For rank one, we obtain

$$(\gamma_i{}^J)^2=(g_i{}^J)^2\Big/1-\sum_k\int dv_k\,|g_k{}^J|^2E_k. \quad \text{(B8)}$$

The proof is simple. Plugging (B7) into (B5) and noting that $C_E{}^J$ and $C_{I^0}{}^J$ commute, we find

$$H_{ij}{}^J=(\bar{g}_i{}^J)^T(1-C_E{}^J)^{-1/2}[1-(1-C_E{}^J)^{-1}C_{I^0}{}^J]^{-1}\times(1-C_E{}^J)^{-1/2}g_j{}^J$$
$$=(\bar{g}_i{}^J)^T(1-C_{I^0}{}^J-C_E{}^J)g_i{}^J,$$

which is (B3) as desired. Since $C_E{}^J=(C_E{}^J)^{T*}$ has no right-hand s cuts, $\bar{\gamma}_i{}^J\to\gamma_i{}^{J*}$ as s becomes physical.

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Regge Slopes and Residues for a Spin-1+Spin-0 System in the Content of $O(4)$ Symmetry and Bethe-Salpeter Model*

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The Bethe-Salpeter equation for a spin-1 + spin-0 system at vanishing total 4-momentum K is separated by means of the $O(4)$ vector spherical harmonics, which are defined and discussed briefly. Perturbation expressions for the $M=1$ residues near $K=0$ are derived and then examined near $J=0$, where it is found that only the nonsense part of the $O(3)$ content of the $M=1$ residues survives. Perturbation expressions are given also for the slopes of Regge trajectories at $s=-K^2=0$, the signs of the slopes are investigated for certain classes of potentials, and a sum rule satisfied by slopes of contiguous daughters is derived.

I. INTRODUCTION

CONSIDERABLE progress has been achieved in understanding the $O(4)$ properties of scattering amplitudes since it was first discovered that these amplitudes acquire $O(4)$ symmetry at vanishing total 4-momentum K of the interacting particles ($K^2=-s=0$ in the s channel corresponds to forward scattering in the t channel for processes with pairwise-equal masses).[1-3] The Bethe-Salpeter (BS) model,[4] which exhibits this type of symmetry, has been popular because of its mathematical tractability. Thus, many attempts have been made to study the $O(4)$ properties of scattering amplitudes in the context of the BS model and several methods have been proposed for the expansion of these amplitudes.[5,6]

This work is devoted to investigating the residues near $J=0$ at small values of K and the signs of the slopes of Regge trajectories at $s=-K^2=0$ for a spin-1 +spin-0 system in the context of the BS model. We start off in Sec. II by separating the BS equation using the $O(4)$ vector spherical harmonics (VSH), which are defined and discussed briefly in the Appendix. The definition of these functions is motivated by the four-dimensional vector character of the BS wave function and the analogy with $O(3)$ VSH.[7,8]

Assuming nondegenerate perturbation theory to hold near $K=0$, we derive, in Sec. III, expressions for the $M=1$ residues at small values of K and investigate the behavior of these residues near $J=0$. The assumption that nondegenerate perturbation theory holds near $K=0$ is consistent with an $M=0$ classification of the pion since the $J=0$, $M=1$ residue chooses nonsense at $K=0$ and continues to choose nonsense near $K=0$.[9] The possibility of an $M=1$ classification of the pion which attracted interest for some time[10] could be realized if nondegenerate perturbation theory failed near $K=0$ and the pion residue were allowed to recieve a considerable contribution from the $M=1$ states.[11] The realization of this situation in the context of the BS model did not produce satisfactory results.[6]

In Sec. IV we study in some detail the signs of the slopes of Regge trajectories at $s=0$. Our tool is again nondegenerate perturbation theory, which we apply to the BS equation after putting it in a simpler form by

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means of a trick. Finally, we conclude our work by commenting on the unequal-mass problem with arbitrary spin.

II. SOLUTIONS OF BS EQUATION AT $K=0$

The BS equation for a bound state of two particles with spins 0 and 1 and a total 4-momentum K is

$$D_{\mu\nu}^{-1}\Psi_K{}^{\nu}(x_1,x_2)=\lambda\int dx_3dx_4\, V_{\mu\nu}(x_1,\ldots,x_4)\Psi_K{}^{\nu}(x_3,x_4)\,,$$

$$D_{\mu\nu}^{-1}\equiv(\Box_1{}^2+m_1{}^2)\times[(\Box_2+m_2{}^2)g_{\mu\nu}-\partial_{2\mu}\partial_{2\nu}]\,, \quad (2.1)$$

where

$$\Box=\partial_\mu\partial^\mu=g_{\mu\nu}\partial_\mu\partial_\nu\,,\quad \mu,\nu=0,1,2,3\,.$$

Following Wick,[12] we rotate the time direction from the real x_0 axis to the imaginary x_4 axis. As a result of this rotation the time component A_0 of any 4-vector A goes into iA_4 and we end up with vectors obeying the Euclidean metric. Then $D_{\mu\nu}^{-1}$ goes into

$$D_{\mu\nu}^{-1}=(-\Box_1+m_1{}^2)[(-\Box_2+m_2{}^2)\delta_{\mu\nu}-\partial_{2\mu}\partial_{2\nu}]\,,$$
$$\Box\equiv\partial_\mu\partial_\mu\,,\quad \mu=1,\ldots,4\,. \quad (2.2)$$

Let us transform to momentum space where it will be easier for us to work. There, (2.1) and (2.2) become

$$D_{\mu\nu}^{-1}(k_1,k_2)\Phi_{K\nu}(k_1,k_2)=\int dk_3dk_4\times V_{\mu\nu}(k_1,\ldots,k_4)\Phi_{K\nu}(k_3,k_4)\,, \quad (2.3)$$

$$D_{\mu\nu}^{-1}(k_1,k_2)=(k_1{}^2+m_1{}^2)\times[(k_2{}^2+m_2{}^2)\delta_{\mu\nu}-k_{2\mu}k_{2\nu}]\,. \quad (2.4)$$

