat{a}_y)$. Thus the same form of Eq. (1) holds for a and b. All relevant contractions are listed in Table I.

In the Lorentz tensor $\mathcal{O}$ which we contract against $|a_\lambda{}^J\rangle|b_{\lambda'}{}^{J'}\rangle$, we may put $\mathcal{P}$,$\mathcal{Q}$, or $g^{\mu\nu}$ (we can, in principle, put in an even number of $\epsilon^{\mu\nu\rho\sigma}$'s, but these require four P_a, P_b, Q_1, or Q_2's for each nontrivial pair included and will disallow coupling to lowest spin, physically accessible, intermediate states). In the rest frame of, say, particle a the state $|J,\lambda\rangle$ is symmetric in all of its J $O(3)$ indices [the index $\mu=0$ is eliminated by $P_{a\mu}\epsilon(a)_\lambda{}^\mu=0$] and traceless in any two indices using a metric $\delta_{\mu\nu}$, $\mu,\nu=1,2,3$. Correspondingly, in an arbitrary frame, the contraction of $g^{\mu\nu}$ with two Lorentz indices from two $|J,\lambda\rangle$ vanishes. $G^{\mu\nu}$ must therefore connect one index of a to one of b. The symmetry in Eq. (1) means that only the number of $\mathcal{P}$'s or $\mathcal{Q}$'s is important in the group of J (J') indices referring to particle M^J ($M^{J'}$). Thus the most general amplitude we can construct has $r\leq\min(J,J')$ $g^{\mu\nu}$'s, one index in M^J and one in $M^{J'}$, i $\mathcal{Q}$'s and $J'-i-r$ $\mathcal{P}$'s in $M^{J'}$, and j $\mathcal{Q}$'s and $J-j-r$ $\mathcal{P}$'s in M^J. When $J'\leq J$, there are $\frac{1}{6}(J'+1)(J'+2)\times(3J-J'+3)$ possible operators. From counting independent helicity amplitudes, however, there are only $(2JJ'+J+J'+1)$ which may be independent. The two formulas agree only for $(J,J')=(J,0)$, $(1,1)$ and there are too many operators in all other cases. For the time being we carry all of these tensor operators, defined as

$$\mathcal{O}_{ij}{}^{J'Jr}=(\mathcal{O}_i{}^{J'-r}G^r\mathcal{O}_j{}^{J-r})^{\mu_1,\ldots,\mu_{J+J'}}, \tag{11}$$

where the symmetrization in $\mathcal{O}^{J'-r}$ is over the first $J'-r$ Lorentz indices, in $\mathcal{O}^{J-r}$ over the last $J-r$ indices, and G^r is a product of r $g^{\mu\nu}$'s, first index in an available J' index, second in J.

We first consider $\mathcal{O}_{ij}{}^{J'J0}$. The analysis of Sec. II can be done directly for the $|a_\lambda{}^J\rangle$ state. Except for an easily accounted for sign, the same can also be done for $|b_u{}^{J'}\rangle$. The substitution

$$I_{ij}{}^0=\sum_{m,n}\frac{(-)^{i+j-m-n}m!n!}{(m-i)!i!(n-j)!j!}D_{mn}{}^0, \tag{12a}$$

$$D_{mn}{}^0=\sum_{i,j}\frac{i!j!}{(i-m)!m!(j-n)!n!}I_{ij}{}^0, \tag{12b}$$

corresponding to a product of two substitutions (4) yields

$$f_{\mu\lambda}=\mathfrak{N}_\mu{}^{J'}\mathfrak{N}_\lambda{}^J \times\sum_{a,b,c,d,e,f,m,n}\frac{(-)^{c+f}2^{-c-f}(J-n)!(J'-m)!}{a!(b-n)!c!d!(e-m)!f!} \times(U_b)^{J-b}(-U_a)^{J'-e}(V_b)^{b-n}(V_a)^{e-m} \times(X_a)^n(X_b)^mD_{mn}{}^0. \tag{13}$$

The μ refers to $M^{J'}$, λ to M^J. Using Eq. (2.7) of Gell-Mann *et al.*,[16] we can construct from (13) the parity-*dominated* helicity amplitudes $\mathfrak{F}_{\mu\lambda}{}^\pm$.

