PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

High-Energy Behavior near Threshold: Potential Scattering*

HUNG CHENG†
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

TAI TSUN WU
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138

(Received 20 April 1970)

In this paper we study the promotion phenomenon in potential scattering. In Sec. 3 we give the general formulas for the scattering amplitude at threshold, and show that the asymptotic form of the scattering amplitude as the momentum transfer squared approaches infinity is determined not only by the right-hand-plane Regge poles [$\alpha(0)>-\frac{1}{2}$], but also by the left-hand-plane zeros of $Y(\lambda)$. In Sec. 3 we study the leading Regge pole $\alpha(k_t^2)$ over the whole region $-\infty<k_t^2<0$ for a weak and attractive Yukawa potential, and find that it is indeed promoted from -1 to $-\frac{1}{2}$ within a small neighborhood of the threshold. We also obtain the leading zero of $Y(\lambda)$ if the potential is repulsive. Together with the results in Sec. 2, we have, therefore, the explicit asymptotic form of the scattering amplitude. In Sec. 4 we study a general nth Born term and find that its leading term is promoted from $s^{-1}(\ln s)^{n-1}$ to $s^{-1/2}(\ln s)^{n-2}$ at threshold. Summing over these leading terms, we obtain results in complete agreement with those in Secs. 2 and 3, both for attractive and for repulsive potentials, off or at threshold.

1. INTRODUCTION

IN the two preceding papers,[1,2] we made a field-theoretic study of the scattering amplitude in the high-energy limit $s\rightarrow\infty$, with t near the two-body threshold value. The cases examined included quantum electrodynamics, scalar electrodynamics, and φ^3 theory. In all of the cases considered, we found that the power of s for the scattering amplitude of a tower diagram or a ladder diagram is promoted when t is near the threshold. Specifically, this power is promoted from 1 to $\frac{3}{2}$ for tower diagrams in both quantum electrodynamics and scalar electrodynamics, and from -1 to $-\frac{1}{2}$ for ladder diagrams in φ^3 theory. In this paper, we shall study the promotion phenomenon in potential theory.

In contrast to the various field theories considered, the potential-scattering case can be treated in a rigorous way. We recall that when we studied the promotion phenomenon in a field theory, we had to rely on the method of perturbation. We extracted the leading term from each perturbation order, and summed over all orders. This process of summing leading terms is, however, without justification, and on occasion is known to lead to an erroneous answer. It is therefore interesting to apply it to the Born series, and see if the results agree with those obtained by the rigorous method. As it turns out, there is complete agreement between the results obtained by the two methods.

We shall adopt the units $2M=\hbar=1$ in the Schrödinger equation, and denote the momentum as k_t. The threshold is therefore at $k_t=0$. To be consistent with the notation in the preceding papers, we shall put

$$s=-2k_t^2(1-\cos\theta), \qquad (1.1)$$

where θ is the scattering angle. Note that this designation of s is exactly opposite to the standard designation in potential scattering. The potential $V(r)$ shall be taken to be a superposition of Yukawa potentials. We are interested in the scattering amplitude in the limit $s\rightarrow\infty$, with k_t fixed in one of the following regions: (1) k_t^2 is nonzero (this case is of course standard and shall be mentioned only briefly); (2) $k_t^2=0$; and (3) k_t^2 is very small.

In Sec. 2, we shall give the general asymptotic forms of the scattering amplitude in the three regions listed above. We shall find, to our surprise, that at the threshold $k_t=0$ this asymptotic form is not entirely determined by the Regge poles. A discussion of the Regge-pole behavior near the threshold is also given in this section. In Sec. 3 we shall study the leading Regge pole in the weak-coupling limit for *all* values of k_t^2. This study confirms that it is the leading Regge pole, located near $\alpha=-1$ if k_t^2 is not too small, which moves to the right of $\alpha=-\frac{1}{2}$ at $k_t=0$ if the potential is attractive. In Sec. 4 we shall extract the leading terms in the Born series and sum them up, and show that they agree with the results of Secs. 2 and 3.

2. GENERAL FORMS OF HIGH-ENERGY AMPLITUDE

Let us denote the scattering amplitude by $f(k_t^2,s)$. The Regge representation for $f(k_t^2,s)$ is

$$f(k_t^2,s)=\sum_{\mathrm{Re}\alpha_n(k_t^2)>-\frac{1}{2}+\epsilon}\beta_n(k_t^2)\times P_{\alpha_n(k_t^2)}(-1-\tfrac{1}{2}sk_t^{-2})/\sin\pi\alpha_n(k_t^2)-\frac{1}{2i}\int_{-\frac{1}{2}+\epsilon-i\infty}^{-\frac{1}{2}+\epsilon+i\infty}dl(2l+1)\times P_l(-1-\tfrac{1}{2}sk_t^{-2})A(l,k_t^2)/\sin\pi l, \qquad (2.1)$$

* Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(30-1)-4101.

† Work supported in part by the National Science Foundation under Grant No. GP-13775.

[1] H. Cheng and T. T. Wu, second preceding paper, Phys. Rev. D 2, 2276 (1970).