If we make use of translational invariance and the following change of variables

$$K=k_1+k_2=k_3+k_4\,,\quad k=ak_1-bk_2\,,$$
$$k'=ak_3-bk_4\,,\quad a+b=1\,, \quad (2.5)$$

we obtain after some algebraic manipulations the following equation for the wave function $\mathbf{\Phi}$:

$$\mathbf{\Phi}(\mathbf{K},\mathbf{k})=\lambda\mathsf{D}(\mathbf{K},\mathbf{k})\cdot\int\mathsf{V}(\mathbf{K},\mathbf{k},\mathbf{k}')\cdot\mathbf{\Phi}(\mathbf{K},\mathbf{k}')d\mathbf{k}'\,, \quad (2.6)$$

where

$$\mathsf{D}(\mathbf{K},\mathbf{k})=\frac{\mathsf{I}-(b\mathbf{K}-\mathbf{k})(b\mathbf{K}-\mathbf{k})/m_2{}^2}{[(a\mathbf{K}+\mathbf{k})^2+m_1{}^2][(b\mathbf{K}-\mathbf{k})^2+m_2{}^2]}\,, \quad (2.7\text{a})$$

$$\mathsf{D}^{-1}(\mathbf{K},\mathbf{k})=[(a\mathbf{K}+\mathbf{k})^2+m_1{}^2]\times\{[(b\mathbf{k}-\mathbf{K})^2+m_2{}^2]\mathsf{I}-\mathbf{kk}\}\,, \quad (2.7\text{b})$$

where I stands for the unit dyad. We have dropped the cumbersome subscript notation in favor of the more compact dyadic and three-dimensional vector notations. We have used sans serif for dyadics and denoted four-dimensional vectors with boldfaced letters which are conventionally used for three-dimensional vectors. However, this will not result in any confusion since we will be dealing with four-dimensional vectors only.

At $K=0$ the BS equation acquires $O(4)$ symmetry. In this special case (2.6) and (2.7a) go into

$$\boldsymbol{\varphi}(\mathbf{k})=\lambda\mathsf{D}_0(\mathbf{k})\cdot\int\mathsf{V}_0(\mathbf{k},\mathbf{k}')\cdot\boldsymbol{\varphi}(\mathbf{k}')d\mathbf{k}'\,, \quad (2.8)$$

where

$$\boldsymbol{\varphi}(k)\equiv\mathbf{\Phi}(\mathbf{K}=0,\mathbf{k})\,,\quad \mathsf{V}_0(\mathbf{k},\mathbf{k}')\equiv\mathsf{V}(\mathbf{K}=0,\mathbf{k},\mathbf{k}')\,,$$

$$\mathsf{D}_0(\mathbf{k})=\frac{\mathsf{I}+\mathbf{kk}/m_2{}^2}{(k^2+m_1{}^2)(k^2+m_2{}^2)}\,, \quad (2.9)$$

where k stands for the length of the 4-vector $\mathbf{k}$. Lengths of 4-vectors will be consistently represented by the corresponding lightfaced italic letters.

The most general form of V_0 is

$$\mathsf{V}_0(k,k')=\sum_i F_i(k,k,'\mathbf{k}\cdot\mathbf{k}')\mathsf{E}_i\,, \quad (2.10)$$

where E_i $(i=1,\ldots,5)$ denote the following symmetry-conserving tensors:

$$\mathsf{E}_i\equiv\mathsf{I},\ \mathbf{kk},\ \mathbf{k}'\mathbf{k}',\ \mathbf{kk}',\ \mathbf{k}'\mathbf{k}\,. \quad (2.11)$$

Similar symmetry-conserving tensors involving the derivatives $\boldsymbol{\partial}$ and $\boldsymbol{\partial}'$ with respect to $\mathbf{k}$ and $\mathbf{k}'$ may also be added to (2.11). Thus, for each term of (2.10) the BS equation (2.8) may be written in the following form:

$$\boldsymbol{\varphi}(\mathbf{k})=\lambda\int\mathsf{U}(\mathbf{k},\mathbf{k}')\cdot\boldsymbol{\varphi}(\mathbf{k}')d\mathbf{k}'\,, \quad (2.12)$$

$$\mathsf{U}(\mathbf{k},\mathbf{k}')\equiv\mathsf{D}_0(\mathbf{k})\cdot\mathsf{E}F(k,k',\mathbf{k}\cdot\mathbf{k}')\,. \quad (2.13)$$

To separate Eq. (2.12), we expand $\boldsymbol{\varphi}(\mathbf{k})$ in the set of $O(4)$ VSH defined by (A4)[13]:

$$\boldsymbol{\varphi}(\mathbf{k})=\sum_{N'J'}\sum_{(n',\epsilon')}\mathbf{\Omega}_{N'(n',\epsilon')}{}^{J'}(\Omega)\gamma_{N'(n',\epsilon')}(k)\,. \quad (2.14)$$

Since J is always conserved, and N and ϵ are good quantum numbers at $K=0$, the matrix elements of U with the angular functions reduce to

$$(\mathbf{\Omega}_{N(n,\epsilon)}{}^J(\Omega)\,\mathsf{U}(k,k')\cdot\mathbf{\Omega}_{N'(n',\epsilon')}{}^J(\Omega'))=U_{nn'}{}^{N\epsilon}(k,k')\delta_{JJ'}\delta_{NN'}\delta_{\epsilon\epsilon'}\,. \quad (2.15)$$

Thus U is diagonal in the $M=1$ states $(\epsilon=\pm)$ while it mixes the $M=0$ states $(\epsilon=0)$. This result is consistent with the $O(4)$ parities and reflection eigenvalues for the states (see the Appendix). If we now substitute (2.15) and (2.14) into (2.12), we end with the following

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[13] The method of $O(4)$ VSH and its use to separate the BS equation are discussed in great detail by the author, J. Math. Phys. 11, 2272 (1970).

matrix integral equation in the radial variable k:

$$\gamma_{N(n,\epsilon)}{}^{(k)}=\sum_{n'}\int k'^3dk'\; U_{nn'}{}^{N\epsilon}(k,k')\gamma_{N(n',\epsilon)}{}^{(k')}. \quad (2.16)$$

Equation (2.16) consists of two one-dimensional equations for the $M=1$ states and a two-dimensional matrix equation for the $M=0$ states. For a given N this equation possesses, in general, four solutions and four corresponding eigenvalues which we indicate with the subscript $r=1, \ldots, 4$. Moreover, for given N and r there is an infinite number of eigenvalues corresponding to different potential strengths $\lambda_{Nr}{}^i$. This infinity, which we indicate with the superscript i, results from solving an integral equation in the radial variable k. It is obvious from the formulas given in the Appendix that only the identity tensor $\mathbf{E}=\mathbf{I}$ gives nonvanishing matrix elements for the $M=1$ angular functions. Therefore, $U_{NN'}{}^{N+}=U_{NN'}{}^{N-}$, which means that the radial functions and the eigenvalues of the $M=1$ wave functions are identical. Let us write down the total wave functions and their eigenvalues:

$$\left.\begin{aligned}\boldsymbol{\varphi}_{N1}{}^{Ji}(\mathbf{k})&=\gamma_N{}^i(k)\boldsymbol{\Omega}_{N(N,+)}{}^J(\Omega)\\ \boldsymbol{\varphi}_{N2}{}^{Ji}(\mathbf{k})&=\gamma_N{}^i(k)\boldsymbol{\Omega}_{N(N,-)}{}^J(\Omega)\end{aligned}\right\}$$

