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Gauge Problem in Quantum Field Theory. III. Quantization of Maxwell Equations and Weak Local Commutativity

F. STROCCHI

Istituto di Fisica dell'Università di Pisa, Istituto Nazionale di Fisica Nucleare, Sezione di Pisa,
and
Scuola Normale Superiore, Pisa, Italy

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The problem of the quantization of the Maxwell equations is analyzed in connection with the basic assumptions of quantum field theory. It is shown that it is impossible to quantize the Maxwell equations by means of a potential $A_\mu(x)$ which is a weakly local field. Thus, a result which was known for the Coulomb gauge is shown to hold in general: The quantization of the Maxwell equations requires the use of a potential $A_\mu(x)$ which is both noncovariant and nonlocal. It is shown that a weakly local and/or covariant operator $A_\mu(x)$ can be introduced only in a Hilbert space in which the vectors corresponding to physical states do not form a dense set, and therefore unphysical states must be present. The connections with the Gupta-Bleuler formulation are discussed.

I. INTRODUCTION

BECAUSE of the interplay between the Poincaré and the gauge groups, the quantization of the Maxwell equations has encountered many and serious difficulties since the beginning of quantum field theory.[1] Several solutions of this problem have been put forward, but the real core of the problem has not been sufficiently clarified. As a matter of fact, almost every year a new paper on this subject is published proposing a solution of the gauge problem in quantum electrodynamics.

The best known solutions of the quantization of electrodynamics are essentially two. One involves indefinite-metric and covariant fields (Gupta-Bleuler formulation[2]), the other uses noncovariant fields[3] (Coulomb- or radiation-gauge formulation). We will comment later on the difficulties connected with the Coulomb gauge. As far as the Gupta-Bleuler formulation is concerned, we do not find it very appealing from a physical point of view, for the following reasons.

(a) Unphysical particles have to be introduced in the theory.

(b) The Hilbert space has to be equipped with an indefinite metric, and one is faced with the problem of giving a meaning to states of negative norm, to negative "probabilities," etc.

(c) The unphysical photons enter into the theory with a gradient-type coupling to the electromagnetic current

$$\mathcal{L}_{\text{int}} \sim j_\mu \partial^\mu \varphi ,$$

and the widespread belief that these particles do not contribute to anything is not correct. As a matter of fact, it has been proved that the above interaction affects the renormalization constants by infinite amounts.[4]

(d) The presence of unphysical particles in the theory gives rise to complications when one tries a Lehmann-Symanzik-Zimmermann (LSZ) formulation of quantum electrodynamics.[5] In fact, the physical photon states do not form a complete set of states, and therefore the asymptotic limit for the interpolating field, which may create unphysical photons, must be combined with a gauge transformation.[5] We will comment on this point later (Secs. VII and VIII) in more detail.

(e) The Maxwell equations are not satisfied when applied to the vacuum state. More precisely, the equation

$$\partial^\mu F_{\mu\nu} \Psi_0 = 0$$

does not hold as an equation in $\mathcal{H}$, the Hilbert space in which $A_\mu(x)$ is defined as a local covariant operator. This last appears to us as an unpleasant feature of the Gupta-Bleuler formulation. For a more detailed discussion of the Gupta-Bleuler formulation see Secs. VII and VIII.

In conclusion, we find it difficult to regard the Gupta-Blueler formulation as a completely satisfactory solution of the problem of quantizing the Maxwell equations. As a matter of fact, the difficulties of quantizing the Maxwell equations have been overcome by changing the equations and, to a certain extent, the physical content of the problem.

A natural question at this point is why the quantization of the Maxwell equations encounters such difficulties. Do the difficulties arise because we insist on asking for unnecessary conditions or is there something

[1] W. Heisenberg and W. Pauli, Z. Physik **56**, 1 (1929); **59**, 169 (1930); E. Fermi, Rend. Accad. Nazl. Lincei **2**, 881 (1929); Rev. Mod. Phys. **4**, 87 (1932); P. A. M. Dirac, V. A. Fock, and B. Podolsky, Z. Physik Sowjetunion **2**, 468 (1932).

[2] S. N. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950); K. T. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A114**, 243 (1927); **A114**, 710 (1927); W. Heitler, *The Quantum Theory of Radiation*, 3rd ed. (Oxford U. P., New York, 1954); P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Oxford U. P., New York, 1958); J. Schwinger, Phys. Rev. **74**, 1439 (1948); **127**, 324 (1964); S. Weinberg, *ibid.* **134**, B882 (1964); **138**, B988 (1965).

[4] S. Okubo, Nuovo Cimento **19**, 574 (1961); A. S. Wightman, in *Cargèse Lectures in Physics*, edited by M. Lévy (Gordon and Breach, New York, 1967), Pt. III, Vol. 1; B. Klaiber, Nuovo Cimento **36**, 165 (1965).

[5] K. Nishijima, Phys. Rev. **119**, 485 (1960); H. Rollnik, B. Stech, and E. Nunnemann, Z. Physik **159**, 482 (1960); R. E. Pugh, Ann. Phys. (N. Y.) **30**, 422 (1964); R. B. Willey, University of Pittsburgh report (unpublished).

fundamental at the roots of this problem? The impression one gets from the literature is that the difficulties connected with the quantization of the Maxwell equations have a rather accidental origin. As a matter of fact, it appears as if all the troubles arise because one insists on the Lorentz condition $\partial_\mu A^\mu=0$ and one requires a positive metric in the Hilbert space. Now, none of the above conditions are really necessary. Even classically, there is no need for imposing the Lorentz condition in the Maxwell equations

$$(\Box g_{\mu\nu}-\partial_\mu\partial_\nu)A^\nu=0.$$

Thus, one may wonder about the possibility of quantizing the Maxwell equations without imposing any of the above conditions.

Unfortunately, as we shall see below, the difficulties connected with the quantization of the Maxwell equations have very deep roots. They have very little to do with the Lorentz condition and the indefinite metric, in contrast with what is generally stated in the literature. Rather, they are connected with some of the basic principles of quantum field theory.

The aim of this paper is to show which of the basic assumptions of quantum field theory conflict with the quantization of the Maxwell equations. The natural framework for the discussion of this problem is the Wightman formulation of quantum field theory.[6] Therefore, even if we do not use all of the Wightman axioms, we shall follow the Wightman formulation as a guide.

II. QUANTIZATION OF MAXWELL EQUATIONS: BASIC ASSUMPTIONS

In order to clarify the crucial points in the quantization of the Maxwell equations, we will proceed by successive steps. We will successively adopt some of the basic assumptions of quantum field theory (Wightman axioms), and we will point out when the contradictions arise. In this way, it will be clear which of the fundamental axioms of quantum field theory cannot be reconciled with the quantization of the Maxwell equations.

(0) *Definition of the problem.* We want to describe massless spin-one particles. According to the representations of the Poincaré group, we have to use an antisymmetric tensor $F_{\mu\nu}$ and to impose the Maxwell equations

$$\partial_\mu F^{\mu\nu}=0, \quad (1)$$

$$\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma}=0. \quad (2)$$

Among the many advantages of these equations, we want to stress the fact that they do not involve unphysical particles and do not suffer the ambiguities connected with the gauge problem.

(1) The fields $F_{\mu\nu}(x)$ may be defined as operator-valued distributions in a Hilbert space $\mathcal{H}$.

(2) There exists a "unitary" representation of the Poincaré group $\{a,\Lambda\}\rightarrow U(a,\Lambda)$ such that one has

$$U(a,\Lambda)F_{\mu\nu}(x)U(a,\Lambda)^{-1}=\Lambda^{-1}{}_\mu{}^\rho\Lambda^{-1}{}_\nu{}^\sigma F_{\rho\sigma}(\Lambda x+a). \quad (3)$$

(3) There exists an invariant state Ψ_0 (vacuum state) such that

$$U(a,\Lambda)\Psi_0=\Psi_0. \quad (4)$$

Now, the second set of Maxwell's equations is equivalent to the statement that $F_{\mu\nu}$ may be written as a four-dimensional curl of a potential A_μ,

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu, \quad (5)$$

so that the quantization of the electromagnetic fields may be reduced to the problem of quantizing the potential A_μ. The use of the potential A_μ is in fact necessary if one wants a connection with conventional quantum electrodynamics. As a matter of fact, we are not able to write down a local interaction Lagrangian in terms of $F_{\mu\nu}$. In addition, the fields $F_{\mu\nu}$ cannot account for the production and absorption of soft photons which are instead characteristic features of electromagnetic interactions (long-range forces):

$$F_{ij}(\mathbf{p})\sim p_i\epsilon_j-p_j\epsilon_i \xrightarrow[\mathbf{p}\to 0]{} 0.$$

Thus, in any case, the fields $F_{\mu\nu}$ cannot reasonably be expected to enter into transition amplitudes or S-matrix elements.

In conclusion, there is little hope of having a reasonable quantum field theory of electrodynamics involving only the fields $F_{\mu\nu}$, and the introduction of the potential A_μ appears to be an unavoidable step in the quantization of the Maxwell equations. Therefore we shall make the following assumptions.

(1′) The fields $A_\mu(x)$, $\mu=0, 1, 2, 3$, may be defined as operator-valued distributions in a Hilbert space $\mathcal{H}$.

(4) The fields $A_\mu(x)$ transform correctly under the space-time translation group

$$U(a,1)A_\mu(x)U(a,1)^{-1}=A_\mu(x+a), \quad (6)$$

and the spectral condition[6] is satisfied.

Before proceeding further, it is convenient to comment on the mildness of the above assumptions. The statements (0), (1), and (1′) are essentially the definition of the problem. It is worthwhile to stress that no restriction has been made on the type of operator-valued distributions, i.e., on the kind of singularities which the fields $F_{\mu\nu}$ and/or A_μ may have. (In the conventional Wightman theory, the fields are assumed to be operator-valued tempered distributions.) Condition (2) is merely the statement that the electromagnetic fields

[6] A. S. Wightman, Phys. Rev. **101**, 860 (1956); Lectures given at the Faculté des Sciences, Université de Paris, 1958 (unpublished), and in *Les Problèmes Mathématiques de la Théorie Quantique des Champs* (Colloques Internationaux du Centre Nationale de la Recherche Scientifique, Paris, 1959); A. S. Wightman and L. Gårding, Arkiv Fysik **28**, 129 (1964); R. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

$F_{\mu\nu}$ are observable quantities and therefore they must transform as the component of a second-rank tensor under the Lorentz group. The situation is different for the potential A_μ, which is not an observable quantity. As a matter of fact, up to now no assumption has been made about the transformation properties of A_μ under the Lorentz group. The four fields $A_\mu(x)$, $\mu=0, 1, 2, 3$, are not restricted to behave as the components of a four-vector. Weak local commutativity is not assumed to hold for the field $F_{\mu\nu}(x)$ and/or $A_\mu(x)$.

It may be worthwhile to remark that the above conditions, and in particular Eq. (6) and the spectral condition, are obviously satisfied in the standard quantizations of the Maxwell equations, as in the Gupta-Bleuler or in the radiation gauge formulation, in spite of the many contradictory statements one may find in the literature.

Finally, we want to stress that no assumption has been made about the positivity of the metric in the Hilbert space $\mathcal{H}$ in which $A_\mu(x)$ is defined. It may very well be that the physically meaningful quantities such as the transition probabilities, vacuum expectation values, etc., have to be defined in terms of a "product" (,) between vectors, which does not coincide with the scalar product $\langle\ ,\ \rangle$ in $\mathcal{H}$. The product (Ψ_1,Ψ_2) between the two vectors Ψ_1, Ψ_2 may be defined as a sesquilinear form

$$(\Psi_1,\Psi_2)=\langle\eta\Psi_1,\Psi_2\rangle,$$

where η is the "metric" operator. In this case, the vacuum expectation values are defined, e.g., in the following way:

$$(\Psi_0,A_\mu(x)A_\nu(y)\Psi_0)=\langle\eta\Psi_0,A_\mu(x)A_\nu(y)\Psi_0\rangle.$$

It is important to remark that the unitarity of the representation of the Poincaré group is defined in terms of the product (,). This last, in fact, occurs in the definition of observable quantities. The results of the present paper are independent of whether $\eta=1$ or $\eta\neq1$.

In the following sections, we will see that inconsistencies arise when additional assumptions such as Lorentz covariance or weak local commutativity, are made on the fields.

III. DIFFICULTIES OF QUANTUM FIELD THEORY OF MAXWELL EQUATIONS: LORENTZ COVARIANCE

The first basic difficulty connected with the quantization of the Maxwell equations is the Lorentz covariance. In fact, if we require that the fields $A_\mu(x)$, $\mu=0, 1, 2, 3$, transform as the components of a four-vector, we get a trivial theory. More precisely, if one adds to the previous conditions the requirement

$$U(0,\Lambda)A_\mu(x)U(0,\Lambda)^{-1}=\Lambda^{-1}{}_\mu{}^\rho A_\rho(\Lambda x),\tag{5}$$

then necessarily one has $A_\mu(x)=\partial_\mu\varphi(x)$, i.e.,

$$F_{\mu\nu}(x)=0.$$

For the details of the proof, we refer the reader to Ref. 7.

Here, we want to remark that Lorentz covariance is one of the building stones of axiomatic field theory. All the mathematical tools, such as analytic continuation of the Lorentz group, analyticity domain of the Wightman functions, etc., are based on the Lorentz covariance of the fields. Thus, the impossibility of having a quantum field theory of the Maxwell equations, defined in terms of a covariant potential $A_\mu(x)$, has rather unpleasant consequences. For example, almost all the interesting results of the Wightman theory, such as the *TCP* theorem, the connection between spin and statistics, etc., have no sound basis in a non-manifestly covariant theory.

Clearly, no difficulty arises if the theory is formulated only in terms of the fields $F_{\mu\nu}(x)$, which may be defined as covariant fields. In this sense, one may have a covariant quantum field theory of Maxwell equations. Difficulties, however, arise when one wants to deal with the interacting case. Local interactions, S-matrix elements, etc., require the introduction of the potential $A_\mu(x)$, which should have an asymptotic limit $A^{\text{in}}{}_\mu(x)$. As shown above, the free field $A^{\text{in}}{}_\mu(x)$ cannot be defined as a covariant field.

IV. DIFFICULTIES OF QUANTUM FIELD THEORY OF MAXWELL EQUATIONS: WEAK LOCAL COMMUTATIVITY

Independently of the above result, one may wonder about the possibility of quantizing the Maxwell equations in the framework of quantum field theory. The point is to understand whether Lorentz covariance is the only difficulty and whether one may hope to get a reasonable quantum field theory of the Maxwell equations by using noncovariant fields. As a matter of fact, it is not obvious whether one may have a Wightman-type formulation of electrodynamics satisfying all of the Wightman axioms except Lorentz covariance.

In order to clarify the above problems, we shall investigate the possibility of imposing one of the basic assumptions of quantum field theory: weak local commutativity. Again we will find that this assumption is incompatible with a nontrivial quantum theory of the Maxwell equations.

We shall now investigate the possibility of a quantum field theory of the Maxwell equations, in which the potential $A_\mu(x)$ satisfies weak local commutativity (WLC), i.e.,

$$(\Psi_0,[A_\mu(f),A_\nu(g)]\Psi_0)=0 \quad\text{if}\quad \operatorname{supp}f\times\operatorname{supp}g\tag{7}$$

($\operatorname{supp}f\times\operatorname{supp}g$ means that the support of f is space-like with respect to the support of g, i.e., that any point

[7] F. Strocchi, Phys. Rev. 162, 1429 (1967).

of supp$f$ is spacelike with respect to every point of supp$g$).

We will show in Sec. V that if conditions (0)–(3), (1′), (4), and (7) are satisfied, the theory is trivial and does not describe physical photons.

V. IMPOSSIBILITY OF LOCAL QUANTUM THEORY OF MAXWELL EQUATIONS

In order to prove the announced theorem we consider the following Lemmas.

Lemma 1. Condition (2) restricts the possible noncovariance of $A_\mu(x)$ to the following kind:

$$\Lambda^{-1}{}_\mu{}^\rho U(0,\Lambda)^{-1} A_\rho(\Lambda x) U(0,\Lambda) = A_\mu(x) + \partial_\mu \mathfrak{F}(x,\Lambda), \qquad (8)$$

where $\mathfrak{F}(x,\Lambda)$ is a field of which we do not specify the transformation properties under the homogeneous Lorentz group.

Proof. By assumption (2) there exists a unitary representation $\{0,\Lambda\} \to U(0,\Lambda)$ of the homogeneous Lorentz group, and therefore we may consider the action of $U(0,\Lambda)$ on $A_\mu(x)$. Without loss of generality, we may write

$$\Lambda^{-1}{}_\mu{}^\rho U(0,\Lambda)^{-1} A_\rho(\Lambda x) U(0,\Lambda) = A_\mu(x) + \mathfrak{F}_\mu(x,\Lambda).$$

Then, by taking the four-dimensional curl of the above equation and comparing the result with condition (2), we get

$$\partial_\mu \mathfrak{F}_\nu(x,\Lambda) - \partial_\nu \mathfrak{F}_\mu(x,\Lambda) = 0, \qquad (9)$$

i.e., $\mathfrak{F}_\mu(x,\Lambda)$ may be written as the μ derivative of a field

$$\mathfrak{F}_\mu(x,\Lambda) = \partial_\mu \mathfrak{F}(x,\Lambda). \qquad (10)$$

Lemma 2. The two-point function

$$\mathfrak{F}_{\mu\rho\sigma}(x-y) \equiv (\Psi_0, A_\mu(x) F_{\rho\sigma}(y) \Psi_0) \qquad (11)$$

transforms covariantly under the homogeneous Lorentz group, i.e.,

$$\mathfrak{F}_{\mu\rho\sigma}(\Lambda\xi) = \Lambda_\mu{}^\lambda \Lambda_\rho{}^\tau \Lambda_\sigma{}^\nu \mathfrak{F}_{\lambda\tau\nu}(\xi).$$

Proof. By inserting $U(0,\Lambda)U(0,\Lambda)^{-1}$ into Eq. (11) and using the transformation porperties (2) and (8), we get

$$\mathfrak{F}_{\mu\rho\sigma}(\xi) = \Lambda_\mu{}^\lambda \Lambda_\rho{}^\tau \Lambda_\sigma{}^\nu [\mathfrak{F}_{\lambda\tau\nu}(\xi) + \partial_\lambda(\psi_0, \mathfrak{F}(x,\Lambda) F_{\tau\nu}(y)\Psi_0)].$$

Hence, in order to prove the lemma we have to show that the two-point function

$$\mathfrak{F}_{\tau\nu}(\xi) \equiv (\Psi_0, \mathfrak{F}(x,\Lambda) F_{\tau\nu}(y) \Psi_0)$$

vanishes.

Now, as a consequence of condition (4), $\mathfrak{F}_{\tau\nu}(\xi)$ may be regarded as the boundary value of an analytic function $\mathfrak{F}_{\tau\nu}(z)$, which is analytic in the forward tube $\mathcal{T}$:

$$\mathcal{T} \equiv \{z \mid -\infty < \operatorname{Re} z < \infty, \operatorname{Im} z \in V_+\},$$

where

$$V_+ \equiv \{x \mid x \text{ is a four-vector, } x^2 > 0, \ x_0 > 0\}.$$

The analyticity domain of $\mathfrak{F}_{\tau\nu}(z)$ can be extended to the extended tube $\mathcal{T}' \equiv \{$union of the open sets obtained from $\mathcal{T}$ by applying all proper complex Lorentz transformations$\}$ as a consequence of condition (7). Then, one may show that an analog of the Bogoliubov-Vladimirov theorem holds[8] and the possible noncovariance of $\mathfrak{F}_{\tau\nu}(z)$ is at most polynomial. This means that $\mathfrak{F}_{\tau\nu}(z)$ can be written in the following form:

$$\mathfrak{F}_{\tau\nu}(z) = \sum_{k=0}^{M} \sum_{i_l=0}^{3} C_{i_1\cdots i_k}{}^{(\tau\nu)} F_{i_1\cdots i_k}{}^{(\tau\nu)}(z^2) z_{i_1} \dots z_{i_k},$$

or, equivalently,

$$\mathfrak{F}_{\tau\nu}(z) = \sum_{k=0}^{N} \sum_{i_l=0}^{3} c_{i_1\cdots i_k}{}^{(\tau\nu)} \frac{\partial}{\partial z_{i_1}} \cdots \frac{\partial}{\partial z_{i_k}} f_{i_1\cdots i_k}{}^{(\tau\nu)}(z^2).$$

Taking the boundary value of the above equation gives

$$\mathfrak{F}_{\tau\nu}(x) = \sum_{k=0}^{N} \sum_{i_l=0}^{3} c_{i_1\cdots i_k}{}^{(\tau\nu)} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} f_{i_1\cdots i_k}{}^{(\tau\nu)}(x), \qquad (12)$$

where $f_{i_1\cdots i_k}{}^{(\tau\nu)}(x)$ are Lorentz invariant distributions. In momentum space we have

$$\mathfrak{F}_{\tau\nu}(p) = \sum_{k=0}^{N} \sum_{i_l=0}^{3} i^k c_{i_1\cdots i_k}{}^{(\tau\nu)} p_{i_1} \cdots p_{i_k} f_{i_1\cdots i_k}{}^{(\tau\nu)}(p),$$

and we may reorder the above sum in such a way that terms belonging to the same irreducible tensor representation are grouped together:

$$\mathfrak{F}_{\tau\nu}(p) = \sum_{k=0}^{N} T_k{}^{(\tau\nu)}(p),$$

where

$$T_k{}^{(\tau\nu)}(p) = \sum_{i_1\cdots i_k} i^k c_{i_1\cdots i_k}{}^{(\tau\nu)} p_{i_1} \cdots p_{i_k} f_{i_1,\cdots i_k}{}^{(\tau\nu)}(p).$$

Furthermore, we reorder in such a way that the above formulas show the "minimal" noncovariance. By this we mean that it does not contain terms that reduce to a covariant function when grouped together. For example, terms like

$$c_{i_3\cdots i_k}{}^{(\tau\nu)} g^{i_1 i_2} p_{i_1} \cdots p_{i_k} f_{i_1\cdots i_k}{}^{(\tau\nu)}(p)$$

($g^{i_1 i_2}$ being the metric tensor) will not appear explicitly in the above expression.

We will now exploit the full consequences of the Maxwell equations

$$\Box F_{\mu\nu}(x)\Psi_0 = 0, \quad \partial^\mu F_{\mu\nu}(x)\Psi_0 = 0.$$

They imply the equations

$$\Box \mathfrak{F}_{\tau\nu}(x) = 0, \quad \partial^\mu \mathfrak{F}_{\mu\nu}(x) = 0,$$

or, in momentum space,

$$p^2 \mathfrak{F}_{\tau\nu}(p) = 0, \quad p^\tau \mathfrak{F}_{\tau\nu}(p) = 0. \qquad (13)$$

The first equation implies the vanishing of each ir-

[8] N. N. Bogoliubov and V. S. Vladimirov, Nauchn. Dokl. Vysshei Shkoly 3, 26 (1958); J. Bros, H. Epstein, and V. Glaser, Commun. Math. Phys. 6, 77 (1967).

reducible tensor

$$t_k^{(\tau\nu)}(p)=\sum c_{i_1\cdots i_k}{}^{(\tau\nu)}p_{i_1}\cdots p_{i_k}h_{i_1\cdots i_k}{}^{(\tau\nu)}(p)\,, \quad (14)$$

where

$$h_{i_1\cdots i_k}{}^{(\tau\nu)}(p)=p^2 f_{i_1\cdots i_k}{}^{(\tau\nu)}(p)\,,$$

and, since the tensors $p_{i_1}\ldots p_{i_k}$ are linearly independent, each coefficient $h_{i_1\cdots i_k}{}^{(\tau\nu)}(p)$ must vanish. Then, one may write

$$f_{i_1\cdots i_k}{}^{(\tau\nu)}(p)=\lambda_{i_1\cdots i_k}{}^{(\tau\nu)}(p)\delta(p^2)\,.$$

Finally, by using microscopic causality or WLC, one gets

$$f_{i_1\cdots i_k}{}^{(\tau\nu)}(p)=\lambda_{i_1\cdots i_k}{}^{(\tau\nu)}\theta(p)\delta(p^2)\,,$$

where the λ's are constants and may be eliminated by a redefinition of the $c_{i_1\cdots i_k}{}^{(\tau\nu)}$'s:

$$\mathfrak{F}_{\tau\nu}(p)=(\sum c_{i_1\cdots i_k}{}^{(\tau\nu)}p_{i_1}\cdots p_{i_k})\theta(p)\delta(p^2)\,.$$

We will now use the second of Eqs. (13). Again, by a reasoning similar to the previous one, each tensor

$$\sum p^\tau c_{i_1\cdots i_k}{}^{(\tau\nu)}p_{i_1}\ldots p_{i_k}\theta(p)\delta(p^2) \equiv \sum c_{i_1\cdots i_k i_\tau}{}^{(\nu)}p_{i_1}\cdots p_{i_k}p_{i_\tau}\theta(p)\delta(p^2)$$

must vanish. This is possible only if the c's are of the form

$$c_{i_1\cdots i_k i_\tau}{}^{(\nu)}\sim g_{i_l}{}^{i_\tau}c_{i_1\cdots i_{l-1}i_{l+1}i_k}{}^{(\nu)}$$

(g being the metric tensor). Therefore, we have

$$\mathfrak{F}_{\tau\nu}(p)=p_\tau(\sum c_{i_1\cdots i_k}{}^{(\nu)}p_{i_1}\cdots p_{i_k})\theta(p)\delta(p^2)\,.$$

The argument may be repeated for the index ν, yielding

$$\mathfrak{F}_{\tau\nu}(p)=p_\tau p_\nu(\sum c_{i_1\cdots i_k}p_{i_1}\cdots p_{i_k})\theta(p)\delta(p^2)\,. \quad (15)$$

Finally, by the antisymmetry of $F_{\mu\nu}$, $\mathfrak{F}_{\tau\nu}$ must also be antisymmetric under the exchange of τ and ν, and, therefore, the above Eq. (15) gives

$$\mathfrak{F}_{\tau\nu}(p)=0\,.$$

We may now prove the following theorem.

Theorem: If conditions (0)–(4) and (7) are fulfilled, one has

$$\mathfrak{F}_{\mu\rho\sigma}(\xi)=(\Psi_0,A_\mu(x)F_{\rho\sigma}(y)\Psi_0)=0\,.$$

Proof. As a result of the previous lemmas, the two-point function $\mathfrak{F}_{\mu\rho\sigma}(\xi)$ yields a representation of the complex Lorentz group $L_+(C)$, when analytically continued to complex z. Then by the Araki-Hepp theorem,[9] one may write $\mathfrak{F}_{\mu\rho\sigma}(z)$ in the following form:

$$\mathfrak{F}_{\mu\rho\sigma}(z)=g_{\mu\rho}z_\sigma F_1(z)+g_{\mu\sigma}z_\rho F_2(z) +g_{\rho\sigma}z_\mu F_3(z)+\epsilon_{\mu\rho\sigma\tau}z^\tau F_4(z)\,, \quad (16)$$

[9] K. Hepp, Helv. Phys. Acta **36**, 355 (1963).

when $F_i(z)$, $i=1,\ldots 4$, are invariant under $L_+(C)$,

$$F_i(\Lambda z)=F_i(z)\,, \quad \Lambda\in L_+(C)$$

and, therefore, they may be written as functions of z^2. Then, by using the same technique discussed elsewhere[10] and the antisymmetry of $F_{\rho\sigma}$, Eq. (16) may be written in the equivalent form

$$\mathfrak{F}_{\mu\rho\sigma}(z)=\left(g_{\mu\sigma}\frac{\partial}{\partial z_\rho}-g_{\mu\rho}\frac{\partial}{\partial z_\sigma}\right)F(z^2)+\epsilon_{\mu\rho\sigma\tau}\frac{\partial}{\partial z_\tau}G(z^2)\,. \quad (17)$$

By going to the boundary value, we get

$$\mathfrak{F}_{\mu\rho\sigma}(x)=\left(g_{\mu\sigma}\frac{\partial}{\partial x_\rho}-g_{\mu\rho}\frac{\partial}{\partial x_\sigma}\right)F(x)+\epsilon_{\mu\rho\sigma\tau}\frac{\partial}{\partial x_\tau}G(x)\,, \quad (18)$$

F and G being Lorentz-invariant distributions. Finally, by using the Maxwell equations on $\mathfrak{F}_{\mu\rho\sigma}$, we get

$$(\Box g_{\mu\sigma}-\partial_\mu\partial_\sigma)F(x)=0\,,$$
$$(\Box g_{\mu\lambda}-\partial_\mu\partial_\lambda)G(x)=0\,.$$

The above equations have been shown[11] to have only the trivial solutions

$$F(x)=\text{const}\,, \quad G(x)=\text{const}\,.$$

Hence, one gets

$$\mathfrak{F}_{\mu\rho\sigma}(x)=0\,. \quad (19)$$

The conclusion of the above theorem implies that the two-point function of the electromagnetic field vanishes:

$$(\Psi_0,F_{\mu\nu}(x)F_{\rho\sigma}(y)\Psi_0)=0\,, \quad (20)$$

and there is little hope of having a nontrivial theory, as is shown in the following corollary.

Corollary. Let D_0 be the set of vectors of $\mathfrak{H}$, which are obtained from the vacuum state by applying polynomials in the smeared fields $F_{\mu\nu}(f)$, and let the metric operator η be non negative on D_0. Then $\mathfrak{F}_{\mu\rho\sigma}=0$ implies that all the Wightman functions of the field $F_{\mu\nu}(x)$ vanish, i.e., the theory is trivial:

$$A_\mu=\partial_\mu\varphi\,.$$

Proof. It is not difficult to see that if $\Psi\in D_0$ and $(\Psi,\Psi)=0$, then $(\Psi,\Phi)=0$ for any $\Phi\in D_0$, provided that η is non-negative on D_0. Now, Eq. (20) implies

$$(F_{\mu\nu}(f)\Psi_0,F_{\mu\nu}(f)\Psi_0)=0$$

($F_{\mu\nu}$ is Hermitian with respect to η). Thus

$$(\Psi_0,F_{\mu\nu}(f_1)\cdots F_{\rho\sigma}(f_n)\Psi_0) (F_{\mu\nu}(f_1)\cdots F_{\lambda\tau}(f_{n-1})\Psi_0,F_{\rho\sigma}(f_n)\Psi_0)=0\,.$$

[10] F. Strocchi, Phys. Rev. **162**, 1429 (1967); **166**, 1302 (1968).
[11] S. Weinberg, Phys. Rev. **134**, B882 (1964); **138**, B988 (1965).

VI. REPRESENTATIONS OF POINCARÉ AND GAUGE GROUP FOR MASSLESS SPIN-ONE PARTICLES

We may now discuss the implications of the previous results in terms of the representations of the Poincaré and gauge group for massless spin-one particles.

A detailed analysis of helicity representations has been given by Weinberg[11] under the following assumptions.

(a) The Hilbert Space is equipped with a positive definite metric, $\eta=1$, and therefore the analysis of the representations of the Poincaré group follows the standard pattern.

(b) The discussion is carried on in the Fock representation and the representation is realized with tempered fields.

(c) The Lorentz condition

$$\partial_\mu A^\mu=0$$

is imposed on the potential A_μ as an operator identity.

(d) The fourth component A_0 of the potential is required to vanish. In this framework, the representation obtained is the Coulomb or radiation gauge, as could have been anticipated:

$$U(0,\Lambda)A_\mu(x)U(0,\Lambda)^{-1}=\Lambda^{-1}{}_\mu{}^\rho A_\rho(\Lambda x)+\partial_\mu\Phi(x;\Lambda),$$
$$A_0=0,\quad \mathrm{div}\mathbf{A}=0.$$

As is known, the Coulomb gauge has two disadvantages: (i) It is not manifestly covariant; (ii) it is nonlocal in the sense that the fields do not satisfy weak local commutativity.

Actually, one has

$$[A_i(x),A_j(y)]=i\delta_{ij}\Delta(x-y;0)-i(\partial_i\partial_j/|\nabla|^2)\Delta(x-y;0)$$

and the second term is clearly nonlocal, i.e., it does not vanish for spacelike separations.

For the above reasons, the Coulomb gauge was not regarded as a good candidate for a Wightman formulation of quantum electrodynamics.

The natural reaction to the difficulties of the Coulomb gauge is to regard them as the price for the very strong assumptions (a)–(d). One might hope to get a more reasonable theory by rejecting some of the above assumptions. For example, by working in a Hilbert space with a non-positive-definite metric, one should look for representations of the Poincaré group which are "unitary" with respect to the metric operator

$$U(a,\Lambda)\eta U(a,\Lambda)^\dagger=\eta,$$

and the analysis of these representations is not known.[12] Moreover, one might hope that a suitable choice of the phases of the helicity representation could give locality if the two conditions

$$A_0(x)=0,\quad \partial^\mu A_\mu(x)=0$$

are relaxed. Unfortunately, this cannot happen.

The results of the previous sections show that the difficulties of the Coulomb gauge, i.e., (i) noncovariance and (ii) nonlocality, have a very general character. As a matter of fact, they appear to be necessary features of *the quantization of the Maxwell equations* by means of a potential $A_\mu(x)$.

[12] A. S. Wightman and L. Gårding, Arkiv Fysik 28, 129 (1964).

VII. WEAK LOCAL COMMUTATIVITY AND UNPHYSICAL STATES

In conclusion, one may say that the quantization of the Maxwell equations, even in the weak form

$$\partial_\mu F^{\mu\nu}\Psi_0=0, \tag{21}$$

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}\Psi_0=0, \tag{22}$$

can be done by means of a potential $A_\mu(x)$, only if $A_\mu(x)$ is (i) noncovariant and (ii) nonlocal.

In order to get a local and covariant theory avoiding the above difficulties, Eqs. (21) and (22) must be abandoned. On the other hand, if the theory must have any contact with quantum electrodynamics, one is forced to have the Maxwell equations satisfied at least in the "mean." This means that one may require that the Maxwell equations are satisfied only when one takes the mean values of Eqs. (1) and (2) on the *physical states.*

Here, and in the following, a vector of $\mathcal{H}$ corresponding to a physical state is a vector belonging to the set D_0 defined, in the corollary of Sec. VII.

Thus, instead of Eqs. (1) and (2), one has

$$\partial^\mu(\Psi,F_{\mu\nu}\Phi)=0,\quad \epsilon^{\mu\nu\rho\sigma}\partial_\nu(\Psi,F_{\rho\sigma}\Phi)=0, \tag{23}$$

where Ψ, $\Phi\in D_0$. It is important to remark that Eqs. (23) are rather weak equations; in fact, D_0 cannot be dense in $\mathcal{H}$, as shown by the following statement.

Statement. In a weakly local theory with the properties (1)–(4), the set of physical states D_0, on which Eqs. (1) and (2) hold in the mean, cannot be dense in $\mathcal{H}$.

In fact, the equations

$$(\Psi,\partial_\mu F^{\mu\nu}\Psi_0)=0,\quad \epsilon^{\mu\nu\rho\sigma}\partial_\nu(\Psi,F_{\rho\sigma}\Psi_0)=0,$$

with Ψ running over a dense subset of $\mathcal{H}$, imply

$$(\Psi_0,A_\tau\partial^\mu F_{\mu\nu}\Psi_0)=0,\quad (\Psi_0,A_\tau\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}\Psi_0)=0.$$

By the previous theorem, the above equations lead to a trivial local theory.

An alternative way of stating the above result is that the *vacuum* Ψ_0 *cannot be a cyclic vector with respect to the polynomial algebra of observable fields* $F_{\mu\nu}$.

Thus, one cannot hope to realize a local quantum field theory of the Maxwell equations by means of a potential $A_\mu(x)$, in a Hilbert space $\mathcal{H}$ in which the physical states form a dense subset of $\mathcal{H}$. Alternatively, one may say that a local field $A_\mu(x)$ cannot be introduced in D_0. The introducting of unphysical states, in an essential way, is a necessary step if one wants to define $A_\mu(x)$ as a weakly local operator.

Finally, we want to recall the following theorem.

Theorem. The vector potential $A_\mu(x)$ can be defined as a weakly local and Lorentz-covariant operator-valued distribution only in a Hilbert space with indefinite metric.

Proof. See Ref. 12.

The results of this section show that the introduction of unphysical states and the indefinite metric are necessary features if $A_\mu(x)$ has to be local and covariant. All this leads essentially to the Gupta-Bleuler formulation.

VIII. COMMENTS ON GUPTA-BLEULER FORMULATION

As anticipated in the previous section, the Gupta-Bleuler formulation is essentially forced by the requirement of locality and Lorentz covariance. It may be useful to check explicitly how the Gupta-Bleuler formulation escapes the difficulties discussed in the previous sections.

To this purpose, we introduce the subspace $\mathcal{H}_1 \subset \mathcal{H}$ defined as the set of vectors Ψ such that

$$\partial^\mu A_\mu{}^+(x)\Psi = 0. \tag{24}$$

[$\partial^\mu A_\mu{}^+(x)$ is the positive-frequency part of the operator $\partial^\mu A_\mu(x)$.]

Equation (24) is the Gupta-Bleuler substitute for the Fermi condition

$$\partial^\mu A_\mu \Psi = 0. \tag{25}$$

This last equation leads to inconsistencies. For example, if one takes the vacuum state Ψ_0 for the physical state Ψ, one has by Eq. (25),

$$(\Psi_0, A_\nu \partial^\mu A_\mu \Psi_0) = 0,$$

whereas the equation

$$(\Psi_0, A_\nu(x) A_\mu(y)\Psi_0) = g_{\mu\nu} D(x-y) \tag{26}$$

(which is assumed to hold in the Fermi theory) gives

$$(\Psi_0, A_\nu \partial^\mu A_\mu \Psi_0) = \partial_\nu D(x-y) \neq 0.$$

To "explain" this paradox, it has sometimes been concluded that the vacuum has infinite norm, that it is not a vector of $\mathcal{H}$, or that it is not a physical state.

Without any reference to Eq. (26), one may immediately see that Eq. (25) leads to a theory that is either trivial or inconsistent. In fact, Eq. (25), together with $\Box A_\mu \Psi = 0$, leads to the Maxwell equations (21) and (22), when Ψ_0 is taken as the physical vector Ψ. As shown in Sec. VI, this leads to a trivial local theory.

The above paradox does not arise if one uses Eq. (24) instead of Eq. (25). As a matter of fact, in the Gupta-Bleuler formulation the Maxwell equations do not hold when applied to vectors of D_0 or when applied to vectors of $\mathcal{H}_1 \supset D_0$. One has instead

$$(\Psi, \partial^\mu F_{\mu\nu}(f)\Phi) = 0, \quad (\Psi, \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}(f)\Phi) = 0$$

for any two vectors Ψ, $\Phi \in D_0$. In particular, by taking

$$\Psi = \partial^\mu F_{\mu\nu}(f)\Phi, \quad \Psi = \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}(f)\Phi,$$

one has that even if the Maxwell equations do not hold when applied to vectors of D_0, they yield vectors of zero length. One may say that the Maxwell equations can be regarded as equations in $\mathcal{H}_1/\mathcal{H}_2$, where $\mathcal{H}_2$ is the set of vectors of $\mathcal{H}_1$ with zero length.

Finally, it is not difficult to see that in the Gupta-Bleuler formulation one has

$$(\Psi_0, A_\mu(x) F_{\rho\sigma}(y)\Psi_0) = (g_{\mu\rho}\partial_\sigma - g_{\mu\sigma}\partial_\rho) D(x-y), \tag{27}$$

in agreement with Lemma 2 and Eq. (17) of Sec VI. Equation (27), however, does not imply a trivial theory, again because the Maxwell equations do not hold as equations in D_0.

From Eq. (27), one has, in fact,

$$(\Psi_0, A_\mu(x)\partial^\rho F_{\rho\sigma}(y)\Psi_0) = \partial_\mu \partial_\sigma D(x-y) \neq 0.$$

This is not in contradiction to Eq. (23), as $A_\mu(f)\Psi_0$ is not, in general, a vector of D_0. One has a physical state if one takes the vector

$$A(f)\Psi_0 \equiv \int A_\mu(x) f^\mu(x)\Psi_0 d^4x,$$

where $f^\mu(x)$ is a test function satisfying

$$\partial_\mu f^\mu(x) = 0.$$

In this case, however, one has

$$(\Psi_0, A(f)\partial^\rho F_{\rho\sigma}(y)\Psi_0) = \int \partial_\sigma D(x-y)\partial_\mu f^\mu(x) d^4x = 0,$$

in agreement with Eqs. (23).