

Nonlinear Effects in Quantum Electrodynamics. Photon Propagation and Photon Splitting in an External Field*

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The effective nonlinear Lagrangian derived by Heisenberg and Euler is used to describe the propagation of photons in slowly varying but otherwise arbitrary electromagnetic fields. The group and the phase velocities for both propagation modes are calculated, and it is shown that the propagation is always causal. The photon splitting processes are also studied, and it is shown that they do not play any significant role even in very strong magnetic fields surrounding neutron stars.

I. INTRODUCTION

IT was recognized in the early thirties that virtual creation and annihilation of electron pairs induces the self-coupling of the electromagnetic field. This self-interaction is in general nonlocal, and its description is further complicated by the possibility of real pair creation. However, for slowly varying¹ but arbitrarily strong electromagnetic fields, the self-interaction energy was computed already in 1936 by Heisenberg and Euler² in the lowest order of the fine-structure constant (i.e., without radiative corrections). The effective Lagrangian which they obtained³ (in units $\hbar=1=c$) is

$$L = S - \frac{1}{8\pi^2} \int_0^\infty ds s^{-3} e^{-m^2 s} \times \left[(es)^2 P \frac{\text{Re coshes} X}{\text{Im coshes} X} - 1 + \frac{2}{3}(es)^2 S \right], \quad (1)$$

where

$$\begin{aligned} S &\equiv -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}, & P &\equiv -\frac{1}{4} f_{\mu\nu} \hat{f}^{\mu\nu}, \\ f^{\mu\nu} &\equiv \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} f_{\lambda\rho}, & X &\equiv -2S + 2iP. \end{aligned} \quad (2)$$

Owing to the universality of electromagnetic interactions,⁴ the same Lagrangian (1) describes the interaction between photons and between photons and an external field.

The self-interaction of photons can also be described in terms of Feynman diagrams. In the lowest order of perturbation theory in α , the only diagrams that contribute to this interaction are those containing one closed electron loop. The leading term in the low-energy

approximation of the Feynman amplitude represented by such a diagram with $2n$ photon lines corresponds to the n th order term in the asymptotic expansion of the Lagrangian (1). This was verified by a direct computation for $n=2$ by Karplus and Neuman.⁵ Thus the calculation based on the effective Lagrangian (1) can replace the complicated S -matrix calculation in all problems with slowly varying external fields and/or low-energy photons.

Out of many nonlinear effects which can be discussed with the help of this Lagrangian we have selected two: the photon propagation and the photon splitting in an external electromagnetic field. These problems have been studied in many papers.⁶⁻¹⁰ None of these papers, however, contains the correct formula for the probability of the photon splitting. Our calculation of the photon splitting was prompted by recent speculations that intense magnetic fields may be produced by neutron stars. We have wondered whether photon-splitting phenomena could influence the spectrum of the electromagnetic radiation from neutron stars. We have found, however, this effect to be exceedingly small. The theoretical explanation of the smallness of the photon-splitting amplitude is that owing to gauge invariance, this amplitude is of a much higher degree in the external field than one would expect on purely dimensional grounds.

II. PHOTON PROPAGATION

The propagation of photons in an external electromagnetic field will be described here as the propagation of weak disturbances in a strong background field. In this approximation the equation for the photon wave function is linear, but the coefficients depend on the external field. For time-independent and homogeneous external fields this propagation problem can be solved explicitly for any relativistic local theory. Such a theory

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¹ These are fields whose variations over the Compton wavelength of the electron ($\lambda = 3.9 \cdot 10^{-11}$ cm) and over the corresponding time interval ($\tau = \lambda/c$) are much smaller than the field itself: $|\partial_\mu f_{\lambda\rho}| \ll m |f_{\lambda\rho}|$.

² W. Heisenberg and H. Euler, *Z. Physik* **98**, 714 (1936).

³ This form of the Lagrangian is taken from the paper of Schwinger [J. Schwinger, *Phys. Rev.* **82**, 664 (1951)], who re-derived the result of Ref. 2 by the proper-time technique.

⁴ The interaction of charges with an external field is the same as the interaction with photons.

⁵ R. Karplus and M. Neuman, *Phys. Rev.* **80**, 380 (1950).

⁶ A. Minguzzi, *Nuovo Cimento* **4**, 476 (1956); **6**, 501 (1957); **9**, 145 (1958); **19**, 847 (1961).

⁷ J. J. Klein and B. P. Nigam, *Phys. Rev.* **135**, B1279 (1964). [*Soviet Phys. JETP* **28**, 371 (1969)].