For the operators in Eq. (11) with $r>0$, the contractions can also be done for the highest few powers of z. Substitutions (12a) and (12b) are also appropriate to these amplitudes and, after tedious combinatorial analysis, an expansion for $f_{\mu\lambda}$ can be obtained and the highest few contributions to $\mathfrak{F}_{\mu\lambda}{}^\pm$ can be tabulated.

In the previous calculation we were able to pick out directly the asymptotic behavior of $A_k{}^J$, $C_k{}^J$ from the form of Eqs. (5). Because we have superfluous amplitudes we cannot, at this point, determine the behavior

[17] Various tests have been described in M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960); A. O. Barut, *Theory of the Scattering Matrix* (Macmillan, New York, 1967).

of $D_{mn}{}^r$; however, ex post facto we shall see that we require only the highest $\nu = t-u$ contributions to $\mathfrak{F}^\pm$ to determine the set of invariant amplitudes. The next to highest contribution to $\mathfrak{F}^+$ will be needed for our factorization requirements. Equation (13) and its analog for $r \neq 0$ yield for the highest powers in ν (or z),

$$\mathfrak{F}_{\mu\lambda}{}^{\{\pm\}} = \mathfrak{N}_\lambda{}^J \mathfrak{N}_\mu{}^{J'} \left(\frac{-(s+a^2-\mu^2)}{4a\phi_a\psi_a}\right)^J \left(\frac{-(s+b^2-\mu^2)}{4b\phi_b\psi_b}\right)^{J'}$$
$$\times \left(\frac{\sqrt{2}a\phi_a\psi_a\phi_b\psi_b}{s^{1/2}(s+a^2-\mu^2)}\right)^\lambda \left(\frac{\sqrt{2}b\phi_a\psi_a\phi_b\psi_b}{s^{1/2}(s+b^2-\mu^2)}\right)^\mu 2(-)^{\mu_m}$$
$$\times \sum_{c,m,f,n} \frac{(J-n)!(J'-m)!}{(\lambda+c)!(J-\lambda-2c-n)!c!(\mu+f)!}$$
$$\times \frac{1}{(J'-\mu-2f-m)!f!} \left(\frac{4\phi_a{}^2\psi_a{}^2}{s+a^2-\mu^2}\right)^n \left(\frac{4\phi_b{}^2\psi_b{}^2}{s+b^2-\mu^2}\right)^m$$
$$\times \left(\frac{a^2 s}{(s+a^2-\mu^2)^2}\right)^c \left(\frac{b^2 s}{(s+b^2-\mu^2)^2}\right)^f \mathcal{G}_{\mu\lambda}{}^{\{\pm\}}, \quad (14a)$$

where $\lambda,\mu \geq 0$, $\mu_m = \min(\lambda,\mu)$, and

$$\mathcal{G}_{\mu\lambda}{}^- \sim -\mu_m z^{\mu_m - 1} \left(1 - \frac{r\lambda_m}{(J-n)(J'-m)}\right)(-4)^r$$
$$\times \nu^{J+J'-\lambda-\mu-m-n-r} D_{mn}{}^r (1 + 1/\nu + \cdots), \quad (14b)$$

$$\mathcal{G}_{\mu\lambda}{}^+ \sim z^{\mu_m} (-4)^r \nu^{J+J'-\lambda-\mu-m-n-r} D_{mn}{}^r$$
$$\times \left\{1 - \frac{1}{\nu}\left[\frac{r(J-\lambda-2c-n)(J'-\mu-2f-m)}{(J-n)(J'-m)}\right.\right.$$
$$\times R + \frac{(J'-\mu-2f-m)(J'-m-r)}{(J'-m)}$$
$$\left.\times S + \frac{(J-\lambda-2c-n)(J-n-r)}{(J-n)} T\right]$$
$$\left. + \frac{1}{\nu^2} + \cdots \right\}, \quad (14c)$$

