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High-Energy Behavior near Threshold: φ^3 Theory*

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In this paper we study the two-body elastic scattering process in φ^3 theory, in the limit $s\to\infty$ with t equal to or near the threshold value 4, where the mass of the scalar particle is taken to be unity. For the scattering amplitude corresponding to any ladder-type diagram, we find that there is a promotion for the power of s by $\frac{1}{2}$. For example, the scattering amplitude for the ladder diagram of $n+1$ rungs is promoted from $O(s^{-1}\ln^n s)$ away from threshold to $O(s^{-1/2}\ln^{n-1}s)$ at threshold. This means that, for small coupling constants, the leading Regge pole is promoted from the neighborhood of $l=-1$ away from threshold to the neighborhood of $l=-\frac{1}{2}$ at threshold. There are, in addition, infinitely many Regge trajectories approaching $l=-\frac{1}{2}$ as t approaches 4. The scattering amplitude for the n-rung diagram is explicitly given in the limit $s\to\infty$, with t at or near the threshold 4, and various scales for $t-4$ are pointed out. The two-rung and three-rung diagrams are especially studied in detail.

1. INTRODUCTION

EIGHT years ago, Gell-Mann and Goldberger[1] suggested that field theory may give rise to Regge behavior $s^{\alpha(t)}$, where as usual s is the square of the c.m. energy and $-t$ is the square of the momentum transfer. In their work and subsequent analysis, attention is concentrated on the ladder diagrams in the t channel. By adding together the leading contributions from these diagrams one finds, for φ^3 theory, indeed the behavior $s^{\alpha(t)}$ for large s. Since this is a perturbation calculation, the $\alpha(t)$ thus obtained is proportional to g^2, where g is the coupling constant. Higher-order terms in g^2 presumably have contributions from other diagrams.

It is realized in the original work[1] that the $\alpha(t)$ so obtained has a singularity at the elastic threshold $t=4m^2$, where m is the mass of the particle under consideration. Furthermore, since such a singularity cannot make sense physically, it must be attributed to the inadequacy of the approximations used. More precisely, in the analysis t is assumed to be fixed at a value away from $4m^2$ while $s\to\infty$, and hence failure near $4m^2$ is not surprising. As a trivial example, consider

$$\int_a^\infty dx\, e^{-\lambda x^2}.$$

For fixed $a>0$ with $\lambda\to\infty$, this is approximately

$$(2\lambda a)^{-1}e^{-\lambda a^2},$$

which has a singularity at $a=0$ where the original integral remains bounded.

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[1] M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962).

In view of the results of the preceding paper[2] for quantum electrodynamics with a massive photon, it is interesting to go back to the simpler φ^3 theory and study the properties of the matrix elements near the elastic threshold. It is the purpose of this paper to carry out this analysis. Compared with the early work away from threshold, the present consideration is much more involved, and many questions cannot be answered without extensive additional work. In Sec. 2 we begin with a detailed analysis of the box diagram, shown in Fig. 1, and in Sec. 3 we study the ladder diagram with three rungs. In the latter case, enormous complications appear because of the presence of two distinct scales for $4m^2-t$, namely, $s^{-1/2}$ and s^{-1}. Some of these results are then generalized in Sec. 4 to ladder diagrams with an arbitrary number of rungs.

2. BOX DIAGRAM

In this section, we discuss the box diagram of Fig. 1 in the case where all the masses are equal. Since there is one loop, we call this matrix element

$$\mathfrak{M}_1=-ig^4(2\pi)^{-4}\int d^4q\,[(r_2-q)^2-1+i\epsilon]^{-1}\times[(r_1+q)^2-1+i\epsilon]^{-1}[(r_1-q)^2-1+i\epsilon]^{-1}\times[(r_3+q)^2-1+i\epsilon]^{-1},\quad(2.1)$$

where we have taken the mass to be 1, and otherwise the notation is that of Sec. 2 of Ref. 3. If Feynman parameters are introduced as usual, then

$$\mathfrak{M}_1=g^4(4\pi)^{-2}I_1,\qquad(2.2)$$

[2] H. Cheng and T. T. Wu, preceding paper, Phys. Rev. D **2**, 2276 (1970).

[3] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1852 (1969).

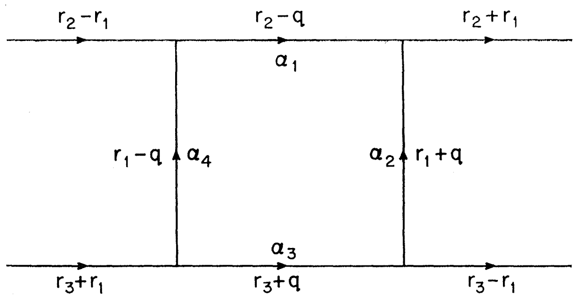

FIG. 1. Box diagram.

where

$$I_1=\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4\,\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)D_1^{-2}, \quad (2.3)$$

with

$$D_1=s\alpha_1\alpha_3+t\alpha_2\alpha_4+(\alpha_1+\alpha_3)(\alpha_2+\alpha_4)-1+i\epsilon. \quad (2.4)$$

We shall study I_1 for large s.

The proper way to analyze the asymptotic behavior is to use Mellin transformation, which has been previously used,[4] and hypergeometric functions.[5] However, since hypergeometric functions, unlike Bessel functions for example, are not familiar to every physicist, we shall avoid them in this section. The more systematic development utilizing Mellin transformation and hypergeometric functions is to be found in Appendix A.

We first review the case where $s\to\infty$ for fixed $t\neq 4$. This has been studied by many persons—for example, Federbush and Grisaru.[6] The result is

$$I_1=s^{-1}\ln s\int_0^1 dx\,[tx(1-x)-1+i\epsilon]^{-1}+O(s^{-1}). \quad (2.5)$$

The term of the order of magnitude s^{-1} is also known.[7] In particular, if the *fixed* t is close to 4, Eq. (2.5) gives

$$I_1\sim -\pi s^{-1}\ln s(4-t-i\epsilon)^{-1/2}. \quad (2.6)$$

Note that the right-hand side of (2.6) has a singularity at $t=4$; this singularity indicates that, at $t=4$, the order of magnitude of I_1 is not $s^{-1}\ln s$. Indeed, we shall see that, at $t=4$, I_1 is *promoted* to the order $s^{-1/2}$.