⁸ N. B. Narozhnyi, *Zh. Eksperim. i Teor. Fiz.* **55**, 714 (1968) [*Soviet Phys. JETP* **28**, 371 (1969)].

⁹ V. G. Skobov, *Zh. Eksperim. i Teor. Fiz.* **35**, 1315 (1958) [*Soviet Phys. JETP* **8**, 919 (1959)].

¹⁰ S. S. Sannikov, *Zh. Eksperim. i Teor. Fiz.* **52**, 1303 (1967) [*Soviet Phys. JETP* **25**, 867 (1967)].

is most generally described by the Lagrangian density $L(x)$ which is a function of two invariants S and P . At the end of this section we concentrate on the Lagrangian derived from quantum electrodynamics.

Let us split the total electromagnetic field $f_{\mu\nu}$ into the strong external constant field $F_{\mu\nu}$ and a weak varying wave field $\phi_{\mu\nu}$:

$$f_{\mu\nu} = F_{\mu\nu} + \phi_{\mu\nu}. \quad (3)$$

The field equations for $f_{\mu\nu}$,

$$\partial_\mu h^{\mu\nu}(x) = 0, \quad (4)$$

where

$$h^{\mu\nu} \equiv \frac{\partial L}{\partial S} f^{\mu\nu} + \frac{\partial L}{\partial P} \hat{f}^{\mu\nu} \quad (5)$$

in the linear approximation with respect to $\phi_{\mu\nu}$, take the form

$$\left. \frac{\partial h^{\mu\nu}}{\partial f_{\alpha\beta}} \right|_{f=F} \partial_\mu \phi_{\alpha\beta}(x) = 0. \quad (6)$$

We seek the solution of these equations in the form

$$\begin{aligned} \phi_{\alpha\beta}(x) &= \epsilon_{\alpha\beta}(k) e^{-ik \cdot x}, \\ \epsilon_{\alpha\beta}(k) &= k_\alpha \epsilon_\beta(k) - k_\beta \epsilon_\alpha(k) \equiv k_{[\alpha} \epsilon_{\beta]}(k) \end{aligned} \quad (7)$$

and we obtain the following set of algebraic equations for $\epsilon_\mu(k)$:

$$M^{\mu\nu} \epsilon_\nu = 0, \quad (8)$$

where

$$\begin{aligned} M^{\mu\nu} &\equiv \left. \frac{\partial h^{\lambda\mu}}{\partial f_{\rho\nu}} \right|_{f=F} k_\lambda k_\rho \\ &= \gamma_S (g^{\mu\nu} k^2 - k^\mu k^\nu) - \gamma_{SS} a^\mu a^\nu \\ &\quad - \gamma_{SP} (a^\mu \hat{a}^\nu + \hat{a}^\mu a^\nu) - \gamma_{PP} \hat{a}^\mu \hat{a}^\nu \end{aligned} \quad (9)$$

and

$$\gamma_S \equiv \left. \frac{\partial L}{\partial S} \right|_{f=F}, \quad \gamma_{SS} \equiv \left. \frac{\partial^2 L}{\partial S^2} \right|_{f=F}, \quad (10)$$

$$\begin{aligned} \gamma_{SP} &\equiv \left. \frac{\partial^2 L}{\partial S \partial P} \right|_{f=F}, \quad \gamma_{PP} \equiv \left. \frac{\partial^2 L}{\partial P^2} \right|_{f=F}, \\ a^\mu &\equiv F^{\mu\nu} k_\nu, \quad \hat{a}^\mu \equiv \hat{F}^{\mu\nu} k_\nu. \end{aligned} \quad (11)$$

The general solution for $\epsilon_\mu(k)$ can be written as a linear combination of any four linearly independent vectors. For a general external field these vectors can be chosen as a_μ , \hat{a}_μ , k_μ , and b_μ ,

$$b_\mu \equiv F_{\mu\nu} a^\nu, \quad (12)$$

$$\epsilon_\mu(k) = \alpha a_\mu + \beta \hat{a}_\mu + \gamma k_\mu + \delta b_\mu. \quad (13)$$

Upon substituting this form of ϵ_μ into Eq. (8), we derive the following conditions:

$$\begin{aligned} (a) \quad &\text{The coefficients } \alpha \text{ and } \beta \text{ obey the equations} \\ &\alpha(\gamma_S k^2 - \gamma_{SS} a^2 + \gamma_{SP} k^2 P) \\ &\quad + \beta[\gamma_{SS} k^2 P - \gamma_{SP}(a^2 + 2k^2 S)] = 0, \\ &\alpha(-\gamma_{SP} a^2 + \gamma_{PP} k^2 P) \\ &\quad + \beta[\gamma_S k^2 + \gamma_{SP} k^2 P - \gamma_{PP}(a^2 + 2k^2 S)] = 0. \end{aligned} \quad (14)$$

- (b) The coefficient γ is arbitrary.
(c) $\delta = 0$.