[2] H. Cheng and T. T. Wu, preceding paper, Phys. Rev. D 2, 2285 (1970).

valid for $k_t^2>0$ and for all s. In (2.1), ϵ is a positive number and $A(l,k_t^2)$ is the analytically continued partial-wave amplitude. We shall keep $\epsilon>0$ for reasons that will become obvious. Also, α_n is a Regge pole of $A(l,k_t^2)$ and

$$\beta_n=-\pi(2\alpha_n+1)\,\mathrm{Res}A(l,k_t^2)]_{l=\alpha_n}. \qquad (2.2)$$

For the convenience of later comparisons, we shall give the well-known asymptotic form of $f(k_t^2,s)$ in the limit $s\to\infty$ with k_t^2 fixed at a nonzero positive value. In this limit, we have

$$P_l(-1-\tfrac{1}{2}sk_t^{-2})\sim\frac{\Gamma(l+\frac{1}{2})}{(\sqrt{\pi})\Gamma(l+1)}\left(\frac{e^{-i\pi}s}{k_t^2}\right)^l. \qquad (2.3)$$

Thus we get

$$f(k_t^2,s)\sim\sum_n\frac{\beta_n(k_t^2)\Gamma(\alpha_n(k_t^2)+\frac{1}{2})}{(\sqrt{\pi})\Gamma(\alpha_n(k_t^2)+1)\sin\pi\alpha_n(k_t^2)}\times\left(\frac{e^{-i\pi}s}{k_t^2}\right)^{\alpha_n(k_t^2)}. \qquad (2.4)$$

Although (2.4) is obtained by restricting $k_t^2>0$, it can be extended[3] to all complex values of k_t^2. Also, the summation over n in (2.4) can include all Regge poles,[4] not only those in the right half-plane as indicated by (2.1).

Next we shall consider $f(k_t^2,s)$ as $k_t^2\to 0$. Before we do this, some properties of the functions $A(l,k_t^2)$, $\alpha_n(k_t^2)$, and $\beta_n(k_t^2)$ necessary for later discussions will be listed.

A. Some Threshold Formulas

The S matrix can be written as[5]

$$S(\lambda,k_t)=\frac{Y(\lambda,k_t)+k_t^{2\lambda}e^{i\pi\lambda}}{Y(\lambda,k_t)+k_t^{2\lambda}e^{-i\pi\lambda}}, \qquad (2.5)$$

where $\lambda=l+\frac{1}{2}$, and where $Y(\lambda,k_t)$ is a meromorphic function of λ and an analytic function of k_t regular at the threshold $k_t=0$, i.e., $Y(\lambda)\equiv Y(\lambda,0)$ is a meromorphic function of λ. From now on, the variable λ will be used instead of l when convenient. We shall also use the notation λ_n defined by

$$\lambda_n=\alpha_n+\tfrac{1}{2},$$

where α_n is the location of a Regge pole in the l plane. Equation (2.5) can also be written as

$$A(l,k_t^2)=[S(\lambda,k_t)-1]/2ik_t=\frac{k_t^{2l}\sin\pi\lambda}{Y(\lambda,k_t)+k_t^{2\lambda}e^{-i\pi\lambda}}. \qquad (2.6)$$

Thus we have

$$\lim_{k_t\to 0}A(l,k_t^2)k_t^{-2l}=(\sin\pi\lambda)/Y(\lambda),\quad \mathrm{Re}\lambda>0. \qquad (2.7)$$

When $\lambda=n$, an integer, the denominator and the numerator of the right-hand side of (2.5) become equal. Since $S(n,k_t)$ cannot be equal to 1 for all k_t, we must have[6]

$$Y(n,k_t)=(-1)^{n-1}k_t^{2n},\quad n=0,\pm1,\pm2,\ldots \qquad (2.8)$$

and, in particular,

$$Y(0,k_t)=-1. \qquad (2.9)$$

By taking the limit $k_t\to 0$ in (2.8), we find that

$$Y(n)=0,\quad n>0$$
$$=\infty,\quad n<0. \qquad (2.10)$$

Since $Y(\lambda)$ is a meromorphic function of λ, (2.10) means that $Y(\lambda)$ has zeros at $\lambda=1,2,\ldots,n,\ldots$, and poles at $\lambda=-1,-2,\ldots,-n,\ldots$.

From (2.6), we see that a Regge pole $\lambda(k_t^2)=\alpha(k_t^2)+\frac{1}{2}$ is implicitly given by

$$Y(\lambda(k_t^2),k_t)+(k_t^2e^{-i\pi})^{\lambda(k_t^2)}=0. \qquad (2.11)$$

By taking the limit $k_t\to 0$ in (2.11), we find that if $\lambda(0)$ is a Regge pole at threshold, we must have

$$Y(\lambda(0))=0,\quad \mathrm{Re}\lambda(0)>0$$
$$=\infty,\quad \mathrm{Re}\lambda(0)<0. \qquad (2.12)$$

Thus the Regge poles $\lambda(0)>0$ are located at the zeros of $Y(\lambda)$ and the Regge poles $\lambda(0)<0$ are located at the poles of $Y(\lambda)$. The converse is not, however, necessarily true. For example, the points $\lambda=n$ $(-n)$ are always zeros (poles) of $Y(\lambda)$.

From (2.2) and (2.7) ,we have

$$\lim_{k_t\to 0}\beta_n(k_t^2)k_t^{-2\alpha_n(0)}=-2\pi\lambda_n(0)\sin[\pi\lambda_n(0)]/Y'(\lambda_n(0)),\quad \mathrm{Re}\lambda_n(0)>0. \qquad (2.13)$$

There are also infinitely many poles approaching $\lambda=0$ at threshold.[7] From (2.9) and (2.11), they are approximately given by

$$\lambda_n\sim 2n\pi i/\ln\eta^2,\quad n=\pm1,\pm2,\ldots \qquad (2.14)$$

where

$$\eta=e^{-\frac{1}{2}i\pi}k_t.$$

Since

$$\mathrm{Res}A(l,k_t^2)\,|_{l=\alpha_n}=\frac{k_t^{2\alpha_n}\sin(\lambda_n\pi)}{\partial Y(\lambda_n,k_t)/\partial\lambda+\eta^{2\lambda_n}\ln\eta^2}, \qquad (2.15)$$

[3] H. Cheng, Studies Appl. Math. **48**, 341 (1969).
[4] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).
[5] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962).
[6] H. Cheng, Nuovo Cimento **44**, 487 (1966).
[7] V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962).

the β_n functions for the poles given by (2.14) are

$$\beta_n \sim 2i\pi^2\lambda_n{}^2/\eta \ln\eta^2 = -8in^2\pi^4/\eta(\ln\eta^2)^3. \quad (2.16)$$

B. Amplitude at Threshold

Let us consider the amplitude $f(k_t{}^2,s)$ at $k_t=0$. We set $k_t=0$ in (2.1). Then from (2.1), (2.3), (2.7), and (2.13), we get

$$f(0,s)=\sum_{\mathrm{Re}\alpha_n(0)>-\frac{1}{2}+\epsilon} c_n(e^{-i\pi}s)^{\alpha_n(0)} + i\int_{-\frac{1}{2}+\epsilon-i\infty}^{-\frac{1}{2}+\epsilon+i\infty} dl \frac{\Gamma(-l)}{(\sqrt{\pi})\Gamma(-l-\frac{1}{2})}\frac{(e^{-i\pi}s)^l}{Y(\lambda)}, \quad (2.17)$$

with

$$c_n=-2(\sqrt{\pi})\Gamma(-\alpha_n(0))[\Gamma(-\alpha_n(0)-\tfrac{1}{2})Y'(\lambda_n(0))]^{-1}. \quad (2.18)$$

Note that (2.7) is applicable since we keep $\epsilon>0$, and (2.3) is applicable since, in the limit $k_t\to 0$ and s fixed, the argument of P_l is infinite. Note also that (2.17) is exact and is not merely an asymptotic formula for $s\to\infty$.

Once we arrive at (2.17), we may now set $\epsilon=0$. In fact, from the asympotoic form of $Y(\lambda)$ in the left-hand plane,[8] we may move the contour of integration in the background term of (2.17) to the left, obtaining

$$f(0,s)=\sum_{\mathrm{Re}A_n>-L} c_n(e^{-i\pi}s)^{a_n} + i\int_{-L-i\infty}^{-L+i\infty} dl \frac{\Gamma(-l)}{(\sqrt{\pi})\Gamma(-l-\frac{1}{2})}\frac{(e^{-i\pi}s)^l}{Y(\lambda)}, \quad (2.19)$$

where L is an arbitrary real constant and c_n is given by (2.18) with $\alpha_n(0)$ replaced by a_n. We observe that a_n is the same as $\alpha_n(0)$ if $\mathrm{Re}a_n>-\frac{1}{2}$; however, if $\mathrm{Re}a_n<-\frac{1}{2}$, then it is a zero of $Y(\lambda)$. In the limit $s\to\infty$, (2.19) gives

$$f(0,s)\sim\sum_n c_n(e^{-i\pi}s)^{a_n}, \quad |s|\to\infty. \quad (2.20)$$

It is interesting that the asymptotic form of $f(0,s)$ is *not* entirely given by the Regge-pole parameters.

Finally, we mention that the background integral in (2.19) does not vanish when $L\to\infty$, and hence (2.20) is an asymptotic series but not a convergent series.

C. Amplitude near Threshold

As long as k_t is nonzero, $f(k_t{}^2,s)$ is given by (2.1) for $k_t{}^2>0$ and the asymptotic form of $f(k_t{}^2,s)$ as $s\to\infty$ is given by (2.4) for all $k_t{}^2$. On the other hand, when $k_t{}^2$ is very small, $f(k_t{}^2,s)$ should approach $f(0,s)$ and its asymptotic form as $s\to\infty$ should approach (2.20). Since (2.4) includes the contribution of an infinite number of Regge poles approaching $\lambda=0$, while (2.20) does not, these two equations do not seem to be consistent with each other at first sight.

We shall show, however, that the contribution of the Regge poles approaching $\lambda=0$ goes to zero as $k_t\to 0$. From (2.4), (2.14), and (2.16), the contribution is equal to

$$-4\pi^2 i(\ln\eta^2)^{-2}s^{-1/2}\sum_n n(sk_t{}^{-2})^{2n\pi i/\ln\eta^2}. \quad (2.21)$$

Since (2.14) is valid only if $|\lambda_n|\ll 1$, or

$$|n|\ll(2\pi)^{-1}|\ln\eta^2|, \quad (2.22)$$

the summation in (2.21) should include only the poles satisfying (2.22). Thus (2.21) vanishes as $k_t\to 0$.

3. LEADING REGGE POLE

It is well known that, when $k_t{}^2$ is nonzero, the Regge poles are in the neighborhood of $l=-1, -2, \ldots, -n, \ldots$, if the potential is very weak. On the other hand, at the threshold value $k_t=0$, there is always a Regge pole on the right of $l=-\frac{1}{2}$ as long as the potential is attractive, no matter how weak it is.[9] This promotion of the Regge pole at threshold has the same origin as that in φ^3 theory. In fact, in Sec. IV we shall see that the Born series in potential scattering is in precisely the same form as the perturbation series we investigated in φ^3 theory. For the sake of simplicity, let us from now on restrict ourselves to the single Yukawa potential $-Ge^{-r}/r$, where $|G|\ll 1$, so that the potential is very weak. If G is positive, then there is always a Regge pole on the right of $l=-\frac{1}{2}$ at threshold. There are two possibilities: (i) This Regge pole is the leading Regge pole located near $l=-1$ for nonzero $k_t{}^2$; (ii) this Regge pole is not the leading Regge pole, in which case it must be located to the left of $l=-1$ for nonzero $k_t{}^2$ and catches up with the leading Regge pole before $k_t{}^2$ reaches zero. In this section we shall show that possibility (i) is the correct one. In so doing we shall also obtain the behavior of the leading Regge pole throughout the range $0\leqslant\eta<\infty$.

For small values of G, we have[10]

$$A(l,k_t{}^2)\sim\tfrac{1}{2}Gk_t{}^{-2}Q_l\left(1+\frac{1}{2k_t{}^2}\right)\times\left\{1-\frac{G}{2\cos l\pi}\left[i\frac{e^{-il\pi}}{k_t}Q_l\left(1+\frac{1}{2k_t{}^2}\right)+\int_0^{\pi/2}\frac{\cos[(2l+1)\theta]d\theta}{(\frac{1}{4}+k_t{}^2\cos^2\theta)^{1/2}}\right]\right\}^{-1}. \quad (3.1)$$

The Regge poles are determined by setting the de-

[8] H. Cheng and T. T. Wu, Phys. Rev. **144**, 1232 (1966).

[9] R. G. Newton, J. Math. Phys. **3**, 867 (1962).

[10] H. Cheng, Phys. Rev. **130**, 1283 (1963).

nominator on the right-hand side of (3.1) equal to zero:

$$1-\frac{G}{2\cos\alpha\pi}\left[i\frac{e^{-i\alpha\pi}}{k_t}Q_\alpha\left(1+\frac{1}{2k_t^2}\right)+\int_0^{\pi/2}\frac{\cos[(2\alpha+1)\theta]\,d\theta}{(\frac{1}{4}+k_t^2\cos^2\theta)^{1/2}}\right]\sim 0. \quad (3.2)$$

In the limit $G\to 0$, the second term in (3.2) is in general small compared to the first term and (3.3) cannot be satisfied unless α is in the neighborhoods of -1, -2, $\ldots$, $-n$, where Q_α has simple poles. Since

$$Q_\alpha(z)\sim\frac{P_n(z)}{\alpha+n+1},\quad |\alpha+n+1|\ll 1,\quad n=0,1,\ldots \quad (3.3)$$

(3.2) is approximately

$$1-\frac{GiP_n(1+1/2k_t^2)}{2k_t(\alpha+n+1)}\sim 0,$$

or

$$\alpha_n(k_t^2)\sim -n-1-GP_n\left(1+\frac{1}{2k_t^2}\right)\Big/2k_ti \quad (3.4)$$

and

$$\beta_n(k_t^2)\sim\pi(n+\tfrac{1}{2})GP_n\left(1+\frac{1}{2k_t^2}\right)\Big/k_t^2,\quad n=0,1,\ldots. \quad (3.5)$$

Equation (3.4) confirms that, if $k_t\neq 0$, there are Regge poles in the neighborhood of $l=-1,-2,\ldots$, as $G\to 0$.

Let us concentrate on the leading Regge pole $\alpha(k_t^2)$ ($n=0$). We have

$$\alpha(k_t^2)\sim -1-G/2k_ti \quad (3.6)$$

and

$$\beta(k_t^2)\sim\tfrac{1}{2}\pi G/k_t^2. \quad (3.7)$$

The perturbation expansion (3.6) is meaningful only if $|G/2k_t|$ is small compared to unity, or

$$|k_t|\gg\tfrac{1}{2}|G|. \quad (3.8)$$

Thus (3.6) fails when $|k_t|$ is very small, and, in particular, it fails at the threshold value $k_t=0$.

Since (3.6) and (3.7) are good approximations if (3.8) is satisfied, it only remains to investigate the region where $|k_t|$ is small. Let us consider (3.2) when $|k_t|\ll 1$. Since

$$Q_l(z)\sim(\sqrt{\pi})(2z)^{-l-1}\Gamma(l+1)/\Gamma(l+\tfrac{3}{2}),\quad |z|\gg 1$$

the Regge poles are given by the solution of

$$1-\tfrac{1}{2}(G/\lambda)[-\pi^{-1/2}\Gamma(1-\lambda)\Gamma(\tfrac{1}{2}+\lambda)e^{-i\pi\lambda}k_t^{2\lambda}+1]\sim 0. \quad (3.9)$$

In particular, since

$$\lim_{k_t\to 0}k_t^{2\lambda}=0\quad\text{if }\lambda>0,$$

the limit of (3.9) as $k_t\to 0$ gives

$$1-\tfrac{1}{2}G/\lambda\sim 0, \quad (3.10)$$

which means that $\lambda(0)$, the Regge pole at threshold, is given by

$$\lambda(0)\sim\tfrac{1}{2}G. \quad (3.11)$$

Equation (3.11) is consistent with (3.10) only if $G>0$. This is in agreement with the earlier statement that at $k_t=0$, there is a Regge pole on the right of $l=-\frac{1}{2}$ if the potential is attractive. Comparing (3.9) with (2.11), we get

$$Y(\lambda)\sim-(1-2\lambda/G)\pi^{1/2}/[\Gamma(1-\lambda)\Gamma(\tfrac{1}{2}+\lambda)]. \quad (3.12)$$

Equations (2.18) and (3.12) give

$$c_0\sim\tfrac{1}{2}\pi G^2. \quad (3.13)$$

Note that, when $G<0$, the right-hand side of (3.11) is still a zero of $Y(\lambda)$, although it is not a Regge pole.

Let us now consider the case $G>0$ so that (3.11) holds. How does this Regge pole move when k_t is small but nonzero? Let us return to (3.9) and restrict ourselves to the region $\eta\geqslant 0$. As η increases, the Regge pole retreats to the left. By setting $\lambda=0$ in (3.9), we find that the Regge pole moves into the left half-plane $\mathrm{Re}\lambda<0$ as $\eta>\eta_0$, where

$$\eta_0^2\sim\exp[-2/G+\gamma-\psi(\tfrac{1}{2})]\sim\exp(-2/G). \quad (3.14)$$

In (3.14), γ is the Euler's constant and $\psi(x)$ is the logarithmic derivative of the gamma function. Note that η_0 is an exponentially small number as $G\to 0$.

When λ is very small, (3.9) is approximated by

$$\lambda\sim\lambda(0)(1-\eta^{2\lambda}), \quad (3.15)$$

where $\lambda(0)$ is given by (3.11). Equation (3.15) is consistent with the well-known threshold behavior of $\lambda(k_t^2)$. It appears unlikely that (3.15) can be explicitly solved for $\lambda(k_t^2)$ as a function of k_t^2. It is easily shown, however, that (3.15) has one and only one real solution. A graphical solution of (3.15) is illustrated in Fig. 1. We see that the solution is near $\lambda=\lambda(0)$ of $\eta\ll\eta_0$, and moves to the left half-plane as $\eta>\eta_0$.

As η increases further and becomes much larger than η_0, $-\lambda(k_t^2)$ also increases and at some point the approximation (3.15) is no longer valid. We have to go back to (3.9). We may solve (3.9) to obtain

$$\eta^2=\{(1-2\lambda/G)\pi^{1/2}/[\Gamma(1-\lambda)\Gamma(\tfrac{1}{2}+\lambda)]\}^{1/\lambda}. \quad (3.16)$$

Equation (3.16) gives η^2 as a function of $\lambda(k_t^2)$. It remains now to investigate (3.16) when $\lambda(k_t^2)$ is negative. We shall put

$$\bar{\lambda}\equiv-\lambda(k_t^2) \quad (3.17)$$

in (3.16). Then (3.16) can be written as

$$1=\eta^{2\bar{\lambda}}(1+2\bar{\lambda}/G)\pi^{1/2}/[\Gamma(1+\bar{\lambda})\Gamma(\tfrac{1}{2}-\bar{\lambda})]. \quad (3.18)$$

Graphic solutions of (3.18) for various η are illustrated

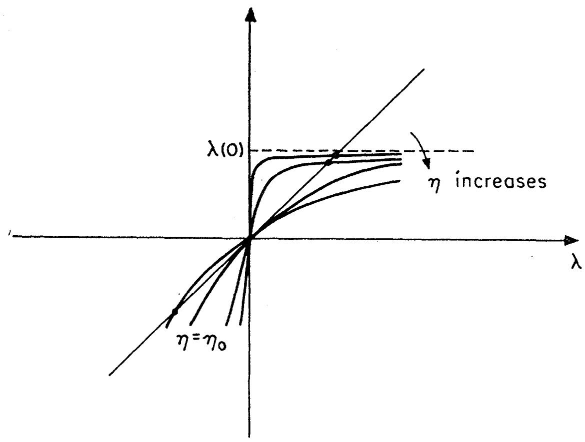

FIG. 1. Right-hand side of (3.15) plotted as a function of λ for various values of η. The intersections with the straight line give the locations of the Regge poles.

in Fig. 2. We note that the right-hand side of (3.18), considered as a function of $\bar{\lambda}$, is equal to 1 at $\bar{\lambda}=0$, and is increasing at $\bar{\lambda}=0$ if $\eta>\eta_0$. On the other hand, the right-hand side of (3.18) vanishes at $\bar{\lambda}=\frac{1}{2}$. Thus (3.18) always has at least one solution. This solution is seen to approach $\bar{\lambda}=\frac{1}{2}$, or $l=-1$ as η increases. Thus this is indeed the leading Regge pole given by (3.6).

Finally, we must show that (3.18) has only one solution. Now the logarithmic derivative of the right-hand side of (3.18) is

$$\frac{2G^{-1}}{1+2\bar{\lambda}G^{-1}}-\left(2\ln\frac{1}{\eta}+\psi(1-\bar{\lambda})-\psi(\tfrac{1}{2}-\bar{\lambda})-\frac{d}{d\bar{\lambda}}\ln\frac{\sin(\bar{\lambda}\pi)}{\bar{\lambda}\pi}\right). \quad (3.19)$$

For $\frac{1}{2}>\lambda>0$, the first term in (3.19) is a decreasing function of $\bar{\lambda}$, and the quantity in large parentheses in (3.19) is an increasing function of $\bar{\lambda}$. Thus (3.19) can vanish only once, and the right-hand side of (3.18) has only one maximum as plotted in Fig. 2. Thus (3.18) has only one solution.

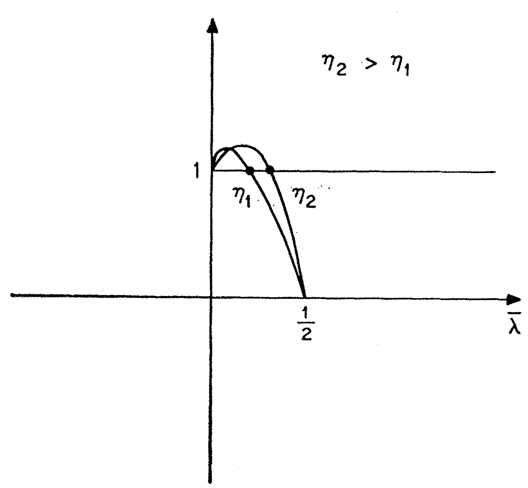

FIG. 2. Graphic solution of (3.18). The right-hand side of (3.18) is plotted for two values of η, and the intersections with the horizontal line give the locations of the Regge poles.

In summary, we have followed the movement of the Regge pole given by (3.11) at threshold. When η is of the order of or smaller than η_0, it is given by the solution of (3.15) and illustrated in Fig. 1. This Regge pole moves very rapidly into the left half-plane as η increases, and when η is of the order of G^n, it arrives at the neighborhood of $\lambda=-(2n)^{-1}$ [see (3.16)]. The precise form of $\lambda(k_t^2)$ is determined by (3.18) and illustrated in Fig. 2. As η continues to increase, it is given explicitly by (3.6). Thus the Regge pole (3.11) is also the leading Regge pole (3.6).

It is interesting to note that $\alpha(k_t^2)$ has perturbation expansion both when k_t^2 is away from threshold [(3.6)] and when k_t is at threshold [(3.11)], while the two expansions do not join into each other. This is not inconsistent since $\alpha(k_t^2)$ has *no* perturbation expansion when k_t is near threshold, and its form can be given only implicitly. The fact that $\alpha(k_t^2)$ has no perturbation expansion near threshold is probably not surprising, since it must satisfy the well-known threshold behavior

$$\alpha(k_t^2)\sim\alpha(0)+ak_t^{2\lambda(0)}, \quad (3.20)$$

which cannot be expanded as a perturbation series near threshold if $\lambda(0)\sim\frac{1}{2}G$.

4. BORN SERIES

In the preceding two sections, we have established the asymptotic form of $f(k_t^2,s)$ as $s\to\infty$, with k_t^2 especially in the region near or at the threshold. For the purpose of testing the legitimacy of summing leading terms of the perturbation series when s and k_t^2 are in the region mentioned above, we shall, in this section, apply this method to the Born series of potential scattering, sum them up, and compare it with the results in the preceding sections.

The potential is again taken to be $-Ge^{-r}/r$, where G can be either positive or negative. Then the nth term in the Born series is

$$f_n=(4\pi)^{n-1}G^nJ_n, \quad (4.1)$$

where

$$J_n=\int\prod_1^{n-1}[d^3k_i\,(2\pi)^{-3}(k_i^2-k_t^2-i\epsilon)^{-1}]\times\prod_1^n[(\mathbf{k}_i-\mathbf{k}_{i-1})^2+1]^{-1}. \quad (4.2)$$

In (4.2), $\mathbf{k}_0$ and $\mathbf{k}_n$ are the initial and the final momenta, respectively, i.e.,

$$\mathbf{k}_0^2=\mathbf{k}_n^2=k_t^2,$$

and

$$s=-(\mathbf{k}_n-\mathbf{k}_0)^2.$$

By introducing Feynman parameters, (4.2) can be written as

$$J_n=\Gamma(2n-1)\int_0^1\prod_1^n d\alpha_i\prod_1^{n-1}d\beta_i$$
$$\times\delta\Big(1-\sum_1^n\alpha_i-\sum_1^{n-1}\beta_i\Big)\int\prod_1^{n-1}[d^3k_i\,(2\pi)^{-3}]$$
$$\times\Big\{\sum_1^{n-1}\beta_i(k_i^2-k_t^2)+\sum_1^n\alpha_i[(k_i-k_{i-1})^2+1]-i\epsilon\Big\}^{G-2n+1}. \quad (4.3)$$

Since

$$\int\prod_1^{n-1}[d^3k_i\,(2\pi)^{-3}]\Big(\sum_{i=1}^{n-1}k_i^2+a\Big)^{-m}=\frac{a^{\frac{3}{2}(n-1)-m}\Gamma(-\frac{3}{2}(n-1)+m)}{(4\pi)^{\frac{3}{2}(n-1)}\Gamma(m)}, \quad (4.4)$$

(4.3) is equal to

$$J_n=\Gamma(\tfrac{1}{2}n+\tfrac{1}{2})(4\pi)^{-\frac{3}{2}(n-1)}\int_0^1\prod_1^n d\alpha_i\prod_1^{n-1}d\beta_i$$
$$\times\delta\Big(1-\sum_1^n\alpha_i-\sum_1^{n-1}\beta_i\Big)D^{-\frac{1}{2}(n+1)}\Lambda^{\frac{1}{2}n-1}. \quad (4.5)$$

In (4.5), Λ is the $(n-1)\times(n-1)$ determinant

$$\Lambda=\begin{vmatrix}\alpha_1+\alpha_2+\beta_1 & -\alpha_2 & 0 & 0 & \cdots & \cdots & \cdots & 0\\ -\alpha_2 & \alpha_2+\alpha_3+\beta_2 & -\alpha_3 & 0 & \cdots & \cdots & \cdots & 0\\ 0 & -\alpha_3 & \alpha_3+\alpha_4+\beta_3 & -\alpha_4 & \cdots & \cdots & \cdots & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_{n-1}+\alpha_n+\beta_{n-1}\end{vmatrix}, \quad (4.6)$$

and D is the $n\times n$ determinant

$$D=\begin{vmatrix}\alpha_1+\alpha_2+\beta_1 & -\alpha_2 & 0 & 0 & \cdots & \cdots & \cdots & -\alpha_1k_0\\ -\alpha_2 & \alpha_2+\alpha_3+\beta_2 & -\alpha_3 & 0 & \cdots & \cdots & \cdots & 0\\ 0 & -\alpha_3 & \alpha_3+\alpha_4+\beta_3 & -\alpha_4 & \cdots & \cdots & \cdots & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\alpha_nk_n\\ -\alpha_1k_0 & 0 & \cdots & \cdots & \cdots & \cdots & -\alpha_nk_n & (\alpha_1+\alpha_n-\sum_1^{n-1}\beta_i)k_t^2+\sum_1^n\alpha_i\end{vmatrix}. \quad (4.7)$$

The determinant in (4.7) is equal to

$$D=-s\prod_1^n\alpha_i+\Lambda\sum_1^n\alpha_i+D_kk_t^2, \quad (4.8)$$

where

$$D_k=\begin{vmatrix}\alpha_1+\alpha_2+\beta_1 & -\alpha_2 & 0 & 0 & \cdots & \cdots & \cdots & -\alpha_1\\ -\alpha_2 & \alpha_2+\alpha_3+\beta_2 & -\alpha_3 & 0 & \cdots & \cdots & \cdots & 0\\ 0 & -\alpha_3 & \alpha_3+\alpha_4+\beta_3 & -\alpha_4 & \cdots & \cdots & \cdots & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -\alpha_n\\ -\alpha_1 & 0 & \cdots & \cdots & \cdots & \cdots & -\alpha_n & \alpha_1+\alpha_n-\sum_1^{n-1}\beta_i\end{vmatrix}$$

$$
=\begin{vmatrix}
\alpha_1+\alpha_2+\beta_1 & -\alpha_2 & 0 & 0 & \cdots & \cdots & \cdots & \beta_1 \\
-\alpha_2 & \alpha_2+\alpha_3+\beta_2 & -\alpha_3 & 0 & \cdots & \cdots & \cdots & \beta_2 \\
0 & -\alpha_3 & \alpha_3+\alpha_4+\beta_3 & -\alpha_4 & \cdots & \cdots & \cdots & \beta_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \beta_{n-1} \\
\beta_1 & \beta_2 & \beta_3 & \cdots & \cdots & \cdots & \beta_{n-1} & 0
\end{vmatrix}
=-\sum_{i,j}\beta_i\beta_j\Lambda(i,j)\,. \tag{4.9}
$$

In the above, $\Lambda(i,j)$ is the determinant (4.6) with the ith row and the jth column deleted. A few examples of D_k are

$$
\begin{aligned}
D_k&=-\beta_1^2,\quad n=2\\
&=-[\beta_1^2(\alpha_2+\alpha_3)+\beta_2^2(\alpha_1+\alpha_2)+2\alpha_2\beta_1\beta_2\\
&\qquad+(\beta_1+\beta_2)\beta_1\beta_2],\quad n=3.
\end{aligned} \tag{4.10}
$$

From (4.5)–(4.9), we observe that J_n is equal to, aside from a multiplicative constant, I_{n-1} in Ref. 2.

Let us now consider the limit of J_n as $s\to\infty$. For this purpose, we make a Mellin transform of (4.5). We have

$$
\begin{aligned}
\tilde{J}_n(\xi)&=\int_0^\infty J_n s^{-\xi}ds\\
&=\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}+\xi)\Gamma(1-\xi)(4\pi)^{-\frac{3}{2}(n-1)}\int_0^1\prod_1^n d\alpha_i\prod_1^{n-1}d\beta_i\\
&\quad\times\delta(1-\sum_1^n\alpha_i-\sum_1^{n-1}\beta_i)(-\prod_1^n\alpha_i)^{-1+\xi}\\
&\quad\times(D_k k_t^2+\Lambda\sum_1^n\alpha_i)^{-\frac{1}{2}(n-1)-\xi}\Lambda^{\frac{1}{2}n-1}.
\end{aligned} \tag{4.11}
$$

A. Away from Threshold

If k_t^2 is nonzero, the calculation is standard. The integral (4.11) has a singularity of $\xi=0$. The integration region which gives the dominant contribution is in the neighborhood $\alpha_i=0$, $i=1,\ldots,n$. Thus

$$
\begin{aligned}
\tilde{J}_n(\xi)&\sim-\Gamma(\tfrac{1}{2}n-\tfrac{1}{2})(4\pi)^{-\frac{3}{2}(n-1)}\xi^{-n}\int_0^1\prod_1^{n-1}d\beta_i\\
&\quad\times\delta(1-\sum_1^{n-1}\beta_i)(e^{-i\pi}k_t^2)^{-\frac{1}{2}(n-1)}(\prod_1^{n-1}\beta_i)^{-1/2}.
\end{aligned} \tag{4.12}
$$

Now

$$
\begin{aligned}
&\int_0^1\prod_1^{n-1}d\beta_i\,\delta(1-\sum_1^{n-1}\beta_i)(\prod_1^{n-1}\beta_i)^{-1/2}\\
&=\frac{1}{2\pi}\int_{-\infty}^\infty dx\,e^{ix}\int_0^\infty\exp(-ix\sum_1^{n-1}\beta_i)(\prod_1^{n-1}\beta_i)^{-1/2}\prod_1^{n-1}d\beta_i\\
&=\frac{1}{2\pi}\int_{-\infty}^\infty dx\,e^{ix}[e^{-i\pi/2}\pi(x-i\epsilon)^{-1}]^{(n-1)/2}\\
&=\pi^{(n-1)/2}[\Gamma(\tfrac{1}{2}n-\tfrac{1}{2})]^{-1}.
\end{aligned}
$$

Thus as $s\to\infty$, t nonzero, we have

$$
J_n\sim-(8\pi e^{-i\pi/2}k_t)^{-n+1}(\ln s)^{n-1}s^{-1}/(n-1)! \tag{4.13}
$$

and

$$
\begin{aligned}
f(k_t^2,s)&=\sum_1^\infty f_n=\sum_1^\infty(4\pi)^{n-1}G^nJ_n\\
&=-Gs^{-1+\frac{1}{2}Gi/k_t}.
\end{aligned} \tag{4.14}
$$

Equation (4.14) agrees with (2.4), (3.6), and (3.7). Note that this agreement is independent of the sign of G. This is quite impressive as successive leading terms in the Born series alternate in sign if G is negative and η is real, and the sum (4.14) is smaller than any individual term in the series.

B. At Threshold

If $k_t^2=0$, then (4.11) gives

$$
\begin{aligned}
\tilde{J}_n(\xi)&=\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}+\xi)\Gamma(1-\xi)(4\pi)^{-\frac{3}{2}(n-1)}\\
&\quad\times\int_0^1\prod_1^n d\alpha_i\prod_1^{n-1}d\beta_i\,\delta(1-\sum_1^n\alpha_i-\sum_1^{n-1}\beta_i)\\
&\quad\times(-\prod_1^n\alpha_i)^{-1+\xi}(\sum_1^n\alpha_i)^{-\frac{1}{2}(n-1)-\xi}\Lambda^{-\frac{1}{2}-\xi}.
\end{aligned} \tag{4.15}
$$

The integral above is already divergent at $\xi=\frac{1}{2}$ if $n>1$. To see this, we make use of the identity

$$\int_0^1 \prod_1^n d\alpha_i \prod_1^m d\beta_i\, \delta(1-\sum_1^n \alpha_i-\sum_1^m \beta_i) \times F(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m)$$
$$=\int_0^1 \prod_1^n d\alpha_i\, \delta(1-\sum_1^n \alpha_i)\int_0^\infty \prod_1^m d\beta_i \times F(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m), \quad (4.16)$$

where $F(\alpha_1,\ldots,\alpha_m,\beta_1,\ldots,\beta_m)$ satisfies

$$F(a\alpha_1,\ldots,a\alpha_n,a\beta_1,\ldots,a\beta_m) = a^{-n-m}F(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m).$$

Thus (4.15) can be written as

$$\tilde{J}_n(\xi)\sim\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2})(4\pi)^{-\frac{3}{2}(n-1)}\int_0^1 \prod_1^n d\alpha_i\, \delta(1-\sum_1^n \alpha_i) \times e^{i\pi/2}(\prod_1^n \alpha_i)^{-1/2}\int_0^\infty \prod_1^{n-1} d\beta_i\, \Lambda^{-\frac{1}{2}-\xi}, \quad \xi\sim\tfrac{1}{2}. \quad (4.17)$$

The divergence at $\xi=\frac{1}{2}$ comes from the region $\beta_i\gg1$, $i=1,\ldots,n-1$. Thus we may make the approximation

$$\Lambda\sim\prod_1^{n-1}\beta_i,$$

and (4.17) becomes

$$\tilde{J}_n(\xi)\sim i\Gamma(\tfrac{1}{2}n)\Gamma(\tfrac{1}{2})(4\pi)^{\frac{3}{2}(n-1)} \times\int_0^1 \prod_1^n d\alpha_i\, \delta(1-\sum_1^n \alpha_i)(\prod_1^n \alpha_i)^{-1/2}(\xi-\tfrac{1}{2})^{-n+1}$$
$$=i\pi(8\pi)^{-(n-1)}(\xi-\tfrac{1}{2})^{-n+1}. \quad (4.18)$$

Thus at $k_t^2=0$, we have, as $s\to\infty$,

$$J_n\sim i\pi(8\pi)^{-(n-1)}(\ln s)^{n-2}s^{-1/2}/(n-2)!, \quad n>1 \quad (4.19)$$

while J_1 is of the order of s^{-1} and will be neglected. Hence

$$f(0,s)\sim\sum_2^\infty (4\pi)^{n-1}G^nJ_n=\tfrac{1}{2}i\pi G^2 s^{-\frac{1}{2}+\frac{1}{2}G}, \quad s\to\infty. \quad (4.20)$$

Equation (4.20) agrees with (2.20), (3.11), and (3.13). Note that when $G<0$, $\lambda(0)$ $(=\frac{1}{2}G)$ is no longer a Regge pole. However, it continues to be a zero of $Y(\lambda)$. Thus (4.20) is still correct and summing the leading terms in the Born series is legitimate even for a repulsive potential, despite the fact that successive terms in this series alternate in sign and the sum (4.20) is smaller than any individual term.

C. Near Threshold

The behavior of the nth Born term in the limit $s\to\infty$ with k_t^2 near the threshold is quite complicated. A discussion has been given in Ref. 2 and will not be repeated here. It suffices to say that there are infinitely many scales for k_t^2. This complicated behavior is related to the existence of infinitely many Regge poles in the neighborhood of $\lambda=0$ as $k_t^2\to0$. The precise relationship between the Born series near threshold and (2.21) appears to be quite difficult to establish and is beyond our power of analysis.