amplitude $a_l(s)$, for $l<O(s^{1/2})$, is independent of l, as is the case for scattering by a black sphere, our assumption would be true. Perhaps the most vulnerable among our assumptions is the one about the universality of ρ coupling to hadrons. It is, however, possible to turn our argument around and use our sum rules as a testing ground for the hypothesis of universality. In the sum rules for $\pi\Sigma$ or $\pi\Xi$ scattering, one is confronted with unknown baryon-pole contributions. But the sum rules for πN and πK scattering, which are free from these difficulties, yield values of $g_{\rho NN}$ and $g_{\rho KK}$ which are in excellent agreement with the hypothesis of universality. Other "experimental" checks on the assumption of universal ρ-meson coupling to hadrons consist in examining our sum rules for Kp scattering (helicity nonflip) and those for the helicity-flip amplitudes B in both πN and Kp scattering. Investigations in these directions have been carried out and the results, to be published elsewhere, justify our assumptions.

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Fredholm Character of the N/D Equations

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We point out that, under rather general assumptions (in particular, that the Pomeranchuk singularity has either nonzero slope or intercept less than 1), unitarity guarantees that the N/D method leads to an integral equation for partial-wave amplitudes which is of Fredholm type, and hence possesses a unique solution, regardless of the behavior of the amplitude as s goes to infinity along the unphysical cut.

THE so-called N/D equations for partial-wave amplitudes have been one of the basic tools in the S-matrix approach to the theory of strong interactions. The starting point in obtaining these equations has, in general, been a dispersion relation for the partial-wave amplitude.[1,2] We define the amplitude $A_l(s)$ to be the usual invariant partial amplitude and let $B_l(s) = A_l(s)/(s-s_0)$; that is, $B_l(s)$ is normalized so that, in the case of elastic unitarity, it is given by $B_l(s) = 2[s/(s-s_0)]^{1/2}(e^{i\delta_l}\sin\delta_l)/(s-s_0)$. (We use the usual Mandelstam variables: s, the c.m. energy squared, and t, the negative square of the four-momentum transfer. s_0 is a subtraction point.) Then $B_l(s)$ is taken to satisfy the dispersion relation

$$B_l(s)=L_l(s)+U_l(s)\,, \qquad (1)$$

where $L_l(s)$ and $U_l(s)$, the dispersion integrals over the unphysical (left-hand) cut and the physical (unitary) cut, respectively, are given by

$$L_l(s)=\frac{A_l(s_0)}{s-s_0}+\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}B_l(s')ds'}{(s'-s)} \qquad (2a)$$

and

$$U_l(s)=\frac{1}{\pi}\int_{4}^{\infty}\frac{\mathrm{Im}B_l(s')ds'}{(s'-s)}. \qquad (2b)$$

(Throughout the paper we assume for simplicity the kinematics corresponding to the elastic scattering of equal-mass spinless particles; this simplification has no effect on our arguments. We choose our units so that the scattering particles have unit mass.) Assuming the validity of Eq. (1), one then carries out the usual decomposition

$$B(s)=N(s)/D(s)\,, \qquad (3)$$

where we have dropped here and for the remainder of the paper the irrelevant subscript l. In Eq. (3), $D(s)$ is the Omnes function[3]

$$D(s)=\exp\left(-\frac{s-s_0}{\pi}\int_4^{\infty}ds'\frac{\delta(s')}{(s'-s)(s'-s_0)}\right), \qquad (4)$$

where $\delta(s)$ is the real part of the phase shift, and has a right-hand (unitary) cut only, while $N(s)=A(s)D(s)$ has only a left-hand cut. In the remainder of our discussion we will consider the approximation of purely elastic unitarity, which is made in most actual calculations. To discuss the situation with inelastic effects included, one would have to proceed in the same way using the Frye-Warnock form of the N/D equations.[2] The functions $N(s)$ and $D(s)$ satisfy the coupled integral equations, in the approximation of elastic unitarity,

$$N(s)=\frac{A(s_0)}{s-s_0}+\frac{1}{\pi}\int_{-\infty}^{0}ds'\frac{f(s')D(s')}{(s'-s)} \qquad (5a)$$