Since γk_μ does not contribute to $\phi_{\mu\nu}(x)$, we shall not consider it any further. Nontrivial solutions for α and β can be obtained, provided the determinant of the system of Eqs. (14) vanishes. This leads to the following dispersion laws¹¹:

$$\begin{aligned} k^2 &= a^2(k) \lambda_\pm = a^2(k) \\ &\times \frac{-\gamma_S(\gamma_{SS} + \gamma_{PP}) + 2S(\gamma_{SS}\gamma_{PP} - \gamma_{SP}^2) \pm \Delta^{1/2}}{2[-\gamma_S^2 + 2\gamma_S(S\gamma_{PP} - P\gamma_{SP}) + P^2(\gamma_{SS}\gamma_{PP} - \gamma_{SP}^2)]}, \end{aligned} \quad (15)$$

where

$$\Delta = [\gamma_S(\gamma_{SS} - \gamma_{PP}) - 2S(\gamma_{SS}\gamma_{PP} - \gamma_{SP}^2)]^2 + [2\gamma_S\gamma_{SP} - 2P(\gamma_{SS}\gamma_{PP} - \gamma_{SP}^2)]^2. \quad (16)$$

In general, the propagation of waves in the external field exhibits the phenomenon of birefringence. We will call λ the birefringence index because to two values of λ (λ_+ and λ_-) there correspond two different polarization states described by polarization vectors $\epsilon_\pm^\mu(k)$, and the waves with different polarizations propagate according to different dispersion laws. It is only when $\Delta = 0$ that the propagation velocities in all directions do not depend on the wave polarization. The Maxwell theory and the nonlinear electrodynamics of Born and Infeld are the only relativistic theories in which this takes place.¹²

Most of our further considerations will be restricted to those cases in which the nonlinear effects are only small corrections as compared to the predictions of the Maxwell theory. This restriction places a limit on the strength of the external field. In particular,

$$|\lambda_\pm| |F_{\mu\nu}|^2 \ll 1. \quad (17)$$

Under this condition the interpretation of Eq. (15) becomes especially transparent. Since

$$\begin{aligned} a^2(k) &= (\mathbf{E} \cdot \mathbf{k})^2 - \mathbf{E}^2 \omega^2(\mathbf{k}) \\ &\quad - (\mathbf{k} \times \mathbf{B})^2 - 2\omega(\mathbf{k}) \mathbf{k} \cdot (\mathbf{B} \times \mathbf{E}), \end{aligned} \quad (18)$$

the solutions for positive frequencies $\omega_\pm(\mathbf{k})$ are

$$\omega_\pm(\mathbf{k}) \simeq |\mathbf{k}| (1 - \frac{1}{2} \lambda_\pm |\mathbf{Q}|^2), \quad (19)$$

where

$$\mathbf{Q} \equiv \mathbf{n} \times \mathbf{E} + \mathbf{n} \times (\mathbf{n} \times \mathbf{B}) \quad (20)$$

and

$$\mathbf{n} \equiv \mathbf{k} / |\mathbf{k}|.$$

The phase velocity \mathbf{u} and the group velocity \mathbf{v} are therefore given by the formulas

$$\mathbf{u}_\pm = c \mathbf{n} (1 - \frac{1}{2} \lambda_\pm |\mathbf{Q}|^2), \quad (21)$$

$$\begin{aligned} \mathbf{v}_\pm &= c \mathbf{n} [1 - \frac{1}{2} \lambda_\pm (\mathbf{E}^2 + \mathbf{B}^2) + (\mathbf{n} \cdot \mathbf{E})^2 + (\mathbf{n} \cdot \mathbf{B})^2] \\ &\quad + c \lambda_\pm [\mathbf{E}(\mathbf{n} \cdot \mathbf{E}) + \mathbf{B}(\mathbf{n} \cdot \mathbf{B}) - (\mathbf{B} \times \mathbf{E})]. \end{aligned} \quad (22)$$

¹¹ The same formula has been obtained independently by G. Boillat [J. Math. Phys. 11, 941 (1970)], who studied the propagation of discontinuities in general nonlinear electrodynamics.