To see this, let

$$t=4(1-T/s), \quad (2.7)$$

fix T, and let $s\to\infty$. The leading term in this limit is easily obtained as follows from (2.3) by using the new variables

$$\alpha_1'=\alpha_1 s,\quad \alpha_3'=\alpha_3 s,$$

and

$$x=s^{1/2}(2\alpha_2-1)\sim s^{1/2}(1-2\alpha_4). \quad (2.8)$$

Therefore, in this limit of fixed T,

$$\begin{aligned}\lim_{s\to\infty} s^{1/2}I_1
&=\frac{1}{2}\int_0^\infty d\alpha_1'\int_0^\infty d\alpha_3'\int_{-\infty}^\infty dx\\
&\qquad\times(\alpha_1'\alpha_3'-T-\alpha_1'-\alpha_3'-x^2+i\epsilon)^{-2}\\
&=\int_0^\infty dx\int_0^\infty d\alpha_1'\,(-\alpha_1'+1-i\epsilon)^{-1}(T+\alpha_1'+x^2-i\epsilon)^{-1}\\
&=\int_0^\infty dx\,(T+1+x^2-i\epsilon)^{-1}[-\ln(T+x^2-i\epsilon)+i\pi]\\
&=-\pi(T+1-i\epsilon)^{-1/2}\{\ln[(T-i\epsilon)^{1/2}\\
&\qquad+(T+1-i\epsilon)^{1/2}]-\tfrac{1}{2}i\pi\}. \quad (2.9)\end{aligned}$$

In particular, when $t=4$, $T=0$ and

$$I_1\sim\tfrac{1}{2}i\pi^2 s^{-1/2} \quad (2.10)$$

as $s\to\infty$. A much more precise formula for I_1 in this case is to be found in Appendix A, Eq. (A22).

On the other hand, when T is large, we can neglect 1 in (2.9) to get

$$I_1\sim -\pi s^{-1}(4-t-i\epsilon)^{1/2}\{\ln[s(4-t-i\epsilon)]-i\pi\}. \quad (2.11)$$

This is more precise than (2.6). Thus the asymptotic behavior of I_1 is given completely by (2.5) and (2.11).

We emphasize that, for the present problem, there is only one scale for $t-4$, namely, s^{-1} as seen from (2.27).

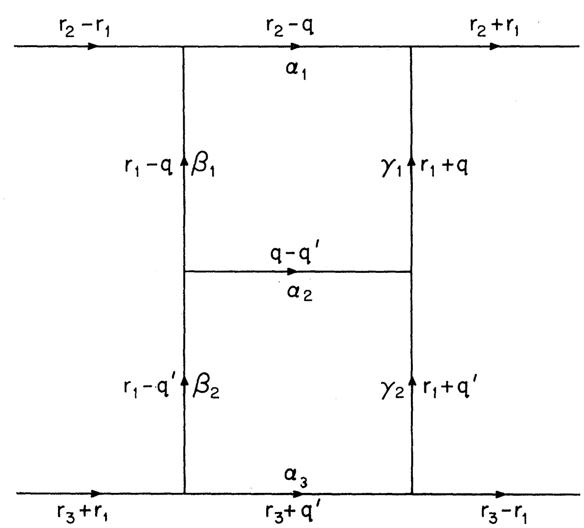

FIG. 2. Ladder diagram with three rungs.

[4] J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963).
[5] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. I, Chap. 2.
[6] P. G. Federbush and M. T. Grisaru, Ann. Phys. (N. Y.) 22, 263 (1963).
[7] T. L. Trueman and T. Yao, Phys. Rev. **132**, 2741 (1963).

As we shall see in Sec. 3, the situation is much more complicated for the ladder diagram with three rungs.

3. LADDER DIAGRAM WITH THREE RUNGS

A. Formulation

We turn our attention to the next more complicated diagram, shown in Fig. 2. Since there are two loops, we call this matrix element

$$\mathfrak{M}_2=g^6(2\pi)^{-8}\int d^4q d^4q' [(r_2-q)^2-1+i\epsilon]^{-1} \times[(r_1+q)^2-1+i\epsilon]^{-1}[(r_1-q)^2-1+i\epsilon]^{-1} \times[(q-q')^2-1+i\epsilon]^{-1}[(r_1+q')^2-1+i\epsilon]^{-1} \times[(r_1-q')^2-1+i\epsilon]^{-1}[(r_3+q')^2-1+i\epsilon]^{-1}. \quad (3.1)$$

Again Feynman parameters are introduced, and

$$\mathfrak{M}_2=-g^6(4\pi)^{-4}I_2, \quad (3.2)$$

where

$$I_2=2\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 d\beta_2 d\gamma_1 d\gamma_2 \times\delta(1-\alpha_1-\alpha_2-\alpha_3-\beta_1-\beta_2-\gamma_1-\gamma_2)\Lambda_2 D_2^{-3}, \quad (3.3)$$

with

$$\Lambda_2=(\beta_1+\gamma_1+\alpha_1+\alpha_2)(\beta_2+\gamma_2+\alpha_2+\alpha_3)-\alpha_2^2 \quad (3.4)$$

and

$$D_2=s\alpha_1\alpha_2\alpha_3+t[\beta_1\gamma_1(\beta_2+\gamma_2+\alpha_2+\alpha_3) +\beta_2\gamma_2(\beta_1+\gamma_1+\alpha_1+\alpha_2)+\alpha_2(\beta_1\gamma_2+\beta_2\gamma_1)] +[\alpha_1(\beta_1+\gamma_1)(\beta_2+\gamma_2+\alpha_2+\alpha_3) +\alpha_3(\beta_2+\gamma_2)(\beta_1+\gamma_1+\alpha_1+\alpha_2) +\alpha_1\alpha_2(\beta_2+\gamma_2)+\alpha_2\alpha_3(\beta_1+\gamma_1)] -\Lambda(\alpha_1+\alpha_2+\alpha_3+\beta_1+\beta_2+\gamma_1+\gamma_2)+i\epsilon. \quad (3.5)$$

In this section, we shall study I_2 for large s.