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963); R. L. Warnock, *ibid.* **131**, 1320 (1963).
[3] R. Omnes, Nuovo Cimento **8**, 316 (1958); **21**, 524 (1961).

and

$$D(s)=1-\frac{(s-s_0)}{\pi}\int_4^\infty ds'\frac{\rho(s')N(s')}{(s'-s_0)(s'-s)},\qquad (5b)$$

where

$$\rho(s')=\tfrac{1}{2}[(s'-4)/s']^{1/2}(s'-s_0)\qquad (6)$$

and we have introduced the abbreviation

$$f(s)=\mathrm{Im}B(s).\qquad (7)$$

The occurrence of the substraction constant $A(s_0)$ can be of importance only in the S wave, since for higher angular momentum one can choose the subtraction point at threshold, and then $A(s_0)$ is known to be zero.

Equations (5) were solved, in Ref. 1, by substituting (5a) into (5b), yielding an equation of the form

$$D(s)=1-\int_{-\infty}^0 K(s,s')f(s')D(s')ds',\qquad (8)$$

which can be shown to be a Fredholm equation provided $f(s)=O((\ln s)^{-1-\epsilon}/s)$, $\epsilon>0$, on the negative real axis.[1,4]

The problem in the justification of the N/D method occurs when $f(s)$ fails to vanish fast enough as $s\to-\infty$ along the negative real axis. One example of such a case is the case that the left-hand cut is approximated by the partial-wave projection of the force due to the exchange of an elementary particle of spin greater than zero. However, even in a theory where the exchanged particles are Reggeized, it is quite possible that $f(s)$ will be nonvanishing at $s=-\infty$. This is particularly likely to be true if Regge trajectories rise indefinitely, since in this case the partial-wave amplitude cannot be expected to be polynomial bounded in the s plane, in particular not along the negative real axis, so that one cannot even write a dispersion relation of the form (1) for $B(s)$. The possible existence of such difficulties has recently been reemphasized by Atkinson and Calogero[5]; these authors propose an alternative to the N/D method for certain types of unbounded discontinuity functions. It is the purpose of the present paper to argue that, under fairly general assumptions, the validity of the N/D equations, in the form in which they are usually used in actual calculations, is independent of the behavior of $f(s)$ on the distant left-hand cut, and is guaranteed by the restrictions imposed by unitarity on the behavior of $B(s)$ and $U(s)$ on the positive real axis. There remains the question of whether particular approximate forms for the input of the N/D equations, used in actual calculations, are reasonable or not; we shall make only a few brief and rather inconclusive remarks on this very difficult question below.

Omnes[6] and Squires[7] have considered the question of the behavior of the discontinuity function $f(s)$ on the left-hand cut in a conventional Regge theory, and are able to show that, because of cancellations in the integral over the angle involved in taking the partial-wave projection, $f(s)=0(s^{\alpha(\infty)-1})$ on the left-hand cut. Squires[7] points out that this proof actually requires an additional assumption which, though plausible in the context of conventional Regge theory, goes beyond it. Much more serious is the fact that, if the Regge trajectories in relativistic problems, unlike those in potential scattering, do not return to the left half j plane, in particular, if they continue to rise indefinitely, then the result $f(s)=O(s^{\alpha(\infty)-1})$ is valueless as far as establishing the convergence of dispersion integrals over the left-hand cut, or the Fredholm character of the integral equations resulting from the N/D method.

We consider the Uretsky form of the N/D equations.[8] For convenience, we will change to the variable $\nu=\frac{1}{4}s-1$, which is zero at threshold. Then N obeys the integral equation[8]

$$N(\nu)=L(\nu)+\frac{1}{\pi}\int_0^\infty\left[L(\nu')-\frac{\nu}{\nu'}L(\nu)\right]\frac{N(\nu')\rho(\nu')d\nu'}{\nu'-\nu}.\qquad (9)$$

$D(\nu)$ is again obtained simply by integration from Eq. (5b).

