Pomeranchuk-Like Theorem that Can Be Proved*

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Assuming that the total cross sections tend to finite nonvanishing limits as $s \to \infty$, it is shown in axiomatic field theory that the scattering amplitudes $f_1(s,t)$ and $f_2(s,t)$ for processes $A+B \to A+B$ and $\bar{A}+B \to \bar{A}+B$ must satisfy either or both of the relations $\sigma_1(s)/\sigma_2(s) \to 1$ and $[d\sigma_1(s,t(s))/dt]/[d\sigma_2(s,t(s))/dt] \to 1$ as $s \to \infty$, where σ_i and $d\sigma_i/dt$ are the total and elastic differential cross sections and $t(s) = -C(\ln s)^{-2}$. Since σ_i and $d\sigma_i/dt$ are both measurable, this result enables us to subject axiomatic field theory to an experimental test.

SINCE Pomeranchuk[1] conjectured the equality of particle-particle and particle-antiparticle total cross sections at infinite energy and proved it under some assumptions, attempts have been made to justify these assumptions within the framework of axiomatic field theory.[2–4] The most crucial assumption, which has thus far defied such a justification, is one on the phase of the scattering amplitude $F(s,t)$ at $t=0$:

$$\mathrm{Re}F(s,0)/[(\ln s)\mathrm{Im}F(s,0)] \to 0 \text{ as } s \to \infty\,, \quad (1)$$

where s and t are the usual Mandelstam variables.[5] What has been found thus far is that (1) can be derived from unitarity and analyticity in the Martin-Lehmann ellipse of the $\cos\theta$ plane[6] if the total cross section $\sigma(s)$ grows indefinitely[3,4]:

$$\sigma(s) \to \infty \text{ as } s \to \infty\,, \quad (2)$$

or if $\sigma(s)$ tends to a nonvanishing finite limit but the elastic cross section $\sigma_{\mathrm{el}}(s)$ tends to zero, so that[4]

$$\sigma_{\mathrm{el}}(s)/\sigma(s) \to 0 \text{ as } s \to \infty\,. \quad (3)$$

But the Pomeranchuk theorem remains unproved in the important case where both $\sigma(s)$ and $\sigma_{\mathrm{el}}(s)$ tend to nonvanishing limits.

Not infrequently, when a theorem is hard to prove, it may be useful to modify it until it can be proved. The purpose of this paper is to show that we may modify the Pomeranchuk theorem in a similar fashion and obtain a new theorem which can be proved in axiomatic field theory. From the physicist's point of view, interest in this theorem lies in that it can be compared with the measurements of high-energy cross sections just as well as the Pomeranchuk theorem itself. Thus it enables us to subject axiomatic field theory to an experimental test. It is amusing to note that an essential ingredient of this theorem can be found in Ref. 4. What was missing is the realization that it can be stated as a theorem somewhat distinct from the Pomeranchuk theorem.

Before describing the theorem, let us recall some results of axiomatic field theory. By this we mean those properties of $F(s,t)$ that can be proved rigorously in all known versions of local field theory: Wightman field theory, Jaffe's theory,[7] and the Araki-Haag theory of local observables.[8]

(A) *Analyticity in s and t.* $F(s,t)$ is analytic in the domain $(|t|<t_0)\otimes$ (*cut s plane*), where t_0 is some positive number, provided the masses of incident particles satisfy certain inequalities.[9] Actually it is sufficient for our purpose to assume the analyticity domain proved for general masses.[10]

(B) *Polynomial boundedness.* For $|t|<t_0$ and some constant N, $F(s,t)$ satisfies the inequality $|F(s,t)| < |s|^N$ for large $|s|$.[8]

(C) *Analyticity in the Martin-Lehmann ellipse.* For fixed real s above threshold, $F(s,t)$ is analytic in an ellipse in the $\cos\theta$ plane with foci at $\cos\theta = +1$ and -1 and semimajor axis of the form $1+a/s$ for large s.[6]

Let f_1 and f_2 be the scattering amplitudes for the processes $A+B \to A+B$ and $\bar{A}+B \to \bar{A}+B$, respectively. Then, for fixed t in $-t_0<t\leq 0$, we can express the crossing and reality relations as

$$f_1(u,t) = f_2(s,t)\,,$$
$$f_i^*(s^*,t) = f_i(s,t)\,, \quad i=1,2\,. \quad (4)$$

We shall also define

$$\rho_i(s,t) = \mathrm{Re}f_i(s,t)/[(\ln s)\,\mathrm{Im}f_i(s,t)]\,, \quad i=1,2. \quad (5)$$

We are now ready to state the theorem.