$$R = \frac{\phi_a{}^2\psi_a{}^2\phi_b{}^2\psi_b{}^2}{s(s+a^2-\mu^2)(s+b^2-\mu^2)},$$

$$S = \frac{\phi_b{}^2\psi_b{}^2(s+a^2-\mu^2)}{s(s+b^2-\mu^2)},$$

$$T = \frac{\phi_a{}^2\psi_a{}^2(s+b^2-\mu^2)}{s(s+a^2-\mu^2)},$$

$$m = 0,\ldots,J'-r, \quad n = 0,\ldots,J-r,$$

and

$$\lambda_m = \max(\lambda,\mu).$$

There are three things for which we must use Eqs. (14): determining the correct set of invariant amplitudes, finding their asymptotics, and requiring the residues to factorize.

To accomplish the first of these, we consider a pair of amplitudes $\mathfrak{F}_{\mu\lambda}{}^+$, $\mathfrak{F}_{\mu\lambda}{}^-$ at a value of s for which there is a pole having highest spin $j = \lambda_m = \max(\lambda,\mu)$. To allow the coupling of this particle first with one and then the other normality (only normality $+$ if $\mu_m = 0$), there must be two (one if $\mu_m = 0$) independent invariant amplitudes which appear, multiplied by ν^0 from the kinematics, in $\mathfrak{F}_{\mu\lambda}{}^+$ and $\mathfrak{F}_{\mu\lambda}{}^-$. Thus, in $\mathfrak{F}_{\mu\lambda}{}^+$, for instance, we seek a $D_{mr}{}^r$ with $J+J'-\lambda-\mu-m-n-r+\mu_m = 0$ [see 14(c)]. By angular momentum conservation, these two amplitudes we seek cannot appear in a $\mathfrak{F}_{\mu\lambda}{}^\pm$ with a $\lambda_m{}'$ greater than λ_m. Thus, from the factorials in (14a), $J-(\lambda_0+1)-2c-n < 0$ and $J'-(\mu_0+1)-2f-m < 0$ for all c, $f \geq 0$, but $J-\lambda_0-2c-n \leq 0$, $J'-\mu_0-2f-m \leq 0$ for at least one c and $f \geq 0$. This requires $n = J-\lambda$, $m = J'-\mu$ and, hence, $r = \mu_m$. From (14b) we see that this $D_{J'-\mu,J-\lambda}{}^{\mu_m}$ does not contribute to $\mathfrak{F}_{\mu\lambda}{}^-$. By an analogous argument we find $D_{J'-\mu,J-\lambda}{}^{\mu_m-1}$ is needed. These latter D's are needed only for $\mu_m > 0$, since $\mathfrak{F}_{\mu\lambda}{}^- = 0$ if $\mu_m = 0$. Either directly counting these D's or observing that one new D is added for each independent $\mathfrak{F}$ shows that we have the correct number of amplitudes.

For simplicity we introduce the notation

$$(-4)^{\min(J-n,J'-m)} E_{m,n} = D_{m,n}{}^{\min(J-n,J'-m)},$$
$$m = 0,\ldots,J'; \quad n = 0,\ldots,J \quad (15a)$$

$$(-4)^{\min(J-n,J'-m)} F_{m,n} = D_{m,n}{}^{\min(J-n,J'-m)-1}$$
$$m = 0,\ldots,J'-1; \quad n = 0,\ldots,J-1. \quad (15b)$$

Schematically this is shown in Fig. 2 for $J=4$, $J'=2$.

