Dual Resonance Theory with Nonlinear Trajectories*

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A dual resonance model with nonlinear trajectories which includes the Veneziano model as a limiting case is discussed. Rules are given for constructing the N-point function in such a model, and an explicit expression for the N-point planar tree graph is derived. A conjectured rule for loop diagrams is also presented. Representations of four-point-function satellite terms which admit straightforward generalization to the N-point function are derived.

I. INTRODUCTION

DURING the past year, the Veneziano representation[1] has been used or interpreted in two ways: (i) as a guide to an improved Regge phenomenology, or (ii) as the first Born term in a perturbation series of a theory containing an infinite number of particles of arbitrarily high spin. To implement this second program, it is necessary to specify a procedure for calculating higher-order "Feynman-like diagrams." Progress in this direction has been made by Kikkawa, Sakita, and Virasoro,[2] Fubini, Gordon, and Veneziano,[3] Bardakci, Halpern, and Shapiro,[4] and others.[5]

In this paper, we seek generalizations of the N-point functions of Veneziano,[1] Bardakci and Ruegg,[6] Chan and Tsou,[7] Goebel and Sakita,[8] and others[9] in order to find a larger class of candidates for an N-point Born approximation. In order to be acceptable candidates, the functions must have the correct pole structure, crossing symmetry, and suitable asymptotic behavior. This problem was solved in the four-point-function case by Coon.[10] Coon looked for a function $F(s,t)$ which was crossing symmetric and had poles in s and t with polynomial residues. Upon imposing these requirements, he found a class of crossing-symmetric meromorphic functions $B_4(s,t)$ which has poles at $s=s_j$, where

$$as_j+b=q^{-j},\quad j=0,1,2,\cdots. \tag{1}$$

The parameters q, a, and b are arbitrary except that $0<q<1$. In the limit $q\to 1$, $B_4(s,t)$ reduces to the Veneziano representation. These generalizations of the Veneziano representation may well constitute the largest class of amplitudes which are good Born terms.

In this article, we first obtain a double power-series expansion in s and t for the Coon four-point function. Use of this series greatly simplifies its generalization to the N-point function. We can then obtain a power-series representation for the N-point function which explicitly exhibits crossing symmetry and has the correct pole structure of planar tree graphs. We have not succeeded in constructing a deductive argument analogous to Coon's argument leading to the four-point function. However, a lemma of Watson concerning power-series representations of certain meromorphic functions has enabled us to extend our representation. Thus, we may, in fact, have the most general form satisfying the above requirements of meromorphy, polynomial residues, and crossing symmetry. We discuss these general forms in Appendices B and C. In Appendix C we show that Veneziano satellite terms are obtained as $q\to 1$ limits of our expressions. The actual proof that our N-point function reduces to the Veneziano N-point function[6–9] in the limit $q\to 1$ will be given in a future paper.[11]

It should be emphasized that our amplitudes are constructed by requiring only that they possess the correct pole structure and crossing symmetry. However,

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[3] S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters **29B**, 697 (1969); S. Fubini and G. Veneziano, Nuovo Cimento **64A**, 811 (1969).

[4] K. Bardakci, M. B. Halpern, and J. A. Shapiro, Phys. Rev. **185**, 1910 (1969); K. Bardakci and M. B. Halpern, *ibid.* **183**, 1456 (1969).

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[6] K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968); Phys. Rev. **181**, 1884 (1969).

[7] H.-M. Chan, Phys. Letters **28B**, 425 (1969); H.-M. Chan and S. T. Tsou, *ibid.* **28B**, 485 (1969).

[8] C. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969).

[9] M. Virasoro, Phys. Rev. Letters **22**, 37 (1969); Z. Koba and H. Nielsen, Nucl. Phys. **B10**, 633 (1969).

[10] D. Coon, Phys. Letters **29B**, 669 (1969). For an application of the Adler self-consistency condition to this four-point function, see D. Coon, Phys. Rev. **186**, 1422 (1969). The results of this application are analogous to those obtained from the Veneziano four-point function by C. Lovelace, Phys. Letters **28B**, 264 (1968), and M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters **22**, 83 (1969).

[11] M. Baker and D. D. Coon (unpublished).

it turns out that all of these amplitudes also possess duality; that is, they can be represented as sums over the set of poles contained in any one of the planar tree graphs. For example, the four-point function can be represented either as a sum over t-channel poles or a sum over s-channel poles.

The fact that any properly behaved crossing-symmetric resonance approximation must possess the above duality property can easily be understood on the basis of general arguments which are reviewed in Sec. II. In that section we also discuss the physical requirements for a good Born term. Of course, our basic hypothesis, namely, that the Born term be well behaved, cannot be justified on any *a priori* physical grounds. One cannot rule out the possibility that the high-spin Born terms have their usual bad properties and that these are somehow canceled out by the higher-order diagrams. However, such a cancellation mechanism is unnecessary if one starts from an unrenormalized Born approximation where the masses are given by Eq. (1). Then the higher-order diagrams would just shift the masses as well as produce unitarity corrections.

In Sec. III, we review the properties of the Coon four-point function and its $q \to 1$ Veneziano limit, and give a double power-series expansion which is valid for $q<1$. One can take the $q \to 1$ limit only after summing the series. In Sec. IV we obtain the explicit form for the five-point function and formulate the rules for the N-point planar tree graphs.

Although our rules for planar tree diagrams suggest natural rules for loop diagrams, the situation is more complicated and will be mentioned only briefly in Sec. V.

The proofs of certain mathematical theorems which are needed in the text are given in Appendix A. Appendix B is a discussion of generalizations of the power-series expansion for the N-point function which have the same poles but different polynomial residues. (In accordance with the usual terminology we can call these generalizations satellite terms.) In Appendix C we introduce explicit forms for four-point-function satellite terms and show that as $q \to 1$, they reduce to the usual Veneziano satellite terms. We also derive the power-series representation of these terms.

An expansion of the nonlinear-trajectory four-point function which is the formal nonlinear analogy of the beta-function integral representation is derived in Appendix D.

II. DUALITY AND BORN TERM

The discussion of this section is qualitative and much of it is perhaps familiar. It also contains no results which we later use. Its purpose here is to help provide physical motivation for the work of the remainder of the paper.

We argue that duality is simply a *consequence* of requiring that the Born term be sufficiently well behaved to describe peripheral collisions at high energy (small fixed t and large s). To satisfy this reasonable and apparently weak requirement in general, we must include an infinite number of particles with arbitrarily high spins. Historically, this rather drastic measure was supposedly avoided by dropping Born terms in favor of Regge poles as peripheral, high-energy pole approximations. But the insight into Regge poles provided by the models of Van Hove,[12] and Blankenbecler, Sugar, and Sullivan,[13] as well as the field-theory approach to the Veneziano representation, all seem to lead us back to the possibility of a good Born term arising from an infinite number of particles with arbitrarily high spin.