$$\text{with eigenvalue } \lambda_{N1}{}^i=\lambda_{N2}{}^i=\lambda_N{}^i,$$

$$\boldsymbol{\varphi}_{N3}{}^{Ji}(\mathbf{k})=\eta_{N(N-1,0)}{}^i(k)\boldsymbol{\Omega}_{N(N-1,0)}{}^J(\Omega)+\eta_{N(N+1,0)}{}^i(k)\times\boldsymbol{\Omega}_{N(N+1,0)}{}^J(\Omega) \text{ with eigenvalue } \lambda_{N3}{}^i,$$

$$\boldsymbol{\varphi}_{N4}{}^{Ji}(\mathbf{k})=\zeta_{N(N-1,0)}{}^i(k)\boldsymbol{\Omega}_{N(N-1,0)}{}^J(\Omega)+\zeta_{N(N+1,0)}{}^i(k)\times\boldsymbol{\Omega}_{N(N+1,0)}{}^J(\Omega) \text{ with eigenvalue } \lambda_{N4}{}^i.$$

It remains now to determine the explicit form of the matrix $U_{nn'}{}^{N\epsilon}$ for the various tensors enumerated in (2.11). This may be accomplished using the basic formulas given in the Appendix and the following expansion:

$$F(k,k',\mathbf{k}\cdot\mathbf{k}')=\sum_{nlm} Z_{nl}{}^m(\Omega)F_n(k,k')Z_{nl}{}^m(\Omega'). \quad (2.17)$$

III. $M=1$ RESIDUES NEAR $K=0$

If we assume that nondegenerate perturbation theory holds near $K=0$, the BS wave functions and their eigenvalues may be expanded in powers of K (the length of $\mathbf{K}$) in this region:

$$\begin{aligned}\boldsymbol{\Omega}_{Nr}{}^{Ji}(\mathbf{k},\mathbf{K})&=\sum_{p=0}^{\infty} K^p\boldsymbol{\Omega}_{Nr}{}^{Ji(p)}(\mathbf{k}),\\ \lambda_{Nr}{}^{Ji}&=\sum_{p=0}^{\infty} K^p\lambda_{Nr}{}^{Ji(p)},\end{aligned} \quad (3.1)$$

where $\boldsymbol{\Phi}_{Nr}{}^{Ji(0)}\equiv\boldsymbol{\varphi}_{Nr}{}^{Ji}$ and $\lambda_{Nr}{}^{Ji(0)}\equiv\lambda_{Nr}{}^i$ are the zeroth-order eigenfunctions and eigenvalues obtained in Sec. II. If we now substitute (3.1) into (2.6), we can in principle find the corrections for the eigenfunctions and to all orders in K. Let us focus on the angular dependence of $\boldsymbol{\Phi}$ only. The perturbed $M=1$ wave functions may then be written in the following form:

$$\boldsymbol{\Phi}_{Nr}{}^J(k,K)=\sum_{N'r'} G_{Nr,N'r'}{}^J(k,K)\boldsymbol{\Omega}_{N'r'}{}^J(\Omega), \quad M=1\ (r=1,2) \quad (3.2)$$

with

$$G_{Nr,N'r'}{}^J(k,K)=\gamma_N(k)\delta_{Nr,N'r'}+\sum_{p=1}^{\infty} K^pH_{Nr,N'r'}{}^{(p)J}(k), \quad (3.3)$$

where γ_N is the $M=1$ radial function at $K=0$ and the index $r'=1, \ldots, 4$ stands for the pair of quantum numbers $(n',\epsilon')=(N',+)$, $(N',-)$, $(N'-1,0)$, and $(N'+1, 0)$, respectively.

If the kernel of Eq. (2.6) is of Fredholm type or may be made so by introducing appropriate cutoff functions, then the resolvent kernel, which is the T matrix, is a meromorphic function of the parameter λ. If we assume the poles in λ to be simple, the T matrix may be proven to have the following form near a simple pole at $\lambda_{Nr}{}^J$:

$$\begin{aligned}\mathbf{T}_{Nr}{}^J(\mathbf{k}',\mathbf{k}'',\mathbf{K}\lambda)&=\frac{\boldsymbol{\Phi}_{Nr}{}^{J\dagger}(\mathbf{k}',\mathbf{K})\boldsymbol{\Phi}_{Nr}{}^J(\mathbf{k}'',\mathbf{K})}{\lambda-\lambda_{Nr}{}^J}\\ &=\frac{1}{\lambda-\lambda_{Nr}{}^J}\sum_{N'r',N''r''} G_{Nr,N'r'}{}^J(k',K)G_{Nr,N''r''}{}^{J\dagger}(k'',K)\boldsymbol{\Omega}_{N'r'}{}^J(\Omega')\boldsymbol{\Omega}_{N''r''}{}^J(\Omega'')\\ &=\frac{1}{\lambda-\lambda_{Nr}{}^J}\sum_{ls,l's'}\sum_{N'r',N''r''} G_{Nr,N'r'}{}^J(k',K)C_{N'r'}{}^{l's'J}\chi_{n'}{}^{l'}(\chi')\\ &\quad\times G_{Nr,N''r''}{}^{J\dagger}(k'',K)C_{N''r''}{}^{l''s''J}\chi_{n''}{}^{l''}(\chi'')\mathbf{Y}_{Jl's'}{}^\dagger(\varphi',\theta')\mathbf{Y}_{Jl''s''}(\varphi'',\theta''). \end{aligned} \quad (3.4)$$

We made use of Eq. (A5) to pass from $O(4)$ to $O(3)$ VSH. To find the residues of the corresponding poles in the J plane, we Reggeize expression (3.4) by continuing N and J into complex values while keeping their difference always equal to a non-negative integer.[5] Thus

$$N\to\alpha_0\,,\quad J\to\alpha_\kappa\,,\quad \text{with}\quad \alpha_0-\alpha_\kappa=\kappa=0,\,1,\,2,\,\ldots,\qquad(3.5)$$

and (3.4) becomes

$$\mathbf{T}_{\kappa r}(\mathbf{k}',\mathbf{k}'',\mathbf{K};\alpha_\kappa)=\frac{1}{(d\lambda_r/d\alpha_\kappa)(\alpha_\kappa-J)}\times\sum_{l',s',l'',s''}\sum_{\kappa'r',\kappa''r''}G_{\kappa r,\kappa'r'}{}^{\alpha_\kappa}C_{\kappa'r'}{}^{l's'\alpha_\kappa}\chi_{n'}{}^{l'}\times G_{\kappa r,\kappa''r''}{}^{\alpha_\kappa\dagger}C_{\kappa''r''}{}^{l''s''\alpha_\kappa}\chi_{n''}{}^{l''}\mathbf{Y}_{\alpha_\kappa l's'}{}^{\dagger}(\varphi'\theta')\times\mathbf{Y}_{\alpha_\kappa l''s''}(\varphi'',\theta'')\,.$$