We now consider the hypothesis $\alpha_\rho = \alpha_B + 1$. The ρ contributes to $\mathfrak{F}_{\mu\lambda}{}^+$ proportional to $z^{\alpha_\rho - \lambda_m}$ and to $\mathfrak{F}_{\mu\lambda}{}^-$ as $z^{\alpha_\rho - \lambda_m - 1}$. The B contributes to $\mathfrak{F}_{\mu\lambda}{}^-$ as $z^{\alpha_B - \lambda_m} = z^{\alpha_\rho - \lambda_m - 1}$. At $\alpha = \alpha_\rho = \lambda_m$ only $E_{J'-\mu,J-\lambda}$ can contribute to $\mathfrak{F}_{\mu\lambda}{}^+$ with a residue proportional to ν^0. Since this corresponds to a particle on the ρ-trajectory of spin λ_m,

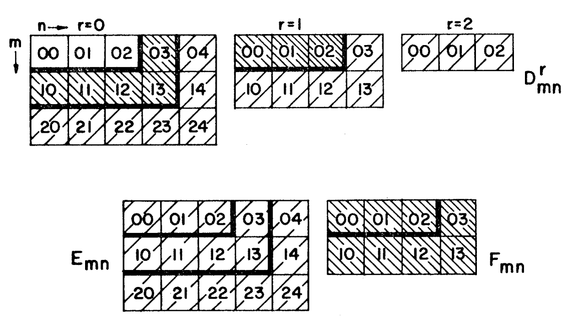

FIG. 2. Substitution (15a) and (15b) illustrated for $J=4$, $J'=2$. Similarly shaded regions of $D_{mn}{}^r$ correspond either to E_{mn} or F_{mn} with the same shading.

we have

(residue of E_{mn} at α=integer)

$$=\frac{E_{mn}{}^0\nu^{\alpha-\max(J'-m,J-n)}}{[\alpha-\max(J'-m,J-n)]!}+\cdots. \quad (16a)$$

We must also have a ν^0 contribution to $\mathfrak{F}_{\mu\lambda}{}^-$ at $\alpha=\lambda_m+1$ from both the ρ trajectory and B trajectory particles. To allow this requires

(residue of F_{mn} at α=integer)

$$=\frac{F_{mn}{}^0\nu^{\alpha-\max(J'-m,J-n)-1}}{[\alpha-\max(J'-m,J-n)-1]!}+\cdots, \quad (16b)$$

where $E_{mn}{}^0$ and $F_{mn}{}^0$ are finite polynomials in α. It is easily checked that these asymptotic behaviors agree with our requirements of $\alpha_\rho=\alpha_B+1$ at all values of α. We can now postulate that the residues implied by these forms factorize. However, the phases of the three-point vertices are already fixed by Eqs. (7) and (10). We allow for this uncertainty in phases by putting

$$\mathfrak{F}_{\mu\lambda}{}^+\to\eta_{\lambda\mu}\mathcal{C}(\alpha,\lambda,\mu)\beta_\lambda{}^{J0}\beta_\mu{}^{J'0}z^{\alpha-\lambda_m},$$

$$\mathfrak{F}_{\mu\lambda}{}^-\to\eta_{\lambda\mu}\left[-\mu_m\left(\frac{\alpha-\lambda_m}{\alpha}\right)\mathcal{C}(\alpha,\lambda,\mu)\beta_\lambda{}^{J0}\beta_\mu{}^{J'0}+\mathcal{C}(\alpha-1,\lambda,\mu)\gamma_\lambda{}^{J0}\gamma_\mu{}^{J'0}\right]z^{\alpha-\lambda_m-1}, \quad (17)$$

where $\eta_{\lambda\mu}$ can be ±1. Note the contribution to $\mathfrak{F}_{\mu\lambda}{}^-$ from the "wrong" normality is of the same asymptotic power as the "right" normality contribution. Using Eqs. (8), (8′), (8″), (10), and (14)–(17), we set up the factorization conditions for leading particles of both normalities. In these equations the choice $\eta_{\lambda\mu}=(-1)^{\mu m}$ appears to be natural and has the simplest solution. Solving these equations and separating $E_{mn}{}^0$ from $F_{mn}{}^0$, we find

