relation to fit the data. In other words, Eq. (9) might be useful if the measured amplitude did not satisfy the usual dispersion relation within experimental error. Then if Eq. (9) were satisfied only for $l$ larger than some $l_0$ it would appear that causality was violated to distances of $l_0$. *However, as long as the usual dispersion relation fits the experiments, Eq. (9) is of no use in bounding a fundamental length.*

In conclusion, the only believable limit on a fundamental length at this time is given by the dimensional argument that since dispersion relations work at energies up to 20 BeV, a fundamental noncausal length is unlikely to be much larger than $\hbar c/(20\text{ BeV})=10^{-15}$ cm. It should be understood that this is a purely dimensional argument and should be viewed with the appropriate caution.[9]

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We would like to thank Sidney Drell for suggesting that we study the role of causality in dispersion relations. We would also like to thank J. Finkelstein for discussions on the nonuniqueness of the solution to the noncausal dispersion relation.

[9] An excellent example of the pitfalls of dimensional analysis occurs in calculation of the muon decay width [c.f. S. Gasiorowicz, *Elementary Particle Physics* (Wiley, New York, 1966)]. Dimensional analysis gives $\Gamma\sim G^2 m_\mu{}^5$, while a detailed calculation in the $V$-$A$ theory gives $\Gamma=(1/196\pi^3)G^2 m_\mu{}^5$, a difference of nearly four orders of magnitude.

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# Structure of an Equal-Time Commutator Occurring in a Theory of Currents

M. A. Ahmed*
*International Centre for Theoretical Physics, Trieste, Italy*

AND

D. B. Fairlie
*Mathematics Department, University of Durham, Durham, United Kingdom*

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The structure of an equal-time commutator involving the time component of the axial-vector current and its divergence is investigated. It is shown that the value of the equal-time commutator, formerly given by a power-series expansion, can be obtained in a closed form.

## I. INTRODUCTION

The model due to Sugawara[1] is a concrete realization of a theory of hadron dynamics envisaged by Dashen and Sharp[2] based on the idea that the vector currents $V_\mu{}^i$ and the axial-vector currents $A_\mu{}^i$ should be considered as local observables in terms of which one can formulate the theory. The dynamical content of Sugawara's model resides in an explicit representation of the energy-momentum tensor of the hadrons $\Theta^{\mu\nu}$ as a bilinear form

$$\Theta^{\mu\nu}(x)=(1/2C)[V_i{}^\mu(x)V_i{}^\nu(x)+V_i{}^\nu(x)V_i{}^\mu(x) -g^{\mu\nu}V_i{}^\rho(x)V_{i\rho}(x)]+(V\leftrightarrow A)\,. \quad (1.1)$$

Dashen and Frishman[3] then pointed out that $\Theta^{\mu\nu}$ as given by Eq. (1.1) can be put in the form

$$\Theta^{\mu\nu}(x)=\Theta_+{}^{\mu\nu}(x)+\Theta_-{}^{\mu\nu}(x)\,, \quad (1.2)$$

where $\Theta_\pm$ are bilinear forms of the same structure as $\Theta^{\mu\nu}$ but constructed out of currents $J_{\pm i}{}^\mu=\frac{1}{2}(V_i{}^\mu\pm A_i{}^\mu)$. They then demonstrated that $\Theta_\pm{}^{\mu\nu}$ are separately conserved, commute with each other at equal times, and transform into one another under parity. This feature then leads to the invariance group $P_+\otimes P_-$, with $P_+$ and $P_-$ being two commuting Poincaré groups which transform into each other under the parity operation. These authors then argued that this aspect of the model, which they called too much symmetry, requires parity doubling of the spectrum of particles.

A way out of this difficulty was proposed by Cronin and Guralnik[4] by insisting that $\Theta_+$ should be identically equal to $\Theta_-$ instead of being related to it by parity. For this equality to hold one must abandon the algebra of fields[5] which the currents $V_i{}^\mu$ and $A_i{}^\mu$ are supposed to satisfy in Sugawara's model. Instead, these authors proposed an algebra with $q$-number Schwinger terms. For $\Theta^{\mu\nu}$ they adopt the same form as (1.1), i.e., they write

$$\Theta^{\mu\nu}=\Theta_+{}^{\mu\nu}+\Theta_-{}^{\mu\nu}\,,$$

* On leave of absence from the Department of Physics, University of Khartoum, Sudan.