Our basic points consist of the following observations: First, we note that Eq. (9) is derived by applying the Cauchy integral formula to the function

$$H(\nu)=U(\nu)D(\nu).\qquad (10)$$

Clearly, since each of its factors has only right-hand cuts, so does $H(\nu)$. Since $f(\nu)$ is bounded, and goes asymptotically as ν^{-1} on the physical cut because of unitarity, the convergence of the integral in Eq. (2b) defining $U(\nu)$ is assured and, moreover, $U(\nu)\sim\nu^{-1}$ apart from logarithmic factors. In addition, if the asymptotic behavior of the phase shift is governed by Levinson's theorem,[2,9] then in a bootstrap theory in which there are no CDD (Castillejo-Dalitz-Dyson) poles, $D(\nu)$ will be bounded by a constant at infinity. Hence $H(\nu)\sim\nu^{-1}$, and the dispersion relation which leads directly to Eq. (9) is valid.

Equation (9) does involve a function $L(\nu)$ which depends on the unphysical discontinuity. If $f(\nu)$ does not vanish as $\nu\to-\infty$, $L(\nu)$ will not, of course, be given by Eq. (2a) since the integral will not converge. However, Eq. (9) follows with $L(\nu)$ simply defined by Eq. (1) in terms of $B(\nu)$ and $U(\nu)$. That is, we write

$$L(\nu)\equiv B(\nu)-U(\nu).\qquad (11)$$

[In the situation that the dispersion integral does not converge, $L(\nu)$ is the contribution to the Cauchy-integral formula from the integral over the left-hand cut

[4] We use the notation $f(s)=O(g(s))$ in the usual way. Thus this statement means that $|f|\leqslant M|(\ln s)^{-1-\epsilon}/s|$ for some constant M and all s less than some negative number S.

[5] D. Atkinson and F. Calogero, Phys. Rev. **185**, 1702 (1969).

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[8] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

[9] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. **25**, No. 9 (1949).

plus the integral over the path at infinity.] Moreover, Eq. (9) involves $L(\nu)$ only on the unitary cut, $\nu>0$. In that region, we will show that under rather general conditions unitarity constraints on $B(\nu)$ and $U(\nu)$ enable one, through (11), to obtain a condition on $L(\nu)$ which guarantees that the integral equation (9) is of Fredholm character and hence has a unique solution. The required assumption is that the high-energy behavior of the amplitude is controlled by a moving Regge singularity, together with weak conditions on the trajectory and residue functions, specified below. Our proof would fail for an amplitude whose asymptotic behavior was controlled by the exchange of a fixed Pomeranchuk trajectory. (The proof also works for an amplitude whose asymptotic behavior is governed by a fixed singularity in the angular momentum plane provided it occurs to the left of, not on, the line $j=1$.) The moving singularity could be either a Regge pole or cut.

To carry out the proof, we first show that, with the above assumption, $B(\nu)=O((\nu\ln\nu)^{-1})$ on the positive real axis. Second, we show that $U(\nu)$ and, therefore, $L(\nu)$ are $O(\ln\ln\nu/\nu)$ on the positive real axis. Finally, we observe that this condition on $L(\nu)$ renders Eq. (9) a Fredholm equation.