$$\frac{E_{mn}{}^0H^+H^-}{\alpha![\alpha-\max(J-n,J'-m)]!}=\frac{8sC_n{}^JC_m{}^{J'}H^++[(J-n)(J'-m)/\alpha-\min(J-n,J'-m)+1]A_n{}^JA_m{}^{J'}H^-}{(\alpha-J+n)!(\alpha-J'+m)!}, \quad (18)$$

$$\frac{F_{mn}{}^0H^+H^-}{4\alpha![\alpha-\max(J-n,J'-m)-1]!}=\frac{8sC_n{}^JC_m{}^{J'}H^++[(J-n)(J'-m)/\alpha-\min(J-n,J'-m)]A_n{}^JA_m{}^{J'}H^-}{(\alpha-J+n)!(\alpha-J'+m)!}. \quad (19)$$

Equation (19) holds only for $n\neq J$, $m\neq J'$.

Let us now consider the hypothesis that there are parity-doubled parent trajectories $\alpha=\alpha_\rho=\alpha_B$. For this to occur, $\mathfrak{F}_{\mu\lambda}{}^\pm$ are the same order, $z^{\alpha-\lambda_m}$ asymptotically, even though Eqs. (14) naturally imply $\mathfrak{F}^+/\mathfrak{F}^-=O(z)$. The depression of $\mathfrak{F}^+$ is accomplished through a careful choice of asymptotic behaviors and correlating them. We determine the behavior by induction.

Let us assume that none of the amplitudes contributing to $\mathfrak{F}_{\mu\lambda}{}^\pm$ except $E,F_{J'-\mu,J-\lambda}$ violate $\mathfrak{F}_{\mu\lambda}{}^\pm\to z^{\alpha-\lambda_m}$. To obtain parity-doubled particles at $\alpha=\lambda_m$ with independent residues we must start both E and F at $\alpha=\lambda_m$, and hence they behave at least like $\nu^{\alpha-\lambda_m}$. Of the two, only $F_{J'-\mu,J-\lambda}$ can contribute to $\mathfrak{F}_{\mu\lambda}{}^-$ and hence behaves exactly as $\nu^{\alpha-\lambda_m}$. However, it then contributes like $\nu^{\alpha-\lambda_m+1}$ to $\mathfrak{F}_{\mu\lambda}{}^-$, so $E_{J'-\mu,J'-\lambda}$ must behave like $\nu^{\alpha-\lambda_m+1}$ to cancel it. It is easily seen in (14c) that cancellation in $\mathfrak{F}_{\mu\lambda}{}^+$ at the first occurrence of an E,F pair continues to all "lower" $\mathfrak{F}_{\mu\lambda}{}^+$'s for the highest ν power. This and the subsequent contribution of F to "lower" $\mathfrak{F}_{\mu\lambda}{}^-$'s provides our initial assumption of "higher" E's and F's not violating "lower" asymptotic restrictions. We have, therefore,

$$\text{residue } E_{mn}=\frac{E_{mn}{}^0\nu^{\alpha-\lambda_m+1}}{(\alpha-\lambda_m)!}+\frac{E_{mn}{}^1\nu^{\alpha-\lambda_m}}{(\alpha-\lambda_m)!}+\cdots, \quad (20a)$$

$$\text{residue } F_{mn}=\frac{F_{mn}{}^0\nu^{\alpha-\lambda_m}}{(\alpha-\lambda_m)!}+\frac{F_{mn}{}^1\nu^{\alpha-\lambda_m-1}}{(\alpha-\lambda_m-1)!}+\cdots, \quad (20b)$$

$$\lambda_m=\max(J-n,J'-m),$$

where E^i and F^i are finite polynomials in α. From (14c) we see that they are restricted by

$$F_{mn}{}^0=4E_{mn}{}^0,\quad m\neq J',\quad n\neq J,$$
$$O=E_{mn}{}^0,\quad m=J'\quad\text{or}\quad n=J. \quad (21)$$