From this point of view, there is no reason why one should expect the Born term to accurately describe nonperipheral processes such as fixed-angle scattering. Thus, for example, one cannot rule out as possible candidates for Born terms the unequal-slope Veneziano model or the nonlinear model of Coon just because they increase at high energy for some physical fixed angle.[10,14] Higher-order effects like cuts should be important here.

We begin by considering a theory in which the t-channel pole terms are given by

$$\sum_{J}^{\infty} \frac{P_J(s)}{t-m_J{}^2}, \tag{2}$$

where the residues are polynomials of order J in s. For simplicity, we assume that there are no u-channel poles and that the amplitude is $s \leftrightarrow t$ crossing symmetric. If the sum (2) itself has no singularities in s, then (2) represents an entire function of s which is not a polynomial and therefore must grow faster than any polynomial as $|s| \to \infty$. This behavior cannot be canceled by a sum of s-channel poles because, in general, Mittag-Leffler expansions do not exist for functions which grow faster than any power at infinity.

This undesirable large-s behavior can be avoided if the sum of t-channel pole terms (2) contains at least some of the s-channel pole terms. Moreover, the number of such s-channel pole terms in (2) must be infinite in order to avoid bad behavior as $|t| \to \infty$. Furthermore if the amplitude contains an additional finite number of s-channel pole terms which are not present in (2), the amplitude will then possess polynomial t dependence. In general, the asymptotic t behavior of (2) is s dependent and cannot cancel an additive polynomial in t. Therefore, if we demand that the Born term be well behaved at small fixed s and large t, the only s-channel pole terms which might not be included in (2) would be s-wave terms. Hence, the requirement of good behavior of our Born term in the peripheral, high-energy region leads to duality. Furthermore, we have examples[1,10] which show that the behavior of such a Born term can be better than one which contains only finite spin.

[12] L. Van Hove, Phys. Letters **24B**, 183 (1967); see also L. Durand, III, Phys. Rev. **161**, 1610 (1967).
[13] R. L. Sugar and J. D. Sullivan, Phys. Rev. **166**, 1515 (1968); R. Blankenbecler and R. L. Sugar, *ibid.* **168**, 1597 (1968).
[14] F. Capra, Phys. Letters **30B**, 53 (1969).

We now turn to the explicit construction of our "good Born terms."

III. FOUR-POINT FUNCTION

Ideally, one would like to know the most general function which satisfies our Born-term requirements of

(i) meromorphy,
(ii) polynomial residues, and
(iii) "acceptable" asymptotic behavior.

Coon[10] found that in addition to the Veneziano representation, there is a one-parameter family[15] of functions which satisfies the three conditions. This more general representation consists of an arbitrary constant times the function

$$B_4(s,t)=\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}=\prod_{l=0}^{\infty}\frac{(1-\sigma\tau q^l)}{(1-\sigma q^l)(1-\tau q^l)},\tag{3}$$

where q is a parameter, $0<q<1$,

$$\sigma=as+b,\quad \tau=ct+d,\tag{4}$$

and

$$G(\sigma)=\prod_{l=0}^{\infty}(1-\sigma q^l)\tag{5}$$

is an entire function. The poles of B_4 are located at $\sigma=q^{-j}$ with polynomial residues of order j, and so the trajectory function is

$$\alpha_s=-(\ln\sigma)/\ln q.\tag{6}$$

If the q dependence of σ and τ near $q=1$ is given by

$$\sigma=1+(1-q)\sigma'(s)+(1-q)^2\sigma''(s)+\cdots,\tag{7}$$

$$\tau=1+(1-q)\tau'(s)+(1-q)^2\tau''(s)+\cdots,\tag{8}$$

then

$$\lim_{q\to1}\alpha_s=\sigma'\tag{9}$$

and

$$\lim_{q\to1}(1-q)\frac{G(q)G(\sigma\tau)}{G(\sigma)G(\tau)}=\frac{\Gamma(-\sigma')\Gamma(-\tau')}{\Gamma(-\sigma'-\tau')}.\tag{10}$$

Thus, we see that in the limit $q\to1$ the logarithmic trajectories become linear, and the representation reduces to the Veneziano formula.

From an asymptotic expansion[16] given in Appendix A or a simple asymptotic estimate,[10] it can be shown that for large $|s|$,

$$B_4(s,t)\sim(as)^{\alpha_t},\tag{11}$$

and for large $|t|$,

$$B_4(s,t)\sim(ct)^{\alpha_s}.\tag{12}$$

[15] In order to obtain his representation from assumptions (i) and (ii), Coon made an additional technical assumption about the cancellation mechanism which produced the polynomial residues required by (ii). However, it can easily be seen that this mechanism is the only one which is consistent with assumption (iii). Thus, his solutions may be the unique solutions.

[16] J. E. Littlewood, Proc. London Math. Soc. 5, 361 (1907).

In order to exhibit the polynomial residues and duality, we give the two partial-fraction expansions

$$G(q)B_4(s,t)=\frac{G(q)G(\sigma\tau)}{G(\sigma)G(\tau)}$$

$$=\frac{1}{1-\sigma}+\sum_{j=1}^{\infty}\frac{1}{1-\sigma q^j}\prod_{l=1}^{j}\frac{\tau-q^l}{1-q^l}\tag{13}$$

$$=\frac{1}{1-\tau}+\sum_{j=1}^{\infty}\frac{1}{1-\tau q^j}\prod_{l=1}^{j}\frac{\sigma-q^l}{1-q^l}.\tag{14}$$

The first converges for $|\tau|<1$ and the second for $|\sigma|<1$, as can be seen from the ratio test.

In Appendix A we obtain the expansion

$$B_4=\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\sigma^n}{(q)_{q,n}}q^{nm}\frac{\tau^m}{(q)_{q,m}},\tag{15}$$

where

$$(q)_{q,n}=(1-q)(1-q^2)\cdots(1-q^n).\tag{16}$$

This expansion converges for $|\sigma|<1$ and $|\tau|<1$. In order to exhibit the poles of the function defined by this series, one can use the summation formula

$$\frac{1}{G(z)}=\sum_{n=0}^{\infty}\frac{z^n}{(q)_{q,n}}\tag{17}$$

derived in Appendix A to do either sum in Eq. (15). After evaluating the n sum, one finds that

$$\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}=\sum_{m=0}^{\infty}\frac{1}{G(\sigma q^m)}\frac{\tau^m}{(q)_{q,m}}.\tag{18}$$

The pole at $\sigma=q^{-j}$ occurs only in the terms with $m\leq j$, so that the polynomial nature of the residues is obvious. It is shown in Appendix B that one can also use Watson's lemma[17] to prove meromorphy and polynomial residues without doing any sums.