[1] H. Sugawara, Phys. Rev. **170**, 1659 (1968); that the Sugawara $\Theta^{\mu\nu}$ should be interpreted as a limit was discussed recently by S. Coleman, D. Gross, and R. Jackiw, *ibid.* **180**, 1359 (1969).

[2] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968); D. H. Sharp, *ibid.* **165**, 1867 (1968).

[3] R. F. Dashen and Y. Frishman, Phys. Rev. Letters **22**, 572 (1969).

[4] J. A. Cronin and G. S. Guralnik, Phys. Rev. **184**, 1803 (1969).

[5] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).

where

$$\Theta_{\pm}{}^{\mu\nu}=(1/f^2)[J_{\pm i}{}^{\mu}(x)J_{\pm i}{}^{\nu}(x)+J_{\pm i}{}^{\nu}(x)J_{\pm i}{}^{\nu}(x)-g^{\mu\nu}J_{\pm i}{}^{\rho}(x)J_{\pm i\rho}(x)], \quad (1.3)$$

and, of course

$$\Theta_{+}{}^{\mu\nu}=\Theta_{-}{}^{\mu\nu}. \quad (1.4)$$

Equation (1.4) leads to a rather strong condition on the bilinear product of currents[4] and, furthermore, imposing the fundamental Schwinger condition[6] as well as the Heisenberg equations of motion leads to eigenvaluelike equations for the Schwinger terms.

Both Sugawara's model and the Cronin-Guralnik model possess $SU(3)\otimes SU(3)$ or $SU(2)\otimes SU(2)$ symmetry, depending on whether one considers the full octets of vector and axial-vector currents or only the isovector currents. Extensions of these models that break the conservation of the axial-vector currents, and hence reduce the symmetry to $SU(3)$ or $SU(2)$ only, have been proposed in Refs. 7 and 8. The basic idea is that one extends the set of independent dynamical variables in terms of which one expresses all quantities occurring in the theory to include, besides the vector and axial-vector currents, the divergences of the axial-vector currents $D_i$. The dynamical content of such models is specified by requiring $\Theta^{\mu\nu}$ to read in the Sugawara case

$$\Theta^{\mu\nu}(x)=\Theta_s{}^{\mu\nu}(x)+\lambda g^{\mu\nu}D_i(x)D_i(x), \quad (1.5)$$

where $\Theta_s{}^{\mu\nu}$ is given by Eq. (1.1), while in the Cronin-Guralnik case $\Theta_{\pm}{}^{\mu\nu}$ should read

$$\Theta_{\pm}{}^{\mu\nu}(x)=\Theta_{\pm s}{}^{\mu\nu}(x)+\tfrac{1}{2}\lambda g^{\mu\nu}D_i(x)D_i(x), \quad (1.6)$$

where $\Theta_{\pm s}{}^{\mu\nu}$ is given by Eq. (1.3) and $\lambda$ is a nonvanishing constant, i.e., the new energy-momentum tensors differ from those of the symmetrical models by the addition of a term proportional to the bilinear product $D_i(x)D_i(x)$. Such models have the merit that the chiral symmetry is broken without appealing to canonical fields,[9] namely, in terms of quantities pertaining to the original dynamical variables on which the symmetrical models are based. The generalized models, of course, still possess $SU(3)$ or $SU(2)$ symmetry. In the extended models the following extra set of equal-time commutation relations are assumed to hold:

$$[V_i{}^0(x),D_j(y)]_{x_0=y_0}=ic_{ijk}D_k(x)\delta^3(\mathbf{x}-\mathbf{y}), \quad (1.7)$$

$$[A_i{}^0(x),D_j(y)]_{x_0=y_0}=-if_{ij}(x)\delta^3(\mathbf{x}-\mathbf{y}), \quad (1.8)$$

$$[V_i{}^a(x),D_j(y)]_{x_0=y_0}=[A_i{}^a(x),D_j(y)]_{x_0=y_0}=[D_i(x),D_j(y)]_{x_0=y_0}=[f_{ij}(x),D_k(y)]_{x_0=y_0}=0, \quad (1.9)$$