Corresponding to Eq. (17), we have

$$\mathfrak{F}_{\mu\lambda}{}^+\to\eta_{\lambda\mu}\mathcal{C}(\alpha,\lambda,\mu)\beta_\lambda{}^{J0}\beta_\mu{}^{J'0}z^{\alpha-\lambda_m},$$
$$\mathfrak{F}_{\mu\lambda}{}^-\to\eta_{\lambda\mu}\mathcal{C}(\alpha,\lambda,\mu)\gamma_\lambda{}^{J0}\gamma_\mu{}^{J'0}z^{\alpha-\lambda_m}. \quad (22)$$

Factorization of the residues in (22) with (8), (8′), (8″), and (10) gives

$$\frac{E_{mn}{}^0H^-}{\alpha![\alpha-\max(J-n,J'-m)]!}=\frac{8sC_n{}^JC_m{}^{J'}}{(\alpha-J+n)!(\alpha-J'+m)!},$$
$$n=0,\ldots,J-1;\quad m=0,\ldots,J'-1 \quad (23)$$

$$\frac{\{E_{mn}{}^1-\frac{1}{4}[\alpha-\max(J-n,J'-m)F_{mn}{}^1]\}H^+}{\alpha![\alpha-\max(J-n,J'-m)]!}$$
$$=\frac{A_n{}^JA_m{}^{J'}}{(\alpha-J+n)!(\alpha-J'+m)!}+\frac{E_{m-1,n-1}{}^0H^+R'}{\alpha![\alpha-\max(J-n+1,J'-m+1)]!}-\frac{E_{m-1,n}{}^0H^+S'}{\alpha![\alpha-\max(J-n,J'-m+1)]!}-\frac{E_{m,n-1}{}^0H^+T'}{\alpha![\alpha-\max(J-n+1,J'-m)]!},$$
$$m=0,\ldots,J';\quad n=0,\ldots,J \quad (24)$$

$$R'=1/16s\,,\quad S'=(s+a^2-\mu^2)/4s\,,\quad T'=(s+a^2-\mu^2)/4s\,.$$

The equations correspond to factorizing the leading particles of both normalities. Compare the $1/s$ in R', S', and T' with the s in Eq. (23).

To consider the hypothesis $\alpha_\rho=\alpha_B-1$ and impose factorization simultaneously on the ρ and B trajectories would require knowledge of the $1/\nu^2$ term in (14c) as well as the z^{μ_m-2} in (14a). Although this calculation is possible with the above procedures, it is extremely complex and not warranted here. The positioning of the trajectories, however, can be accomplished by setting $A_n{}^J=A_m{}^{J'}=0$, all n,m in Eq. (24). The factorization conditions on $\mathfrak{F}_{\mu\lambda}{}^+$, then imply restrictions on $E_{mn}{}^{(2)}$ and $F_{mn}{}^{(2)}$.

IV. CONCLUSIONS

In the preceding sections we have presented the s-channel requirements on Veneziano models for scattering of particles $M^J=\pi$, A_1, etc., on pions. The analysis is also appropriate for $M^J=\omega$, A_2, etc. In either of these reactions the dominant trajectories appear to be separated by one unit in angular momentum and it is thus possible to assume that only one trajectory function α appears in the Veneziano form. We have obtained requirements on the residues of the invariant amplitudes so that parental particles have residues which factorize for the reactions considered. (By parental particle, we have meant consistently that particle of a particular normality and mass with highest spin. Some authors refer to parent particle as the highest-spin particle of either normality.) We have done this for a variety of hypotheses about trajectory spacing: $\alpha_c=\alpha_u+1$, where $c,u=+,-$ the product of normalities of the incoming particles (appropriate to $M^J=\pi$, $A_1,\ldots$ and $\alpha_\rho=\alpha_B+1$), $\alpha_c=\alpha_u-1$ (appropriate to $M^J=\omega$, $A_2,\ldots$ and $\alpha_\rho=\alpha_B+1$), and $\alpha_c=\alpha_u$ (appropriate if the ρ trajectory begins to be parity-doubled at higher J).