A very important feature of the power-series representation is the way in which duality manifests itself. If the q^{nm} factor were not present, then both summations could be done using Eq. (17), and the result would be

$$1/G(\sigma)G(\tau),\tag{19}$$

which has poles in both σ and τ simultaneously. Thus, we see that the q^{nm} factor prevents simultaneous poles and replaces singular residues with polynomial residues. One can therefore think of q^{nm} as a "simultaneous-pole eliminator" or a "duality factor." This factor can be inserted in more complicated multiple sums to obtain representations for amplitudes with more complicated pole structure.

IV. N-POINT TREE GRAPHS

The simple way in which duality manifests itself in our representation (15) for the four-point function will

[17] G. N. Watson, Trans. Cambridge Phil. Soc. 11, 281 (1910).

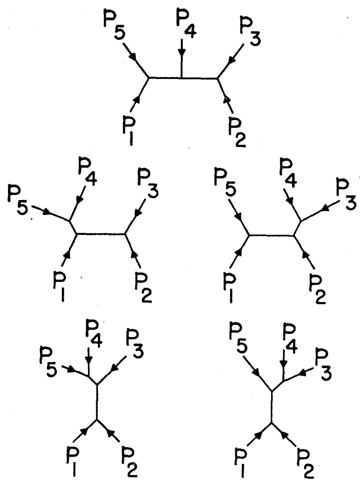

FIG. 1. Feynman-like diagrams which are represented by the five-point function.

enable us to immediately write down representations which incorporate duality for the N-point tree graphs. That is, each of the N-point functions will have many distinct partial-fraction expansions which correspond to summing different subsets of lowest-order "Feynman diagrams." We say that these various sums of Feynman diagrams are related by duality because including any one set of diagrams is equivalent to including any one of the other sets of Feynman diagrams. Each sum of diagrams can be denoted by a single, typical diagram where a sum over particles with all spins in the internal lines is understood. Such a diagram is called a Feynman-like diagram.[18]

A. Five-Point Function

Suppose that we wish to relate the five planar Feynman-like diagrams of Fig. 1 by duality. The external particles have zero spin. We can introduce five quantities, σ_{12}, σ_{23}, σ_{34}, σ_{45}, and σ_{51}, which are linearly related to the five kinematic variables $s_{ij}=(p_i+p_j)^2$ by

$$\sigma_{ij}=a_{ij}s_{ij}+b_{ij}. \tag{20}$$

We can introduce poles corresponding to all the "propagators" by simply writing the multiple sum

$$\sum_{\text{all } n_{ij}=0}^{\infty} \frac{\sigma_{12}^{n_{12}}}{f_{n_{12}}} \frac{\sigma_{23}^{n_{23}}}{f_{n_{23}}} \frac{\sigma_{34}^{n_{34}}}{f_{n_{34}}} \frac{\sigma_{45}^{n_{45}}}{f_{n_{45}}} \frac{\sigma_{51}^{n_{51}}}{f_{n_{51}}} =[G(\sigma_{12})G(\sigma_{23})G(\sigma_{34})G(\sigma_{45})G(\sigma_{51})]^{-1}, \tag{21}$$

where we have used the abbreviation

$$f_n=(q)_{q,n}=(1-q)(1-q^2)\cdots(1-q^n) \tag{22}$$

and Eq. (17) to do the sums. The expression (21) has simultaneous poles in all channels. It is obvious from the graphs in Fig. 1 that we cannot have simultaneous poles in adjacent channels. That is, no diagram with a pole in the s_{23} channel can also have a pole in the s_{12} or the s_{34} channels. This requirement can be enforced by inserting the factors $q^{n_{12}n_{23}}$ and $q^{n_{23}n_{34}}$ under the sums in (21). By cyclic symmetry, we must include other such factors, and then we obtain

$$B_5=\sum_{\text{all } n_{ij}=0}^{\infty} \frac{\sigma_{12}^{n_{12}}}{f_{n_{12}}} q^{n_{12}n_{23}} \frac{\sigma_{23}^{n_{23}}}{f_{n_{23}}} q^{n_{23}n_{34}} \times \frac{\sigma_{34}^{n_{34}}}{f_{n_{34}}} q^{n_{34}n_{45}} \frac{\sigma_{45}^{n_{45}}}{f_{n_{45}}} q^{n_{45}n_{51}} \frac{\sigma_{51}^{n_{51}}}{f_{n_{51}}} q^{n_{51}n_{12}}, \tag{23}$$

where $\sigma_{ij}=a_{ij}s_{ij}+b_{ij}=a_{ij}(p_i+p_j)^2+b_{ij}$. The sums converge for $|\sigma_{ij}|<1$ and the relevant range $(0<q<1)$ of q. We are not able to do all of the sums. However, we will show that with the summation formulas (15) and (17), one can easily evaluate any two sums and verify that only pairs of poles exist and that these pairs correspond to the poles in the diagrams of Fig. 1. With successive application of Eqs. (15) and (17), four of the five summations in Eq. (23) can be performed.

If we use the summation formula (17) to evaluate the n_{23} sum in Eq. (23), we obtain a factor

$$\frac{1}{G(\sigma_{23}q^{n_{12}+n_{34}})}=\prod_{l=0}^{\infty}\frac{1}{(1-\sigma_{23}q^{l+n_{12}+n_{34}})} \tag{24}$$

under the remaining multiple sum. From Eq. (24) we see that the pole at $\sigma_{23}=1$ is contained only in the $n_{12}=n_{34}=0$ terms in Eq. (23). Thus, the residue of the $\sigma_{23}=1$ pole of B_5 is independent of σ_{12} and σ_{34}. The surviving σ_{ij} dependence of the residue is easily seen to be

$$\sum_{n_{45}=0}^{\infty}\sum_{n_{51}=0}^{\infty} \frac{\sigma_{45}^{n_{45}}}{f_{n_{45}}} q^{n_{45}n_{51}} \frac{\sigma_{51}^{n_{51}}}{f_{n_{51}}}, \tag{25}$$

which is just a four-point function (15). This proves the consistency between our four-point and five-point functions and factorization at the $\sigma_{23}=1$ pole. The pole at $\sigma_{23}=1$ corresponds to spin zero because there are no dot products involving p_2 or p_3 in the residue (25). Factorization requires that this residue be proportional to our four-point function for spin-zero external lines.

In addition to the n_{23} sum in Eq. (23), we can also use Eq. (17) to do the σ_{45} summation to obtain

$$B_5=\sum_{n_{12}=0}^{\infty}\sum_{n_{34}=0}^{\infty}\sum_{n_{51}=0}^{\infty} \frac{\sigma_{12}^{n_{12}}}{f_{n_{12}}} \frac{1}{G(\sigma_{23}q^{n_{12}+n_{34}})} \times \frac{\sigma_{34}^{n_{34}}}{f_{n_{34}}} \frac{1}{G(\sigma_{45}q^{n_{34}+n_{51}})} \frac{\sigma_{51}^{n_{51}}}{f_{n_{51}}} q^{n_{51}n_{12}}, \tag{26}$$

which shows that we have poles at $\sigma_{23}=q^{-j}$ and $\sigma_{45}=q^{-k}$ with j, $k=0, 1, 2, \ldots$. Furthermore, all terms in the triple sum (26) with $n_{12}+n_{34}>j$ or $n_{34}+n_{51}>k$ do not possess a pair of σ_{23}, σ_{45} poles and therefore the residues of the σ_{23}, σ_{45} pair of poles are polynomials in σ_{12}, σ_{34},

[18] This is the terminology of Ref. 2.

and σ_{51}. This is the general structure which one expects from Feynman diagrams or pole models. Since Eq. (23) is cyclicly symmetric, our conclusions apply to any pair of poles in nonadjacent channels.

In order to find out what happens in adjacent or overlapping channels such as s_{12} and s_{23}, we use our four-point-function expansion (15) as a summation formula. Evaluating the n_{12} and n_{23} sums in Eq. (23) gives

$$B_5=\sum_{n_{34}=0}^{\infty}\sum_{n_{45}=0}^{\infty}\sum_{n_{51}=0}^{\infty}\frac{G(\sigma_{12}\sigma_{23}q^{n_{51}+n_{34}})}{G(\sigma_{12}q^{n_{51}})G(\sigma_{23}q^{n_{34}})}\times\frac{\sigma_{34}{}^{n_{34}}}{f_{n_{34}}}q^{n_{34}n_{45}}\frac{\sigma_{45}{}^{n_{45}}}{f_{n_{45}}}q^{n_{45}n_{51}}\frac{\sigma_{51}{}^{n_{51}}}{f_{n_{51}}}. \quad (27)$$

From Eq. (27), we see that simultaneous poles in the σ_{12} and σ_{23} channels are prevented by the vanishing of the $G(\sigma_{12}\sigma_{23}q^{n_{51}+n_{34}})$ factor. Thus, our five-point function is free of poles in adjacent channels. Therefore, we have a five-point function with only that pole structure which is found in the trilinear coupling graphs of Fig. 1. Furthermore, we see that only trivial applications of Eqs. (15) and (17) are required in order to exhibit these properties of B_5 if we start with our defining representation (23). This fact together with the symmetry of Eq. (23) and the ease with which one can guess the form (23) on the basis of Eq. (15) for the four-point function are all appealing features of the power-series representation.

We will now show that it is possible to express B_5 as a single infinite sum. This can be accomplished by successive applications of Eqs. (15), (17), and (A11). Using Eq. (17) to do the n_{45} sum in Eq. (27), we find that

$$B_5=\sum_{n_{34}=0}^{\infty}\sum_{n_{51}=0}^{\infty}\frac{G(\sigma_{12}\sigma_{23}q^{n_{51}+n_{34}})}{G(\sigma_{12}q^{n_{51}})G(\sigma_{23}q^{n_{34}})}\times\frac{1}{G(\sigma_{45}q^{n_{51}+n_{34}})}\frac{\sigma_{34}{}^{n_{34}}}{f_{n_{34}}}\frac{\sigma_{51}{}^{n_{51}}}{f_{n_{51}}}. \quad (28)$$

Equation (A11) can be used to write

$$\frac{G(\sigma_{12}\sigma_{23}q^{n_{51}+n_{34}})}{G(\sigma_{45}q^{n_{51}+n_{34}})}=\sum_{n_{45}=0}^{\infty}\left(\frac{\sigma_{12}\sigma_{23}}{\sigma_{45}}\right)_{q,n_{45}}\frac{\sigma_{45}{}^{n_{45}}}{f_{n_{45}}}q^{n_{45}(n_{51}+n_{34})} \quad (29)$$

and Eq. (17) can be used to expand $1/G(\sigma_{12}q^{n_{51}})$ and $1/G(\sigma_{23}q^{n_{34}})$. Substituting these expansions in Eq. (28) gives

$$B_5=\sum_{\text{all }n_{ij}=0}^{\infty}\frac{\sigma_{12}{}^{n_{12}}}{f_{n_{12}}}\frac{\sigma_{23}{}^{n_{23}}}{f_{n_{23}}}q^{n_{23}n_{34}}\frac{\sigma_{34}{}^{n_{34}}}{f_{n_{34}}}q^{n_{34}n_{45}}\times\left(\frac{\sigma_{12}\sigma_{23}}{\sigma_{45}}\right)_{q,n_{45}}\frac{\sigma_{45}{}^{n_{45}}}{f_{n_{45}}}q^{n_{45}n_{51}}\frac{\sigma_{51}{}^{n_{51}}}{f_{n_{51}}}q^{n_{51}n_{12}}. \quad (30)$$

By means of our manipulations, we have eliminated the factor $q^{n_{12}n_{23}}$. Again we use our four-point function expansion (15) as a summation formula for the n_{51}, n_{12} and n_{23}, n_{34} double sums and find that

$$B_5=\sum_{n_{45}=0}^{\infty}\frac{G(\sigma_{51}\sigma_{12}q^{n_{45}})}{G(\sigma_{51}q^{n_{45}})G(\sigma_{12})}\times\frac{G(\sigma_{23}\sigma_{34}q^{n_{45}})}{G(\sigma_{23})G(\sigma_{34}q^{n_{45}})}\left(\frac{\sigma_{12}\sigma_{23}}{\sigma_{45}}\right)_{q,n_{45}}\frac{\sigma_{45}{}^{n_{45}}}{f_{n_{45}}}. \quad (31)$$

A virtue of this representation is that one can now take the limit $q\to 1$ of $(1-q)^2G^2(q)B_5$ to obtain a five-point function with linear trajectories. In Ref. 11, we show that this limiting case is the Bardakci-Ruegg five-point function. In Ref. 11, we also verify that B_5 has the correct Regge asymptotic behavior.

B. Rules for Feynman-like Diagrams Compatible with Duality

We can now see the simplicity of the structure of our mermorphic functions with which we represent a given set of Feynman-like diagrams related by duality. For tree graphs with spinless external particles, the rules are as follows[19]:

(a) For each internal line of momentum p, define a quantity $\sigma=ap^2+b$, where a and b are constants which may be different for different lines.

(b) Write a multiple sum for the product which contains one factor

$$\sum_{n=0}^{\infty}\frac{\sigma^n}{f_n} \quad (32)$$

for each internal line of distinct momentum. The constants f_n are given by Eq. (22).

(c) Under the multiple sum, introduce a factor of q^{nm} for each pair of internal lines which are dual to each other, i.e., for each pair of "propagators" which never occur in the same Feynman-like diagram.

These rules give the correct pole structure for nonplanar as well as planar tree graphs. The nonplanar case differs only in that there are more diagrams and internal lines.

Following the methods used in the discussion of the five-point function, one can show that the residue of each of the $\sigma=1$ (spin zero) poles factorizes into two lower N-point functions which are consistent with the same rules. Factorization of the residues of spin-zero poles just relates over-all constants which can multiply our B_N.

Simple considerations[7] show that the N-point planar tree graphs can have poles in the variables

$$s_{ij}=(p_i+p_{i+1}+\cdots+p_j)^2,$$

[19] A brief treatment of the procedure for constructing our N-point functions is also contained in another article by M. Baker and D. Coon (unpublished).

where $1 \leq i \leq N-2$ and $i+1 \leq j \leq N-1$, with the exception of $i=1$, $j=N-1$. A given planar tree graph which has a pole in a particular variable s_{ij} cannot also have a pole in a "partially overlapping" channel s_{kl} with $i+1 \leq k \leq j$ and $j+1 \leq l \leq N-1$.[7] With our rules it is a simple matter to write down a dual amplitude having this pole structure.

The explicit expression for the planar-tree-graph approximation to the N-point function which one obtains from the above rules is the following:

$$B_N = \sum_{\text{all } n_{ij}=0}^{\infty} \prod_{i=1}^{N-2} \prod_{j=i+1}^{N-1} \frac{\sigma_{ij}{}^{n_{ij}}}{f_{n_{ij}}} \times \prod_{k=i+1}^{j} \Big(\prod_{l=i+1}^{N-1} q^{n_{ij}n_{kl}} \Big/ \prod_{l=i+1}^{j} q^{n_{ij}n_{kl}} \Big), \tag{33}$$

where

$$\sigma_{ij} = a_{ij}(p_i + p_{i+1} + \cdots + p_j)^2 + b_{ij}, \tag{34}$$

except for $\sigma_{1,N-1}=0$ which is introduced to make Eq. (33) more compact. The limit as $q \to 1$ of the function defined by B_N has been shown[11] to be the Veneziano N-point function.[6–9]

V. LOOP DIAGRAMS

Although loop diagrams will be treated in detail in another paper,[20] we think it might be useful to make a few comments on them here.

Since the problem of consistency of factorization of loop diagrams and planar tree graphs has proved to be difficult,[3–5] it may be that extra considerations are required in constructing loop diagrams. However, the most natural guess for the rules appears to be the rules of Sec. IV plus the following rule:

(d) One $\int d^4k$ for each loop.

To clarify the ambiguities of this prescription would require further discussion and examples. Here we only want to note two features of the integrals. First, they do not correspond to the rules of Kikkawa, Sakita, and Virasoro[2] although the rules as applied to planar tree graphs do give the appropriate generalization of the Veneziano N-point function.[6–9] Secondly, if one tries to take the limit $q \to 1$ before doing the loop integrals, a divergence is usually encountered. Thus, to get meaningful results in this limit it may be necessary to carry out the integrals before taking the limit $q \to 1$. It therefore appears that when loop diagrams are considered, basic differences arise between the linear and nonlinear theories. These are not present for tree graphs. Since dual theories of the higher-order diagrams in the Veneziano limit are still in an incomplete form and detailed discussion of higher-order diagrams in the nonlinear theory has not yet been presented, it is clearly still an open question as to which if either of these possibilities can be made into a consistent theory.

[20] M. Baker and D. D. Coon (unpublished).

VI. CONCLUSION

We have deduced rules for constructing a one-parameter family of N-point-function Born terms B_N which satisfy duality, have nonlinear trajectories, and include the Veneziano N-point function as a limiting case. These rules yield power-series representations for B_N in which the role of duality is transparent. As shown in Appendix C, the structure of the power-series representations of satellite terms is similar enough to allow straightforward generalization of the rules to the case of satellite terms.

As illustrated in the example of B_5, many properties of B_N can easily be deduced from the power-series representations using the expansion (15) of the four-point function as a summation formula.

Because the nonlinear Born term B_N goes smoothly into the Veneziano Born term as $q \to 1$, it is not easy to distinguish the case where $q=1$ from the case where q is near one by direct comparison with experiment. Thus, in order to determine whether the nonlinear dual resonance theory is physically interesting, one must probably await the study of the higher-order diagrams. In a forthcoming paper[20] we will present some attempts in this direction based on the ideas mentioned in Sec. V.

APPENDIX A: MATHEMATICAL RESULTS

Here we shall derive some results which are needed in the development of the $q<1$ theory. The relevant special functions have been thoroughly studied in the mathematical literature. They are known as basic functions and their theory has been developed by Heine,[21] Jackson,[22] Watson,[17] Bailey,[23] Sears,[24] and others.[25] We will use their notation in this Appendix and in Appendix B.

We define an entire function

$$G(z) = \prod_{l=0}^{\infty} (1 - zq^l), \tag{A1}$$

where $0<q<1$ and q is called the base. A convenient and frequently used symbol is defined by

$$(a)_{q,n} = G(a)/G(aq^n), \tag{A2}$$

so that $(a)_{q,0}=1$ and for positive integer n

$$(a)_{q,n} = (1-a)(1-qa)\cdots(1-aq^{n-1}). \tag{A3}$$

[21] E. Heine, *Handbuch der Kugelfunktionen* (G. Riemer, Berlin, 1878).

[22] F. H. Jackson, Am. J. Math. **32**, 305 (1910); Quart. J. Pure Appl. Math. **41**, 193 (1910). A more extensive list of Jackson's articles is given by L. J. Slater (Ref. 25).

[23] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge U. P., London, 1935).

[24] D. B. Sears, Proc. London Math. Soc. (2) **53**, 158 (1951); **53**, 181 (1951).

[25] For further references and a summary of results see L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge U. P., London, 1966); see also *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1, p. 195.

In order to derive a useful summation formula, we consider the series

$$\Phi(a;z)=\sum_{n=0}^{\infty}\frac{(a)_{q,n}}{(q)_{q,n}}z^n \tag{A4}$$

which converges for $|z|<1$. Using Eq. (A3), it is easily seen that

$$\frac{(a)_{q,n}}{(q)_{q,n}}(1-q^n)=\frac{(a)_{q,n}}{(q)_{q,n-1}}=(1-a)\frac{(aq)_{q,n-1}}{(q)_{q,n-1}}. \tag{A5}$$

If we multiply Eq. (A5) by z^n and sum over n, we find that

$$\Phi(a;z)-\Phi(a;qz)=(1-a)z\Phi(aq;z). \tag{A6}$$

Again using Eq. (A3), we see that

$$(a)_{q,n}(1-aq^n)=(a)_{q,n+1}=(1-a)(aq)_{q,n}. \tag{A7}$$

Multiplying Eq. (A7) by z^n and summing over n gives

$$\Phi(a;z)-a\Phi(a;qz)=(1-a)\Phi(aq;z). \tag{A8}$$

If we eliminate the series $\Phi(aq;z)$ in Eqs. (A6) and (A8), we find that

$$\Phi(a;z)=\frac{(1-az)}{1-z}\Phi(a;qz). \tag{A9}$$

Repeated application of Eq. (A9) gives

$$\Phi(a,z)=\frac{(1-az)(1-qaz)\cdots(1-q^{m-1}az)}{(1-z)(1-qz)\cdots(1-q^{m-1}z)}\times\Phi(a;q^m z). \tag{A10}$$

If we let $m\to\infty$, $\Phi(a;q^m z)\to 1$ and we obtain Heine's formula:

$$\frac{G(az)}{G(z)}=\sum_{n=0}^{\infty}\frac{(a)_{q,n}}{(q)_{q,n}}z^n. \tag{A11}$$

A useful special case is found by setting $a=0$:

$$\frac{1}{G(z)}=\sum_{n=0}^{\infty}\frac{z^n}{(q)_{q,n}}, \tag{A12}$$

which is a formula of Euler.

We will now use Eqs. (A11), (A2), and (A12), in that order, to obtain a power-series expansion of the function

$$\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}=\frac{1}{G(\tau)}\sum_{n=0}^{\infty}\frac{(\tau)_{q,n}}{(q)_{q,n}}\sigma^n$$
$$=\sum_{n=0}^{\infty}\frac{\sigma^n}{(q)_{q,n}}\frac{1}{G(\tau q^n)}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\sigma^n}{(q)_{q,n}}q^{nm}\frac{\tau^m}{(q)_{q,m}}, \tag{A13}$$

which converges for $|\sigma|<1$ and $|\tau|<1$. We frequently use the abbreviation

$$f_n=(q)_{q,n},$$

which suppresses the dependence on the parameter q.

Asymptotic Expansion

In order to demonstrate the Regge behavior (11) and (12) of our four-point function (3) and to aid in estimating the asymptotic behavior of our N-point functions, it is helpful to have an analog of Stirling's series for the gamma function. Such a formula can be derived from the identity[16]

$$G(z)G(q/z)=\exp\left[-\frac{\ln^2(-z)}{2\ln q}+\tfrac{1}{2}\ln(-z)+H(z)\right], \tag{A14}$$

where

$$H(z)=-\frac{\pi^2}{6\ln q}-\frac{\ln q}{12}+\sum_{m=1}^{\infty}\frac{\cos[2m\pi\ln(-z)/\ln q]}{2m\sinh(2m\pi^2/\ln q)}. \tag{A15}$$

The quantity $H(z)$ remains finite as $|z|\to\infty$ for $|\arg(-z)|<\pi$. By expanding $\ln G(q/z)$ for large z, one obtains[16] the useful asymptotic formula

$$G(z)=\exp\left[-\frac{\ln^2(-z)}{2\ln q}+\tfrac{1}{2}\ln(-z)+H(z)+\sum_{l=1}^{\infty}\frac{q^l}{l(1-q^l)(z)^l}\right], \tag{A16}$$

in which the leading asymptotic behavior is displayed in much the same way as Stirling's formula displays the asymptotic behavior of the gamma function.

APPENDIX B: GENERALIZATIONS

The power-series representation of the four-point function (15) is an expansion of a function possessing simple infinite-product and partial-fraction representations in which it is easy to see the spectrum of poles and polynomial residues. These features are not manifestly exhibited in the power-series expansion, but they can be proved without evaluating any sums by application of a lemma given by Watson.[17] Likewise, without performing any sums, all of the power-series representations of our N-point functions (Sec. IV) can easily be shown to be meromorphic with polynomial residues. Thus, we see that we could generalize our rules to include power series in which we are not able to evaluate any sums and in which Watson's lemma is sufficient to prove the required properties.

We now state Watson's lemma[17] without proof:

Given a series of the form

$$\Psi(z)=\sum_{n=0}^{\infty} u_0u_1\cdots u_nK_nz^n \tag{B1}$$

in which u_n and K_n can be expanded in the convergent series

$$u_n=1+A_1q^n+A_2q^{2n}+\cdots, \tag{B2}$$

$$K_n=q^{nm}(B_0+B_1q^n+B_2q^{2n}+\cdots) \tag{B3}$$

for all integer values of n greater than a certain finite n_0, where the A_i and B_i are independent of n and m is not necessarily an integer, the only possible singularities of the analytic function $\Psi(z)$ in the finite part of the z plane are simple poles at the points $z=q^{-m}$, q^{-m-1}, q^{-m-2}, $\cdots$.

The rules of Sec. IV generate multiple sums to which Watson's lemma applies. If a given residue is supposed to be a polynomial in certain variables, we can look at various powers of those variables in the multiple sum and use the lemma to check that the pole is contained only in the coefficients of finite powers of the relevant variables.

It is clear that we could define new functions by means of multiple power series in which each sum satisfies the requirements of Watson's lemma. Thus, we could construct a more general class of functions with properties required of a Born term.

For example, instead of including a sum such as

$$\sum_{n=0}^{\infty} \frac{\sigma^n}{(q)_{q,n}} \tag{B4}$$

for each internal line [see rule (b) of Sec. IV], we could include the more general term

$$\sum_{n=0}^{\infty} u_0u_1\cdots u_n\sigma^n, \tag{B5}$$

with

$$u_i=A(q^i), \tag{B6}$$

where $A(0)=1$ and $A(x)$ has no singularities in some finite circle about $x=0$.

Similarly, the q^{nm} dual propagator factor [see rule (c) of Sec. IV] could be replaced by the more general form

$$K_{nm}=q^{nm}\sum_{i,j=0}^{\infty} C_{ij}q^{in+jm}, \tag{B7}$$

provided that the i, j sum converges for all non-negative n and m. The generalizations encompassed by (B5) and (B7) will have the same spectrum of poles as the N-point functions of Sec. IV, but the polynomials in the residues will be modified.

A class of functions which satisfy the requirements of Watson's lemma and which have been thoroughly studied are the generalized basic hypergeometric functions[17,21–25] defined for $|z|<1$ by

$$_A\Phi_B(a_1,a_2,\ldots,a_A;b_1,b_2,\ldots,b_B;z)=\sum_{n=0}^{\infty}\frac{(a_1)_{q,n}(a_2)_{q,n}\cdots(a_A)_{q,n}}{(b_1)_{q,n}(b_2)_{q,n}\cdots(b_B)_{q,n}}\frac{z^n}{(q)_{q,n}}. \tag{B8}$$

The sums in Eqs. (A11) and (A12) are simple examples:

$$G(az)/G(z)={}_1\Phi_0(a;z), \tag{B9}$$

$$1/G(z)={}_0\Phi_0(z). \tag{B10}$$

Besides being possible generalizations of the internal line factor (B4), generalized basic hypergeometric series are encountered when we begin doing sums in our N-point-function formulas (Sec. IV) and in satellite terms (Appendix C).

Various asymptotic limits of many basic hypergeometric series can be deduced from nonlinear identities given by Sears.[24] These identities are the basic analogs of the Thomae relations[25] for hypergeometric series.

APPENDIX C: SATELLITE TERMS

If, in Eq. (3), we make the replacements $\sigma\to q^N\sigma$ and $\tau\to q^M\tau$, where N and M are positive integers, we obtain a similar function

$$G(\sigma\tau q^{N+M})/G(\sigma q^N)G(\tau q^M) \tag{C1}$$

in which poles occur at $\sigma=q^{-N}$, q^{-N-1}, $\ldots$ and $\tau=q^{-M}$, q^{-M-1}, $\ldots$. The residues of these poles are polynomials. Multiplying (C1) by the $\sigma\tau$ polynomial

$$\prod_{l=1}^{L}(1-\sigma\tau q^{N+M-l}) \tag{C2}$$

gives us a meromorphic function

$$G(\sigma\tau q^{N+M-L})/G(\sigma q^N)G(\tau q^M) \tag{C3}$$

with polynomial residues if $L\geq 0$. There are poles at $\sigma=q^{-j}$, where $j\geq N$, with polynomial residues of order $j+L-N$. There are poles at $\tau=q^{-j}$, where $j\geq M$, with polynomial residues of order $j+L-M$. The order of each polynomial residue cannot be greater than the maximum angular momentum content of the pole. This is the no-ancestor requirement. Since we can always define σ so that j is the maximum angular momentum content of the pole at $\sigma=q^{-j}$, we have

$$j+L-N\leq j. \tag{C4}$$

Equation (C4) and the same argument for τ poles give

$$M\geq L\geq 0 \tag{C5}$$

and

$$N\geq L\geq 0. \tag{C6}$$

Any linear combination of terms of the form (C3) with indices satisfying Eqs. (C5) and (C6) will satisfy our Born-term requirements. The extra terms which we are free to add are called "satellite terms." For some processes low angular momentum poles are missing, and this can be taken into account through the indices N and M.

Linear Limit

In order to take the limit of (C3) as the trajectories become linear ($q \to 1$), we must first multiply (C3) by some appropriate constants:

$$(1-q)^{1-L}\frac{G(q)G(\sigma\tau q^{N+M-L})}{G(\sigma q^N)G(\tau q^M)} = \left[(1-q)\frac{G(q)G(\sigma\tau q^{N+M})}{G(\sigma q^N)G(\tau q^M)}\right]\frac{(\sigma\tau q^{N+M-L})_{q,L}}{(1-q)^L}. \quad \text{(C7)}$$

The equality follows from Eq. (A2). To take the limit as $q \to 1$ of the term in square brackets, we use Eq. (A1) and rearrange the infinite products:

$$\Lambda \equiv \lim_{q\to 1}\left[(1-q)\frac{G(q)G(\sigma\tau q^{N+M})}{G(\sigma q^N)G(\tau q^M)}\right] = \left[\lim_{q\to 1}\frac{(1-q)(1-\sigma\tau q^{N+M})}{(1-\sigma q^N)(1-\tau q^M)}\right] \times\left[\prod_{l=1}^{\infty}\lim_{q\to 1}\frac{(1-q^l)(1-\sigma\tau q^{N+M+l})}{(1-\sigma q^{N+l})(1-\tau q^{M+l})}\right]. \quad \text{(C8)}$$

Substituting Eqs. (7) and (8) and taking limits gives

$$\Lambda = \frac{N+M-\sigma'-\tau'}{(N-\sigma')(M-\tau')}\prod_{l=1}^{\infty}\frac{(N+M-\sigma'-\tau'+l)l}{(N-\sigma'+l)(M-\tau'+l)}. \quad \text{(C9)}$$

Comparison with the infinite-product representation[26] for the beta function,

$$B(x,y) = \frac{x+y}{xy}\prod_{m=1}^{\infty}\frac{(x+y+m)m}{(x+m)(y+m)}, \quad \text{(C10)}$$

yields

$$\Lambda = \Gamma(N-\sigma')\Gamma(M-\tau')/\Gamma(N+M-\sigma'-\tau'). \quad \text{(C11)}$$

From Eqs. (A3), (7), and (8), we can see that

$$\lim_{q\to 1}\frac{(\sigma\tau q^{N+M-L})_{q,L}}{(1-q)^L} = \prod_{p=0}^{L-1}\lim_{q\to 1}\left[\frac{1-\sigma\tau q^{N+M-L+p}}{1-q}\right] = \prod_{p=0}^{L-1}(-\sigma'-\tau'+N+M-L+p) = \Gamma(-\sigma'-\tau'+N+M)/\Gamma(-\sigma'-\tau'+N+M-L). \quad \text{(C12)}$$

[26] This representation can easily be derived from the infinite-product representation of the gamma function which is given in *Higher Transcendental Functions* (Ref. 25), Vol. 1, p. 1.

Using this result and Eq. (C11), we can write the limit of our general satellite term (C7) as

$$\lim_{q\to 1}(1-q)^{1-L}\frac{G(q)G(\sigma\tau q^{N+M-L})}{G(\sigma q^N)G(\tau q^M)} = \Gamma(N-\sigma')\Gamma(M-\tau')/\Gamma(N+M-L-\sigma'-\tau'), \quad \text{(C13)}$$

which is just a Veneziano satellite term.

Power-Series Representation

We will now derive the power-series representation of the satellite term (C3). We begin by considering the function

$$\frac{G(\sigma\tau q^{-L})}{G(\sigma)G(\tau)} = \frac{G(\sigma\tau q^{-L})}{G(\sigma\tau)}\left[\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}\right]. \quad \text{(C14)}$$

The expansion

$$\frac{G(\sigma\tau q^{-L})}{G(\sigma\tau)} = \sum_{l=0}^{\infty}(q^{-L})_{q,l}\frac{(\sigma\tau)^l}{(q)_{q,l}} \quad \text{(C15)}$$

follows from Heine's formula (A11). From Eq. (A3), it is obvious that $(q^{-L})_{q,l}=0$ for $l \geq L+1$, so that the sum terminates automatically.

Substituting Eq. (C15) and

$$\frac{G(\sigma\tau)}{G(\sigma)G(\tau)} = \sum_{n,m=0}^{\infty}\frac{\sigma^n}{(q)_{q,n}}q^{nm}\frac{\tau^m}{(q)_{q,m}} \quad \text{(A13)}$$

in Eq. (C14) gives

$$\frac{G(\sigma\tau q^{-L})}{G(\sigma)G(\tau)} = \sum_{l,n,m=0}^{\infty}\frac{\sigma^{n+l}q^{nm}\tau^{m+l}}{(q)_{q,n}(q)_{q,m}}\frac{(q^{-L})_{q,l}}{(q)_{q,l}} = \sum_{l,n,m=0}^{\infty}\frac{\sigma^n q^{(n-l)(m-l)}\tau^m}{(q)_{q,n-l}(q)_{q,m-l}}\frac{(q^{-L})_{q,l}}{(q)_{q,l}}. \quad \text{(C16)}$$

To obtain the last expression, we have redefined n and m and extended the sums down to $n=0$ and $m=0$ using the fact that

$$1/(q)_{q,-i}=0 \quad \text{for} \quad i=1,2,\ldots, \quad \text{(C17)}$$

which follows from Eqs. (A1) and (A2).

Our next step will be to derive a certain identity[27] relating basic hypergeometric symbols. Using Eq. (A3), we find that

$$(q)_{q,n-l} = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q^{n-l+1})\cdots(1-q^n)} = (q)_{q,n}\frac{q^{-n}q^{-n+1}\cdots q^{-n+l-1}}{(q^{-n}-1)(q^{-n+1}-1)\cdots(q^{-n+l-1}-1)} = (q)_{q,n}\frac{(-1)^l}{q^{nl}}\frac{q^{l(l-1)/2}}{(q^{-n})_{q,l}}. \quad \text{(C18)}$$

[27] A table of such identities is given in L. J. Slater, Ref. 25, p. 241.

This equation can be used to eliminate $(q)_{q,n-l}$ and $(q)_{q,m-l}$ in Eq. (C16). This gives

$$\frac{G(\sigma\tau q^{-L})}{G(\sigma)G(\tau)}=\sum_{l,n,m=0}^{\infty}\frac{\sigma^n q^{nm}\tau^m}{(q)_{q,n}(q)_{q,m}}\frac{(q^{-L})_{q,l}(q^{-n})_{q,l}(q^{-m})_{q,l}}{(q)_{q,l}}q^l$$

$$=\sum_{n,m=0}^{\infty}\frac{\sigma^n q^{nm}\tau^m}{(q)_{q,n}(q)_{q,m}}{}_3\Phi_0(q^{-L},q^{-n},q^{-m};q)\,, \quad \text{(C19)}$$

where the last expression is obtained by comparing the sum over l with the definition (B8) of generalized basic hypergeometric series.

To obtain the power-series expansion of our general satellite term (C3), we make the replacements $\sigma\to\sigma q^N$ and $\tau\to q^M$ in Eq. (C19). This gives

$$\frac{G(\sigma\tau q^{N+M-L})}{G(\sigma q^N)G(\tau q^M)}=\sum_{n,m=0}^{\infty}\frac{\sigma^n}{(q)_{q,n}}K_{nm}\frac{\tau^m}{(q)_{q,m}}\,, \quad \text{(C20)}$$

where

$$K_{nm}=q^{nm+nN+mM}{}_3\Phi_0(q^{-L},q^{-n},q^{-m};q)\,. \quad \text{(C21)}$$

The ${}_3\Phi_0$ series terminates after $1+\min(L,n,m)$ terms so that K_{nm} involves only a finite sum of terms. The dependence of K_{nm} on the satellite indices N, M, and L is suppressed.

An important feature of Eq. (C20) is that we still have the same structure as in Eq. (15), so that we can easily generalize the rules of Sec. IV in order to construct satellite terms for N-point tree graphs. The only change will be that the factors $q^{nm}\to K_{nm}$, where each K_{nm} is labeled by a different set of three satellite indices.

APPENDIX D: BASIC INTEGRAL REPRESENTATION

Here we will derive the basic analog of the beta-function integral representation. This is largely for the sake of completeness since this representation appears to be far less useful than one might expect.

Jackson[22] introduced an operation called basic integration which is defined by

$$\mathsf{S}_0^b F(x)\,d(qx)=b(1-q)\times[F(b)+qF(qb)+q^2F(q^2b)+\cdots] \quad \text{(D1)}$$

and is the inverse of the "q-difference operation" defined by

$$\mathbf{D}_q f(x)=\frac{f(x)-f(qx)}{x-qx}. \quad \text{(D2)}$$

In the limit $q\to 1$, a basic integral becomes a Riemann integral.

Using Heine's formula (A11) and the definition (A2) of $(a)_{q,n}$, it can easily be shown that

$$\frac{G(\sigma\tau)}{G(\sigma)G(\tau)}=\frac{1}{G(q)}\sum_{n=0}^{\infty}\sigma^n\frac{G(q^{n+1})}{G(\tau q^n)}$$

$$=\frac{1}{G(q)}\sum_{n=0}^{\infty}q^{-n\alpha_s}\frac{G(q^{n+1})}{G(q^{n-\alpha_t})}, \quad \text{(D3)}$$

where we have made the substitutions

$$\sigma=q^{-\alpha_s},\quad \tau=q^{-\alpha_t} \quad \text{(D4)}$$

which follow from Eq. (6). If we multiply Eq. (D3) by $(1-q)G(q)$ and compare the result with Eq. (D1), we find that

$$(1-q)\,G(q)G(\sigma\tau)/G(\sigma)G(\tau)=\mathsf{S}_0^1 x^{-\alpha_s-1}(1-qx)_{-\alpha_t-1}\,d(qx)\,, \quad \text{(D5)}$$

where the notation[22]

$$(1-x)_a\equiv G(x)/G(xq^a)\equiv(x)_{q,a} \quad \text{(D6)}$$

is motivated by the fact that for integer l

$$(1-x)_l=(1-x)(1-qx)\cdots(1-q^{l-1}x) \quad \text{(D7)}$$

and

$$\lim_{q\to 1}(1-qx)_l=(1-x)^l. \quad \text{(D8)}$$

Thus, we see that Eq. (D5) resembles the integral representation of the beta function

$$B(-\alpha_s,-\alpha_t)=\int_0^1 dx\,x^{-\alpha_s-1}(1-x)^{-\alpha_t-1}. \quad \text{(D9)}$$

However, the symmetry under the interchange $\alpha_s\leftrightarrow\alpha_t$ is not apparent.