As in Sec. 2, consider first the case where $s\to\infty$ with fixed $t\neq4$. Mellin transformation applied to I_2 of (3.3) gives[4]

$$\bar{I}_2(\xi)=\int_0^\infty ds\, s^{-\xi}I_2 =\pi\xi(1+\xi)\csc\pi\xi\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 d\beta_2 d\gamma_1 d\gamma_2\, e^{-i\pi\xi} \times\delta(1-\alpha_1-\alpha_2-\alpha_3-\beta_1-\beta_2-\gamma_1-\gamma_2) \times(\alpha_1\alpha_2\alpha_3)^{-1+\xi}\Lambda_2 D_2'^{-2-\xi}, \quad (3.6)$$

where

$$D_2'=-D_2|_{s=0}. \quad (3.7)$$

With $t\neq4$, let ξ be small; then (3.6) gives approximately

$$\bar{I}_2(\xi)\sim\xi^{-3}\int_0^1 d\beta_1 d\beta_2 d\gamma_1 d\gamma_2\,\delta(1-\beta_1-\beta_2-\gamma_1-\gamma_2) \times(\beta_1+\gamma_1)(\beta_2+\gamma_2)\{-t[\beta_1\gamma_1(\beta_2+\gamma_2) +\beta_2\gamma_2(\beta_1+\gamma_1)]+(\beta_1+\gamma_1)(\beta_2+\gamma_2)-i\epsilon\}^{-2} =\xi^{-3}\left\{\int_0^1 dx\,[tx(1-x)-1+i\epsilon]^{-1}\right\}^2. \quad (3.8)$$

Accordingly, as $s\to\infty$,

$$I_2=\tfrac{1}{2}s^{-1}\left\{\ln s\int_0^1 dx\,[tx(1-x)-1+i\epsilon]^{-1}\right\}^2 +O(s^{-1}\ln s). \quad (3.9)$$

When the fixed t is close to 4, a comparison with (2.5) and (2.6) shows that

$$I_2\sim\tfrac{1}{2}\pi^2 s^{-1}(\ln s)^2(4-t-i\epsilon)^{-1}. \quad (3.10)$$

It is interesting to speculate at this stage about the order of magnitude of I_2 at $t=4$. For I_1, a comparison of (2.6) with (2.10) shows that a factor $(\ln s)(4-t-i\epsilon)^{-1/2}$ away from threshold turns into $\frac{1}{2}i\pi s^{1/2}$ at threshold. If this is also true for I_2, then it follows from (3.10) that I_2 is of order of magnitude 1 at $t=4$. However, a moment's reflection indicates that this is not possible because of (3.3), where s appears only in the denominator. In fact, as we show below, at $t=4$, I_2 is of the order of magnitude $s^{-1/2}\ln s$. In other words, for I_2 a factor $(\ln s)(4-t-i\epsilon)^{-1}$ away from threshold turns into constant $s^{1/2}$. This implies that in the present case there is a second scale for $4-t$, namely, $s^{-1/2}$. It is this presence of two scales for $4-t$ that makes the problem both complicated and interesting.

B. Behavior at Threshold

In this section, let $t=4$ and study the behavior of I_2 as $s\to\infty$. There are many different ways of analyzing this case at threshold; one way is to use Mellin transformation again and note that (3.6) also holds here. If we set $t=4$ in (3.7), then by (3.5) and (3.6) we need to study explicitly

$$\bar{I}_2(\xi)=\pi\xi(1+\xi)\csc\pi\xi\, e^{-i\pi\xi}\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 d\beta_2 d\gamma_1 d\gamma_2$$
$$\times\delta(1-\alpha_1-\alpha_2-\alpha_3-\beta_1-\beta_2-\gamma_1-\gamma_2)(\alpha_1\alpha_2\alpha_3)^{-1+\xi}$$
$$\times[(\beta_1+\gamma_1+\alpha_1+\alpha_2)(\beta_2+\gamma_2+\alpha_2+\alpha_3)-\alpha_2^2]$$
$$\times\{(\alpha_1+\alpha_2+\alpha_3)[(\beta_1+\gamma_1+\alpha_1+\alpha_2)$$
$$\times(\beta_2+\gamma_2+\alpha_2+\alpha_3)-\alpha_2^2]+(\beta_1-\gamma_1)^2$$
$$\times(\beta_2+\gamma_2+\alpha_2+\alpha_3)+(\beta_2-\gamma_2)^2(\beta_1+\gamma_1+\alpha_1+\alpha_2)$$
$$+2\alpha_2(\beta_1-\gamma_1)(\beta_2-\gamma_2)\}^{-2-\xi}. \quad (3.11)$$