What we can obtain from this analysis are the restrictions implicit in Eqs. (18), (19), (23), and (24) on the structure of $H^\pm$, $A_m{}^J$, $C_m{}^J$, E_{mn}, and F_{mn}. Before we do this, however, we should mention a word about H^-, the residue of $\pi\delta\to{}_B\pi\delta$.

The δ [or perhaps $\pi_n(1016)$] has been listed in the Particle Data tables[18] for some time, although the final word on both its existence and quantum numbers is far from having been written. The requirement that δ be given equal footing with π, ω, etc., as a valid external particle whose residues factorize is a bit questionable. Now, the δ appeared in two types of reactions, the $M^J\pi\to\delta\pi$ and $\delta\pi\to\delta\pi$. The use of the first of these was in obtaining the lowest possible occurrence of $\gamma_\lambda{}^J$, and the second had then to be considered to obtain $\bar{\beta}_0{}^{00}$. These difficulties can be avoided by requiring the factorization of the normality nonconserving couplings in

$$(M^J\pi\to M^{J'}\pi)(M^1\pi\to M^1\pi)=(M^J\pi\to M^1\pi)(M^1\pi\to M^{J'}\pi)\,.$$

The kinematics are considerably more complicated but can be shown to be equivalent to calling

$$E_{00}{}^0\to H^-\qquad (J=J'=1)$$
$$C_n{}^J\to E_{n0}{}^0\qquad (J'=1,\ J\text{ arbitrary})$$

in Eqs. (18), (19), (23), and (24). Thus, pragmatically, we may as well allow the δ to exist computationally. Differences, however, may occur in "daughter" calculations.

From the form of Eqs. (18), (19), (23), (24) we see that the factorials are such as not to change the "natural" starting values of the invariant amplitudes. Lovelace[19] has shown some interesting results when H^+, the $\pi\pi\to\pi\pi$ form is particularly simple [in our form $H^+(\alpha)=\alpha$] and there are reasons to believe that the forms for $A_m{}^J$ and $C_m{}^J$ should be as simple as possible.[20] With this assumption and barring accidental polynomial square roots in $H^\pm$, we have (a) a factor of H^- in $C_m{}^J$, (b) a factor of H^+ in $A_m{}^J$, (c) a factor of $[\alpha-\alpha(0)]H^-\alpha/[\alpha-\min(J-n,J'-m)]$ in $E_{mn}{}^0$, and (d) a factor of $\alpha!/[\alpha-\min(J-n,J'-m)]$ in $\{E_{mn}{}^1-[\alpha-\max(J-n,J'-m)]F_{mn}{}^1\}$. With these factors as shown, the factorization equations become considerably simplified. It is consistent, for instance, to have the remaining portions of $H^\pm$, $A_m{}^J$, $C_m{}^J$, and $E_{mn}{}^0$ constants and the remainder of $\{E_{mn}{}^1-[\alpha-\max(J-n,J'-m)]F_{mn}{}^1\}$ a polynomial of degree 2 in α. The fact that E^1, F^1 residues are generally of degree 2 higher than E^0, F^0 is entirely consistent with the Veneziano formula. This is, then, the minimum complexity allowed by factorization in the s channel.

[18] Particle Data Group, Rev. Mod. Phys. **41**, 109 (1969).
[19] See, e.g., C. Lovelace, Phys. Letters **28B**, 265 (1968); D. Y. Wong (Ref. 8); and K. V. Vasavada, Phys. Rev. D **1**, 88 (1970).
[20] Besides the aesthetic reasons for this simplicity, there is also the point that one can be sure that sums of Veneziano terms converge only if they are finite sums.

