PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Dynamical Approach to Higher Symmetries

H. BANERJEE
Saha Institute of Nuclear Physics, Calcutta, India

AND

B. DUTTA-ROY
Stevens Institute of Technology, Hoboken, New Jersey 07030
(Received 26 January 1970)

It is shown that the family of sum rules recently proposed by us provides a very convenient framework for the realization of higher-symmetry relations among the trilinear hadron couplings. These sum rules are derived assuming (i) unsubtracted dispersion relations for the forward as well as the backward elastic hadronic scattering amplitudes corresponding to t-channel isospin and G-parity $I^G=1^+$, and (ii) ρ dominance in the $I^G=1^+$ channel. The remarkable feature of our sum rule is that for the scattering of hadrons of intrinsic parities η_1 and η_2, contributions from states with parity $\eta_1\eta_2$ are suppressed (for meson-meson scattering they are entirely absent). Therefore, it is argued that the sum rules for meson-meson and meson-baryon scattering considered here will be very well saturated by the states belonging to the $SU(6)$ 35-plet of negative-parity mesons and the 56-plet of positive-parity baryons, respectively. The consistency of this principle of saturation may be easily checked if one further assumes the universality of ρ-meson coupling to hadrons. To substantiate the validity of the last assumption, we have verified that its predictions are in excellent agreement with the values of the ρNN and ρKK coupling constants obtained from our sum rules. The hypothesis of universal ρ coupling leads to relations among the trilinear meson couplings which may be identified as the "broken"-$SU(6)$ predictions. In the meson-baryon sector, assuming $SU(3)$ symmetry, we obtain the $SU(6)$ results in the static limit $m_\rho/M_B \to 0$, where m_ρ and M_B are the ρ-meson and baryon masses, respectively. It is emphasized that, in contrast with similar derivations based on other sum rules, the present derivation is relatively free from ambiguities arising from arbitrary and sometimes untenable saturation assumptions.

I. INTRODUCTION

IN this paper we address ourselves to the central problem of higher symmetries in the physics of strong interactions, to wit, the formulation of a suitable dynamical framework from which the symmetry relations emerge in some well-defined approximation. To be specific, we present here a family of sum rules which provides a convenient and sensible framework for a dynamical derivation of "broken"-$SU(6)$ relations among the trilinear meson-meson (MMM) and meson-baryon ($MB\bar{B}$) coupling constants. These sum rules not only are consistent with the available experimental data, but possess the distinctive feature of being amenable to saturation, to a very good approximation, by the states belonging to the lowest-lying multiplets of the $SU(6)$ symmetry group. This remarkable advantage over other sum rules used in this context stems, as we shall see, from a parity selection rule which allows only a small subset of low-lying states (of a definite parity) to contribute. Thus the element of arbitrariness and the consequent ambiguities inherent in the usual saturation procedure are reduced to a considerable extent.

The sum rules on which the present investigation is based were first formulated[1] in an attempt to interpret theoretically the empirical success of the hypothesis of ρ dominance in low-energy pion-nucleon s-wave scattering. It was later shown[2] that for pion scattering on other hadron targets (K,Σ,Ξ) the sum rules give very satisfactory results. The physical content of our sum rule may be stated as follows. For any two hadrons each with nonzero isospin, we choose the particular combination of elastic scattering amplitudes which correspond to $I^G=1^+$ in the t channel, where I and G denote isospin and G parity, respectively. These amplitudes can be made free from kinematic singularities in the forward ($\cos\theta=1$) and in the backward ($\cos\theta=-1$) directions. The forward amplitude D and the backward amplitude F obey unsubtracted dispersion relations in the variables ω_L^2 and ω^2, ω_L and ω being the laboratory and c.m. energies, respectively, of one of the hadrons. The absence of subtraction in the dispersion relations follows, for the amplitude D from the Pomeranchuk theorem, and for the amplitude F from Regge asymptotics. The amplitudes D and F can always be normalized[3] so that at threshold $D_{\rm thr}=F_{\rm thr}$. Assuming that the $I^G=1^+$ t-channel singularities are dominated by the ρ pole, the sum rule then reads

$$\rho \text{ contribution to } F_{\rm thr}\approx 2\times(\text{contributions to } D_{\rm thr} \text{ from the direct-channel states with parity } P=-\eta_1\eta_2), \quad (1)$$

where η_1 and η_2 are the intrinsic parities of the two hadrons. In the case of meson-baryon scattering, the right-hand side of Eq. (1) gets contributions also from the "wrong"-parity states $P=\eta_1\eta_2$ but their contributions are relatively suppressed[1,2] by the factor $(E-M)/(E+M)$, where E is the c.m. energy of the

[1] H. Banerjee and S. Mallik, Nuovo Cimento 1, 87 (1969).
[2] H. Banerjee, B. Dutta-Roy, and S. Mallik, Nuovo Cimento 66A, 475 (1970).

[3] In our units $m_\pi=c=\hbar=1$. In the following the Mandelstam variables will be denoted, as usual, by s, t, and u.

baryon. For meson-meson scattering, the wrong-parity states are entirely absent. The usefulness of sum rule (1) actually stems from the fact that if one assumes the universality of the coupling of ρ mesons to hadrons, then the left-hand side is known.

The sum rule (1) is very well satisfied by the available experimental data.[4] Its superiority, in the present context, becomes evident when one considers the scattering of one negative-parity meson by another. The right-hand side of Eq. (1) then gets contributions only from the odd-parity meson states, all of which (at least for masses $\lesssim 1.6$ BeV) are contained in the 35-plet of $SU(6)$. Thus there is no arbitariness or ambiguity regarding truncation and, as we shall see, the (broken) $SU(6)$ symmetry emerges from the interrelationships of the different terms in the sum rule. The implication is that the predictions of the higher symmetry in so far as these relations are concerned will obtain in reality to a high degree of accuracy. In the case of scattering of pseudoscalar mesons by baryons, the contributions from the positive-parity states outside the 56-plet of baryons is found to be negligible. Thus here also our sum rules provide a satisfactory framework for the derivation of the higher-symmetry relations.