Thus the $O(3)$ residue matrix is given by

$$\beta_{\kappa r}{}^{l's',l''s''}(k,'\chi';k'',\chi'';K;\alpha_\kappa)=\sum_{\kappa'r',\kappa''r''}G_{\kappa r,\kappa'r'}{}^{\alpha_\kappa}C_{\kappa'r'}{}^{l's'\alpha_\kappa}\times\chi_{n'}{}^{l'}G_{\kappa r,\kappa''r''}{}^{\alpha_\kappa\dagger}C_{\kappa''r''}{}^{l''s''\alpha_\kappa}\chi_{n''}{}^{l''}\Big/\frac{d\lambda_r}{d\alpha_\kappa},\qquad r=1,2\ (M=1)\qquad(3.6)$$

where $G_{\kappa r,\kappa'r'}{}^{\alpha_\kappa}$ and $C_{\kappa'r'}{}^{l's'\alpha_\kappa}$ are the Reggeized versions of the coefficients $G_{Nr,N'r'}{}^{J}$ and $C_{N'(n',\epsilon')}{}^{l's'J}$ given by Eq. (3.3) and tabulated in the Appendix, respectively. $\kappa=0$ corresponds to what is usually called the parent pole, while $\kappa=1,\,2,\,\ldots$ correspond to the daughter poles.[2]

Let us investigate β near $\alpha_\kappa=0$, where some of the $O(4)$ and $O(3)$ states are unphysical or nonsense, as they are usually called.

(i) The matrix $C_{\kappa'r'}{}^{l's'\alpha_\iota}\chi_{n'}{}^{l'}(\chi)$ may be written down explicitly using the tabulation in the Appendix, Eq. (A2), and the Reggeization prescription (3.5). If we do this and examine the product near $\alpha_\kappa=0$, we find the following to be true: All the matrix elements that connect sense (nonsense) $O(4)$ states to nonsense (sense) $O(3)$ states contain a factor of $\sqrt{\alpha_\kappa}$ if they do not vanish for some other reason.[14]

Let us illustrate this result on the matrix element $C_{N(N-1,0)}{}^{J-1\,1J}\chi_{N-1}{}^{J-1}$ which relates the $O(4)$ VSH $\Omega_{N(N-1,0)}{}^{J}$ to the $O(3)$ VSH $\mathbf{Y}_{J\,J-1\,1}$:

$$C_{N(N-1,0)}{}^{J-1\,1J}\chi_{N-1}{}^{J-1}=\left[\frac{J(N+J)(N+J\times1)}{2(2J+1)N^2}\,\frac{2^{-2J+1}N\Gamma(N+J)}{\Gamma^2(J+\frac{1}{2})\Gamma(N-J\times1)}\right]^{1/2}\times\sin\chi^{J-1}\,{}_2F_1(N+J,\,-N+J,\,J-\tfrac{1}{2};\,\tfrac{1}{2}-\tfrac{1}{2}\cos\chi)\,.$$

If we now Reggeize using (3.5) and isolate the factors that determine the behavior near $\alpha_\kappa=0$ [we do not have to worry about the factor $\sin\chi^l\,{}_2F_1(\cdots)$ because it is finite at $\alpha_\kappa=0$], we find that the matrix element behaves as

$$[\alpha_\kappa/(\alpha_\kappa+\kappa)]^{1/2}.$$

The $O(3)$ state obviously is a nonsense state since $l=J-1=\alpha_\kappa-1=-1$, while the $O(4)$ state is nonsense at $\kappa=0$ $(n=N-1=-1)$ and sense at $\kappa=1$ $(n=0)$. In the first case the matrix element does not vanish while in the second case it behaves like $\sqrt{\alpha_\kappa}$, as expected.

(ii) In $G_{Nr,N'r'}{}^{J}$ we encounter matrix elements of the symmetry-conserving tensors enumerated in (5.11) and the symmetry-breaking tensors

$$\mathbf{k}\cdot\mathbf{K},\ \mathbf{l},\ \mathbf{kK},\ \mathbf{Kk},\ \mathbf{KK}.\qquad(3.7)$$

Tensors constructed from the derivative $\boldsymbol{\partial}$ and the vector $\mathbf{K}$ may also be added to (3.7). If we reduce the matrix elements of these operators with $O(4)$ angular functions, then Reggeize according to (3.5), we find the following result to be true: Near $\alpha_\kappa=0$ the matrix element of any tensor, symmetry-conserving and otherwise, with any two $O(4)$ VSH contains a factor of $\sqrt{\alpha_\kappa}$ if one of the $O(4)$ states is sense while the other is nonsense. Thus $G_{\kappa r,\kappa'r'}{}^{\alpha_\kappa}$ contains *at least* one factor of $\sqrt{\alpha_\kappa}$ near $\alpha_\kappa=0$ if κr correspond to a sense $O(4)$ state while $\kappa'r'$ correspond to a nonsense $O(4)$ state or vice versa.

Let us illustrate this result on the following two matrix elements which may be calculated from formulas (A6) and (A10):

$$(\mathbf{\Omega}_{N(N-1,0)}{}^{J}\cdot\mathbf{kk}\cdot\mathbf{\Omega}_{N(N+1,0)}{}^{J})=k^2[N(N+2)]^{1/2}/2(N+1)\to k^2[(\alpha_\kappa+\kappa)(\alpha_\kappa+\kappa+2)]^{1/2}/2(\alpha_\kappa+\kappa+1)\approx\sqrt{\alpha_\kappa}\quad\text{at}\quad\kappa=0\quad\text{and}\quad\alpha_\kappa\approx0\,,$$

$$(\mathbf{\Omega}_{N(N-1,-)}{}^{J}(\mathbf{k}\cdot\mathbf{K})\cdot\mathbf{\Omega}_{N(N-1,0)}{}^{J})=-kK[J(J+1)/2N(N+1)^3]^{1/2}\to-kK[\alpha_\kappa(\alpha_\kappa+1)/2(\alpha_\kappa+\kappa)(\alpha_\kappa+\kappa+1)^3]^{1/2}\approx[\alpha_\kappa/(\alpha_\kappa+\kappa)]^{1/2}=\begin{cases}1 & \text{at}\quad\kappa=0,\,\alpha_\kappa\approx0\\ \sqrt{\alpha_\kappa} & \text{at}\quad\kappa=1,\,\alpha_\kappa\approx0.\end{cases}$$