where $c_{ijk}$ are the structure constants of the $SU(3)$ or $SU(2)$ group and $a=1,2,3$. The operator function $f_{ij}(x)$ is a Lorentz scalar function that commutes with $D_k(y)$ at equal times and that is subject to the constraint

$$2\lambda f_{ij}(x)D_j(x)=D_i(x). \quad (1.10)$$

This equation states that in the space of internal quantum number the operator function $f_{ij}$ when acting on the operator $D_j$ leaves it essentially invariant. Stated differently, the column vector $\mathbf{D}$ whose elements are the $D_i$'s is an eigenvector of the matrix $\mathbf{f}$ whose elements are the $f_{ij}$'s with an eigenvalue $1/2\lambda$.

A power-series representation for $f_{ij}$ that satisfies the constraint (1.10) was obtained in Refs. 7 and 8 in the case of $SU(2)\otimes SU(2)$. One assumes that $f_{ij}$ is a functional of the divergences $D_i$ and writes

$$f_{ij}=f_{ij}(\mathbf{D}). \quad (1.11)$$

With such a structure, the last commutator in Eq. (1.9) is automatically satisfied if the one that immediately precedes it holds and it is not necessary to require it to hold separately. Now, since $f_{ij}(\mathbf{D})$ has even parity, its most general form is[10]

$$f_{ij}(\mathbf{D})=f(\mathbf{D}^2)\delta_{ij}+g(D^2)D_iD_j, \quad (1.12)$$

with $f$ and $D$ being isotopic scalar functions of $\mathbf{D}^2$. The Jacobi identity between $A_i{}^0(x')$, $A_j{}^0(x)$, and $D_k(y)$ then leads to the relation

$$2ff'+2\mathbf{D}^2f'g-fg+1=0. \quad (1.13)$$

In terms of (1.12), Eq. (1.10) reads

$$f+\mathbf{D}^2g=1/2\lambda. \quad (1.14)$$

A power-series solution for $f$ and $g$ that satisfies Eqs. (1.13) and (1.14) was given as (writing $z=\mathbf{D}^2$)

$$f(z)=\eta-\frac{z}{3\eta}-\frac{z^2}{5\eta}\left(\frac{1}{3\eta}\right)^2-\frac{2z^3}{35\eta^2}\left(\frac{1}{3\eta}\right)^3-\cdots, \quad (1.15)$$

$$g(z)=\frac{1}{3\eta}+\frac{z}{5\eta}\left(\frac{1}{3\eta}\right)^2+\frac{2z^2}{35\eta^2}\left(\frac{1}{3\eta}\right)^3+\cdots, \quad (1.16)$$

with $\eta=1/2\lambda$.

In this paper we wish to show that a solution to Eqs. (1.13) and (1.14) exists in a closed form. This is the subject of Sec. II. In Sec. III we state some conclusions.

## II. CLOSED FORM FOR TRANSFORMATION FUNCTION $f_{ij}$

The equations we wish to solve are

$$2ff'+2zf'g-fg+1=0, \quad (2.1)$$

$$f+zg=\eta. \quad (2.2)$$

[6] J. Schwinger, Phys. Rev. **130**, 406 (1963); **130**, 800 (1963).
[7] M. A. Ahmed and M. O. Taha, Phys. Rev. **188**, 2517 (1969).
[8] M. A. Ahmed, Nuovo Cimento **69A**, 47 (1970).
[9] K. Bardakci, Y. Frishman, and M. B. Halpern, Phys. Rev. **170**, 1353 (1968).
[10] S. Weinberg, Phys. Rev. **166**, 1568 (1968).