¹² This was proved by G. Boillat (Ref. 11) and independently by J. Plebanski, *Lectures on Nonlinear Electrodynamics* (NORDITA, Copenhagen, 1970).

In our approximation the moduli of both velocities are equal,

$$|\mathbf{v}_\pm| = |\mathbf{u}_\pm| = c(1 - \frac{1}{2}\lambda_\pm |\mathbf{Q}|^2). \quad (23)$$

Both the phase and the group velocities do not depend on the frequency but only on the transverse components of the wave vector relative to the external fields \mathbf{E} and \mathbf{B} .

The necessary and sufficient condition for a causal propagation in a not too strong external field is that both birefringence indices λ_+ and λ_- be positive. In that case the larger value of λ (λ_+) corresponds to the smaller velocity. Since in the lowest approximation

$$\lambda_\pm \simeq \frac{1}{2} \{ (\gamma_{SS} + \gamma_{PP}) \mp [(\gamma_{SS} - \gamma_{PP})^2 + 4\gamma_{SP}^2]^{1/2} \}, \quad (24)$$

the causality requirement ($\lambda_\pm \geq 0$) is equivalent to the following conditions for partial derivatives of the Lagrangian:

$$\gamma_{SS} \geq 0, \quad \gamma_{PP} \geq 0, \quad (\gamma_{SS}\gamma_{PP} - \gamma_{SP}^2) \geq 0. \quad (25)$$

The nonlinear Lagrangian (1) derived from quantum electrodynamics can be expanded into an asymptotic series whose first four terms¹³ are

$$\begin{aligned} L \simeq S + \frac{2\alpha^2}{45m^4}(4S^2 + 7P^2) + \frac{32\pi\alpha^3}{315m^8}(8S^3 + 13SP^2) \\ + \frac{128\pi^2\alpha^4}{945m^{12}}(446S^4 + 201S^2P^2 + 38P^4) \\ = L^{(0)} + L^{(2)} + L^{(3)} + L^{(4)}. \end{aligned} \quad (26)$$

The use of the first few terms of this expansion is justified if the dimensionless expansion parameter

$$4\pi \frac{\alpha \hbar^3}{m^4 c^5} |F|^2 \simeq \begin{cases} 5 \times 10^{-28} (\mathbf{B} \text{ in G})^2 \\ 6 \times 10^{-33} (\mathbf{E} \text{ in V/cm})^2 \end{cases} \quad (27)$$

is much smaller than unity. This is indeed the case even for strong magnetic fields which one may expect to find¹⁴ in the vicinity of a neutron star. To describe the propagation of photons in an external field in quantum electrodynamics we need only the first correction term $L^{(2)}$. Higher terms will be used in Sec. III to study the photon splitting. In this approximation we obtain

$$\begin{aligned} \gamma_S = 1 + 8\kappa S, \quad \gamma_{SS} = 8\kappa, \quad \gamma_{PP} = 14\kappa, \quad \gamma_{SP} = 0, \\ \lambda_+ = 14\kappa, \quad \lambda_- = 8\kappa, \end{aligned} \quad (28)$$

where

$$\kappa = 2\alpha^2 \hbar^3 / 45m^4 c^5 \simeq 2.1 \times 10^{-29} \text{ G}^{-2}, \quad (29)$$

and we have consistently kept only terms linear in κ . Since both λ 's are positive, the propagation of photons in an external field is causal.¹⁵ The photon polarization

vectors in the lowest approximation are

$$\begin{aligned} \epsilon_+^\mu(k) = \alpha(k) \hat{a}^\mu(k), \\ \epsilon_-^\mu(k) = \alpha(k) a^\mu(k), \end{aligned} \quad (30)$$

where

$$\alpha^2(k) = -1/a^2(k).$$

The polarization vectors $\epsilon_\pm^\mu(k)$ correspond to dispersion laws $k^2 = a^2(k)\lambda_\pm$.

III. PHOTON SPLITTING

Before we will carry out an explicit calculation of the transition amplitude for the photon splitting in quantum electrodynamics, we will make a few general remarks about the phase space and the invariance requirements for this process.