In this analysis we have allowed the ρ and B trajectories to begin at spin 0, although in reality, the ρ certainly cannot begin there. The key to the resolution of this problem lies in the remark in Sec. II that the separate channels of a reaction were almost separable. The elimination of the $\alpha=0$ ghost in reactions with only one trajectory function, α, such as ours, is usually accomplished by crossed-channel restrictions on asymptotic behavior. Since we restrict ourselves to a one-channel solution only, this effect is beyond the scope of this paper. In reactions like $\rho\pi \to \rho\pi$ where the π trajectory certainly contributes, there is also the ω trajectory contributing. As the π and ω trajectories are separated by approximately $\frac{1}{2}$ of a canonical unit, there must be (at least) two separate trajectory functions in the s channel of that reaction. Thus, the $\alpha=0$ particle-elimination mechanism we have mentioned should not be thought of as precluding the π.

We mention that implicit in the solution of the inter-reaction factorization conditions are the simultaneous solution of the intra-reaction conditions of factorization. In addition, when $J=J'$, time-reversal invariance is satisfied in that the amplitudes $E_{mn}=E_{nm}$ and $F_{mn}=F_{nm}$ to whatever powers of ν have been determined. From Eqs. (12) we see that this corresponds to the correct symmetry between initial and final states and also reduces the number of amplitudes to $(J+1)^2$, which is the correct number.

ACKNOWLEDGMENT

The author would like to acknowledge discussions with Dr. G. P. Canning during the course of this work.

PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Broken $SU(3)\times SU(3)$ Symmetry*

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We discuss, in detail, the meaning of a proposed unitary transformation which exhibits an inherent ambiguity of the broken $SU(3)\times SU(3)$ symmetry. It is suggested that invariance under this transformation be imposed upon the Hamiltonian of the hadrons. A model consistent with this requirement is proposed. We also discuss some interesting consequences of such a model.

I. INTRODUCTION

RECENTLY a unitary transformation in the $SU(3)\times SU(3)$ space was found[1] which renders the breakdown of chiral symmetry nonunique. In particular, for the GOR model,[2] where

$$H=H_0+H',\quad H'=\alpha(u_0+\sqrt{2}ru_8),\qquad (1)$$

we find that, under the discrete unitary transformation

$$W=\exp(i\tfrac{3}{2}\pi Y_5),\qquad (2)$$

H' remains of the same form, but the parameter r becomes $\tilde{r}$, where

$$\tilde{r}=(1-2r)/(1+4r).\qquad (3)$$

Since this proposal was made, considerable confusion has arisen, especially concerning the meaning and interpretation of such a transformation. We wish now to discuss an example of symmetry breaking in a familiar setting—that of rotational symmetry. This example will be designed so as to resemble, as closely as possible, the more abstract problem considered in I. In so doing, we hope that the ideas in I will become clear. We will also discuss the difficulties of theories which are not invariant under W.

A second motivation of this work is to generalize the results obtained in I. We recall that in I, only the cases when $H'\sim(\bar{3},3)+(3,\bar{3})$ and $(1,8)+(8,1)$ were treated in detail. It was pointed out, but not explicitly proven, that we can actually extend the results so that any transformation properties of H' are allowed. We will now prove that, for a general H', the transformation W always leaves it invariant in form, while changing its parameters.

Finally, we wish to propose a Hamiltonian model of broken $SU(3)\times SU(3)$ symmetry. By requiring explicit invariance under W, we shall arrive at an H' different from the usual ones. Some interesting consequences will be discussed.

This paper is organized as follows. In Sec. II, an example in broken spherical symmetry is analyzed, paving the way for an understanding of broken chiral symmetry. This will be taken up in Sec. III, where we offer, in detail, the interpretation of the transformation

* Supported in part by the U. S. Atomic Energy Commission.
[1] T. K. Kuo, Phys. Rev. D 2, 342 (1970), hereafter referred to as I.
[2] M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968), hereafter referred to as GOR. See also S. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968).