Eliminating $g$ from (2.1) by means of (2.2), we obtain

$$2\eta f' - \eta f/z + f^2/z + 1 = 0. \tag{2.3}$$

This equation is recognized to be of the form of a generalized Riccati differential equation.[11] The substitution

$$f = 2\eta(\omega'/\omega)z, \tag{2.4}$$

where $\omega$ is some function of $z$, transforms the differential equation (2.3) into a homogeneous linear second-order differential equation

$$4\eta^2(\omega''/\omega)z + 2\eta^2\omega'/\omega + 1 = 0. \tag{2.5}$$

It is easily verified that Eq. (2.5) can be cast into the form

$$z^{1/2}\frac{d}{dz}\omega' z^{1/2} + \frac{1}{4\eta^2}\omega = 0. \tag{2.6}$$

We now transform from the variable $z$ to a new variable $u$ simply related to $z$ by

$$u^2 = z. \tag{2.7}$$

In terms of $u$ the differential equation (2.6) becomes

$$\frac{d^2\omega}{du^2} + \frac{1}{\eta^2}\omega = 0, \tag{2.8}$$

which is nothing other than the standard equation for the simple harmonic oscillator. In terms of $\lambda$, (2.8) reads

$$\frac{d^2\omega}{du^2} + 4\lambda^2\omega = 0. \tag{2.9}$$

Writing the solution of Eq. (2.9) as

$$\omega = A\cos 2\lambda u + B\sin 2\lambda u, \tag{2.10}$$

and going back to the original variables, we can now write

$$f(z) = \frac{z^{1/2}(-A + B\cot 2\lambda z^{1/2})}{A\cot 2\lambda z^{1/2} + B}, \tag{2.11}$$

$$g(z) = \frac{1}{2\lambda z} - \frac{z^{-1/2}(-A + B\cot 2\lambda z^{1/2})}{A\cot 2\lambda z^{1/2} + B}. \tag{2.12}$$

Note that whereas in Refs. 7 and 8 the parameter $\lambda$ was required to be small to ensure convergence of the series (1.15) and (1.16), no such requirement on $\lambda$ is needed in the present case.

[11] See, e.g., G. Birkhoff and G. C. Rota, *Ordinary Differential Equations* (Blaisdell, Waltham, Mass., 1962).

## III. CONCLUSIONS

We now write the equal-time commutator (1.8) between $A_i^0$ and $D_j$ in its final form:

$$\begin{aligned}[A_i^0(x), D_j(y)]_{x^0=y^0} = -i\Big[&\frac{(\mathbf{D}^2)^{1/2}[-A + B\cot 2\lambda(\mathbf{D}^2)^{1/2}]}{A\cot 2\lambda(\mathbf{D}^2)^{1/2} + B}\delta_{ij} \\ &+\left(\frac{1}{2\lambda\mathbf{D}^2} - \frac{(\mathbf{D}^2)^{-1/2}[-A + B\cot 2\lambda(\mathbf{D}^2)^{1/2}]}{A\cot 2\lambda(\mathbf{D}^2)^{1/2} + B}\right)D_iD_j\Big] \\ &\times\delta^3(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{3.1}$$

We thus see that of the class of possible functions $f(\mathbf{D}^2)$ and $g(\mathbf{D}^2)$ entering the transformation law of the divergences under the time components of the axial-vector currents and satisfying Eq. (1.13), the constraint (1.14) helps to select the set of harmonic functions. Various simple choices of the constants $A$ and $B$ lead to simple expressions for the functions $f$ and $G$. Thus, for example, the choice $A=0$, $B\neq 0$ gives for the function $f$

$$f(\mathbf{D}^2) = (\mathbf{D}^2)^{1/2}\cot 2\lambda(\mathbf{D}^2)^{1/2}. \tag{3.2}$$

If we now make a tentative link with field theory by assuming PCAC in the operator form $D_i = C\pi_i$, with $\pi_i$ being the pseudoscalar pion field and $C$ a constant, then we see that a suitable choice of $\lambda$ leads essentially to the function $f(\boldsymbol{\pi}^2)$ occurring in the transformation law for the pion field in chiral Lagrangian theories as written down by Callan *et al.*,[12] by Isham,[13] and by Charap.[14]

One may assume a general validity for (3.1), thus abstracting it from the model in which it was established. Extension of these considerations to the case of chiral $SU(3)\otimes SU(3)$ should be interesting.[15]

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[12] S. Coleman, J. Wess, and B. Zumino, Phys. Rev. **177**, 2239 (1969); C. G. Callan, Jr., S. Coleman, J. Wess, and B. Zumino, *ibid.* **177**, 2247 (1969).

[13] C. J. Isham, Nuovo Cimento **59A**, 356 (1969).

[14] J. M. Charap, Can. J. Phys. (to be published).

[15] In this connection, see remarks in Ref. 7.