Combining (i) and (ii) and remembering that the $M=1$ states are nonsense at $\alpha_\kappa=0$ (since $M>J$), we obtain the following behavior for the residue matric near $\alpha_\kappa=0$:

$$\beta_{\kappa r}{}^{l's',l''s''}\big|_{\alpha_\kappa\approx0}\approx\text{finite for nonsense }l's',\text{ sense }l''s''\;\approx\sqrt{\alpha_\kappa}\text{ for nonsense }l's',\text{ sense }l''s''\text{ or vice versa}\;\approx\alpha_\kappa\text{ for sense }l's',\text{ sense }l''s''.\qquad(3.8)$$

Thus only the nonsense part of the $O(3)$ content of the $M=1$ residues survives at $\alpha_\kappa=0$. This result, which is true for the $M=1$ residues only because the zeroth-order $M=1$ wave functions belong to $O(4)$ states which are pure nonsense at $J=0$, is not expected to be true for the $M=0$ residues where the zeroth-order wave func-

[14] This result was proved for the $N\bar{N}$ case by W. R. Frazer, F. R. Halpern, H. M. Lipinski, and D. R. Snider, Phys. Rev. 176, 2047 (1969).

tions belong to a mixture of sense and nonsense $O(4)$ states at $J=0$ and $N=0$, and pure sense at $J\geqslant 0$ and $N\geqslant 1$.

Finally we note that (3.8) gives the minimal behavior in α_κ; β may vanish as a higher power of α_κ since, as we mentioned above, G contains *at least* one factor of $\sqrt{\alpha_\kappa}$ when it relates two $O(4)$ states that disagree in sense or nonsense.

IV. SLOPES OF REGGE TRAJECTORIES

We may substitute the power-series expansions (3.1) into Eq. (2.6) and, in principle, find the corrections for λ and Φ to all orders in K. However, we expect the perturbation formulas obtained in this manner to be quite lengthy and complicated due to the complicated dependence of Eq. (2.6) on K and $\cos\chi$, where χ is the angle between the total 4-momentum K and the relative 4-momentum k. (We choose a coordinate system in which K lies along the fourth axis.) Fortunately, Eq. (2.6) may be simplified considerably if we notice that its eigenvalue spectrum does not depend on the particular choice of the parameters a and b as long as $a+b=1$, a fact that may be proven by examining Eqs. (2.3), (2.5), and (2.6): A change of a and b into $a-c$ and $b+c$ changes k into $k-cK$ and k' into $k'-cK$. If we now define new variables, $p=k-ck$ and $p'=k'-cK$, the eigenvalue equation will look the same as before, while $\Phi(K,k)$ changes into $\Phi(K,\,p+cK)\equiv F(K,p)$.

If we choose $a=1$ and $b=0$ (in the c.m. system $a=b=\frac{1}{2}$ for the equal-mass case), Eq. (2.6) reduces to a simpler form:

$$\left(1+\frac{\cos\chi}{k^2+m_1^2}K+\frac{1}{k^2+m_2^2}K^2\right)\mathbf{\Phi}(\mathbf{K},\mathbf{k})=\lambda\int\mathbf{U}(\mathbf{k},\mathbf{k}')\cdot\mathbf{\Phi}(\mathbf{K},\mathbf{k}')d\mathbf{k}'+\lambda\int\mathbf{W}(\mathbf{K},\mathbf{k},\mathbf{k}')\cdot\Phi(\mathbf{K},\mathbf{k}')d\mathbf{k}',\quad(4.1)$$

where $\mathbf{U}$ is the K-independent (zeroth-order) kernel defined by (2.13) while $\mathbf{W}$ is the K-dependent kernel which may be expanded in a power series in K near $K=0$. Further simplifications occur if we are interested only in classes of s-independent potentials ($s=-K^2$). In this case the second term on the right-hand side of (4.1) drops out and the perturbation formulas for λ and $\mathbf{\Phi}$ assume very simple forms.

To obtain the slopes of Regge trajectories, first we find the second-order correction for λ [the first-order correction vanishes owing to $O(4)$ parity considerations] by inserting (3.1) into (4.1); we restrict ourselves to the case of s-independent potentials:

$$\lambda_{Nr}{}^{Ji}=\lambda_{Nr}{}^{i}+K^2\lambda_{Nr}{}^{Ji(2)}+\cdots,$$

$$\frac{\lambda_{Nr}{}^{Ji(2)}}{\lambda_{Nr}{}^{i}}=\left(\varphi_{Nr}{}^{Ji}\cdot\frac{1}{k^2+m_1^2}\varphi_{Nr}{}^{Ji}\right)+\sum_{N'r'i'}\left(\varphi_{Nr}{}^{Ji}\cdot\frac{2k\cos\chi}{k^2+m_1^2}\varphi_{N'r'}{}^{Ji'}\right)\times\left(\varphi_{N'r'}{}^{Ji'}\cdot\frac{2k\cos\chi}{k^2+m_1^2}\varphi_{Nr}{}^{Ji}\right)\frac{1}{\lambda_{Nr}{}^{i}\lambda_{N'r'}{}^{i'}-1}.\quad(4.2)$$

Then we Reggeize N and J as prescribed in (3.5). Thus

$$\lambda_{Nr}{}^{Ji}=\lambda_r{}^i(s,\alpha_\kappa)\,,\quad\lambda_{Nr}{}^{i}\rightarrow\lambda_r{}^i(\alpha_0)\,,$$
$$\lambda_{Nr}{}^{Ji(2)}\rightarrow\lambda_r{}^{i(2)}(\alpha_\kappa)\,.$$

The Regge trajectories $\alpha_{\kappa r}{}^i(s)$ are found by setting the strength of the potential $\lambda_r{}^i(s,\alpha_\kappa)$ equal to a constant and solving the resulting equations for α_κ as functions of s. The slopes of the trajectories, however, may be found without having to solve the equations:

$$\left.\frac{d\alpha_{\kappa r}{}^i(s)}{ds}\right|_{s=0}=-\left.\frac{\partial\lambda_r{}^i(s,\alpha_\kappa)/\partial s}{\partial\lambda_r{}^i(s,\alpha_\kappa)/\partial\alpha_\kappa}\right|_{s=0}=\frac{\lambda_r{}^{i(2)}(\alpha_\kappa)/\lambda_r{}^i(\alpha_0)}{-[\partial U_r{}^i(k,k';\alpha_0)/\partial\alpha_0]/U_r{}^i(k,k';\alpha_0)},\quad(4.3)$$

where

$$U_r{}^i(k,k';\alpha_0)=(\boldsymbol{\varphi}_r{}^i(k,\alpha_0)\cdot\mathbf{U}(k,k')\cdot\boldsymbol{\varphi}_r{}^i(\mathbf{k},\alpha_0))\,,\quad r=1,\ldots,4.$$