It is of some interest to discuss in this context the Adler-Weisberger–type sum rules[5,6] derived on the basis of the partial conservation of the axial-vector current (PCAC) and current algebra. Here one considers the scattering of π mesons on any hadron target. If, as before, one constructs the forward amplitude D which behaves as $I_t=1$ in the t channel, then the sum rule may be written as

$$I_T/(4\pi f_\pi^2) \approx \text{contributions to } D(\omega_L=1) \text{ from all direct-channel states}, \quad (2)$$

in terms of the pion decay constant f_π and the isospin I_T of the target. In order to facilitate the comparison of this sum rule with (1) one may at this stage use the KSRF relation[7] to rewrite sum rule (2) in the form[8–10]

$$I_T(g_{\rho\pi\pi}^2/4\pi m_\rho^2) \approx \text{contribution to } D(\omega_L=1) \text{ from all direct-channel states}, \quad (3)$$

where $g_{\rho\pi\pi}$ gives the coupling of the ρ meson to pions and m_ρ is the mass of the ρ meson. It is immediately clear that the physical content of sum rules (1) and (3) are quite different. The possibility that both of them are valid cannot be ruled out. Shaw[11] has, however, concluded on the basis of available experimental data that except for the πN case the sum rule (2) seems to fail in most other cases. Unitarity corrections may improve the situation but the various ambiguities[12] in such a scheme hardly restores one's confidence. In order to arrive at an unambiguous conclusion regarding the relative merits of the two sum rules, one should consider such a case where the contributions to the forward amplitude from the wrong-parity (positive parity in the case of meson-meson scattering and negative parity in the case of meson-baryon scattering) states is negligible and where unitarity corrections are expected to be small. In this respect the most favorable case is πK scattering. Here the prediction of sum rule (1) is in excellent agreement with experiment, whereas in the resonance approximation, the right-hand side of sum rule (2) is too small roughly by a factor of 2 compared to the left-hand side. It is now possible to understand why, as has been observed by Gilman and Harari,[6] the current-algebra sum rule when applied to $\pi\rho$ scattering yields two sum rules (corresponding to helicity 1 and 0 states of the target), each of which has a discrepancy of a factor of 2, but when one combines the two wrong-parity sum rules the resultant sum rule agrees with experiment. In some cases it may be possible to get around these difficulties by including contributions from more massive states.[6] But then one has to go outside the low-lying $SU(6)$ multiplets and to that extent the desired symmetry becomes obscure.

Another approach which has met with partial success in the dynamical derivation of the results of $SU(6)$ symmetry is the use of superconvergent dispersion relations.[13] The domain of applicability of this method is very limited compared to that of the sum rules mentioned above. The sum rules which one obtains from the assumption of superconvergence of the amplitudes which correspond to $I=2$ (or 27-plet) t-channel exchanges[14] are suspicious on theoretical grounds.[15] What, however, seems to be more relevant in the present context is the observation that these sum

[4] Quantitative verification of sum rule (1) may be made for πN (see Refs. 1 and 2) and KN scattering, where detailed experimental information is available. One may also write sum rules analogous to (1) for the usual A and B amplitudes for meson-baryon scattering. All these sum rules are in excellent agreement with the available experimental data [S. Mallik and S. Baba Pundari (private communication)].

[5] There have been numerous works along these lines. See, e.g., Ref. 6 and other works mentioned therein.

[6] F. J. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968).

[7] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 225 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966).

[8] For πN scattering, sum rule (3) may be easily recognized to be equivalent to the so-called ρ-dominant result for the s-wave scattering lengths proposed by Sakurai (Ref. 9). The threshold symmetry exhibited through this sum rule has been exploited by Bég (Ref. 10) in an attempt towards dynamical derivation of higher symmetry results.

[9] J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960); Phys. Rev. Letters **17**, 552 (1966).

[10] M. A. B. Bég, Phys. Rev. Letters **19**, 767 (1967).

[11] Graham Shaw, Phys. Rev. Letters **18**, 1025 (1967).

[12] F. Von Hippel and J. K. Kim, Phys. Rev. Letters **20**, 1303 (1968).

[13] V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, Phys. Letters **21**, 576 (1966).

[14] H. Harari, Phys. Rev. Letters **17**, 1303 (1966); B. Sakita and K. C. Wali, *ibid.* **18**, 29 (1967); G. Altarelli, F. Bucella, and R. Gatto, Phys. Letters **24B**, 57 (1967); P. Babu, F. J. Gilman, and M. Suzuki, *ibid.* **24B**, 65 (1967).

[15] R. J. N. Phillips, Phys. Letters **24B**, 342 (1967); I. J. Muzinich, Phys. Rev. Letters **18**, 381 (1967).

rules suffer from the same ambiguities with respect to the truncation procedure as those based on current algebra. Indeed, saturation of all the superconvergence sum rules consistently by a finite number of states [which may be restricted to an $SU(6)$ multiplet] may lead, as in the case of $\pi\rho$ scattering, to unrealistic results.[16]

In Sec. II we apply our sum rules to meson-meson scattering and obtain broken $SU(3)$ and $SL(6,C)$ results.[17] Section III is devoted to the meson-baryon sector wherein we show how in the static limit one obtains the $SU(6)$ relations.[18] In Sec. IV we present a critique of the approaches to higher-symmetry results based on current algebra and superconvergent dispersion relations.

II. SUM RULES FOR MESON-MESON SCATTERING

A. Sum Rules for Scattering of Pseudoscalar Mesons

The general procedure for deriving our sum rule has been developed elsewhere.[1,2] However, for the sake of completeness let us illustrate this in the example of πK scattering. Being free from spin complications, inessential for the present purpose, this case elucidates our method. Consider the Feynman amplitude M for πK scattering which corresponds to $I^G=1^+$ in the t channel $[M=\frac{1}{2}(M_{\pi^-K^+}-M_{\pi^+K^+})]$ normalized through the relation

$$M=8\pi(\sqrt{s})\sum_l (2l+1)f_l(s)P_l(\cos\theta)\,, \tag{4}$$

where f_l is one-third of the difference of the partial-wave amplitudes in the isospin $\frac{1}{2}$ and $\frac{3}{2}$ channels. Making the identifications

$$[M/4\pi(s-u)]_{\cos\theta=+1}=D(\omega_L^2)\,, \tag{5}$$

$$[M/4\pi(s-u)]_{\cos\theta=-1}=F(\omega^2)\,, \tag{6}$$

where ω_L and ω are the laboratory and c.m. pion energies, respectively, we note that $D(\omega_L^2)$ and $F(\omega^2)$ are free from kinematic singularities and, in accordance with our assumptions, obey unsubtracted dispersion relations in the variables ω_L^2 and ω^2, respectively. Thus

$$D(\omega_L^2=1)=\frac{1}{\pi}\int_{mk+1}^{\infty}\frac{dW}{\omega^2-1}\sum_l (2l+1)\,\mathrm{Im}f_l\,, \tag{7}$$

$$F(\omega^2=1)=\frac{1}{\pi}\int_{mk+1}^{\infty}\frac{dW}{\omega^2-1}\sum_l (-1)^l(2l+1)\,\mathrm{Im}f_l+\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}F(\omega^2)d\omega^2}{\omega^2-1}\,, \tag{8}$$

where W is the total c.m. energy. From the definitions for the amplitude F and D [see Eqs. (5) and (6)], and the fact that the invariant amplitude M depends on $\cos\theta$ only through the Mandelstam variable t, there follows the threshold identity:

$$F(\omega^2=1)=D(\omega_L^2=1)\,, \tag{9}$$

whence we obtain

$$\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}F(\omega^2)d\omega^2}{\omega^2-1}=\frac{1}{\pi}\int_{mk+1}^{\infty}\frac{dW}{\omega^2-1}\times\sum_l[1-(-1)^l](2l+1)\,\mathrm{Im}f_l\,. \tag{10}$$

The assumption of ρ dominance enables us to replace the left-hand side of Eq. (10) by the ρ-exchange contribution to the backward amplitude. It is important to note that the right-hand side of Eq. (10) gets contributions only from odd partial waves, and the only known resonance that can contribute is $K^*(890)$ for which we use the narrow-width approximation to obtain

$$\tfrac{3}{4}\Gamma_\rho/q_\rho^3=\Gamma_{K^*}/q_{k^*}^3\,, \tag{11}$$

where Γ_ρ (Γ_{K^*}) and q_ρ (q_{K^*}) are the total widths and the c.m. momentum of the decay products in ρ (K^*) decay, respectively. This relation between the ρ and K^* widths given by Eq. (11) is in very good agreement with experiment[19] and is a broken-$SU(3)$ result in the sense that it reduces to the $SU(3)$ relation if the masses of ρ and K^* are taken to be equal.

For the scattering of self-conjugate mesons, e.g., pion-pion scattering, our sum rule will reduce to trivial identities because the crossed and the direct channels are identical. The requirement that both the hadrons must have nonzero isospin, therefore, means that the only other nontrivial sum rule in the sector of pseudoscalar mesons is that for KK scattering. Here the amplitude $M=(M_{K^-K^+}-M_{K^0K^+})-(M_{K^+K^+}-M_{K^0K^+})$ corresponds to $I^G=1^+$ in the t channel, Following our method, spelled out in detail for πK scattering, and saturating the resulting sum rule by ρ-, ω-, and ϕ-meson states in the direct channel, we arrive at the result

$$\frac{3g_{\rho KK}^2}{4\pi m_\rho^2}=\frac{g_{\omega KK}^2}{4\pi m_\omega^2}+\frac{g_{\phi KK}^2}{4\pi m_\phi^2}\,. \tag{12}$$

The relationship[17] among these coupling constants[20] in the "relativistic version" of $SU(6)$ symmetry is consistent with our result (12) if we put all the masses equal.

[16] F. E. Low, in *Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, 1966* (California U. P., Berkeley, 1967), p. 244.

[17] B. Sakita and K. C. Wali, Phys. Rev. **139**, B1355 (1965).

[18] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964).

[19] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[20] The coupling scheme used in the meson sector is defined through the interaction Lagrangian

$$\mathcal{L}_{\mathrm{int}}=g_{\rho\pi\pi}\boldsymbol{\rho}_\mu\cdot\boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi}+ig_{\rho KK}\boldsymbol{\rho}_\mu\cdot\bar{K}\boldsymbol{\tau}\overleftrightarrow{\partial}_\mu K+ig_{\omega KK}\omega_\mu\bar{K}\overleftrightarrow{\partial}_\mu K$$
$$+ig_{\phi KK}\phi_\mu\bar{K}\overleftrightarrow{\partial}_\mu K+ig_{K^*K\pi}\bar{K}_\mu^*\boldsymbol{\tau}K\overleftrightarrow{\partial}_\mu\cdot\boldsymbol{\pi}+if_{\omega\rho\pi}\epsilon_{\mu\nu\sigma\lambda}\partial_\mu\omega_\nu\partial_\sigma\boldsymbol{\rho}_\lambda\cdot\boldsymbol{\pi}$$
$$+if_{K^*K^*\pi}\epsilon_{\mu\nu\sigma\lambda}\partial_\mu\bar{K}_\nu^*\boldsymbol{\tau}\partial_\sigma K_\lambda^*\cdot\boldsymbol{\pi}+if_{K^*\rho K}\epsilon_{\mu\nu\sigma\lambda}\partial_\mu\bar{K}_\nu^*\boldsymbol{\tau}K\cdot\partial_\sigma\boldsymbol{\rho}_\lambda.$$

B. Sum Rules for Pseudoscalar Meson Vector Meson Scattering

Here the Feynman amplitude M may be written in terms of the amplitudes A_i ($i=1$ to 4) which are free from kinematic singularities:

$$M=A_1 e\cdot e'+A_2 e\cdot Pe'\cdot P+A_3 e\cdot Qe'\cdot Q \\ +\tfrac{1}{2}A_4(e\cdot Pe'\cdot Q+e\cdot Qe'\cdot P)\,, \quad (13)$$

where e (e') is the polarization vector for the initial (final) vector meson and $P=\frac{1}{2}(p+p')$, $Q=\frac{1}{2}(k+k')$, where p (p') and k (k') are the four-momenta of the initial (final) vector and pseudoscalar meson, respectively. We consider, as before, the combination of amplitudes corresponding to $I^G=1^+$ in the t channel. It may easily be verified that the helicity-nonflip amplitudes

$$D_\lambda=M_\lambda(s,\cos\theta=+1)/4\pi(s-u)\,, \quad (14)$$

$$F_\lambda=M_\lambda(s,\cos\theta=-1)/4\pi(s-u)\,, \quad (15)$$

where λ denotes the helicity of the vector meson, are free from kinematic singularities. In terms of the scattering amplitude, defined as $f_\lambda(s,\cos\theta)=M_\lambda/8\pi\sqrt{s}$, our sum rule then reads

$$[F_\lambda(\omega=1)]_\rho=\frac{1}{\pi}\int_L^\infty \frac{dW}{\omega^2-1} \\ \times \mathrm{Im}[f_\lambda(s,\cos\theta=1)-f_\lambda(s,\cos\theta=-1)]\,, \quad (16)$$

where the left-hand side gives the ρ contribution to the background amplitude at threshold. The lower limit (L) of the dispersion integral extends to the lowest singularity in the s channel so as to include the relevant pole terms. As emphasized before, only negative-parity states are able to contribute on the right-hand side. For the case $\lambda=0$, in the resonance approximation, the pseudoscalar mesons are thus seen to be the only candidates that can contribute, and we get the relation

$$2\mu_V+3Q_V m_V^2=2\,, \quad (17)$$

where μ_V and Q_V are the strengths of the "magnetic" and "quadrupole" couplings of the ρ meson to the target vector meson of mass m_V. These quantities are defined through the matrix element of the ρ current between the initial and final vector-meson states thus:

$$\langle e'p'|J_\mu{}^\rho|ep\rangle=(2p_0 2p_0')^{-1/2}g_{\rho VV'}[e\cdot e'(p+p')_\mu \\ +\mu_V(e_\mu e_\nu'-e_\nu e_\mu')(p-p')_\nu+Q_V e\cdot p'e'\cdot p(p+p')_\mu]. \quad (18)$$

Relation (17) between the magnetic and quadrupole coupling strengths in the VVV vertex can be compared with the values given by Sakita and Wali[17] in their relativistic formulation of $SU(6)$ symmetry. They obtain $\mu_V=2$ and $Q_V m_V^2=-\frac{2}{3}$, which is consistent with our sum rule (17).

Considering the amplitudes for $\pi\rho$, πK^*, and ρK scattering which correspond to $I^G=1^+$ t-channel exchange, we note that in all these cases the right-hand side of our sum rule for the helicity state $\lambda=1$ can get contributions only from the negative-parity vector mesons. In $\pi\rho$ scattering, for example, the only candidate is the ω meson. We need not consider the ϕ meson because its coupling to the $\rho\pi$ system is negligible.[19] We thus obtain the following sum rules:

$$\frac{g_{\rho\pi\pi}{}^2}{4\pi m_\rho{}^2}=\frac{1}{4}\frac{f_{\omega\rho\pi}{}^2}{4\pi} \quad (19)$$

$$=\frac{f_{K^*K^*\pi}{}^2}{4\pi}=\frac{f_{K^*\rho K}{}^2}{4\pi}. \quad (20)$$

Relations (19) and (20) are exactly the same as those one obtains in broken $SU(6)$ symmetry.[17] Moreover, since the truncation procedure in this case is quite unambiguous, we expect these relations to hold to a very good approximation. Unfortunately, these relations must be compared rather indirectly with experiment, through the use of the Gell-Mann, Sharp, and Wagner[21] (GSW) model. In this model, using the decay width $\Gamma(\rho\to 2\pi)=122\pm 7$ MeV,[22] the $\rho\gamma$-coupling constant determined from the known $\rho\to e^+e^-$ branching ratio,[19] and Eq. (19), we obtain the decay width $\Gamma(\omega\to\pi^0\gamma)=1.03\pm 0.25$ MeV, which is in very good agreement with the experimental[19] value 1.16 ± 0.23 MeV. As for the process $\omega\to 3\pi$, the use of Eq. (19) yields in the GSW model $\Gamma(\omega\to 3\pi)=7.5\pm 0.7$ MeV, whereas the experimental[19] value is 11.0 ± 1.2 MeV. Here, however, one could attribute the discrepancy to a direct $\omega 3\pi$ coupling as has been suggested by some authors.[17] Again, for the decay $\pi^0\to 2\gamma$ in the GSW model we obtain $\Gamma(\pi^0\to 2\gamma)=6.8\pm 0.3$ eV, compared to the experimental value $\Gamma(\pi^0\to 2\gamma)=7.4\pm 1.5$ eV. Thus, as far as we are able to tell, the sum rule (19) is in very good agreement with experiment. Insofar as the sum rule (20) is concerned, all that can be said is that if we take the decay $K^*\to K\pi\pi$ to occur predominantly through the diagrams corresponding to $K^*\to(K^*\pi)\to K\pi\pi$ and $K^*\to(K\rho)\to K\pi\pi$, then using sum rule (20) we obtain $\Gamma(K^*\to K\pi\pi)\approx 4.5$ keV, whereas experiment merely places an upper bound $\Gamma(K^*\to K\pi\pi)\lesssim 100$ keV with which our result is consistent.

III. SUM RULES FOR SCATTERING OF PSEUDOSCALAR MESONS ON BARYONS

It has already been remarked that from the point of view of truncation, the sum rules for the scattering of pseudoscalar mesons from baryons are not as clean as those for meson-meson scattering. It is true that the wrong-(negative-) parity states contribute here but

[21] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters 8, 261 (1962).
[22] Matts Roos and Jan Pisut, Nucl. Phys. B10, 563 (1969).

at least their contributions are relatively suppressed by a factor $(E-M)/(E+M)$ which is quite small in the low-energy region. What makes matters worse is the presence, as in πN or $\bar{K}N$ scattering, of many "right-" (positive-) parity baryonic resonances outside the lowest occurring $SU(6)$ 56-plet of baryons. It would nevertheless be necessary for our investigation of higher symmetry to assume that the net contribution from these states would be negligible compared to those from the low-lying states in the 56-plet. The best ground for testing the validity of this assumption is the sum rule for πN scattering. Here the sum rule reads[1,2]

$$\frac{g_{\rho\pi\pi}g_{\rho NN}}{4\pi m_\rho^2}\left(1+K_N{}^V\frac{m_\rho^2}{4m_N^2}\right) = \frac{g_r^2}{8\pi m_N^2}-\left[-\frac{1}{\pi}\int_1^\infty \frac{\mathrm{Im}D(\omega_L)d\omega_L}{\omega_L^2-1} +\frac{1}{\pi}\int_1^\infty\frac{\mathrm{Im}F(\omega)\omega d\omega}{\omega^2-1}\right], \quad (21a)$$

where

$$D(\omega_L)=(1/4\pi)[A^-(s,t=0)+\omega_L B^-(s,t=0)], \quad (21b)$$

$$F(\omega)=(1/4\pi)[(E/m\omega)A^-(s,t=-4(\omega^2-1)) +B^-(s,t=-4(\omega^2-1))], \quad (21c)$$

with A^- and B^- in the notation of Ref. 1. In the above $K_N{}^V$ is the isovector anomalous nucleon magnetic moment and the meaning of the other parameters is self-evident. The F integral appearing within the square brackets on the right-hand side of Eq. (21a) has been evaluated by Lovelace.[23] The D integral can be obtained by using the s-wave πN scattering length[23] $a_1-a_3=0.263$ and the value $g_r^2/4\pi=14.6$ for the coupling constant. Thus one finally gets the value 0.096 for the expression within the square brackets. If one instead approximates this entire expression just by the contribution from the P_{33} wave [which is dominated by the $N^*(1236)$ resonance], using the parametrization given by Roper *et al.*,[24] one gets the value +0.095. Thus the approximation of the entire right-hand side of our sum rule for meson-baryon scattering just by the contributions from the baryons in the $SU(6)$ 56-plet is extremely good, at least for πN scattering. The same state of affairs has been found to hold for KN scattering as well.

Some comments on the validity of the hypothesis of the universality of ρ-meson coupling to hadrons (H) are relevant here. This hypothesis, which actually follows from the conservation of the isospin current and the dominance in the low-energy region of the $I=1$, even G-parity channel by the ρ meson, implies that for any hadron H with isospin I_H

$$g_{\rho HH}^2/4\pi=I_H^2\times g_{\rho\pi\pi}^2/4\pi=I_H^2\times 2.4. \quad (22)$$

From our sum rule (11) for πK scattering, we obtain $g_{\rho KK}^2/4\pi=0.60$. As for the ρN coupling constant, the sum rule (21a) gives $g_{\rho NN}^2/4\pi=0.63$. Thus within the limits of experimental errors as well as errors arising from the narrow-width approximations, our sum rule puts the universality hypothesis on a firm footing. In view of the fact that the naive ρ-dominance hypothesis of Sakurai[9] leads to a value[25] for the ρN coupling which is difficult to reconcile with the universality of ρ-meson coupling, the above result may indeed be considered as another triumph of our sum rule.

The sum rule (21a) for πN scattering may be supplemented by other meson-baryon sum rules. It is, however, clear that in order to establish contact with the results of $SU(6)$ symmetry, we have to make narrow-width approximations for the resonance contributions, and assume $SU(3)$ symmetry so as to reduce the multiplicity of parameters in the sum rules. A note of warning is in order at this stage. It is well known that the narrow-width approximation for a broad resonance like $N^*(1236)$ overestimates its contribution to the sum rule by about 30% and that the $SU(3)$ predictions for the interrelationships between the different resonance widths are not in good agreement with the experimental data. Thus the main advantage of our sum rule, viz., the nice property of their being well saturated by the low-lying octet of baryons and the decuplet of $\frac{3}{2}^+$ resonances, would be vitiated to the extent that the above approximations are inaccurate.

At the level of $SU(3)$ symmetry we have three independent sum rules for pseudoscalar meson-baryon scattering. If we choose the sum rules for πN, $\pi\Sigma$, and $\pi\Xi$ scattering, we get

$$\frac{g_{\rho\pi\pi}^2}{4\pi m_\rho^2}\left[1+\kappa_p(1-\chi)\frac{m_\rho^2}{4m_B^2}\right]=\frac{g_r^2}{4\pi m_B^2}-\frac{8}{3}\frac{g^{*2}}{4\pi m^{*2}}, \quad (23a)$$

$$2\frac{g_{\rho\pi\pi}^2}{4\pi m_\rho^2}\left[1+\kappa_p(1+\tfrac{1}{2}\chi)\frac{m_\rho^2}{4M_B^2}\right] =\frac{g_r^2}{4\pi M_B^2}[\tfrac{2}{3}(1-\alpha)^2+2\alpha^2]+\frac{2}{3}\frac{g^{*2}}{4\pi M^{*2}}, \quad (23b)$$

$$\frac{g_{\rho\pi\pi}^2}{4\pi m_\rho^2}\left[1+\kappa_p(1+2\chi)\frac{m_\rho^2}{4M_B^2}\right] =\frac{g_r^2}{4\pi M_B^2}(1-2\alpha)^2+\frac{4}{3}\frac{g^{*2}}{4\pi M^{*2}}, \quad (23c)$$

where κ_p ($\chi\kappa_p$) is the anomalous magnetic moment in nuclear magnetons of the proton (neutron); M_B and

[23] C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles, Heidelberg, 1967*, edited by H. Filthuth (North-Holland, Amsterdam, 1968), p. 79.

[24] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).

[25] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966); P. Signell and J. W. Durso, *ibid.* **18**, 185 (1967).

M^* are the mean masses of the baryons in the $SU(3)$ octet and decuplet, respectively. In the above, g^* is the dimensionless πNN^* coupling constant,[26] and the parameter $\alpha \equiv F/(F+D)$ determines the F/D ratio for the pseudoscalar meson-baryon coupling. It is interesting to note that the quantities within the square brackets on the left-hand side of Eqs. (23) are actually the Sachs-type "charge" form factor with $G_E(0)=1$, for the ρ-meson coupling to baryons. The sum rules (23) provide fewer equations than the unknowns present. However, if one considers the static limit, viz., $m_\rho^2/4M_B^2 \to 0$, one indeed recovers the predictions of static $SU(6)_\sigma$ symmetry[18]:

$$\alpha = 2/5, \tag{24a}$$

$$\frac{g_{\rho\pi\pi}^2}{4\pi m_\rho^2} = \frac{9}{25}\frac{g_r^2}{4\pi M_B^2} = \frac{3}{2}\frac{g^{*2}}{4\pi M^{*2}}, \tag{24b}$$

as one of the solutions of the sum rules. The other solution is an unphysical one corresponding to $\alpha=1$ which forbids $N^* \to N\pi$.

IV. COMMENTS ON OTHER APPROACHES

In their search for a dynamical approach to higher symmetry, several authors[6] have considered the sum rules which result from combining the commutator of chiral charges

$$2I_3 = [\chi^+, \chi^-], \tag{25a}$$

where

$$\chi^\pm = \int d^3x\, A_0^\pm(x), \tag{25b}$$

with the PCAC relation

$$\partial_\mu A_\mu^\pm = f_\pi \phi_\pi^\pm. \tag{25c}$$

For πN scattering, the resulting sum rule may be written as

$$\frac{1}{8\pi f_\pi^2} = \frac{g_r^2}{16\pi m_N^2} + \frac{1}{\pi}\int \frac{\mathrm{Im}D(\omega_L)}{\omega_L^2 - 1} d\omega_L, \tag{26}$$

which is very well satisfied by the experimental numbers.[27,28] However, the moment one attempts to saturate the contributions from the continuum by the N_{33}^* resonance, the agreement disappears, the right-hand side in this approximation being about 60% less than the left-hand side. In extending this sum rule to other processes such as $\pi\Sigma$, $\pi\Xi$ scattering, one is confronted with unknown constants in the baryon-pole contributions on the right-hand side. If, as seems reasonable from the recent analysis of Kim,[29] one assumes $SU(3)$ symmetry and takes $\alpha \equiv F/(F+D) \approx 0.4$ for the meson-baryon coupling, then the agreement of the two sides, as noted by Shaw,[11] even in the full sum rules for $\pi\Sigma$ and $\pi\Xi$ scattering is very bad. As for the truncated version, the situation is still worse, the left-hand side being as large as twice the right-hand side. One might indeed have anticipated this large discrepancy on the basis of the sum rule proposed by us. Thus even if some higher-symmetry result emerges from such badly satisfied sum rules, its significance in the real physical world is obscure.

The situation is no better, if not worse, in the case of meson-meson scattering. As an illustration let us consider the current-algebra sum rule for πK scattering. If one makes the resonance approximation for the contributions from the continuum on the right-hand side of the sum rule, one obtains in the narrow-width limit, which is quite justified for the $K^*(890)$ resonance, the result

$$1/4\pi f_\pi^2 = 2g_{K^*K\pi}^2/4\pi m_{K^*}^2. \tag{27}$$

The contributions from the higher resonances may for all practical purposes be neglected either because they do not decay[19] in the elastic πK channel or because they occur with large energy denominators and have small elastic widths. Thus, *a priori*, it seems that truncation here is justified. Numerically, however, the left-hand side is approximately twice the right-hand side in relation (27). Again this is exactly what one expects in view of our sum rule and the KSRF relation. In the current-algebra sum rule it is difficult to see how the neglected resonance contributions can compensate for the discrepancy, let alone to prove analytically that their net contribution is equal to that from $K^*(890)$. The claim by Riazuddin and Fayyazuddin[7] that relation (11) follows from current algebra and PCAC is, therefore, open to doubt.[30] There seems to be an underlying order in the way the truncated current-algebra sum rules, at least those for meson-meson scattering, seem to fail. Indeed the same discrepancy by a factor of 2 between the two sides of the sum rule has also been noted by Gilman and Harari[6] in their consideration of $\pi\rho$ scattering. From the two sum rules corresponding to the target helicity states 0 and 1, they obtain

$$\frac{1}{2\pi f_\pi^2} = \frac{1}{m_\rho^2}\frac{g_{\rho\pi\pi}^2}{4\pi} \tag{28}$$

$$= \frac{1}{4}\frac{f_{\omega\rho\pi}^2}{4\pi}. \tag{29}$$

[26] We adopt the coupling scheme defined through the interaction Lagrangian

$$\mathcal{L}_{\mathrm{int}} = ig_r \bar{N}\gamma_5 \boldsymbol{\tau} N \cdot \boldsymbol{\pi} + i(g^*/M^*)\bar{N}^+ N_\mu^{*++}(\partial\pi^-/\partial x_\mu) + \cdots.$$

[27] Whereas, in principle, one should write for the right-hand side in Eq. (26) its "analog" for zero-mass external pions, in actual calculation one often uses, as has been done here, the physical pion mass (see, for example, Ref. 11). The latter procedure may be justified in several models (see Ref. 28). Indeed one can identify the sum rule (26) with the ρ-dominant sum rules considered in Refs. 9 and 10 only if this modification is permissible.

[28] S. L. Adler, Phys. Rev. **140**, B736 (1965); G. Hohler and R. Strauss, Phys. Letters **24B** 409 (1967).

[29] J. K. Kim, Phys. Rev. Letters **19**, 1079 (1967); C. H. Chan and F. T. Meire, *ibid.* **20**, 568 (1968).

[30] D. Geffen, Phys. Rev. Letters **19**, 770 (1967); S. G. Brown and G. B. West, *ibid.* **19**, 812 (1967).

Relation (28) is in obvious disagreement with the KSRF relation by a factor of 2. It is, however, interesting to note that if for the moment one disregards this systematic discrepancy in each of the sum rules, one obtains from the three sum rules (27)–(29) the result

$$\frac{f_{\omega\rho\pi}^2}{4\pi}=\frac{4}{m_\rho^2}\frac{g_{\rho\pi\pi}^2}{4\pi}=\frac{16}{m_{K^*}^2}\frac{g_{K^*K\pi}^2}{4\pi}, \tag{30}$$

which is consistent with our result in Sec. II and hence with broken $SU(6)$ symmetry.

Gilman and Harari[6] invoke the A_1 meson to restore the agreement of the two sides in sum rule (28). According to them the decay $A_1(1070)\to\rho\pi$ takes place only in the longitudinal mode and thus the A_1 contribution leaves sum rule (29) unaltered. The symmetry result (30) is, therefore, no longer true in this model. On the other hand, according to Geffen,[30] A_1 decays both by the longitudinal and the transverse modes and contributes equally to the sum rules (28) and (29), leaving the symmetry result (19) intact.

The work of de Alfaro *et al.*[13] has provided considerable impetus to a systematic search[14] for higher-symmetry results within the framework of sum rules based on superconvergent dispersion relations. These authors[13] argued that according to Regge asymptotics the amplitudes A_3^1 and A_4^2 [see Eq. (13)] for $\pi\rho$ scattering, where the superscripts denote the t-channel isospin, should obey superconvergent dispersion relations. The resultant sum rules, when saturated by the π, ω, and ϕ mesons, directly yield the broken $SU(6)$ relation (19) between the coupling constants $f_{\omega\rho\pi}$ and $g_{\rho\pi\pi}$. In view of our previous experience with the current-algebra sum rules, the neglect of $A_1(1070)$ and $A_2(1300)$ mesons in the saturation procedure is quite arbitrary. This fact is clearly brought home when, following Low,[16] one observes that the saturation of the superconvergence sum rule for the amplitude sA_2^2 which, in accordance with the previous assumptions, should behave asymptotically as $s^{-1-\epsilon}$ $(\epsilon>0)$, by π, ω, and ϕ mesons yields the unrealistic result $f_{\omega\rho\pi}=g_{\rho\pi\pi}=0$. These ambiguities are present in the sum rules for the meson-baryon sector[14] as well where one usually assumes that the (helicity-flip) B amplitude corresponding to the $SU(3)$ 27-plet exchange in the t channel is superconvergent. Indeed, saturation of these sum rules by the members of the $SU(6)$ 56-plet only, i.e., the $\frac{1}{2}^+$ baryons and the $\frac{3}{2}^+$ baryonic resonances, is difficult to justify *a priori* since contributions from the d-wave resonances are heavily weighted here. Apart from these practical difficulties, objections on theoretical grounds have also been raised[15] recently. It has been argued that Regge cuts arising from double ρ exchange may invalidate the superconvergence assumption for amplitudes corresponding to t-channel isospin $I=2$, for which one usually assumes that the zero-energy intercept of the relevant Regge trajectory is less than zero.

V. CONCLUSIONS

In any dynamical approach to higher-symmetry relations within the framework of sum rules, truncation is unavoidable. The element of arbitrariness involved in the procedure of truncation, unless minimized, reduces the significance of the results of such an approach. The truncation errors in our sum rules, in contract to those discussed in Sec. IV, are indeed minimal. The reason is that due to the operation of a parity-selection rule, our sum rule does not get contributions from many of the low-lying states, whose neglect otherwise would be quite arbitrary. The situation is most favorable in the case of sum rules for meson-meson scattering. Here only the negative-parity mesons contribute, all of which (at least those with mass $\lesssim 1.6$ BeV) are contained in the 35-plet of $SU(6)$. One would, therefore, expect that the predictions of our sum rule, which are also consistent with (broken) $SU(6)$ symmetry, should hold to a good approximation.

In the case of meson-baryon scattering, low-lying positive-parity baryonic states dominate our sum rule. Here, one would expect the sum rule for πN scattering to be most unfavorable from the point of view of truncation. This is because the nucleon pole and the $N^*(1236)$ contributions are of opposite signs and there exists a low-lying resonance (at 1470 MeV) in the P_{11} state. But even here the principle of saturating the sum rules only by the baryonic states in the $SU(3)$ octet and decuplet works very well, the error due to the truncation being of the order of 2% only. But the narrow-width approximation for the $N^*(1236)$ contribution and the assumption of $SU(3)$ symmetry introduce errors in our sum rules. It is, however, interesting to note that our results that the $SU(6)$ relations, namely, $\alpha=0.4$ and relation (24b), among the coupling constants emerge only in the static limit $(m_\rho^2/M_B^2\to 0)$ is compatible with predictions[31] of the inhomogeneous $SL(6,C)$ group.

The sum rules for trilinear hadronic coupling constants presented in Secs. II and III by no means exhaust the list of all those that may be obtained by our method. Following our procedure one can, in principle, write down a sum rule for the coupling of any three hadrons allowed by strong interactions, provided only the product of their intrinsic parities is odd and two of the hadrons have nonvanishing isospin.

To conclude, let us make a few comments about the basic assumptions underlying the derivation of our sum rules. It should be emphasized that Regge asymptotics, though sufficient, is not necessary to justify our assumption of an unsubtracted dispersion relation for the backward amplitude. Our assumption would be valid under wider conditions. If, for example, the asymptotically leading terms in each partial-wave

[31] W. Ruhl, Nuovo Cimento **37**, 319 (1965).

amplitude $a_l(s)$, for $l<O(s^{1/2})$, is independent of l, as is the case for scattering by a black sphere, our assumption would be true. Perhaps the most vulnerable among our assumptions is the one about the universality of ρ coupling to hadrons. It is, however, possible to turn our argument around and use our sum rules as a testing ground for the hypothesis of universality. In the sum rules for $\pi\Sigma$ or $\pi\Xi$ scattering, one is confronted with unknown baryon-pole contributions. But the sum rules for πN and πK scattering, which are free from these difficulties, yield values of $g_{\rho NN}$ and $g_{\rho KK}$ which are in excellent agreement with the hypothesis of universality. Other "experimental" checks on the assumption of universal ρ-meson coupling to hadrons consist in examining our sum rules for Kp scattering (helicity nonflip) and those for the helicity-flip amplitudes B in both πN and Kp scattering. Investigations in these directions have been carried out and the results, to be published elsewhere, justify our assumptions.

ACKNOWLEDGMENTS

We are indebted to S. Mallik and S. Baba Pundari for communicating to us the results of their investigations.

PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Fredholm Character of the N/D Equations

Allen E. Everett*

Department of Physics, Tufts University, Medford, Massachusetts 01803

(Received 18 June 1970)

We point out that, under rather general assumptions (in particular, that the Pomeranchuk singularity has either nonzero slope or intercept less than 1), unitarity guarantees that the N/D method leads to an integral equation for partial-wave amplitudes which is of Fredholm type, and hence possesses a unique solution, regardless of the behavior of the amplitude as s goes to infinity along the unphysical cut.

The so-called N/D equations for partial-wave amplitudes have been one of the basic tools in the S-matrix approach to the theory of strong interactions. The starting point in obtaining these equations has, in general, been a dispersion relation for the partial-wave amplitude.[1,2] We define the amplitude $A_l(s)$ to be the usual invariant partial amplitude and let $B_l(s) = A_l(s)/(s-s_0)$; that is, $B_l(s)$ is normalized so that, in the case of elastic unitarity, it is given by $B_l(s) = 2[s/(s-s_0)]^{1/2}(e^{i\delta_l}\sin\delta_l)/(s-s_0)$. (We use the usual Mandelstam variables: s, the c.m. energy squared, and t, the negative square of the four-momentum transfer. s_0 is a subtraction point.) Then $B_l(s)$ is taken to satisfy the dispersion relation

$$B_l(s)=L_l(s)+U_l(s)\,, \tag{1}$$

where $L_l(s)$ and $U_l(s)$, the dispersion integrals over the unphysical (left-hand) cut and the physical (unitary) cut, respectively, are given by

$$L_l(s)=\frac{A_l(s_0)}{s-s_0}+\frac{1}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}B_l(s')ds'}{(s'-s)} \tag{2a}$$

and

$$U_l(s)=\frac{1}{\pi}\int_{4}^{\infty}\frac{\mathrm{Im}B_l(s')ds'}{(s'-s)}. \tag{2b}$$

(Throughout the paper we assume for simplicity the kinematics corresponding to the elastic scattering of equal-mass spinless particles; this simplification has no effect on our arguments. We choose our units so that the scattering particles have unit mass.) Assuming the validity of Eq. (1), one then carries out the usual decomposition

$$B(s)=N(s)/D(s)\,, \tag{3}$$

where we have dropped here and for the remainder of the paper the irrelevant subscript l. In Eq. (3), $D(s)$ is the Omnes function[3]

$$D(s)=\exp\left(-\frac{s-s_0}{\pi}\int_4^{\infty}ds'\frac{\delta(s')}{(s'-s)(s'-s_0)}\right), \tag{4}$$

where $\delta(s)$ is the real part of the phase shift, and has a right-hand (unitary) cut only, while $N(s)=A(s)D(s)$ has only a left-hand cut. In the remainder of our discussion we will consider the approximation of purely elastic unitarity, which is made in most actual calculations. To discuss the situation with inelastic effects included, one would have to proceed in the same way using the Frye-Warnock form of the N/D equations.[2] The functions $N(s)$ and $D(s)$ satisfy the coupled integral equations, in the approximation of elastic unitarity,

$$N(s)=\frac{A(s_0)}{s-s_0}+\frac{1}{\pi}\int_{-\infty}^{0}ds'\frac{f(s')D(s')}{(s'-s)} \tag{5a}$$

* Supported in part by the U. S. Atomic Energy Commission.

[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963); R. L. Warnock, *ibid.* **131**, 1320 (1963).

[3] R. Omnes, Nuovo Cimento **8**, 316 (1958); **21**, 524 (1961).