Theorem. Let $f_i(s,t)$ be the scattering amplitudes which satisfy (A), (B), (C), and unitarity. Suppose the total cross sections tend to finite nonvanishing limits

$$\sigma_i(s) \to C_i\,, \quad i=1,2, \quad s \to \infty\,. \quad (6)$$

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[1] I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **34**, 725 (1958) [Soviet Phys. JETP **7**, 499 (1958)].
[2] A. Martin, Nuovo Cimento **39**, 704 (1965).
[3] R. J. Eden, Phys. Rev. Letters **16**, 39 (1966).
[4] T. Kinoshita, in *Perspectives in Modern Physics,* edited by R. E. Marshak (Wiley, New York, 1966), p. 211.
[5] Throughout this paper, $F(s,t)$ will be understood as $F(s+i0,\,t)$ for real positive s and small negative t.
[6] A. Martin, Nuovo Cimento **42A**, 930 (1966).
[7] A. M. Jaffe, Phys. Rev. Letters **17**, 661 (1966).
[8] H. Epstein, V. Glaser, and A. Martin, Commun. Math. Phys. **13**, 257 (1969).
[9] G. Sommer, Nuovo Cimento **52**, 850 (1967); **52**, 866 (1967).
[10] J. Bros, H. Epstein, and V. Glaser, Commun. Math. Phys. **1**, 240 (1965).

Then, for $s \to \infty$, we obtain

$$\sigma_1(s)/\sigma_2(s) \to 1 \quad (7)$$

or

$$[d\sigma_1(s,t(s))/dt]/[d\sigma_2(s,t(s))/dt] \to 1 \quad (8)$$

according as

$$\rho_i(s,0) \to 0\,, \quad i=1,\,2 \quad (9)$$

or

$$\rho_i(s,0) \not\to 0\,, \quad i=1,\,2 \quad (10)$$

where $t(s) = -C(\ln s)^{-2}$, $0 \leq C < C_0$, C_0 being a positive constant.

Before we give the proof let us make a few remarks.

(a) Unlike the Pomeranchuk theorem, (9) and (10) are not assumptions of the theorem but are simply devices for classifying possible alternatives.

(b) If $f_i(s,0)$ satisfies (10), then (7) is not possible. But (8) is not ruled out under (9). Thus (7) and (8) are not incompatible. Our theorem simply maintains that (7) and (8) cannot fail simultaneously insofar as the axiomatic field theory is valid.

(c) When (10) holds, unitarity demands that $d\sigma_i(s,t)/dt$ be peaked forward with a width which shrinks as $(\ln s)^{-2}$ for large s.[4] Thus in this case we must abandon the idea of Pomeranchuk-Regge trajectory, at least in the usual sense.

(d) In general, $\sigma_i(s)$ may become infinite as $s \to \infty$, its growth rate being restricted only by the Froissart bound,

$$\sigma_i(s) < \text{const} \times (\ln s)^2, \quad s \to \infty\,.$$

As was mentioned in the beginning, if $\sigma_i(s) \to \infty$ as $s \to \infty$, the possibility (10) is never realized and our theorem reduces to the Pomeranchuk theorem. On the other hand, if $\sigma_i(s) \to 0$ as $s \to \infty$, the latter is not expected to hold in general.[11] This is why we concentrate on the case of finite nonzero limit of $\sigma_i(s)$ in our theorem. A somewhat weaker assumption will be a finite but infinitely oscillating $\sigma_i(s)$. Such an oscillation cannot be ruled out within axiomatic field theory.[12] Thus, strictly speaking, the assumption (6) is not based on field theory. Rather it must be regarded as an ansatz whose validity is to be checked empirically. Meanwhile, if we are willing to accept a weaker theorem similar to the weak Pomeranchuk theorem of Meiman,[13] the assumption (6) may be replaced by one which allows infinite oscillation of $\sigma_i(s)$.

(e) A theorem of the form (8) for the differential cross sections has been obtained previously.[14] However, our result (8) is derived under different circumstances and the resemblance is only superficial.

(f) We wish to emphasize that asymptotic theorems such as Pomeranchuk's and ours will never be tested experimentally in an unambiguous manner. The only thing experiments can do is to indicate whether certain theoretical ideas are on the right track or not. Indeed, recent inquiries of the Pomeranchuk theorem[11,15] arise from the experiments at Serpukhov[16] which have cast some doubt about its validity. Evidently experimental violation of this theorem is not necessarily a disaster for field theory. However, as is made clear by our theorem, if further measurements of high-energy cross sections indicate that both (7) and (8) are violated, there will be some real trouble with the axioms of quantum field theory. We are assuming in this discussion that the spin-flip cross sections can be ignored at high energy, an assumption which may have to be examined more carefully.

We shall now prove the theorem.

Proof. The derivation of (7) from (9) constitutes the content of usual proof of the Pomeranchuk theorem and can be found in many places in the literature.[2–4] Thus it will not be repeated here.

In order to prove (8), it is sufficient to show that

$$\begin{aligned} &\mathrm{Re} f_i(s,t) = O(s \ln s)\,, \\ &\mathrm{Im} f_i(s,t) = O(s)\,, \quad i=1,\,2, \quad s \to \infty \end{aligned} \quad (11)$$

and

$$\mathrm{Re} f_1(s,t) = -\mathrm{Re} f_2(s,t) + O(s/\ln s)\,, \quad s \to \infty \quad (12)$$

hold for appropriate values of t. In fact, when (11) holds, $f_i(s,t)$ becomes predominantly real at high energy and thus

$$\begin{aligned} d\sigma_i(s,t)/dt &\propto |f_i(s,t)|^2 \\ &= [\mathrm{Re} f_i(s,t)]^2[1+O((\ln s)^{-2})], \quad i=1,2. \end{aligned} \quad (13)$$

The result (8) follows immediately from (11), (12), and (13).[17]

Actually, whereas (12) can be proved for $-t_0 < t \leq 0$, (11) can be proved only for (almost all) t in the small interval

$$-C_0(\ln s)^{-2} < t \leq 0\,, \quad (14)$$

where C_0 is a positive constant to be determined later. Although this interval shrinks to zero as $s \to \infty$, it is all we need in view of remark (c).

To derive (11), let us define a real function $t(s)$ with the properties

$$\begin{aligned} -t_0 < t(s) \leq 0 \quad &\text{for all } s \geq (m_A+m_B)^2, \\ -t(s) \sim C(\ln s)^{-2} \quad &\text{for } s \to \infty\,, \end{aligned} \quad (15)$$

[11] See, e.g., W. S. Lam and Tran N. Truong, Phys. Letters **31B**, 307 (1970).

[12] J. D. Bessis and T. Kinoshita, Nuovo Cimento **50A**, 156 (1967).

[13] N. N. Meiman, Zh. Eksperim. i Teor. Fiz. **43**, 2277 (1962) [Soviet Phys. JETP **16**, 1609 (1963)].

[14] A. A. Logunov, Nguyen van Hieu, I. T. Todorov, and O. A. Khrustalev, Phys. Letters **7**, 69 (1963); L. Van Hove, Rev. Mod. Phys. **36**, 655 (1964).

[15] A. Martin, in *High Energy Collisions* (Gordon and Breach, New York, 1969), p. 227; R. J. Eden, Phys. Rev. D **2**, 529 (1970).

[16] J. V. Allaby *et al.*, Phys. Letters **30B**, 500 (1969).

[17] The prototype of this argument for the case $t=0$ can be found in Ref. 4.

and expand $f_i(s,t(s))$ into partial waves:

$$f_i(s,t(s))=(s^{1/2}/2k)\sum_{l=0}^{\infty}(2l+1)\times a_l^{(i)}(s)P_l(1+t(s)/2k^2),\quad(16)$$

where k is the momentum in the c.m. frame.[18] We shall also introduce

$$D_i^{\pm}(s,t(s))=(s^{1/2}/2k)\sum_{l=0}^{\infty}(2l+1)\times[\pm\mathrm{Re}\,a_l^{(i)}(s)]P_l(1+t(s)/2k^2),$$

where $[x]=x$ or 0 according as $x>0$ or ≤ 0. Obviously we have

$$\mathrm{Re}f_i(s,t(s))=D_i^+(s,t(s))-D_i^-(s,t(s)).$$

Now, since $\mathrm{Re}f_i(s,0)\sim\mathrm{const}\times s\ln s$ according to (6) and (10), we may assume that $D_i^{\pm}(s,0)\sim a_i^{\pm}s\ln s$, where at least one of a_i^+ and a_i^- is nonzero. Since $P_n(z)$ has no zero in the interval $(1-\pi^2/8n^2, 1)$,[19] and since $\lim_{n\to\infty}P_n(\cos(x/n))=J_0(x)$, we obtain

$$P_l(1+t(s)/2k^2)>\epsilon>0 \text{ for all } l\leq b(\epsilon)k/(-t(s))^{1/2},$$

where $b(\epsilon)$ depends on ϵ. For instance $b(\epsilon)$ may be chosen as 2.4 if ϵ equals 0.0025 or less. Meanwhile, the contribution of partial waves for $l>b(\epsilon)k/(-t(s))^{1/2}$ to $D_i^{\pm}(s,t(s))$ can be made less than s by choosing a small enough C in (15). For such C we have

$$D_i^{\pm}(s,t(s))\geq\epsilon a_i^{\pm}s\ln s.$$

Hence $\mathrm{Re}f_i(s,t(s))$ is of order $s\ln s$. In general, $\mathrm{Re}f_i(s,t(s))$ will be of order $s\ln s$ for larger values of C, except possibly for some values of C for which $D_i^+(s,t(s))$ and $D_i^-(s,t(s))$ cancel accidentally.[20] Thus, in order to obtain (11), we have only to define C_0 in (8) and (14) as the least upper bound of all C for which $D_i^{\pm}(s,t(s))$ is of order $s\ln s$ and $\mathrm{Im}f_i(s,t(s))$ is of order s. This result is closely related to the fact that the logarithmic derivative of $f_i(s,t)$ at $t=0$ is bounded by const $\times(\ln s)^2$.[21]

In order to prove (12), it is convenient to introduce the crossing-symmetric amplitude

$$f(s,t)=f_1(s,t)+f_2(s,t),\quad(17)$$

which satisfies the twice-subtracted fixed-t dispersion relation

$$f(s,t)=c_0(t)+\frac{s^2}{\pi}\int\frac{A(s',t)ds'}{s'^2(s'-s)}+\frac{u^2}{\pi}\int\frac{A(u',t)du'}{u'^2(u'-u)}\quad(18)$$

for $-t_0<t\leq 0$, where $u=u(t)=2m_A^2+2m_B^2-s-t$. Subtracting (18) from the corresponding equation for $f(s,0)$, we obtain

$$f(s,0)-f(s,t)=g(s,t)+h(s,t),\quad(19)$$

where

$$g(s,t)=\frac{s^2}{\pi}\int\frac{[A(s',0)-A(s',t)]ds'}{s'^2(s'-s)}+\frac{(2m_A^2+2m_B^2-s)^2}{\pi}\times\int\frac{[A(u',0)-A(u',t)]du'}{u'^2(u'-2m_A^2-2m_B^2+s)},\quad(20)$$

$h(s,t)$ being the remainder. Noting that $|A(u',t)|<A(u',0)\sim C'u'$ [assumption (6)] for negative t and large positive u', we find easily that

$$|h(s,t)|=O(t\ln s)\quad\text{for } s\to\infty.\quad(21)$$

On the other hand, for real t in $-t_0<t\leq 0$, $g(s,t)$ has the property

$$g^*(s^*,t)=g(s,t),\qquad g(s,t)=g(2m_A^2+2m_B^2-s,\,t).\quad(22)$$

Furthermore, for the same t and real $s\geq(m_A+m_B)^2$, we have

$$0\leq\mathrm{Im}g(s,t)=A(s,0)-A(s,t)\leq C''s.\quad(23)$$

Now, analytic functions in the cut s plane satisfying (22) and (23) are known to have the property[22]

$$|\mathrm{Re}g(s,t)/\mathrm{Im}g(s,t)|\leq\mathrm{const}\times(\ln s)^{-1}\quad\text{for } s\to+\infty.\quad(24)$$

From (19), (21), and (24), we find

$$|\mathrm{Re}f(s,0)-\mathrm{Re}f(s,t)|\leq\mathrm{const}\times(s/\ln s)$$

for t in the range $-t_0<t\leq 0$. Combining this with[22]

$$|\mathrm{Re}f(s,0)|\leq\mathrm{const}\times(s/\ln s),$$

we finally arrive at

$$|\mathrm{Re}f(s,t)|\leq\mathrm{const}\times(s/\ln s)\quad\text{for}\quad s\to+\infty,$$

which is equivalent to (12). Q.E.D.

[18] For simplicity, we have assumed that both A and B are scalar particles. The general case may be treated by the method of G. Mahoux and A. Martin, Phys. Rev. **174**, 2140 (1968).

[19] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939), p. 118.

[20] There is an interesting possibility that, when $D_i^+(s,t(s))$ and $D_i^-(s,t(s))$ cancel, $f_i(s,t(s))$ may satisfy the condition (1) and give $\mathrm{Im}f_1(s,t(s))/\mathrm{Im}f_2(s,t(s))\to 1$ instead of (12). This may lead us again to the equality (8) of differential cross sections, but for different reasons.

[21] J. D. Bessis, Nuovo Cimento **45**, 974 (1966); T. Kinoshita, Phys. Rev. **152**, 1266 (1966).

[22] N. N. Khuri and T. Kinoshita, Phys. Rev. **137**, B720 (1965); **140**, B706 (1965); Y. S. Jin and S. W. MacDowell, *ibid.* **138**, B1279 (1965).