Let us investigate the signs of the slopes:

(a) The investigation of the sign of the numerator of (4.3) is far from easy because the second term in (4.2) involves a sum over an infinite number of radial states. It is possible, however, to make a rough approximation that estimates the size of the second term in (4.2), which we denote by B, relative to the first term, which we denote by A:

$$B<\max\left|\frac{\lambda_{Nr}{}^i}{\lambda_{N'r'}{}^{i'}}-1\right|^{-1}\left(\varphi_{Nr}{}^{Ji}\cdot\frac{4k^2\cos^2\kappa}{(k^2+m_1^2)^2}\varphi_{Nr}{}^{Ji}\right)<C\left(\varphi_{Nr}{}^{Ji}\cdot\frac{4\cos^2\kappa}{k^2+m_1^2}\varphi_{Nr}{}^{Ji}\right)<4C\left(\varphi_{Nr}{}^{Ji}\cdot\frac{1}{k^2+m_1^2}\varphi_{Nr}{}^{Ji}\right)D<4CDA\,,\quad(4.4)$$

where

$$C\equiv\max|\lambda_{Nr}{}^i/\lambda_{N'r'}{}^i-1|^{-1},$$

and D is the result of reducing the matrix elements of

$\cos^2\chi$. Using (A10), we find the following values of D for the first few values of N and J:

	$r=1$	$r=2$	$r=3,4$
$N=0,\ J=0$			$\frac{1}{12}$
$N=1,\ J=0$			$\frac{1}{4}$
$N=1,\ J=1$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{4}$

(4.5)

For the numerator of (4.3) to be positive, A has to be larger than B. In the language of the rough approximation (4.4), this means

$$4CD<1$$

or

$$\min|\lambda_{Nr}{}^{i}/\lambda_{N'r'}{}^{i'}-1|>4D. \tag{4.6}$$

From (4.5) we have for the ground state $4D=\frac{1}{3}$. Thus if the potential is such that the ground state is well separated from the others (note that λ for the ground state is the smallest eigenvalue), which is a reasonable assumption, (4.6) will hold for the ground state. Therefore, the numerator of (4.4) is positive for the ground state. We expect this result to hold also for at least the first few excited states since approximation (4.3) was very rough and was directed toward overestimation of B at every step.

(b) The sign of the denominator of (4.3) may be shown to be positive for a positive-definite potential and negative for a negative-definite potential. Using expansion (2.16) and formula (A6), we find, taking $\mathbf{E}=\mathbf{I}$ [see (2.13)],

$$(k^2+m_1{}^2)(k^2+m_2{}^2)U_{nn'}{}^{N\epsilon}(k,k') =(\delta_{nn'}+\epsilon_{\epsilon0}(k^2/m_2{}^2)B_{Nn}B_{Nn'})F_{n'}(k,k'). \tag{4.7}$$

We choose the tensor $\mathbf{E}=\mathbf{I}$ because it is the only symmetry-conserving tensor that has nonvanishing matrix elements with the $M=1$ wave functions. Now if F is positive definite, for the expansion (2.16) to converge it is necessary that F_n decrease monotonically in n. From Eqs. (4.7), (3.5), and (A7), we find that $\ln U_r{}^i(\alpha_0)$ decreases montonically as α_0 increases and therefore the denominator of (4.3),

$$-\frac{dU_r{}^i(\alpha_0)}{d\alpha_0}\Big/ U_r{}^i(\alpha_0)=-\frac{d}{d\alpha_0}\ln U_r{}^i(\alpha_0), \tag{4.8}$$

is positive. Similarly, we may prove that (4.8) is negative for a negative-definite potential.

Combining a and b, we can assert that the slopes for the ground state and very possibly the first few excited states, *whose eigenvalues are expected to be well separated*, are positive (rising trajectories) for a positive-definite potential and negative for a negative-definite potential.

We will end this section by deriving a sum rule satisfied by the slopes. If we use Eq. (A10) to reduce expression (4.2) for the $M=1$ case ($t=1,2$), then Reggeize, we obtain (we drop the radial superscript i)

$$\lambda_r{}^{(2)}(\alpha_\kappa)=A_r(\alpha_0)+\kappa(2\alpha_0-\kappa+1)B_r(\alpha_0)+(\kappa+1)\times(2\alpha_0-\kappa+2)C_r(\alpha_0)+(\alpha_0-\kappa)(\alpha_0-\kappa+1)D_r(\alpha_0)\delta_{r2},\quad r=1,2.$$

Thus

$$\lambda_r{}^{(2)}(\alpha_{\kappa+1})-\lambda_r{}^{(2)}(\alpha_\kappa)=2(\alpha_0-\kappa)[A_r(\alpha_0)+B_r(\alpha_0)+C_r(\alpha_0)-D_r(\alpha_0)\delta_{r2}],$$
$$\lambda_r{}^{(2)}(\alpha_\kappa)-\lambda_r{}^{(2)}(\alpha_{\kappa-1})=2(\alpha_0-\kappa+1)[\cdots]. \tag{4.9}$$

The same relations, except for the fact that there are more terms inside the square brackets, may be shown to be true for the $M=0$ case ($r=3,4$). Relations (4.9) and Eq. (4.3) give for the slopes $\alpha'(s=0)$ of three contiguous daughters the following sum rule:

$$(\alpha_0-\kappa+1)\alpha_{\kappa+1}'(0)-(2\alpha_0-2\kappa+1)\alpha_\kappa'(0)+(\alpha_0-\kappa)\alpha_{\kappa-1}'(0)=0. \tag{4.10}$$

It is easy to prove that the slopes for the spin-0 case satisfy the same sum rule.

V. CONCLUSION

(1) To determine the behavior of the $M=1$ residues near $\alpha_\kappa=0$, we made use of two facts: (i) When an $O(4)$ state and its $O(3)$ content do not agree in sense or nonsense at $\alpha_\kappa=0$, the matrix element which connects their angular functions behaves like $\sqrt{\alpha_\kappa}$ near $\alpha_\kappa=0$. (ii) the matrix elements of all the tensor operators (including symmetry-breaking ones which contain K) with two angular functions belonging to $O(4)$ states that disagree in sense or nonsense behave like $\sqrt{\alpha_\kappa}$ near $\alpha_\kappa=0$. We have illustrated these facts in Sec. II on the spin-1 case. They are deduced solely from the behavior of $\chi_n{}^l(\kappa)$ and the Wigner coefficients $C_{N(n,\epsilon)}{}^{lsJ}$; and if higher-spin spherical harmonics are defined we expect a similar behavior in α_κ near $\alpha_\kappa=0$.

We should also mention that the behavior in α_κ of the $M=1$ residues derived in Sec. III is by no means unique to the BS model. The same results may be obtained in a similar way outside the BS model provided factorization is imposed.

(2) When we go on the mass shell, where we must evaluate the residues eventually since off-shell generalizations are arbitrary, $\cos\chi$, given by

$$\cos\chi=(m_2{}^2-m_1{}^2)/2kK,\quad K=-\sqrt{-s}$$

on the mass shell, blows up as K approaches zero. This in turn induces the Ω's contained in the perturbed wave function [see (3.2)] to blow up. The nondiagonal matrix elements $G_{Nr,N'r'}{}^J$ [see (3.3)] do not decrease fast enough to overcome the increase in the Ω's.[5] Thus we are left at $K=0$ with an infinite series which requires a test for convergence before anything can be said about the vanishing of its sense-nonsense part through the sense-nonsense factor $\sqrt{\alpha_\kappa}$. Thus it seems as though the results obtained for the residues in Sec. III are uncertain except for the equal-mass case where $\cos\chi$ vanishes on the mass shell. However, if we allow for coupling of unequal-mass to equal-mass channels and remember that the BS model obeys factorization, it will be easy to establish similar results for the unequal-mass case. Several authors have proposed off-shell generalizations applicable to the unequal-mass case.[15,16]

(3) The results obtained in Sec. IV for the slopes are true for the unequal-mass case as well as the equal-mass case since the masses are hidden in the parameters a and b, and λ does not depend on the particular choice of a and b as long as $a+b=1$. Also the results of Sec. IV may be proven to be true for a system of two particles one of which has a spin equal to zero while the other may have an arbitrary spin. This is so because the BS equation for such a system reduces to an equation exactly analogous to Eq. (4.1) if we choose $a=1$ and $b=0$.

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APPENDIX

By analogy with $O(3)$,[13] where VSH are defined by combining the orbital space spanned by the Y_l^m's with the spin space e_μ ($\mu=0, \pm1$), we construct $O(4)$ VSH by combining the orbital space $(\frac{1}{2}n,\frac{1}{2}n)$ spanned by the functions

$$Z_{nl}^m(\varphi,\theta,\chi)=Y_l^m(\varphi,\theta)\chi_n^l(\kappa)\,, \tag{A1}$$

$$\chi_n^l(\chi)=\left(\frac{(n+1)\Gamma(n+l+2)}{2^{2l+1}\Gamma^2(l+\frac{3}{2})\Gamma(n-l+1)}\right)^{1/2} \times\sin\chi_2^l F_1(n+l+2,\,-n+l;\,l+\tfrac{3}{2};\,\tfrac{1}{2}-\tfrac{1}{2}\cos\chi)\,, \tag{A2}$$

with the spin space $(\frac{1}{2},\frac{1}{2})$ represented by the 4-vectors

$$\mathbf{e}_{00}=\mathbf{e}_4\,,\quad \mathbf{e}_{1-1}=(\mathbf{e}_1-i\mathbf{e}_2)/\sqrt{2}\,,$$
$$\mathbf{e}_{10}=\mathbf{e}_3\,,\quad \mathbf{e}_{11}=-(\mathbf{e}_1+i\mathbf{e}_2)/\sqrt{2}\,,$$

where $e_1, \ldots, e_4$ are unit vectors directed along the four Cartesian axes. Since we are interested in $O(4)$ representations, i.e., representations belonging to $(j,j')\oplus(j',j)$ rather than (j,j') alone, our functions must be eigenfunctions of the reflection operation

$$\hat{Q}\mathbf{F}(\varphi,\theta,\chi;\,e_{s\mu})=\mathbf{F}(\pi+\varphi,\,\pi-\theta,\,\chi;\,(-1)^s\mathbf{e}_{s\mu})\,, \tag{A3}$$

which leaves the BS equation [Eq. (2.8)] invariant. Thus we define the $O(4)$ VSH as follows:

$$\mathbf{\Omega}_{N(n,\epsilon)}{}^{J\bar{M}}=\sum_{l,s,m,\mu} C_{N(n,\epsilon)}{}^{lms\mu J\bar{M}}\,Z_{nl}{}^m\mathbf{e}_{s\mu}\,,\qquad \epsilon=0,\ \pm1. \tag{A4}$$

For given total $O(4)$ and $O(3)$ angular momenta N and J and azimuthal quantum number $\bar{M}$, there are four possible VSH with $(n,\epsilon)=(N,\pm)$, $(N\pm1,0)$. The reflection eigenvalues (eigenvalues of $\hat{Q}$) are $(-1)^{J+1}$ for the state $(N,+)$ and $(-1)^J$ for the other three. The first two VSH $(N,\pm)$ belong to $M=1$ representations while the other two VSH $(N\pm1,0)$ belong to $M=0$ representations. The values of the $O(4)$ quantum number (Toller quantum number) M fall in the range $0\leq M\leq\max S$, where S is the total $O(3)$ spin which can be either 1 or 0. We note that in addition to a part that describes a system with total spin equal to 1, the BS equation contains a part that describes a system with total spin equal to 0. The latter part can be removed only by imposing subsidiary conditions which are bound to lead to a more complicated equation. The spin-1 and spin-0 parts separate when we look at the $O(3)$ content of $O(4)$ VHS:

$$\Omega_{N(n,\epsilon)}{}^{J\bar{M}}=\sum_{ls} C_{N(n,\epsilon)}{}^{lsJ\bar{M}}\chi_n{}^l(\chi)\mathbf{Y}_{Jls}{}^{\bar{M}}(\varphi,\theta)\,,$$

$$\mathbf{Y}_{Jl1}{}^{\bar{M}}\equiv\sum_{m+\mu=\bar{M}}\langle lm,s\mu|ls,J\bar{M}\rangle Y_l{}^m(\varphi,\theta)\mathbf{e}_{1\mu}\,,\qquad l=J,\ J\pm1 \tag{A5}$$

$$\mathbf{Y}_{JJ0}{}^{\bar{M}}\equiv Y_J{}^{\bar{M}}(\varphi,\theta)\mathbf{e}_{00}\,.$$

We have calculated the coefficients $C_{N(n,\epsilon)}{}^{lsJ}$ which involve Wigner $9j$ symbols. The results are tabulated at the end of this Appendix.

The following basic formulas are useful in reducing the angular parts of the tensor operators:

$$\begin{aligned}
&\mathbf{k}=\sum k_{s\mu}e_{s\mu}=\mathbf{k}^\dagger=\sum k_{s\mu}{}^*e_{s\mu}{}^\dagger\,,\\
&k_{s\mu}=\mathbf{e}_{s\mu}{}^\dagger\cdot\mathbf{k}=(\pi k/\sqrt{2})Z_{1s}{}^{\mu*}(\varphi,\theta,\chi)\,,\\
&k_{s\mu}{}^*=\mathbf{k}\cdot\mathbf{e}_{s\mu}\,,\\
&\mathbf{k}\cdot\mathbf{\Omega}_{N(n,\epsilon)}{}^{J\bar{M}}=\delta_{\epsilon0}kB_{Nn}Z_{NJ}{}^{\bar{M}}\,,
\end{aligned} \tag{A6}$$

where

$$B_{Nn}=\begin{cases}\left[\dfrac{N}{2(N+1)}\right]^{1/2}, & n=N-1\\[2ex] \left[\dfrac{N+2}{2(N+1)}\right]^{1/2}, & n=N+1\end{cases} \tag{A7}$$

$$\mathbf{k}Z_{NJ}=k\sum_{(n,\epsilon)}\delta_{\epsilon0}B_{Nn}\mathbf{\Omega}_{N(n,\epsilon)}{}^J =kB_{NN-1}\mathbf{\Omega}_{N(N-1,0)}{}^J+kB_{NN+1}\mathbf{\Omega}_{N(N+1,0)}{}^J\,, \tag{A8}$$

$$\begin{aligned}
&\mathbf{K}\cdot\mathbf{\Omega}_{N(n,\epsilon)}\sigma^{\bar{M}}=KC_{N(n,\epsilon)}{}^{J0J}Z_{nJ}{}^{\bar{M}}\,,\\
&\mathbf{K}=K\mathbf{e}_{00}=K\mathbf{e}_4\,,
\end{aligned} \tag{A9}$$

$$\begin{aligned}
(\mathbf{\Omega}_{N(n,\epsilon)}{}^J\cdot(Kk)\mathbf{\Omega}_{N'(n',\epsilon')}{}^J)&=\delta_{n'n-1}Kk\times\sum_{l,s} C_{N(n,\epsilon)}{}^{lsJ}C_{N'(n-1,\epsilon')}{}^{lsJ}\mathcal{K}_{nl}+\delta_{n'n+1}Kk\\
&\quad\times\sum_{l,s} C_{N(n,\epsilon)}{}^{lsJ}C_{N'(n+1,\epsilon')}{}^{lsJ}\mathcal{K}_{n+1l}\,,
\end{aligned}$$

$$\mathcal{K}_{nl}\equiv\left(\frac{(n-l)(n+l-1)}{4n(n+1)}\right)^{1/2}. \tag{A10}$$

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TABLE I. The coefficients $C_{N(n,\epsilon)}{}^{lsJ}$.

$(l,s) \backslash (n,\epsilon)$	$M=1$ $(N,+)$	$(N,-)$	$M=0$ $(N-1,0)$	$(N+1,0)$
$(J,0)$	0	$\left[\dfrac{J(J+1)}{(N+1)^2}\right]^{1/2}$	$\left[\dfrac{(N-J)(N+J+1)}{2N^2}\right]^{1/2}$	$\left[\dfrac{(N-J+1)(N+J+2)}{2(N+2)^2}\right]^{1/2}$
$(J-1,1)$	0	$-\left[\dfrac{(J+1)(N-J+1)(N+J+1)}{(2J+1)(N+1)^2}\right]^{1/2}$	$\left[\dfrac{J(N+J)(N+J+1)}{2(2J+1)N^2}\right]^{1/2}$	$\left[\dfrac{J(N-J+1)(N-J+2)}{2(2J+1)(N+2)^2}\right]^{1/2}$
$(J,1)$	1	0	0	0
$(J+1,1)$	0	$-\left[\dfrac{J(N-J)(N+J+2)}{2(2J+1)(N+1)^2}\right]^{1/2}$	$-\left[\dfrac{(J+1)(N-J)(N-J-1)}{2(2J+1)N^2}\right]^{1/2}$	$\left[\dfrac{(J+1)(N+J+2)(N+J+3)}{2(2J+1)(N+2)^2}\right]^{1/2}$

Table I gives the coefficients of $C_{N(n,\epsilon)}{}^{lsJ}$.

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Meson-Baryon Interactions with Baryons Belonging to the Nonlinear Representations of Chiral $U(3)\otimes U(3)$

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We derive a general chiral $U(3)\otimes U(3)$ phenomenological Lagrangian for the meson-baryon system. The meson-baryon derivative coupling is discussed in detail, and K-N scattering is studied. Some renormalized meson-baryon coupling constants are computed and compared with experiments. A suitable canonical transformation is introduced, and conditions under which it yields the Yukawa coupling are discussed.

I. INTRODUCTION

IN this paper we formulate chiral $U(3)\otimes U(3)$ phenomenological Lagrangians with both pseudoscalar mesons and baryons transforming nonlinearly under the chiral $U(3)\otimes U(3)$ group. Our main aim is to study meson-baryon scattering and, in particular, to derive the $\mathcal{L}_{\text{int}}$ for π-N and K-N systems. Our formulation is in a way unconventional, since we do not use Weinberg's covariant derivative formalism,[1] which recently has been used quite successfully in formulating chiral $U(3)\otimes U(3)$ phenomenological Lagrangians by Turner.[2] Instead, we first formulate a $U(3)\otimes U(3)$ chiral-invariant quark-meson Lagrangian, which is then used as a guide for the formulation of chiral $U(3)\otimes U(3)$ invariant meson-baryon interactions.

Thus in Sec. II, we start with two triplets of quarks transforming according to the six-dimensional linear representation of $U(3)\otimes U(3)$. Assuming that there is only one "physical" quark triplet in nature, we express the two triplets of quarks in terms of one quark triplet (the "physical" one) and a unitary meson matrix with definite linear transformation properties under the group. The physical quark fields, however, transform nonlinearly under the group.

In Sec. III, in a similar way, we formulate the meson-baryon chiral-$U(3)\otimes U(3)$-invariant Lagrangian, expressing the two baryon nonets (or octets) in terms of the physical baryon nonet (or octet) and the unitary meson matrix. While the two sets of baryons and the unitary meson matrix obey the linear transformation laws under the group, the physical baryons will obey the nonlinear one. In this formulation, only the derivative meson-baryon coupling constants appear, which are completely determined by means of chiral $U(3)\otimes U(3)$ invariance of the theory. No particular symmetry breaking is needed in the theory, except the one which splits the masses in the observable way.

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