Unlike that of (3.8), but rather similar to the $\bar{I}_1(\xi)$ of (A18) and (A19) in Appendix A, $\bar{I}_2(\xi)$ has a singularity at $\xi=\frac{1}{2}$. This singularity comes from the region where α_1, α_2, α_3, $(\beta_1-\gamma_1)^2$, and $(\beta_2-\gamma_2)^2$ are all small and of the same order of magnitude. In order to get the leading term near $\xi=\frac{1}{2}$, we make the approximation of integrating $\beta_1-\gamma_1$ and $\beta_2-\gamma_2$ over the entire real axis. When these two integrations are carried out, we get

$$\bar{I}_2(\xi)\sim -i\tfrac{1}{8}\pi^2\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 dx_1 dx_2$$
$$\times\delta(1-\alpha_1-\alpha_2-\alpha_3-x_1-x_2)$$
$$\times(\alpha_1\alpha_2\alpha_3)^{-1+\xi}(\alpha_1+\alpha_2+\alpha_3)^{-1-\xi}$$
$$\times[(x_1+\alpha_1+\alpha_2)(x_2+\alpha_2+\alpha_3)-\alpha_2^2]^{-\xi-1/2}, \quad (3.12)$$

where $x_i=\beta_i+\gamma_i$ for $i=1,2$. Because of homogeneity in the integrand of (3.12), this can be easily reduced by changing the argument of the δ function:

$$\bar{I}_2(\xi)\sim -i\tfrac{1}{8}\pi^2\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3\int_0^\infty dx_1 dx_2\,\delta(1-\alpha_1-\alpha_2-\alpha_3)$$
$$\times(\alpha_1\alpha_2\alpha_3)^{-1+\xi}(\alpha_1+\alpha_2+\alpha_3)^{-1-\xi}$$
$$\times[(x_1+\alpha_1+\alpha_2)(x_2+\alpha_2+\alpha_3)-\alpha_2^2]^{-\xi-1/2}$$
$$=-i\tfrac{1}{8}\pi^2\int_0^1 d\alpha_1 d\alpha_2 d\alpha_2\,\delta(1-\alpha_1-\alpha_2-\alpha_3)(\alpha_1\alpha_2\alpha_3)^{-1+\xi}$$
$$\times[(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)]^{-\xi+1/2}$$
$$\times F(\xi-\tfrac{1}{2},\xi-\tfrac{1}{2};\xi+\tfrac{1}{2};\alpha_2^2/[(\alpha_1+\alpha_2)(\alpha_2+\alpha_3)])$$
$$\sim -i\tfrac{1}{8}\pi^2\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3\,\delta(1-\alpha_1-\alpha_2-\alpha_3)$$
$$\times(\alpha_1\alpha_2\alpha_3)^{-1/2}(\xi-\tfrac{1}{2})^{-2}$$
$$=-i\tfrac{1}{4}\pi^3(\xi-\tfrac{1}{2})^{-2}. \quad (3.13)$$

Therefore, by inverting the Mellin transformation, we get

$$I_2=-i\tfrac{1}{4}\pi^3 s^{-1/2}[\ln s+O(1)], \quad (3.14)$$

valid for $t=4$ and large s. This is the desired answer.[8]

C. Behavior near Threshold

In order to connect (3.10) to (3.14), we consider the case

$$t=4(1-\Delta'/\sqrt{s}), \quad (3.15)$$

with Δ' fixed and $s\to\infty$. For $\Delta'\neq 0$, this case is most easily studied by a change of variable:

$$\alpha_1=\alpha_1' s^{-1/2},\quad \alpha_2=\alpha_2' s^{-1/2},\quad \alpha_3=\alpha_3' s^{-1/2},$$
$$\beta_1=\tfrac{1}{2}(x+y_1 s^{-1/4}),\quad \gamma_1=\tfrac{1}{2}(x-y_1 s^{-1/4}),$$

and

$$\beta_2=\tfrac{1}{2}(1-x+y_2 s^{-1/4}),\quad \gamma_2\sim\tfrac{1}{2}(1-x-y_2 s^{-1/4}). \quad (3.16)$$

The substitution into (3.3) then gives

$$\lim_{s\to\infty} s^{1/2}I_2=2\int_0^1 dx\int_0^\infty \frac{x(1-x)d\alpha_1' d\alpha_2' d\alpha_3' dy_1 dy_2}{[\alpha_1'\alpha_2'\alpha_3'-(\Delta'+\alpha_1'+\alpha_2'+\alpha_3')x(1-x)-(1-x)y_1^2-xy_2^2+i\epsilon]^3}. \quad (3.17)$$

Note that this procedure makes sense because the integral on the right-hand side of (3.17) is convergent. As shown in Appendix D, this sixfold integral can be reduced to a single integral[9]

$$\lim_{s\to\infty} s^{1/2}I_2=\pi^2\int_0^{\Delta'} dx\,(\cos^{-1}x)(\Delta'^2-x^2)^{-1/2}(1-x^2)^{-1/2}$$
$$-\tfrac{1}{2}i\pi^3 K'(\Delta'), \quad (3.18)$$

where K is the complete elliptic integral of the first kind.[10]

It is also of some use to have the Mellin transformation of the right-hand side of (3.18):

$$\int_0^\infty d\Delta'\,\Delta'^{-\xi}\lim_{s\to\infty} s^{1/2}I_2$$
$$=\tfrac{1}{8}\pi^2(\cos\tfrac{1}{2}\pi\xi)e^{-i\pi\xi/2}[\Gamma(\tfrac{1}{2}\xi)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\xi)]^2. \quad (3.19)$$

This is derived in Appendix E.

It follows from either (3.18) or (3.19) that

$$\lim_{s\to\infty} s^{1/2}I_2\sim\tfrac{1}{2}\pi^2\Delta'^{-1}(\ln\Delta')^2 \quad (3.20)$$

as $\Delta'\to\infty$, and

$$\lim_{s\to\infty} s^{1/2}I_2\sim\tfrac{1}{2}i\pi^3\ln\Delta' \quad (3.21)$$

[8] The term of order $s^{-1/2}$ can also be found explicitly, but we have failed to find any elegent expression for it.

[9] When Δ' is negative, it should be interpreted as $\Delta'-i\epsilon$.

[10] Reference 5, Vol. II, Chap. 13. See especially pp. 314 and 317. Note that $K'(\Delta')$ is defined to be $K((1-\Delta'^2)^{1/2})$.

as $\Delta'\to 0$. Equation (3.20) agrees with (3.10); this means that, for $t-4$, $s^{-1/2}$ is the largest scale. On the other hand, the right-hand side of (3.21) fails to make sense for $\Delta'=0$; this means that $s^{-1/2}$ is not the only scale for $t-4$.