The relativistic phase-space volume $\rho_N(k)$ for N free photons of total four-momentum k is

$$\begin{aligned} \rho_N(k) &\equiv \int \prod_{A=1}^N d^4 k_A \theta(k_{A0}) \delta(k_A^2) \delta^{(4)}(k - \sum_{B=1}^N k_B) \\ &= C_N \theta(k^2) (k^2)^{N-2}, \end{aligned} \quad (31)$$

where

$$C_N = \frac{(\frac{1}{2}\pi)^{N-1}}{(N-2)!(N-1)!}. \quad (32)$$

Therefore, if the initial and final photons are assumed to propagate along the light cone, then from the phase-space argument alone we can exclude the decay into more than two photons. However, if we take into account the influence of the external electromagnetic field on the propagation of photons, this conclusion is no longer valid. The calculation of the N -photon phase space in this case—say, $\tilde{\rho}_N(k)$ —is easy only when all N photons have the same birefringence index λ_f . We will restrict our discussion to this simple case, expecting that it contains all essential features of the general case. All photon momenta satisfy now the condition

$$0 = k_A^2 - \lambda_f a^2(k_A) = k_A \cdot b \cdot k_A = k_A^\mu b_{\mu\nu} k_A^\nu, \quad A = 1, 2, \dots, N, \quad (33)$$

where

$$b_{\mu\nu} = g_{\mu\nu} + \lambda_f F_{\mu\kappa} F_{\nu\kappa}. \quad (34)$$

The phase-space volume is

$$\begin{aligned} \tilde{\rho}_N(k) &\equiv \int \prod_{A=1}^N d^4 k_A \theta(k_{A0}) \\ &\quad \times \delta(k_A \cdot b \cdot k_A) \delta^{(4)}(k - \sum_{B=1}^N k_B). \end{aligned} \quad (35)$$

By linear transformation of the integration variables

$$k_A^\mu = U^\mu{}_\nu J_A^\nu, \quad (36)$$

6 were of a rather general nature. In Ref. 7 only the propagation in a uniform static electric field was studied. Owing to an error in their calculations, the incorrect statement was made that the phase velocity may exceed the speed of light. In Ref. 8, only the special case of crossed fields was studied.

¹³ The term which can be identified as $L^{(1)}$ contributes to charge renormalization only.

¹⁴ According to T. Gold [Nature 218, 731 (1968)], the magnetic induction \mathbf{B} may be as large as 10^{12} G.

¹⁵ The photon propagation in external field in quantum electrodynamics was studied in Refs. 6-8. The results obtained in Ref.

where the matrix U diagonalizes b ,

$$U^T \cdot b \cdot U = g, \quad (37)$$

we can reduce the phase-space integral $\tilde{\rho}_N$ to ρ_N :

$$\tilde{\rho}_N(k) = |\det b|^{-(N+1)/2} \rho_N(U^{-1}k). \quad (38)$$

This can be further simplified by noticing that

$$(U^{-1}k)^2 = k \cdot (U^T)^{-1} \cdot g \cdot U^{-1} \cdot k = k \cdot b \cdot k, \quad (39)$$

so that the final formula for $\tilde{\rho}_N$ reads

$$\tilde{\rho}_N(k) = |\det b|^{-(N+1)/2} C_N \theta(k \cdot b \cdot k) (k \cdot b \cdot k)^{N-2}. \quad (40)$$

Denoting the birefringence index of the initial photon by λ_i , we obtain

$$k \cdot b \cdot k = (\lambda_i - \lambda_f) a^2(k). \quad (41)$$

Since in the lowest approximation

$$a^2(k) = -|\mathbf{k}|^2 |\mathbf{Q}|^2, \quad (42)$$

the sign of $k \cdot b \cdot k$ is that of $(\lambda_f - \lambda_i)$.

If the initial photon is faster than the decay products ($\lambda_f > \lambda_i$), then it can decay into any number of photons, but the phase-space volume $\tilde{\rho}_N(k)$ contains the $(N-2)$ nd power of the small parameter $(\lambda_f - \lambda_i) |\mathbf{Q}|^2$.

If $\lambda_i = \lambda_f$, only the decay into two photons is possible.

If $\lambda_i > \lambda_f$, the phase-space volume is zero for all N .

We now consider restrictions placed on the photon-splitting amplitudes by the requirements of relativistic invariance, gauge invariance, and charge-conjugation invariance. Let us consider the photon splitting into N photons. The corresponding transition amplitude must be a scalar which is linear in the polarization vector of the initial photon and antilinear in the polarization vectors of the final photons. In addition it will depend on the external field tensor $F_{\mu\nu}$ and photon four-momenta. If we assume that the propagation of the initial and final photons is not affected by the presence of the external field, then it follows from the conservation of energy-momentum that all four-momenta must be parallel. Under these assumptions the most general form of the amplitude compatible with invariance requirements must be of at least $(N+1)$ st degree in $F_{\mu\nu}$. We will confirm this prediction later by a direct calculation in the case $N=2$.¹⁶ This conclusion also holds when the change in the propagation of photons due to the external field is taken into account, because these corrections bring again additional powers of $F_{\mu\nu}$. Even for the strongest external fields which we may expect to find in the cosmos, the dominant photon-splitting mode will be that into two photons. The splitting into a larger number of photons is suppressed, owing to the smallness of both the amplitude and the available phase space.

¹⁶ All results for the photon-splitting amplitude (and the amplitude for the inverse process of the photon fusion) obtained in Refs. 6, 9, and 10 are not only in mutual disagreement but also do not comply with the invariance requirements discussed above.

We will evaluate now the total rate for the process $\gamma \rightarrow \gamma + \gamma$ using the effective nonlinear Lagrangian (1). In this Lagrangian we substitute for $f_{\mu\nu}$ the sum $F_{\mu\nu} + \Phi_{\mu\nu}$ of the external field and the quantized electromagnetic field:

$$\Phi_{\mu\nu}(x) = \frac{1}{(2\pi)^3} \int d^3k \{ [2\omega_+(\mathbf{k})]^{-1} \epsilon_{+\mu\nu}(k) e^{-ik \cdot x} a_+(\mathbf{k}) + [2\omega_-(\mathbf{k})]^{-1} \epsilon_{-\mu\nu}(k) e^{-ik \cdot x} a_-(\mathbf{k}) \} + \text{H.c.}, \quad (43)$$

where the creation and annihilation operators a_{\pm}^{\dagger} and a_{\pm} obey the following commutation rules:

$$\begin{aligned} [a_{\pm}(\mathbf{k}), a_{\pm}^{\dagger}(\mathbf{k}')]_{\pm} &= 2\omega_{\pm}(\mathbf{k}) (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \\ [a_+(\mathbf{k}), a_-^{\dagger}(\mathbf{k}')]_{\pm} &= 0. \end{aligned} \quad (44)$$

The terms trilinear in $\Phi_{\mu\nu}$ arising from $L^{(2)}$ and $L^{(3)}$ are

$$L_3^{(2)} = \kappa: [(\Phi_{\mu\nu} \Phi^{\mu\nu})(\Phi_{\alpha\beta} F^{\alpha\beta}) + (7/4)(\Phi_{\mu\nu} \hat{\Phi}^{\mu\nu})(\Phi_{\alpha\beta} \hat{F}^{\alpha\beta})]:, \quad (45)$$

$$L_3^{(3)} = -\frac{1}{16} \mu: [16(\Phi_{\mu\nu} F^{\mu\nu})^3 + 24(F_{\mu\nu} F^{\mu\nu})(\Phi_{\alpha\beta} \Phi^{\alpha\beta})(\Phi_{\lambda\kappa} F^{\lambda\kappa}) + 13(F_{\mu\nu} F^{\mu\nu})(\Phi_{\alpha\beta} \hat{\Phi}^{\alpha\beta})(\Phi_{\lambda\kappa} \hat{F}^{\lambda\kappa}) + 26(F_{\mu\nu} \Phi^{\mu\nu})(\Phi_{\alpha\beta} \hat{F}^{\alpha\beta})^2]:, \quad (46)$$

where

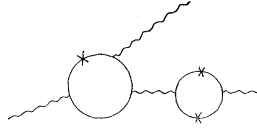
$$\mu = 32\pi\alpha^3/315m^8.$$

The first expression is linear in $F_{\mu\nu}$ and it will contribute to the transition amplitude only when the corrections to the photon propagation are taken into account. The second expression, however, will contribute even when the photons are considered to be free. The corresponding two transition amplitudes will be denoted, respectively, by T_1 and T_2 .

In the approach based on Feynman diagrams, the leading contributions to T_1 would have come from the diagrams containing corrections to external lines (Fig. 1) whereas the leading part in T_2 comes from the diagrams containing only one closed electron loop (Fig. 2). One can infer from these diagrams that the amplitude T_1 will contain an additional factor α as compared to T_2 . The direct calculation of T_1 is given in the Appendix, and it confirms the expectation that the contribution from T_1 to the total decay rate is negligible. The leading term in the amplitude T_2 is