The asymptotic behavior of partial-wave amplitudes in a Regge theory has been discussed, for example, in Ref. 7. The partial-wave amplitude $A(\nu)$ is given by

$$A(\nu)=\int_{-1}^{1}A(\nu,x)P_l(x)dx. \tag{12}$$

Substituting in the asymptotic form $A(\nu,x)\sim\beta(t)s^{\alpha(t)}$ (a similar argument applies to the contribution from a u-channel trajectory), we rewrite (12) as

$$A(\nu)=2\int_{-(s-4)}^{-t_0}\beta(t)s^{\alpha(t)}P_l\left(1+\frac{2t}{s-4}\right)\frac{dt}{s-4}+2\int_{-t_0}^{0}\beta(t)s^{\alpha(t)}P_l\left(1+\frac{2t}{s-4}\right)\frac{dt}{s-4}. \tag{13}$$

In estimating the order of magnitude of integrals in (13), the Legendre polynomials are of no importance, since their arguments lie in the range -1 to 1 and hence they are of order 1. Then from the Froissart bound $\alpha(0)\leq 1$ and from our assumption that $\alpha(t)$ is a moving singularity, i.e., that near $t=0$

$$\alpha(t)=a+bt, \tag{14}$$

it easily follows that the second term in (13) is $O(1/\ln s)$. In order to treat the first term, one must make some assumption about the behavior of the amplitude for $t\approx -s$. A number of plausible assumptions will lead to the conclusion that the first term, and hence $A(\nu)$, also satisfies

$$A(\nu)=O(1/\ln s) \tag{15}$$

on the right-hand cut. For example, taking only the contribution of the leading trajectory,[10] if one makes the very weak assumption, true in almost any version of Regge theory, that $\alpha(t)=O(t^{-\alpha_0})$, $\alpha_0>0$, on the negative real axis, then, provided $\beta(t)=O(t^{\gamma})$, the first integral is equal to $O(s^{\gamma-\alpha_0})$ or $O(s^{-\alpha_0-1})$, whichever is the weaker condition, so that (15) would hold provided $\gamma<\alpha_0$. This would, for example, be true of the residue functions used in phenomenological fits, which invariably decrease for large negative t, as well as for those found in potential scattering with Coulomb or Yukawa potentials. If $\alpha(t)$, rather than simply being bounded by $t^{-\alpha_0}$, becomes more and more negative at large negative t, then even weaker conditions on $\beta(t)$ are required to ensure the validity of Eq. (15).[11] To continue our discussion, we must make the assumption that the behavior of the trajectory and residue functions is such that Eq. (15) is valid. Hence $B(\nu)=A(\nu)/\nu$ satisfies

$$B(\nu)=O((\nu\ln\nu)^{-1}) \tag{16}$$

as $\nu\to\infty$ on the physical cut.

We next turn to the behavior of $U(\nu)$ and $L(\nu)$. From Eqs. (2b) and (16), we have

$$U(\nu)=P\int_{0}^{\infty}\frac{f(\nu')d\nu'}{(\nu'-\nu)}, \tag{17}$$

where $f(\nu')=O((\nu'\ln\nu')^{-1})$. From a theorem of Frye and Warnock[2] it then follows that

$$U(\nu)=\frac{1}{\nu}\int_{0}^{\nu}f(\nu')d\nu'+O((\nu\ln\nu)^{-1})=O((\ln\ln\nu)/\nu). \tag{18}$$

It then follows from (11), (16) [or the weaker unitarity

[10] Of course, in that part of the range of integration in (12) or (13) in which both t and u are comparable with s, and hence the cosine of the scattering angle in neither the t nor u channel is large, one has no justification for representing the amplitude by the contribution of the leading trajectory in either the t or u channel. However, experimentally, differential cross sections are found to fall off extremely rapidly as one goes to regions away from the forward of backward direction. Hence it seems reasonable to hope that the contribution to (12) from the amplitude in these regions is not important in any event, and that our conclusions would not be altered if we could treat it more correctly, rather than adopting, in our present state of ignorance of the detailed behavior of the amplitude for t and u both comparable to s, the procedure of assuming that the amplitude is given everywhere by the sum of the contributions of the leading trajectories in the t and u channels. It is perhaps worth noting that in one case, that of the Coulomb amplitude, where we have an exact expression valid at all s and t [e.g., see S. C. Frautschi, *Regge Poles and S-Matrix Theory* (Benjamin, New York, 1963), p. 124], the conclusions as to the behavior of the second term of (13) are the same whether one uses the exact expression or represents the amplitude throughout the range of integration by the contribution of the leading trajectory.

[11] For example, it is easy to show by a straightforward application of the Stirling approximation that Eq. (15) would hold for the case of an amplitude given by the Veneziano representation with a trajectory having the Pomeranchuk behavior $\alpha(0)=1$.

bound $B(\nu)=O(\nu^{-1})$], and (18) that, on the right-hand cut, that is, in the range of integration in the integral equation (9),

$$L(\nu)=O(\ln\ln\nu/\nu). \quad (19)$$

Finally, we consider the implications of (19) for the N/D integral equation (9). The familiar Fredholm theory of integral equations guarantees that (9) has a unique solution provided

$$\int_0^\infty |L(\nu)|^2 d\nu < \infty \quad (20)$$

and

$$\int_0^\infty \int_0^\infty |K(\nu,\nu')|^2 d\nu d\nu' < \infty , \quad (21)$$

where

$$K(\nu,\nu')=\rho(\nu')[L(\nu')-(\nu/\nu')L(\nu)]/(\nu'-\nu). \quad (22)$$

There is clearly no problem in satisfying the condition (20) on the square integrability of the inhomogeneous term. [The usual behavior, $\delta_l \sim \nu^{l+1/2}$, of the phase shift at threshold gives $\mathrm{Im}B_l \sim \nu^l$, and hence $U_l(\nu)$ is finite at the origin for $l\geq 1$. Since $B_l(\nu)$ is also finite at the origin for $l\geq 1$, so is $L_l(\nu)$, and hence, of course, no problem arises with condition (20) from the behavior at the lower limit.] The problem arises from the condition (21) that the kernel be square integrable. It is well known and easy to show that (21) will be satisfied if, e.g., $L(\nu)=C/\nu+O((\nu\ln\nu)^{-1})$. On the other hand, if $L(\nu)\sim\ln\nu/\nu$, then (9) will not in general have a unique solution.[12] The result for the intermediate case which we have here, in which $L(\nu)$ is governed by Eq. (19), is probably less well known. However, in fact Eq. (19) is enough to guarantee (21), and hence the existence of a unique solution to Eq. (9); the Appendix contains a straightforward proof of this assertion. The fact that (19) is sufficient to guarantee (21) is required, independently of any problems with the convergence of the dispersion integral over the left-hand cut, if one is to be certain of the Fredholm character of the N/D equations for amplitudes governed by Pomeranchuk exchange, since Eq. (11), coupled with the preceding discussion, suggests that any such amplitude will have the asymptotic behavior (19).

In summary, then, provided that the asymptotic behavior of the amplitude is governed by a moving singularity in the angular momentum plane, that the phase shift at infinity obeys Levinson's theorem, and that the behavior of the trajectory and residue functions combine to ensure Eq. (15), then one may write $B=N/D$, where N obeys a Fredholm integral equation [Eq. (9)] and D is given by Eq. (5b). Hence, under these conditions, the requirements that there be no CDD poles, together with the specification of the function $L(\nu)$ for positive ν, determines the partial-wave amplitude uniquely, without arbitrary parameters, except for a possible subtraction in the S wave. In a Regge theory, the latter would presumably be determined by analytic continuation in j.

There still remains the question of to what extent the N/D equations are useful as a practical calculational tool. This depends on the extent to which the equations yield useful results when approximate forms are used for the function $L(\nu)$, since, of course, one does not have available the exact $L(\nu)$. The original hope, when the theory was developed, was that $L(\nu)$ would be given by a convergent dispersion relation of the form (2a) and, moreover, that the dispersion integral would be dominated by the "nearby" part of the left-hand cut, i.e., by the exchange of a few low-mass particles in the crossed channel. What we have here established is that one can write the N/D equations even if the integral in (2a) does not converge, i.e., even if $L(\nu)$ on the physical cut includes contributions from the integral over the infinite semicircle. However, in that situation, there would certainly be no grounds for believing that $L(\nu)$ could be well approximated by the contributions from the nearby part of the dispersion integral. And in any event, if one takes the nearby-singularities approach, and these singularities are particles or resonances of spin ≥ 1, then one gets an approximate L which does not obey (21) and which requires the introduction of an arbitrary cutoff parameter. An alternative approach is to take L to be the unphysical cut contributions to the leading Regge trajectories, in which case our preceding discussion will guarantee the Fredholm nature of the equation. This is essentially the approach taken in the various calculations based on the "new form of the strip approximation."[13] In these calculations, U is taken to include only the contribution of that portion of the physical cut between threshold and the "strip width" s_1. As in our preceding discussion, one can then define $L=A-U$ even if the dispersion integral over the unphysical cut fails to converge. Actually, in this approach, there is no problem with (21) because of the finite range of integration; there is an artificial logarithmic singularity introduced by the approximation procedure, so that the equation obtained is non-Fredholm, but it is possible to show that the solution may still be obtained by standard numerical methods.[14] Squires[7] has pointed out that one cannot expect to be able to calculate the left-hand discontinuity of the partial-wave amplitude from that of the leading Regge trajectories, because of the cancellations which occur in doing the partial-wave projection, but this need not concern us because we do not need the left-hand discontinuity, but rather the function $L(\nu)$ (the dispersion integral over the discontinuity if the integral converges) in the physical region. Since $A(\nu)$ is dominated by the leading Regge contributions at large

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[13] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964); P. D. B. Collins and R. C. Johnson, *ibid.* **182**, 1755 (1969), and references therein.

[14] C. E. Jones and G. Tiktopoulos, J. Math. Phys. **7**, 311 (1966).

positive ν, then indeed for ν large, at least, we will have $L(\nu)\approx L_{\text{Regge}}(\nu)$; whether this is also a good approximation for ν in the low-energy physical region is less clear. We might also note that, given the existence of N/D equations with a unique solution, one might hope that calculational procedures in which solutions are obtained by representing the contribution of distant unphysical singularities by poles whose parameters are determined by imposing some desired property on the amplitude[15] will give reasonable results.

Note added in proof. In a recent paper [Phys. Rev. D **2**, 786 (1970)] Park and Desai have shown that due to large cancellations the partial wave amplitude in the Veneziano representation satisfies an unsubstracted dispersion relation even though the trajectories are linearly rising, although their proof is limited to the case $\alpha(0)<1$. This may be an indication that conventional methods, using partial wave dispersion relations, can be applied to the discussion of the N/D equations, even if trajectories prove to be infinitely rising, though this is not clear. This does not, of course, affect the proof, subject to our assumptions, given here of the Fredholm character of the N/D equations.

APPENDIX

We here demonstrate that the kernel $K(\nu,\nu')$ as given by Eq. (22) satisfies the inequality (21) provided $L(\nu)$ obeys (19). Since $\rho(\nu')$, as seen from Eq. (6), vanishes as $\nu'^{3/2}$ at $\nu'=0$, while from the discussion following Eq. (22) $L(\nu)$ is finite at the origin, it is easy to see from Eq. (22) that $K(\nu,\nu')$ is finite at $\nu=\nu'$, even if both ν and ν' go to zero. Hence the only possible problem in satisfying (21) comes from the region at infinity, so that in investigating the validity of (21) we can, for simplicity, omit any finite region about the origin, e.g., the region with ν and ν' both <2. Since, from (6), $\rho(\nu')\sim\nu'$, we have from (19) and (22)

$$K(x,x')=O((\ln\ln x'-\ln\ln x)/(x'-x)),\qquad \text{(A1)}$$

so that to verify (21) we can consider the expression

$$\left(\int_0^2 dx'\int_2^\infty dx+\int_0^2 dx\int_2^\infty dx'+\int_2^\infty\int_2^\infty dx dx'\right)\times(\ln\ln x'-\ln\ln x)^2/(x'-x)^2.\qquad \text{(A2)}$$

We observe in passing that the weak singularity in the asymptotic form [Eq. (A1)] of the kernel at x or $x'=1$ is, of course, not present in the actual amplitude. It is easy to see that the contributions from the first two terms in (A2), corresponding to integrals over strips of infinite length but finite width, give finite results because of the factor $(x'-x)^{-2}$. Hence we need to consider the last term in (A2), which we denote by I. Transforming to polar coordinates, we have

$$I=\int_4^\infty dr\, r[I_1(r)+I_2(r)],\qquad \text{(A3)}$$

with

$$I_1(r)=\int_{\sin^{-1}(2/r)}^{\pi/4}\frac{d\theta[\ln\ln(r\sin\theta)-\ln\ln(r\cos\theta)]^2}{r^2(\sin\theta-\cos\theta)^2}\qquad \text{(A4)}$$

and I_2 defined similarly but with limits of integration $\frac{1}{4}\pi$ and $\frac{1}{2}\pi-\sin^{-1}(2/r)$. Writing the numerator in (A4) as $[\ln\ln r+\ln(1+\ln\sin\theta/\ln r)-(\ln\ln r+\ln(1+\ln\cos\theta/\ln r))]^2$, I_1 becomes

$$I_1(r)=\frac{1}{r^2}\left(\int_{\sin^{-1}(2/r)}^{\sin^{-1}(1/\ln^n r)}+\int_{\sin^{-1}(1/\ln^n r)}^{\pi/4}\right)\times d\theta\frac{[\ln(1+\ln\sin\theta/\ln r)-\ln(1+\ln\cos\theta/\ln r)]^2}{(\sin\theta-\cos\theta)^2},\qquad \text{(A5)}$$

where, as we shall see, n can be any number greater than 4. Denoting the first and second terms in (A5), respectively, by I_1' and I_1'', we have

$$|I_1'(r)|\leq M(1/\ln^n r)[\ln(1-\ln\tfrac{1}{2}r/\ln r)]^2/r^2,\qquad \text{(A6)}$$

where M is a finite constant. Since, for $0<y\leq 1$, $|\ln(1-y)|\leq y/(1-y)$, as follows immediately from $\ln(1-y)=\int_1^{1-y}dy/y$, one then has

$$|I_1'(r)|\leq M\frac{1}{\ln^n r}\frac{[(\ln\frac{1}{2}r)/(\ln 2)]^2}{r^2}.\qquad \text{(A7)}$$

To estimate I_1'', we note that, in the range of integration,

$$\left[\ln\left(1+\frac{\ln\sin\theta}{\ln r}\right)-\ln\left(1+\frac{\ln\cos\theta}{\ln r}\right)\right]^2=\left[\ln\left(1+\frac{\ln\sin\theta}{\ln r}\right)\right]^2F(\theta),\qquad \text{(A8)}$$

where $|F(\theta)|\leq 1$, and $[F(\theta)/\sin\theta-\cos\theta]^2$ is continuous at $\theta=\frac{1}{4}\pi$. Since in the range of integration, $|\ln(1+\ln\sin\theta/\ln r)|\leq|\ln(1-\ln\ln^n r/\ln r)|$, using the same inequality for $|\ln(1-y)|$ as previously yields

$$|I_1''(r)|\leq(M'/\ln^2 r)\times[\ln\ln^n r/(1-\ln\ln^n r/\ln r)]^2/r^2,\qquad \text{(A9)}$$

and hence

$$|I_1''(r)|\leq M''/(r^2\ln^p r),\qquad \text{(A10)}$$

where p is any number less than 2, e.g., 1.9. Therefore, taking n in (A7) greater than 4, and substituting (A7) and (A10), along with corresponding results for $I_2(r)$ into (A3), one obtains

$$|I|<C\int_4^\infty\frac{dr}{r\ln^{1.9}r}=C\ln^{-0.9}4\qquad \text{(A11)}$$

for some constant C, which establishes the validity of (21).

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