A comparison with the case of I_1 suggests that for I_2 the two scales of $t-4$ are $s^{-1/2}$ and s^{-1}. There are many ways to see this. For example, in (3.13) $\alpha_1+\alpha_2$ and $\alpha_2+\alpha_3$ act, respectively, as cutoffs for x_1 and x_2. On the other hand, from (3.17) with small Δ', the largest contribution comes from the region where α_1', α_2', α_3', and $x^{1/2}$ are of the same order of magnitude Δ'. A comparison of these two cutoffs shows that

$$\Delta'\sim x/\sqrt{x}\sim\alpha_1/\alpha_1'=s^{1/2}, \tag{3.22}$$

and hence

$$4-t=O(s^{-1}). \tag{3.23}$$

When (3.23) holds, (3.21) agrees with (3.14).

D. Summary

We summarize the asymptotic behavior of I_2 for $s\to\infty$ and real t: Eq. (3.9) holds when $s^{1/2}|4-t|\gg 1$, Eq. (3.18) holds when $1\gg|4-t|\gtrsim s^{-1}$, where Δ' is defined by (3.15), and Eq. (3.14) holds when $s|4-t|$ is not large. These formulas cover all the possible cases. In Appendix F, we study in more detail the case $4-t=O(s^{-1})$.

4. GENERAL LADDER DIAGRAM

A. Formulation

We are now in a position to generalize some of the results of the previous two sections to a ladder diagram of $n+1$ rungs, as shown in Fig. 3. Unfortunately, the present generalization is quite incomplete and a great deal of future work is needed. The case of the box diagram in Sec. 2 corresponds to the special case $n=1$, while that in Sec. 3 corresponds to the case $n=2$. The matrix element for an arbitrary n is

$$\begin{aligned}\mathfrak{M}_n=&-i^n g^{2(n+1)}(2\pi)^{-4n}\int d^4q_1d^4q_2\cdots d^4q_n\\ &\times[(r_2-q_1)^2-1+i\epsilon]^{-1}[(r_3+q_n)^2-1+i\epsilon]^{-1}\\ &\times\{\prod_{i=1}^{n}[(r_1+q_i)^2-1+i\epsilon]\}^{-1}\\ &\times\{\prod_{i=1}^{n}[(r_1-q_i)^2-1+i\epsilon]\}^{-1}\\ &\times\{\prod_{i=1}^{n-1}[(q_i-q_{i+1})^2-1+i\epsilon]\}^{-1}.\end{aligned} \tag{4.1}$$

In terms of Feynman parameters, this is

$$\mathfrak{M}_n=(-1)^{n+1}g^{2(n+1)}(4\pi)^{-2n}I_n, \tag{4.2}$$

where

$$\begin{aligned}I_n=n!\int_0^1 &d\alpha_1d\alpha_2\cdots d\alpha_{n+1}d\beta_1d\beta_2\cdots d\beta_nd\gamma_1d\gamma_2\cdots d\gamma_n\\ &\times\delta(1-\sum_{i=1}^{n+1}\alpha_i-\sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n}\gamma_i)\Lambda_n^{n-1}D_n^{-n-1}.\end{aligned} \tag{4.3}$$

In (4.3), Λ_n is the $n\times n$ determinant

$$\Lambda_n=\begin{vmatrix}\beta_1+\gamma_1+\alpha_1+\alpha_2 & -\alpha_2 & 0 & \cdots & 0\\ -\alpha_2 & \beta_2+\gamma_2+\alpha_2+\alpha_3 & -\alpha_3 & \cdots & 0\\ 0 & -\alpha_3 & \beta_3+\gamma_3+\alpha_3+\alpha_4 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_n+\gamma_n+\alpha_n+\alpha_{n+1}\end{vmatrix}, \tag{4.4}$$

and

$$\begin{aligned}D_n=&s\prod_{i=1}^{n+1}\alpha_i+(\tfrac{1}{4}t-1)[2\prod_{i=1}^{n+1}\alpha_i+\alpha_1^2\Lambda_n^{(1,1)}+\alpha_{n+1}^2\Lambda_n^{(n,n)}-(\alpha_1+\alpha_{n+1})\Lambda_n]-(\sum_{i=1}^{n+1}\alpha_i+\sum_{i=1}^{n}\beta_i+\sum_{i=1}^{n}\gamma_i)\Lambda_n\\ &+\tfrac{1}{4}t\begin{vmatrix}\beta_1+\gamma_1+\alpha_1+\alpha_2 & -\alpha_2 & 0 & \cdots & 0 & -\beta_1+\gamma_1\\ -\alpha_2 & \beta_2+\gamma_2+\alpha_2+\alpha_3 & -\alpha_3 & \cdots & 0 & -\beta_2+\gamma_2\\ 0 & -\alpha_3 & \beta_3+\gamma_3+\alpha_3+\alpha_4 & \cdots & 0 & -\beta_3+\gamma_3\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \beta_n+\gamma_n+\alpha_n+\alpha_{n+1} & -\beta_n+\gamma_n\\ -\beta_1+\gamma_1 & -\beta_2+\gamma_2 & -\beta_3+\gamma_3 & \cdots & -\beta_n+\gamma_n & \sum_{i=1}^{n}(\beta_i+\gamma_i)\end{vmatrix}+i\epsilon,\end{aligned} \tag{4.5}$$

where $\Lambda_n^{(i,j)}$ denotes the ijth minor of Λ_n.

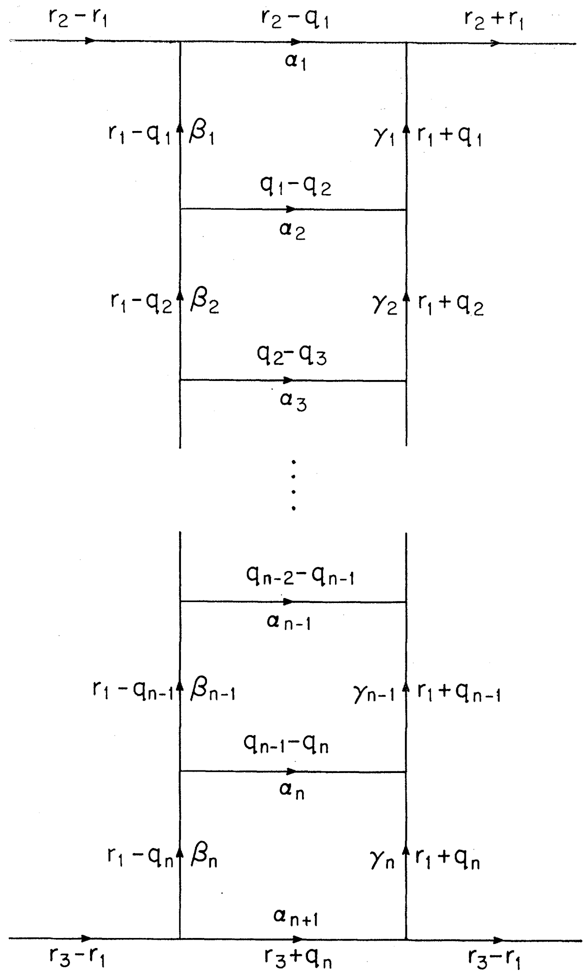

FIG. 3. General ladder diagram.

With any fixed t, we write down the Mellin transform of I_n as

$$\bar{I}_n(\xi)=\int_0^\infty ds\, s^{-\xi} I_n$$
$$=\Gamma(1-\xi)\Gamma(n+\xi)\int_0^1 d\alpha_1 d\alpha_2\cdots d\alpha_{n+1} d\beta_1 d\beta_2\cdots d\beta_n$$
$$\times d\gamma_1 d\gamma_2\cdots d\gamma_n \delta\Big(1-\sum_{i=1}^{n+1}\alpha_i-\sum_{i=1}^{n}\beta_i-\sum_{i=1}^{n}\gamma_i\Big)e^{-i\pi(n+\xi)}$$
$$\times\Big(\prod_{i=1}^{n+1}\alpha_i\Big)^{-1+\xi}\Lambda_n^{n-1}D_n'^{-n-\xi}, \quad (4.6)$$

where

$$D_n'=-D_n|_{s=0}. \quad (4.7)$$

Equation (4.6) is the generalization of (3.6). When the fixed t is not equal to 4, then the behavior of I_n as $s\to\infty$ is given by[4,6,7]

$$I_n=(n!)^{-1}s^{-1}\Big\{\ln s\int_0^1 dx\,[tx(1-x)-1+i\epsilon]^{-1}\Big\}^n$$
$$+O[s^{-1}(\ln s)^{n-1}]. \quad (4.8)$$

In particular, when this fixed t is close to, but not equal to, the threshold value 4, Eq. (4.8) implies that

$$I_n\sim(n!)^{-1}(-\pi)^n s^{-1}(\ln s)^n(4-t+i\epsilon)^{-n/2} \quad (4.9)$$

as $s\to\infty$. In general, therefore, I_n is much larger than $s^{-1}(\ln s)^n$ at $t=4$.

Since, at $t=4$, I_1 is of the order $s^{-1/2}$ while I_2 is of the order $s^{-1/2}\ln s$, it is reasonable to expect that I_n is of the order $s^{-1/2}(\ln s)^{n-1}$. We shall see shortly that this is indeed the case. A comparison with (4.9) then shows that, in the case of I_n, one important scale for $4-t$ must be $s^{-1/n}$. We shall return to the question of other scales later in this section.

B. Behavior at Threshold

Let $t=4$ and study the asymptotic behavior of I_n as $s\to\infty$. At this threshold value, D_n', as given by (4.5) and (4.7), simplifies to

$$D_n'=\Big(\sum_{i=1}^{n+1}\alpha_i\Big)\Lambda+\sum_{i,j=1}^{n}(\beta_i-\gamma_i)(\beta_j-\gamma_j)\Lambda_n^{(i,j)}. \quad (4.10)$$

From the experience we gained from Sec. 3, we can use the approximation of integrating the variables $\beta_i-\gamma_i$ from $-\infty$ to ∞. Since

$$\det\Lambda_n^{(i,j)}=\Lambda_n^{n-1}, \quad (4.11)$$

it follows from (4.6) that

$$\bar{I}_n(\xi)\sim(\tfrac{1}{2}\sqrt{\pi})^n\Gamma(1-\xi)\Gamma(\tfrac{1}{2}n+\xi)e^{-i\pi(n+\xi)}$$
$$\times\int_0^1 d\alpha_1 d\alpha_2\cdots d\alpha_{n+1}dx_1dx_2\cdots dx_n$$
$$\times\delta\Big(1-\sum_{i=1}^{n+1}\alpha_i-\sum_{i=1}^{n}x_i\Big)\Big(\prod_{i=1}^{n+1}\alpha_i\Big)^{-1+\xi}$$
$$\times\Big(\sum_{i=1}^{n+1}\alpha_i\Big)^{-\xi-n/2}\Lambda_n^{-\xi-1/2}, \quad (4.12)$$

where

$$x_i=\beta_i+\gamma_i \quad (4.13)$$

for $i=1, 2, \ldots, n$. Like (3.12), (4.12) is valid near the singularity at $\xi=\frac{1}{2}$; thus the coefficient in front of the integral on the right-hand side of (4.12) can be replaced by

$$-2i(-1)^n(\tfrac{1}{2}\sqrt{\pi})^{n+1}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}).$$

The leading term in the vicinity of $\xi=\frac{1}{2}$ can be obtained from (4.12) as follows:

$$\bar{I}_n(\xi)\sim 2i(-\tfrac{1}{2}\sqrt{\pi})^{n+1}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\times\int_0^1 d\alpha_1 d\alpha_2\cdots d\alpha_{n+1}\int_0^\infty dx_1dx_2\cdots dx_n \times\delta(1-\sum_{i=1}^{n+1}\alpha_i)(\prod_{i=1}^{n+1}\alpha_i)^{-1/2}\Lambda_n^{-\xi-1/2}$$

$$\sim 2i(-\tfrac{1}{2}\sqrt{\pi})^{n+1}\Gamma(\tfrac{1}{2}n+\tfrac{1}{2})\int_0^1 d\alpha_1 d\alpha_2\cdots d\alpha_{n+1}\times\delta(1-\sum_{i=1}^{n+1}\alpha_i)(\prod_{i=1}^{n+1}\alpha_i)^{-1/2}\times\int_1^\infty dx_1dx_2\cdots dx_n(\prod_{i=1}^{n}x_i)^{-\xi-1/2}$$

$$=2i(-\tfrac{1}{2}\pi)^{n+1}(\xi-\tfrac{1}{2})^{-n},\quad(4.14)$$

Consequently

$$I_n=2i(-\tfrac{1}{2}\pi)^{n+1}[(n-1!)]^{-1}s^{-1/2}\times\{(\ln s)^{n-1}+O[(\ln s)^{n-2}]\},\quad(4.15)$$

valid for $t=4$ and large s. When $n=1$ and 2, Eq. (4.15) reduces properly to (2.10) and (3.14), respectively.

C. Behavior near Threshold

The next problem is to generalize the considerations of Sec. 3 C. Let

$$t=4(1-\Delta' s^{-1/n}),\quad(4.16)$$

with Δ' fixed and $s\to\infty$. For $\Delta'\neq 0$, change the variables by

$$\alpha_i=\alpha_i' s^{-1/n}\quad(4.17)$$

for $i=1, 2, \ldots, n+1$,

$$\beta_i=\tfrac{1}{2}[x_i+y_i s^{-1/(2n)}]\quad\text{and}\quad\gamma_i=\tfrac{1}{2}[x_i-y_i s^{-1/(2n)}]\quad(4.18)$$

for $i=1, 2, \ldots, n$; then it follows from (4.3) that

$$\lim_{s\to\infty}s^{1/2}I_n=n!\int_0^1 dx_1dx_2\cdots dx_n\,\delta(1-\sum_{i=1}^{n}x_i)(\prod_{i=1}^{n}x_i)^{n-1}\times\int_0^\infty d\alpha_1'd\alpha_2'\cdots d\alpha_{n+1}'dy_1dy_2\cdots dy_n\times[(\prod_{i=1}^{n+1}\alpha_i')-(\Delta'+\sum_{i=1}^{n+1}\alpha_i')\prod_{i=1}^{n}x_i-\sum_{i=1}^{n}y_i^2\prod_{j\neq i}x_j+i\epsilon]^{-n-1}.\quad(4.19)$$

It is seen that (3.17) is a special case of (4.19).

Since we are unable to find a generalization of (3.18), we concentrate on the Mellin transform of (4.19) with respect to Δ'. This is carried out in Appendix G, and the result is

$$\int_0^\infty d\Delta'\,\Delta'^{-\xi}\lim_{s\to\infty}s^{1/2}I_n=\frac{-i\pi}{n}(-\tfrac{1}{2}\sqrt{\pi})^n e^{i\pi(1-\xi)/n}\times\sec\left[\frac{(1-\xi)\pi}{n}\right]\left[\Gamma\left(\frac{1-\xi}{n}\right)\Gamma\left(\frac{1}{2}-\frac{1-\xi}{n}\right)\right]^n.\quad(4.20)$$

In particular, it follows from (4.20) that

$$\int_0^\infty d\Delta'\,\Delta'^{-\xi}\lim_{s\to\infty}s^{1/2}I_n\sim(-\tfrac{1}{2}n\pi)^n(\xi-1+\tfrac{1}{2}n)^{-n-1}\quad(4.21)$$

as $\xi\to(1-\frac{1}{2}n)+$, and hence

$$\lim_{s\to\infty}s^{1/2}I_n\sim(n!)^{-1}(-\tfrac{1}{2}n\pi)^n\Delta'^{-n/2}(\ln\Delta')^n\quad(4.22)$$

for large Δ'. Note that, by (4.16), the right-hand side of (4.22) is

$$(n!)^{-1}(-\tfrac{1}{2}\pi)^n s^{-1/2}(1-\tfrac{1}{4}t)^{-n/2}\{\ln[s(1-\tfrac{1}{4}t)^n]\}^n.\quad(4.23)$$

Therefore (4.22) is consistent with (4.9). It also follows from (4.20) that

$$\int_0^\infty d\Delta'\,\Delta'^{-\xi}\lim_{s\to\infty}s^{1/2}I_n\sim -i\pi(-\tfrac{1}{2}\pi)^n n^{n-1}(1-\xi)^{-n}\quad(4.24)$$

as $\xi\to 1-$, and hence

$$\lim_{s\to\infty}s^{1/2}I_n\sim -i\pi[(n-1)!]^{-1}\times(-\tfrac{1}{2}\pi)^n n^{n-1}(-\ln\Delta')^{n-1}\quad(4.25)$$

for small Δ'.[9]

For $n>1$, we cannot take the limit $\Delta'\to 0$ in (4.25) to recover (4.15). Instead, the right-hand sides of (4.15) and (4.25) agree when

$$\Delta'^n=O(s^{-1})\quad(4.26)$$

or

$$4-t=O(s^{-2/n}).\quad(4.27)$$

5. DISCUSSION

The results in Sec. 4 are summarized as follows: For $t-4=0$, we have, from (4.2) and (4.15),

$$\mathfrak{M}\sim\sum_{n=1}^\infty\mathfrak{M}_n\sim\tfrac{1}{32}ig^4 s^{-1/2+g^2(32\pi)^{-1}},\quad s\to\infty.\quad(5.1)$$

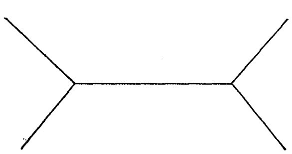

FIG. 4. Lowest-order scattering diagram.

Note that all $\mathfrak{M}_n$ are of the same sign and (5.1) is exactly the same as the attractive case in potential scattering.[11] Note also that $\mathfrak{M}_0$, the scattering amplitude corresponding to the diagram in Fig. 4, is of the order of s^{-1} and does not contribute. This is not the case if $t-4\neq 0$. For $t-4\neq 0$, we have

$$\mathfrak{M}\sim\sum_{n=0}^{\infty}\mathfrak{M}_n\sim -g^2 s^{-1+\alpha(t)}\,,\quad s\to\infty \tag{5.2}$$

where

$$\alpha(t)=-g^2(4\pi)^{-2}\int_0^1 dx\,[tx(1-x)-1+i\epsilon]^{-1}. \tag{5.3}$$

Note that the summation in (5.2) starts from $n=0$. Equation (5.1) means that as $t=4$, there is a Regge pole at $l=-\frac{1}{2}+g^2(32\pi)^{-1}$. By (5.2), this Regge pole, for $t\neq 4$, is located at $-1+\alpha(t)$. Observe that if the coupling constant g is very small, this Regge pole is "promoted" from -1 to $-\frac{1}{2}$ as t approaches 4. This phenomenon is discussed in more detail in the following paper.

Direct interpretation for the case when t is near the threshold value 4 is more difficult. However, we observe that at high energies, the amplitudes $\mathfrak{M}_n$ or, more specifically, $\bar{I}_n(\xi)$ as given by (4.12), are *exactly* the same as the Born amplitudes f_n in potential scattering studied in the following paper. Thus in the high-energy limit $s\to\infty$, there is no distinction between the φ^3 case and the potential-scattering case. The results in the potential-scattering case therefore lead us to the conclusion that there are, in addition to the Regge pole exhibited in (5.1), infinitely many Regge poles approaching $l=-\frac{1}{2}$ as t approaches 4.[12]

The promotion of Regge poles at threshold can be extended to more general diagrams. Let us consider a diagram with a two-body intermediate state in the t channel. As $s\to\infty$ there is, among others, a contribution from the region where these two particles carry transverse momenta only. As $t=(m_a+m_b)^2$, where m_a and m_b are the masses of these two intermediate-state particles, respectively, the propagators for these two particles blow up simultaneously and a square-root divergence occurs. This means that at $t=(m_a+m_b)^2$ the power of s for this diagram is promoted by $\frac{1}{2}$.

When t is at a three-particle threshold, it appears that the divergence for the high-energy amplitudes, if it occurs, can only be logarithmic; and there is no promotion at this point.

APPENDIX A

In Sec. 2, we discuss the asymptotic behavior as $s\to\infty$ of the box diagram. Since it has been known for a number of years that the ladder diagrams are most conveniently discussed by means of Mellin transforms,[4] we shall in this appendix treat the box diagram more systematically in this way.

1. Mellin Transform

Let t be fixed and define the Mellin transform of I_1 by

$$\bar{I}_1(\xi)=\int_0^\infty ds\,s^{-\xi}I_1; \tag{A1}$$

then it follows from (2.3) that

$$\begin{aligned}\bar{I}_1(\xi)=\pi\xi\csc\pi\xi\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4\\ \times\delta(1-\alpha_1-\alpha_2-\alpha_3-\alpha_4)\alpha_1^{-1+\xi}\alpha_3^{-1+\xi}\\ \times[t\alpha_2\alpha_4+(\alpha_1+\alpha_3)(\alpha_2+\alpha_4)-1+i\epsilon]^{-1-\xi}.\end{aligned} \tag{A2}$$

The right-hand side of (A2) can be reduced to a single integral as follows, with the obvious changes of variables:

$$\begin{aligned}\bar{I}_1(\xi)&=\pi\xi\csc\pi\xi\int_0^1 d\alpha_1\int_0^{1-\alpha_1}d\alpha_3\\ &\quad\times\int_0^{1-\alpha_1-\alpha_3}d\alpha_2\,\alpha_1^{-1+\xi}\alpha_3^{-1+\xi}[t\alpha_2(1-\alpha_1-\alpha_2-\alpha_3)\\ &\qquad+(\alpha_1+\alpha_3)(1-\alpha_1-\alpha_3)-1+i\epsilon]^{-1-\xi}\\ &=\pi\xi\csc\pi\xi\int_0^1 d\alpha_1\int_{\alpha_1}^1 dy\int_0^1 dx\,\alpha_1^{-1+\xi}(y-\alpha_1)^{-1+\xi}\\ &\quad\times(1-y)[t(1-y)^2x(1-x)+y(1-y)-1+i\epsilon]^{-1-\xi}\\ &=\pi\xi\csc\pi\xi[\Gamma(\xi)]^2[\Gamma(2\xi)]^{-1}\int_0^1 dx\int_0^1 dy\,y^{-1+2\xi}\\ &\quad\times(1-y)[t(1-y)^2x(1-x)+y(1-y)-1+i\epsilon]^{-1-\xi}\\ &=\pi\xi\csc\pi\xi[\Gamma(\xi)]^2[\Gamma(2\xi)]^{-1}\\ &\quad\times\int_0^1 dx\int_0^\infty dy'\,y'[tx(1-x)y'^2\\ &\qquad-(y'^2+y'+1)+i\epsilon]^{-1-\xi}.\end{aligned} \tag{A3}$$

In order to evaluate the integral over y', consider

$$F(A)=\int_0^\infty dy'\,y'^\lambda[Ay'^2+y'+1]^{-\tau}, \tag{A4}$$

which is convergent for $2\tau>\lambda+1>0$ and $A>0$. Note that $F(A)$ is analytic at $A=\frac{1}{4}$, where

$$F(\tfrac{1}{4})=2^{\lambda+1}\Gamma(1+\lambda)\Gamma(2\tau-\lambda-1)/\Gamma(2\tau). \tag{A5}$$

[11] H. Cheng and T. T. Wu, following paper, Phys. