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Construction of Convergent Dual Loops

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Starting with a generalized representation for the multiparticle dual amplitudes, we present a construction of convergent dual loops. This is done at the expense of a simple condition to be satisfied by the universal functions which enter into our representation. The case of nondegenerate trajectories is discussed, and it is shown that the convergence of the loops *does not* depend on any condition on the Regge intercepts.

I. INTRODUCTION

A POSSIBLE procedure for the unitarization of dual resonance models has been to associate the multiparticle dual amplitudes[1] with the "tree graphs" or Born terms in a perturbation series. For this purpose, expressions for higher-order loops were constructed,[2] using as a guide the so-called "tree theorem," namely, that the residue of the loop amplitude at a pole corresponding to an internal line must coincide with the corresponding double factorized tree graph. Starting with the usual dual amplitudes, it was found that the loops diverge exponentially and hence do not exist. This was attributed to the large number of resonances in the particle spectrum involved in the dual amplitudes.

Recently a generalization of the multiparticle dual amplitudes[3,4] has been presented, in which the particle spectrum may be richer than before, but with the difference that the couplings of the various resonances is to a large extent arbitrary. The natural thing to ask then is whether it is possible, starting with these generalized amplitudes, to construct convergent loops—the hope of success being based on the idea that in spite of the richness of the spectrum, one may use the arbitrariness in the resonance couplings to suppress a great portion of it.

This question is discussed in this paper and answered in the affirmative. It is possible to construct convergent loops if a simple condition is satisfied by one of the universal functions that enter into the generalized representation of dual amplitudes with which we start.

We should point out here that recently Olesen[5] has discussed this problem and arrived at a convergent loop in the case of nondegenerate trajectories. This is done at the expense of a certain condition that the Regge intercepts must satisfy. We briefly touch on this case and show that it is possible to construct convergent loops *without* imposing any conditions on the intercepts. Thus Olesen's conditions are a reflection of his use of a particular form for the tree graphs and *not* duality as such.

In Sec. II we discuss the single and double factorization of our generalized tree graphs using a five-dimensional oscillator formalism. In Sec. III the construction of the dual loops is done following the method of Amati, Bouchiat, and Gervais[6] and the condition of convergence is obtained. Section IV deals with this condition and Sec. V contains a discussion of the case of nondegenerate trajectories. In Sec. VI we present a discussion and conclusions.

II. GENERALIZED TREE AMPLITUDE IN A FIVE-DIMENSIONAL OSCILLATOR FORMALISM

The essentially new input into the construction of convergent dual loops is a generalized expression[3,4] for the tree graph shown in Fig. 1. Following the notation of Ref. 4, this expression takes the following form:

$$B_{0,\ldots,N+2}{}^{(N+3)}=\int dx_1\cdots\int dx_N \prod_{i=1}^{N} x_i^{-\alpha_{0i}-1}$$
$$\times\prod_{i=1}^{N} f(x_i)^{-\alpha_{i,i+1}-1}R(x_i)$$
$$\times\prod_{N\geqslant j>i\geqslant 1} f(\rho_{ij})^{\alpha_{i+1,j+1}+\alpha_{ij}-\alpha_{i,j+1}-\alpha_{i+1,j}}R(\rho_{ij})\,. \quad (1)$$

The two universal functions $f(z)$ and $R(z)$ must satisfy some simple conditions and equations which are also discussed in Ref. 3. We only remark here that to a large extent they are arbitrary.

The choice $f(z)=1-z$ and $R(z)=1$ leads to the expressions for the tree graphs usually employed.

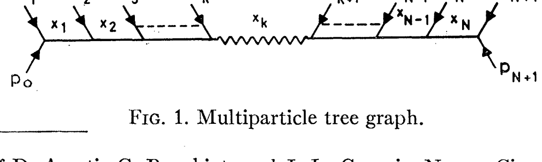

FIG. 1. Multiparticle tree graph.

[1] The five-point function was given by K. Bardakci and H. Ruegg, Phys. Letters **28B**, 342 (1968), and by M. A. Virasoro, Phys. Rev. Letters **22**, 37 (1969). The generalization to multiparticle amplitudes was given by H. M. Chan, Phys. Letters **28B**, 425 (1969); H. M. Chan and T. S. Tsun, *ibid.* **28B**, 485 (1969); C. Goebel and B. Sakita, Phys. Rev. Letters **22**, 257 (1969); K. Bardakci and H. Ruegg, Phys. Rev. **181**, 1884 (1969); Z. Koba and H. B. Nielsen, Nucl. Phys. **B10**, 633 (1969).

[2] K. Bardakci, M. B. Halpern, and J. Shapiro, Phys. Rev. **185**, 1910 (1969); K. Kikkawa, B. Sakita, and M. Virasoro, *ibid.* **184**, 1701 (1969).

[3] Khalil M. Bitar, Phys. Rev. **186**, 1424 (1969).

[4] Khalil M. Bitar, Phys. Rev. D **1**, 3319 (1970).

[5] P. Olesen, Nucl. Phys. **B18**, 473 (1970).

[6] D. Amati, C. Bouchiat, and J. L. Gervais, Nuovo Cimento Letters **2**, 399 (1969).

In this section we shall use a five-dimensional oscillator formalism to reexpress Eq. (1) as the vacuum expectation value of the factorized product of operator vertices and propagators. We are only extending here the usual four-dimensional oscillator formalism. The extra modes introduced reflect, as we shall see, the arbitrariness in our universal functions.

A. Single Factorization

One may write Eq. (1) as[7]

$$B_{0,\ldots,N+2}{}^{(N+3)}=I_{1,k-1}I_{k+1,N}\int dx_k x_k{}^{-\alpha_{0k}-1}$$
$$\times f(x_k)^{+2ap_k\cdot p_{k+1}+b-1}R(x_k)$$
$$\times \prod_{N\geqslant j\geqslant k\geqslant i\geqslant 1,j\neq i} f(\rho_{ij})^{+2ap_i\cdot p_{j+1}}R(\rho_{ij}), \quad (2)$$

where

$$I_{1,k-1}=\int dx_1\cdots\int dx_{k-1}\prod_{i=1}^{k-1}x_i{}^{-\alpha_{0i}-1}$$
$$\times\prod_{i=1}^{k-1}f(x_i)^{-\alpha_{i,i+1}-1}R(x_i)$$
$$\times\prod_{k-1\geqslant j>i\geqslant 1}f(\rho_{ij})^{+2ap_i\cdot p_{j+1}}R(\rho_{ij}) \quad (3)$$

and

$$I_{k+1,N}=\int dx_{k+1}\cdots\int dx_N\prod_{i=k+1}^{N}x_i{}^{-\alpha_{0i}-1}$$
$$\times\prod_{i=k+1}^{N}f(x_i)^{-\alpha_{i,i+1}-1}R(x_i)$$
$$\times\prod_{N\geqslant j>i\geqslant k+1}f(\rho_{ij})^{-2ap_i\cdot p_{j+1}}R(\rho_{ij}). \quad (4)$$

Define

$$F=f(x_k)^{-\alpha_{k,k+1}-1}\prod_{N\geqslant j\geqslant k\geqslant i\geqslant 1,j\neq i}f(\rho_{ij})^{2ap_i\cdot p_{j+1}}, \quad (5)$$

$$G=R(x_k)\prod_{N\geqslant j\geqslant k\geqslant i\geqslant 1,j\neq i}R(\rho_{ij}). \quad (6)$$

One then has

$$F=f(x_k)^{+b-1}\exp[2a\sum_{N\geqslant j\geqslant k\geqslant i\geqslant 1}p_i\cdot p_{j+1}\ln f(\rho_{ij})], \quad (7)$$

$$G=\exp[\sum_{N\geqslant j\geqslant k\geqslant i\geqslant 1}R(\rho_{ij})]. \quad (8)$$

In these last expressions when $j=i=k$, one has $\rho_{kk}\to x_k$.

We now define the four-vectors $P^{(n)}$ and $Q^{(n)}$ and the quantities $L^{(n)}$ and $K^{(n)}$ as follows:

$$P^{(n)}=p_k+\sum_{r=1}^{k-1}\rho_{r,k-1}{}^n p_r, \quad (9)$$

$$Q^{(n)}=p_{k+1}+\sum_{r=k+1}^{N}\rho_{k+1,r}{}^n p_{r+1}, \quad (10)$$

and

$$L^{(n)}=1+\sum_{r=1}^{k-1}\rho_{r,k-1}{}^n, \quad (11)$$

$$K^{(n)}=1+\sum_{r=k+1}^{N}\rho_{k+1,r}{}^n. \quad (12)$$

One also expands $\ln f(z)$ and $\ln R(z)$ about the point[8] $z=0$:

$$\ln f(z)=\sum_{n=1}^{\infty}c_n z^n, \quad (13)$$

$$\ln R(z)=\sum_{n=1}^{\infty}d_n z^n. \quad (14)$$

Combining Eqs. (7)–(14), one is led to the following expressions for F and G of Eqs. (5) and (6):

$$F=f(x_k)^b\exp[\sum_n c_n x_k{}^n(2aP^{(n)}\cdot Q^{(n)})], \quad (15)$$

$$G=\exp[\sum_n d_n x_k{}^{(n)}(L^{(n)}\cdot K^{(n)})]. \quad (16)$$

Thus the tree amplitude takes the form

$$B_{0,\ldots,N+2}{}^{(N+3)}=I_{1,k-1}I_{k+1,N}\int dx_k x_k{}^{-\alpha_{0k}-1}f(x_k)^{b-1}$$
$$\times\exp\{\sum_n x_k{}^n[2ac_nP^{(n)}\cdot Q^{(n)}+d_nL^{(n)}\cdot K^{(n)}]\}. \quad (17)$$

Introducing the five-vectors, $\alpha=1,\ldots,5$,

$$J_\alpha{}^{(n)}=((2a)^{1/2}P^{(n)}{}_\mu,(d_n/c_n)^{1/2}L^{(n)}), \quad (18)$$

$$H_\alpha{}^{(n)}=((2a)^{1/2}Q_\mu{}^{(n)},(d_n/c_n)^{1/2}K^{(n)}), \quad (19)$$

the exponential in Eq. (17) takes the form

$$\exp\sum_n(c_n)x_k{}^nJ_\alpha{}^{(n)}\cdot H^{(n)\alpha}. \quad (20)$$

The metric is such that $g_{\alpha\beta}=g_{\mu\nu}$ for $\alpha,\beta=1,\ldots,4$; $g_{55}=1$; and all other elements are zero.

Introduce now the oscillators

$$a_\alpha{}^{(n)},a_\alpha{}^{(n)\dagger},\quad \alpha=1,\ldots,5$$

such that

$$[a_\alpha{}^{(n)},a_\beta{}^{(m)\dagger}]=\delta_{nm}g_{\alpha\beta} \quad (21)$$

[7] Our metric is (1, 1, 1, −1). $\alpha_{ij}=-a(p_i+\cdots+p_j)^2+b$.

[8] $f(z)\sim 1-z$ and $R(z)\sim 1$ and $z\to 0$. Thus, the expansions start with $n=1$.

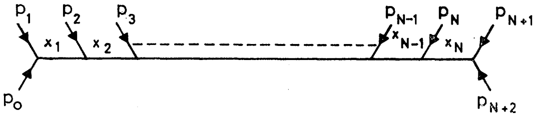

FIG. 2. Single factorization of the multiparticle tree graph.

and the complete set of states

$$|\lambda,\alpha\rangle=|\lambda_1,\lambda_2,\ldots,\lambda_i\cdots\rangle$$
$$=\prod_{i=1}^{\infty}\frac{[a^{(i)\dagger}]^{\lambda_i}}{\sqrt{(\lambda_i!)}}\bigg|0\bigg\rangle. \quad (22)$$

It is straightforward then, using techniques similar to those of Fubini, Gordon, and Veneziano[9] and others,[5,6] to show that each term in the exponential of Eq. (20) may be written as

$$\exp(c_n x_k{}^n J^{(n)}\cdot H^{(n)})=\langle 0|\exp[J^{(n)}\cdot a^{(n)}\sqrt{c_n}]$$
$$\times(x_k{}^n)^{a^{(n)\dagger}\cdot a^{(n)}}\exp[H^{(n)}\cdot a^{(n)\dagger}\sqrt{c_n}]|0\rangle. \quad (23)$$

Thus Eq. (17) becomes

$$B_{0,\ldots,N+2}{}^{(N+3)}=I_{1,k-1}I_{k+1,N}\langle 0|\exp(\sum_n c_n{}^{1/2}J^{(n)}\cdot a^{(n)})$$
$$\times\int dx_k x_k{}^{-\alpha_{0k}+\Sigma_n n a^{(n)\dagger}a^{(n)}-1}$$
$$\times f(x_k)^{b-1}\exp(\sum_n c_n{}^{1/2}H^{(n)}\cdot a^{(n)\dagger})|0\rangle, \quad (24)$$

where all indices α have been suppressed.

The factorization of the amplitude is then clearly exhibited as the momentum dependence is explicitly divided into two separate contributions, one to the left and the other to the right of x_k in Fig. 2. The appearance of a fifth oscillator mode here leads, as pointed out earlier,[4] to a spectrum possibly richer than that with the usual amplitudes.

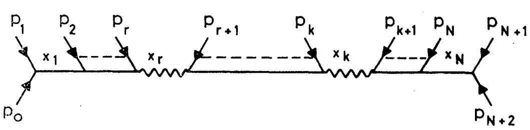

FIG. 3. Double factorization of the multiparticle tree graph.

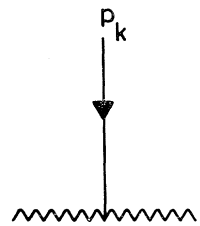

FIG. 4. Simple vertex.

B. Double Factorization

We now study the double factorization of the tree graph at x_r and x_k as shown schematically in Fig. 3. This is the first step towards the N-fold factorization and leads us to an expression for the vertex of Fig. 4 (for $r=k-1$) needed in the construction of the simple loop.

We consider expression (17) and define the following five-vectors:

$$U_\alpha{}^{(n)}=[(2a)^{1/2}(p_r+\sum_{i=1}^{r-1}\rho_{i,r-1}{}^n p_i)_\mu,$$
$$(d_n/c_n)^{1/2}(1+\sum_{i=1}^{r-1}\rho_{i,r-1}{}^n)], \quad (25)$$

$$V_\alpha{}^{(n)}=[(2a)^{1/2}(p_k+\sum_{i=r+1}^{k-1}\rho_{i,k-1}{}^n p_i)_\mu,$$
$$(d_n/c_n)^{1/2}(1+\sum_{i=r+1}^{k-1}\rho_{i,k-1}{}^n)]. \quad (26)$$

One then has

$$J_\alpha{}^{(n)}=V_\alpha{}^{(n)}+z^n x_r{}^n U_\alpha{}^{(n)}, \quad (27)$$

where

$$z=\prod_{i=r+1}^{k-1}x_i=\rho_{r+1,k-1}. \quad (28)$$

With these definitions, one may write the tree amplitude as follows:

$$B_{0,\ldots,N+2}{}^{(N+3)}=I_{1,r-1}\int dx_r x_r{}^{-\alpha_{0r}-1}f(x_r)^{b-1}\exp(\sum_n c_n x_r{}^n U^{(n)}\cdot V^{(n)})I_{r+1,k-1}$$
$$\times\int dx_k x_k{}^{-\alpha_{0k}-1}f(x_k)^{b-1}\exp[\sum_n c_n(x_k{}^n V^{(n)}\cdot H^{(n)}+x_k{}^n z^n x_r{}^n U^{(n)}\cdot H^{(n)})]I_{k+1,N}$$
$$=I_{1,r-1}I_{r+1,k-1}I_{k+1,N}\int dx_r x_r{}^{-\alpha_{0r}-1}f(x_r)^{b-1}\int dx_k x_k{}^{-\alpha_{0k}-1}f(x_k)^{b-1}$$
$$\times\exp[\sum_n c_n(x_k{}^n V^{(n)}\cdot H^{(n)}+x_r{}^n U^{(n)}\cdot V^{(n)}+x_k{}^n z^n x_r{}^n U^{(n)}\cdot H^{(n)})]. \quad (29)$$

[9] S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters 29B, 679 (1969).

Introducing again the five-dimensional oscillators of Sec. II A, we may rewrite Eq. (29) as follows:

$$\begin{aligned} B_{0,\ldots,N+2}{}^{(N+3)} = \langle 0|\, & \mathcal{G}_{1,r-1}(p_0,\ldots,p_r,U_\alpha{}^{(n)},a_\alpha)\mathcal{P}(H,\alpha_{0r}) \\ & \times\Gamma(p_{r+1},\ldots,p_k,V_\alpha{}^{(n)},a_\alpha{}^{(n)\dagger}a_\alpha{}^{(n)})\mathcal{P}(H,\alpha_{0k}) \\ & \times\mathcal{G}_{k+1,N}(p_{k+1},\ldots,p_{N+3},H_\alpha{}^{(n)},a_\alpha{}^{(n)\dagger})|0\rangle , \end{aligned} \quad (30)$$

where

$$\mathcal{G}_{1,r-1}(p_0,\ldots,p_r,U_\alpha{}^{(n)},a_\alpha) = I_{1,r-1}\exp[\sum_n c_n{}^{1/2}K^{(n)}\cdot a^{(n)}] \quad (31)$$

and

$$\begin{aligned} \Gamma(p_{r+1},\ldots,p_k,V_\alpha{}^{(n)},a_\alpha{}^{(n)\dagger},a_\alpha) &= I_{r+1,k-1}\exp(\sum_n c_n{}^{1/2}V^{(n)}\cdot a^{(n)\dagger})z^H \\ &\quad\times\exp(\sum_n c_n{}^{1/2}V^{(n)}\cdot a^{(n)}) , \end{aligned} \quad (32a)$$

$$\mathcal{P}(H,\alpha_{0r}) = \int_0^1 dx_r x_r{}^{-\alpha_{0r}+H-1}f(x_r)^{b-1} , \quad (32b)$$

$$H = \sum_n n a^{(n)\dagger}\cdot a^{(n)} . \quad (32c)$$

The vertex of Fig. 4 is then easily obtained by further factorization or, more directly, by looking at Γ of (32a) for $r=k-1$. Thus the vertex operator in that case becomes

$$\Gamma(p_k,a_\alpha{}^{(n)\dagger},a_\alpha) = \exp[\sum_n c_n{}^{1/2}V_k{}^{(n)}\cdot a^{(n)\dagger}] \times\exp[\sum_n c_n{}^{1/2}V_k{}^{(n)}\cdot a^{(n)}] . \quad (33)$$

This is true because now $z=1$, so that $I_{r+1,k-1}$ reduces to 1 (no integration), and the five-vector $V_k{}^{(n)}$ becomes simply [see Eq. (26)]

$$V_k{}^{(n)} = ((2a)^{1/2}p_k,(d_n/c_n)^{1/2}) . \quad (34)$$

The main ingredients then entering into the construction of the simple loop are expressions (33) and (34) for the vertex and expression (32b) for the propagator.

We point out here that if $R(z)=1$ [and hence $f(z)=1-z$], then all the d_n's will be zero, and the five-vector of (34) will be the four-vector $(2a)^{1/2}p_k$, and Eq. (33) will be the vertex usually[2] used to construct dual loops. In Olesen's model the fifth component of the vector is related to the unequal intercepts of the trajectories. If in our case we use unequal intercepts, the effect would be to modify the fifth component by adding to it the one used by Olesen. We shall say more about this later.

III. CONSTRUCTION OF CONVERGENT SIMPLE LOOPS

Starting with expressions (33) and (32b) for the vertex and propagator, respectively, one may write an expression for the simple loop of Fig. 5 following the method of Amati, Bouchat, and Gervais.[6] If $p_1,p_2,\ldots,p_N$ denote the incoming external momenta and $k_1,\ldots,k_{N-1}$ the internal independent momenta, then

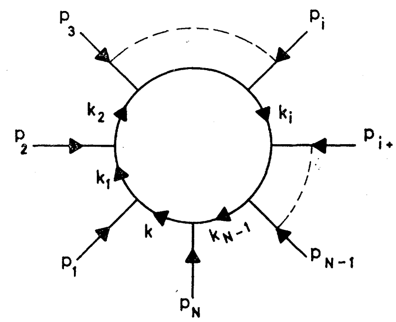

FIG. 5. Multiparticle loop diagram.

$$F^N(p_1,p_2,\ldots,p_N) = \int d^4k\, M(k,k_1,\ldots,k_{N-1}) . \quad (35)$$

In Eq. (35) we have

$$\begin{aligned} M = \int dx_1\cdots\int dx_N (\prod_{i=1}^N f(x_i)^{b-1}) & \times\prod_{n=1} \mathrm{Tr}[x_1{}^{h(n)}\Gamma^{(n)}(p_1)\cdots x_N{}^{h(n)}\Gamma^{(n)}(p_N)] \\ & \times\prod_{i=1}^N x_i{}^{-\alpha(k_{i-1}{}^2)-1} , \end{aligned} \quad (36)$$

where

$$\begin{aligned} \Gamma^{(n)} &= \exp[c_n{}^{1/2}V^{(n)}\cdot a^{(n)\dagger}]\exp[\sum_n c_n{}^{1/2}V^{(n)}\cdot a^{(n)}] , \\ h^{(n)} &= n a^{(n)\dagger}\cdot a^{(n)} , \end{aligned} \quad (37)$$

and $V_\alpha{}^{(n)}$ is given by Eq. (34). The trace in Eq. (36) has been calculated in Ref. 6 and using that result we find, with $w=\prod_{i=1}{}^N x_i$,

$$\mathrm{Tr}[\;] = \prod_{n=1}^{\infty}\left\{\left(\frac{1}{1-w^n}\right)^5 \times\exp\left[-\sum_{i,j=1}^N \frac{c_n V_i{}^{(n)}V_j{}^{(n)}c_{ij}}{(1-w^n)}\right]\right\} , \quad (38)$$

where

$$\begin{aligned} c_{ij} &= (x_1\cdots x_i)^n && \text{if } j=N \\ &= (x_1\cdots x_i)^n(x_{j+1}\cdots x_N)^n && \text{if } i<j<N-1 \\ &= (x_{j+1}\cdots x_i)^n && \text{if } i>j . \end{aligned} \quad (39)$$

Note that as $x_i\to 1$, $c_{ij}\to 1$ and that $V_i{}^{(n)}\cdot V_j{}^{(n)}$ is the scalar product of five-vectors. Thus the loop amplitude becomes

$$F^{(n)}(p_1,p_2,\ldots,p_N)$$
$$=\int d^4k\int_0^1 dx_1\cdots\int_0^1 dx_N\Big(\prod_{i=1}^N f(x_i)^{b-1}\Big)$$
$$\times\prod_{i=1}^N x_i^{-\alpha(k_{i-1}^2)-1}\prod_{n=1}^\infty\left\{\left(\frac{1}{1-w^n}\right)^{-5}\right.$$
$$\left.\times\exp\left[-\frac{1}{1-w^n}\sum_{i,j=1}^N c_nV_i^{(n)}V_j^{(n)}c_{ij}\right]\right\}. \quad (40)$$

The usual difficulty arises when in the integration over x_i the integration region where $w\to 1$ is approached (all $x_i\to 1$). In the usual case, all $V_i^{(n)}$ are simply the four-vectors p_i and $c_{ij}\to 1$. In this case the integrand becomes

$$\prod_{n=1}^\infty\left(\frac{1}{1-w^n}\right)\sim\exp\left(\frac{4\pi^2}{6}\frac{1}{1-w}\right), \quad (41)$$

which is exponentially divergent.

In our case the situation is different. The $V_i^{(n)}$ are five-vectors as in Eq. (34), and as $w\to 1$, the integrand approaches

$$\prod_{n=1}^\infty\left[\left(\frac{1}{1-w^n}\right)^{-5}\exp\left(\frac{-N^2d_n}{(1-w^n)}\right)\right]$$
$$=\exp\left(\frac{5\pi^2}{6}\frac{1}{1-w}\right)\exp\left(-N^2\sum_n\frac{d_n}{(1-w^n)}\right). \quad (42)$$

Although the first term is divergent, the possibility still exists that by a proper choice of the constants d_n, i.e., a proper choice of the function $R(z)$, one may damp this divergence by the exponential multiplying it.

Now for $w\sim 1$ we have

$$\exp\left(-N^2\sum_{n=1}^\infty\frac{d_n}{1-w^n}\right)=\exp\left(\frac{-N^2}{1-w}\sum_{n=1}^\infty\frac{d_n}{n}\right). \quad (43)$$

If we put

$$\xi=\sum_{n=1}^\infty\frac{d_n}{n}, \quad (44)$$

then the integrand takes the form

$$\exp[(\tfrac{5}{6}\pi^2-\xi N^2)(1-w)]. \quad (45)$$

This is convergent as $w\to 1$ (from below) if

$$(\tfrac{5}{6}\pi^2-\xi N^2)<0, \quad (46)$$

i.e.,

$$\xi>5\pi^2/6N^2. \quad (47)$$

This condition is the basic condition to be satisfied by our universal function $R(z)$ so that the simple N-particle loop is convergent. Since this function is to a great extent arbitrary, it is not a very restrictive condition. In the following section, we shall discuss the possible choices of $R(z)$ for which Eq. (47) is satisfied and consequently for which the loop amplitude of Eq. (40) is well defined and convergent.

Notice that the simplest loop has $N=2$. If condition (47) is satisfied for $N=2$ it will be satisfied as well for all other values of N.

IV. FUNCTION $R(z)$

We recall from Eq. (14) that

$$\ln R(z)=\sum_{n=1}^\infty d_nz^n.$$

This is possible if $R(0)=1$, which we take it to be.[7] Using this series expansion, one may easily write

$$\sum_{n=1}^\infty\frac{d_n}{n}=\int_0^1 dz\frac{\ln R(z)}{z}. \quad (48)$$

Thus the condition of Eq. (47) now becomes

$$\int_0^1 dz\frac{\ln R(z)}{z}>\frac{5\pi^2}{6N^2}. \quad (49)$$

This may be easily satisfied by taking $R(z)$ large enough between zero and 1. Since the convergence of the loop does not depend critically on the choice of $f(z)$ (although of course its value does), we may, for the purpose of simplicity, then take $f(z)=1-z$, which is the usual choice. In this case, $R(z)$ is such that

$$R(1-z)=R(z), \quad (50)$$

which is the function that is symmetric under reflection about the line $z=\frac{1}{2}$. Our tree graph then is the simple sum of a leading term plus satellite terms. It must be quite evident that there are many choices of $R(z)$ for which Eq. (49) is satisfied. It also must be clear why the usual tree graph also leads to divergent loops, for in that case $R(z)=1$ and condition (49) is evidently not satisfied.

For more general choices of $f(z)$, we do know[3] that there are infinitely many choices for $R(z)$. Therefore, again satisfying Eq. (49) should not prove to be difficult.

V. NONDEGENERATE TRAJECTORIES

The above discussion has been carried out for degenerate trajectories in all channels. Recently Olesen[5] discussed the problem of constructing dual loops for trajectories with nondegenerate intercepts. He finds that the condition of convergence leads to some conditions on these intercepts. We attribute these conditions to the particular choice of tree graph, namely, the usual one he uses in constructing the loops.

If we start with nondegenerate trajectories, in our case the effect is the modification of the five-vectors of Eq. (34) to the form

$$V_k^{(n)} = ((2a)^{1/2}p_k; (d_n/c_n)^{1/2} + (2c'')^{1/2}b_k''), \quad (51)$$

where c'' and b_k'' are defined by Olesen. This then leads to the same loop amplitude as in Eq. (40) with the modified five-vector of Eq. (51). The convergence condition then reads

$$\left\{\tfrac{5}{6}\pi^2 - N^2 \sum_{n=1}^{\infty} \frac{d_n}{n} - 2N[(2c'')^{1/2} \sum_{i=1} b_i''] \times \sum_{n=1}^{\infty} (d_n c_n)^{1/2} - \sum_{n=1}^{\infty} \frac{c_n}{n} (2c'' \sum_{i=1}^{N} b_i'')^2 \right\} < 0. \quad (52)$$

For $f(z) = 1 - z$ and $R(z) = 1$, all $d_n = 0$ and $c_n = -1/n$, and Olesen's condition is obtained.

It is clear in this case that the convergence condition is more complicated than before. It involves the intercepts b_i'' and the constant c'', *as well as* the coefficients d_n and c_n. Thus in this case both universal functions $f(z)$ and $R(z)$ are involved.

It should be clear that Eq. (52) is not a condition on the intercepts in this more general formulation. One may choose b_i'' and c'' at will and *then* make a proper choice of $R(z)$ and/or $f(z)$ to satisfy Eq. (52). As an example we choose again $f(z) = 1 - z$. In this case $c_n = -1/n$. The third term on the left is pure imaginary (we take, say, all d_n's positive) and may be discarded. The condition becomes

$$\tfrac{5}{6}\pi^2 - N^2 \sum_{n=1}^{\infty} \frac{d_n}{n} + \frac{2c''\pi^2}{6} (\sum_i b_i'')^2 < 0, \quad (53)$$

which is Olesen's condition modified by

$$-N^2 \sum_{n=1}^{\infty} \frac{d_n}{n}.$$

It is clear that this may be satisfied for any choice of b_i'' by simply making $\sum (d_n/n)$ or the integral of Eq. (48) as positive as necessary.

From the above discussion, we see that in our more general formalism one may write convergent dual loop amplitudes when the trajectories are nondegenerate, and that this leads to *no* restrictions on these intercepts, but only makes considerably more complicated the conditions that both our universal functions $f(z)$ and $R(z)$ must now satisfy [Eq. (52)].

VI. DISCUSSION AND CONCLUSIONS

The main result of this discussion has been the construction of convergent dual loop amplitudes as given in Eq. (40). This amplitude depends on the two universal functions $f(z)$ and $R(z)$ which enter into the expression for the generalized dual tree graphs with which we start. In particular, the convergence of the loop leads to a condition on the function $R(z)$ for degenerate trajectories [Eq. (49)] and $f(z)$ and $R(z)$ in the nondegenerate case [Eq. (52)]. These conditions do not restrict our universal functions in a strong way but point out the possibility of further restrictions as one constructs more complicated higher-order graphs. This is in line with the view expressed before,[4] that the construction of these higher-order graphs may lead not only to a unitary amplitude but to a unique one as well. At this stage, however, we are far from uniqueness. There is an infinity of loops, each determined by a choice of an $f(z)$ and an $R(z)$.

The expression for the tree graphs with which we start leads to a possibly richer resonance spectrum than the expressions usually employed. Nevertheless, this did not interfere with our construction, for the relative couplings of the various resonances were arbitrarily determined by our universal functions, and hence convergence could be achieved by a proper choice of such functions. In the special case of $f(z) = 1 - z$ and $R(1-z) = R(z)$, which corresponds to a simple sum over leading and satellite terms, the condition on $R(z)$ is simply a condition on the relative strengths of these terms. We point out here that the compact form we use for such a sum [i.e., via $R(z)$] made it much more apparent that the construction of convergent loops is possible.

APPENDIX

In this Appendix, we make further comments about the generalized amplitudes we use for the tree graphs. This may be also considered as an erratum for Ref. 4. It was claimed there that the generalized amplitude is cyclic under permutation of the external momenta. This is not generally true. The discussion of Ref. 4 amounts to saying that there is a case where this is so. We show here that it is the usual case. We demonstrate this for the line point amplitude.

Consider Eq. (24) of Ref. 4, namely,

$$f\left(\frac{f(x_2)}{f(x_1 x_2)}\right) = x_2 \frac{f(x_1)}{f(x_1 x_2)}. \quad (A1)$$

This implies that

$$f(1) = 0 \quad (x_2 = 0),$$
$$f(0) = 1 \quad (x_2 = 1),$$

as is required.

Putting $x_1 = 0$, we have

$$f(f(x_2)) = x_2, \quad (A2)$$

which is again equivalent to Eq. (23) of Ref. 4. This generality, however, is spoiled for arbitrary values of x_1 as Eq. (24) of Ref. 4 becomes very restrictive and leads essentially to the unique solution $f(x_2) = 1 - x_2$.

We now demonstrate this. We take the derivative of Eq. (A1) with respect to x_1. One has then

$$-f'\left(\frac{f(x_2)}{f(x_1x_2)}\right)x_2\frac{f(x_2)}{f^2(x_1x_2)}f'(x_1x_2)=x_2\frac{f'(x_1)}{f(x_1x_2)}-\frac{x_2{}^2f(x_1)}{f^2(x_1x_2)}f'(x_1x_2). \quad (A3)$$

Putting $x_1=0$ and using $f(0)=1$, we get

$$f'(f(x_2))f(x_2)=-1+x_2. \quad (A4)$$

Differentiating Eq. (A2) leads also to

$$f'(f(x_2))f'(x_2)=1. \quad (A5)$$

Therefore one has, by combining Eq. (A5) and (A4),

$$f'(x_2)/f(x_2)=-1/(1-x_2). \quad (A6)$$

In other words,

$$\frac{d}{dx}\ln f(x_2)=\frac{-1}{1-x_2} \quad (A7)$$

and

$$\ln f(x_2)=\ln(1-x_2), \quad (A8)$$

or, in general,

$$f(x_2)=1-x_2. \quad (A9)$$

This is the usual choice for $f(z)$. If we add one to all α_{ij} in Ref. 4, then the only choice for $R(z)$ may be shown to be a constant. The condition on $f(z)$ is basic, because $f(z)$ carries all the dependence on the kinematic invariants S_{nm}. The condition on $R(z)$ is a consequence of the product of R-functions that we have chosen. This choice, however, maintains factorization, as is evident in the discussion in the present text and Sec. V of Ref. 4.

Our amplitudes are, of course, still dual in the sense that they may be written as an infinite sum over narrow-width resonances in any channel and have the characteristic Regge behavior.

One, of course, may construct amplitudes invariant under cyclic permutations by taking the sum

$$B_{0,\ldots,N+3}{}^{(N+3)}+B_{N+3,0,\ldots,N+2}{}^{(N+3)}+\cdots+B_{1,\ldots,N+3,0}{}^{(N+3)}.$$

We point out here that our amplitudes have, of course, the important property of symmetry under reflection, namely,

$$B_{0,\ldots,N+3}{}^{(N+3)}=B_{N+3,\ldots,0}{}^{(N+3)}.$$

Thus, the only disadvantage of our amplitudes relative to the usual ones is that upon writing an amplitude for the N-particle process, one has to add not only terms for all possible permutations of the external legs but all the cyclic variations as well. Their advantage is, of course, that they have arbitrary residues and, as is evident in the text, lead to convergent loops.

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Covariant Time-Ordered Products of N Nonconserved Currents*

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The covariant time-ordered products of arbitrary number of nonconserved currents are shown to exist. We formulate the rules for constructing such products, and show explicitly that they are covariant under Lorentz transformations.

I. INTRODUCTION

RECENTLY the existence of covariant time-ordered products has been an interesting subject for investigation. The construction of covariant time-ordered products has been studied by Brown and others[1–3] within the framework of canonical field theory. Dashen and Lee[4] have demonstrated that covariant time-ordered products of conserved currents exist and can be constructed by algebraic methods. In their approach, one assumes only the following: (1) The equal-time commutators of two time components of the currents are the usual ones, and the equal-time commutators of their time and space components contain terms which are no more singular than the first derivatives of a δ function. (2) The Schwinger terms are assumed to be well-defined operators so that Jacobi identities for currents are satisfied. Possible c numbers can be taken care of by subtracting the vacuum expectation value.

In this paper, we generalize the work of Dashen and Lee. We find the following: (1) In addition to the above

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[1] D. G. Boulware and L. S. Brown, Phys. Rev. **156**, 1724 (1967).
[2] S. G. Brown, Phys. Rev. **158**, 1444 (1967).
[3] S. L. Adler and R. F. Dashen, *Current Algebra* (Benjamin, New York, 1968).
[4] R. F. Dashen and S. Y. Lee, Phys. Rev. **187**, 2017 (1969).