$$\begin{aligned} T_2 &= i \langle 0 | a(\mathbf{k}) \int d^4x L^{(3)}(x) a^{\dagger}(\mathbf{k}_1) a^{\dagger}(\mathbf{k}_2) | 0 \rangle \\ &\simeq -i\mu (2\pi)^4 \delta^{(4)}(k - k_1 - k_2) \epsilon_{\mu\nu}(k) \\ &\quad \times [6F^{\mu\nu} F^{\alpha\beta} F^{\gamma\delta} + (13/4)(F^{\mu\nu} \hat{F}^{\alpha\beta} \hat{F}^{\gamma\delta} \\ &\quad + \hat{F}^{\mu\nu} F^{\alpha\beta} \hat{F}^{\gamma\delta} + \hat{F}^{\mu\nu} \hat{F}^{\alpha\beta} F^{\gamma\delta})] \epsilon_{\alpha\beta'}(k_1) \epsilon_{\gamma\delta''}(k_2) \\ &= 8i\mu (2\pi)^4 \delta^{(4)}(k - k_1 - k_2) \epsilon_{\mu}(k) \{ 6a^{\mu}(k) a^{\alpha}(k_1) a^{\beta}(k_2) \\ &\quad + (13/4)[a^{\mu}(k) \hat{a}^{\alpha}(k_1) \hat{a}^{\beta}(k_2) + \hat{a}^{\mu}(k) a^{\alpha}(k_1) \hat{a}^{\beta}(k_2) \\ &\quad + \hat{a}^{\mu}(k) \hat{a}^{\alpha}(k_1) a^{\beta}(k_2)] \} \epsilon_{\alpha'}(k_1) \epsilon_{\beta''}(k_2). \end{aligned} \quad (47)$$

FIG. 1. Example of diagrams containing corrections to external lines.



The total transition rate for unpolarized photons is

$$\begin{aligned}
 W &= -\frac{8\mu^2(2\pi)^4}{\omega(\mathbf{k})} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6 2\omega(\mathbf{k}_1) 2\omega(\mathbf{k}_2)} \delta^{(4)}(k - k_1 - k_2) \\
 &\quad \times \{6a^\mu(k)a^\alpha(k_1)a^\beta(k_2) + (13/4)[a^\mu(k)\hat{a}^\alpha(k_1)\hat{a}^\beta(k_2) \\
 &\quad + \hat{a}^\mu(k)a^\alpha(k_1)\hat{a}^\beta(k_2) + \hat{a}^\mu(k)\hat{a}^\alpha(k_1)a^\beta(k_2)]\} \\
 &\quad \times \{6a_\mu(k)a_\alpha(k_1)a_\beta(k_2) + (13/4)[a_\mu(k)\hat{a}_\alpha(k_1)\hat{a}_\beta(k_2) \\
 &\quad + \hat{a}_\mu(k)a_\alpha(k_1)\hat{a}_\beta(k_2) + \hat{a}_\mu(k)\hat{a}_\alpha(k_1)a_\beta(k_2)]\} \\
 &\simeq \frac{-1083\mu^2}{2(4\pi)^2\omega(\mathbf{k})} \int \frac{d^3k_1 d^3k_2}{\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)} \\
 &\quad \times \delta^{(4)}(k - k_1 - k_2) a^2(k) a^2(k_1) a^2(k_2) \\
 &= \frac{1083\mu^2}{2(4\pi)^2} |\mathbf{k}| |\mathbf{Q}|^6 \int \frac{d^3k_1 d^3k_2}{\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)} \\
 &\quad \times \delta^{(4)}(k - k_1 - k_2) |\mathbf{k}_1|^2 |\mathbf{k}_2|^2. \quad (48)
 \end{aligned}$$

After the integration over \mathbf{k}_1 and \mathbf{k}_2 , we obtain

$$\begin{aligned}
 W &= \frac{361}{160\pi} \mu^2 |\mathbf{k}|^5 |\mathbf{Q}|^6 \\
 &\simeq 0.07\alpha^3 (|\mathbf{k}|/m)^4 (\alpha|\mathbf{Q}|^2/m^4)^3 |\mathbf{k}|. \quad (49)
 \end{aligned}$$

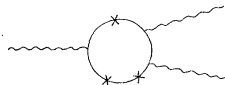
Even for hard γ rays ($E_\gamma = 50$ keV, $|\mathbf{k}|/m \simeq 10^{-1}$) and a magnetic field of 10^{12} G ($|\mathbf{Q}| = 10^{12}$ G/ $\sqrt{4\pi}$), we get an exceedingly small transition rate. The photon mean free path with respect to photon splitting is in this case of the order of 10^{16} cm.

Thus our result shows that even under most favorable conditions which one may expect to find in the vicinity of a neutron star, we can safely disregard the photon-splitting processes.

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FIG. 2. Example of diagrams containing only one closed electron loop.



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APPENDIX

We will calculate here the amplitude T_1 in the special (most favorable) case when the faster photon decays into two slower ones in the presence of an external magnetic field. The polarization vectors of the initial and final photons are then $\epsilon_-(k)$, $\epsilon_+(k_1)$, and $\epsilon_+(k_2)$. We find

$$\begin{aligned}
 T_1 &= i\langle 0 | a_-(\mathbf{k}) \int d^4x L^{(2)}(x) a_+^\dagger(\mathbf{k}_1) a_+^\dagger(\mathbf{k}_2) | 0 \rangle \\
 &= i\kappa(2\pi)^4 \delta^{(4)}(k - k_1 - k_2) [2\epsilon_{+\mu\nu}(k_1)\epsilon_+^{\mu\nu}(k_2)\epsilon_-^{\alpha\beta} F_{\alpha\beta} \\
 &\quad + \frac{7}{2}\epsilon_{-\mu\nu}(k)\hat{\epsilon}_+^{\mu\nu}(k_1)\epsilon_+^{\alpha\beta}(k_2)\hat{F}_{\alpha\beta} \\
 &\quad + \frac{7}{2}\epsilon_{-\mu\nu}(k)\hat{\epsilon}_+^{\mu\nu}(k_2)\epsilon_+^{\alpha\beta}(k_1)\hat{F}_{\alpha\beta}], \quad (A1)
 \end{aligned}$$

where we used the following properties of polarization vectors in the pure magnetic field:

$$\epsilon_+^{\mu\nu} F_{\mu\nu} = 0, \quad \epsilon_-^{\mu\nu} \hat{F}_{\mu\nu} = 0.$$

With the use of Eq. (30), this further reduces to

$$\begin{aligned}
 T_1 &= -2i\kappa(2\pi)^4 \delta^{(4)}(k - k_1 - k_2) \alpha(k)\alpha(k_1)\alpha(k_2) \\
 &\quad \times \{4a^2(k)[k_1 \cdot k_2 \hat{a}(k_1) \cdot \hat{a}(k_2) - k_1 \cdot \hat{a}(k_2) k_2 \cdot \hat{a}(k_1)] \\
 &\quad + 7a^2(k_1)[k \cdot k_2 a(k) \cdot a(k_2) + (k_2 \cdot a(k))^2 - k_2^2 a^2(k)] \\
 &\quad + 7a^2(k_2)[k \cdot k_1 a(k) \cdot a(k_1) \\
 &\quad + (k_1 \cdot a(k))^2 - k_1^2 a^2(k)]\}. \quad (A2)
 \end{aligned}$$

It shows that the amplitude T_1 does not contain a zero-order term since all products like k^2 , k_1^2 , k_2^2 , $k_1 \cdot k_2$, $k \cdot k_1$, $k \cdot k_2$, $k_1 \cdot a(k_2)$, $k_2 \cdot a(k_1)$, $k_1 \cdot a(k)$, $k_2 \cdot a(k)$ are of higher order. The leading term in the amplitude T_1 calculated in the case when $\mathbf{k} \perp \mathbf{B}$ in the special coordinate frame in which $\mathbf{k} = |\mathbf{k}|(1, 0, 0)$, $\mathbf{B} = |\mathbf{B}|(0, 0, 1)$, and $\mathbf{k}_1 = (\gamma, \beta \sin\phi, \beta \cos\phi)$, is

$$\begin{aligned}
 T_1 &= \kappa(2\pi)^4 \delta(\omega_-(\mathbf{k}) - \omega_+(\mathbf{k}_1) - \omega_+(\mathbf{k}_2)) \delta^{(3)}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \\
 &\quad \times 2\kappa |\mathbf{k}| |\mathbf{B}|^3 [27|\mathbf{k}|^2 - 8|\mathbf{k}|\gamma - 42\gamma^2 \\
 &\quad - \sin^2\phi(36|\mathbf{k}|^2 - 42|\mathbf{k}|\gamma + 84\gamma^2)]. \quad (A3)
 \end{aligned}$$

The corresponding transition rate W is

$$W = 121\pi^{-1}\kappa^4 |\mathbf{B}|^6 |\mathbf{k}|^5 \simeq 10^{-4}\alpha^5 (|\mathbf{k}|/m)^4 (\alpha|\mathbf{B}|^2/m^4)^3 |\mathbf{k}|.$$

It is much smaller than the transition rate for the amplitude T_2 owing not only to the additional factor of α^2 but also to a much smaller numerical factor.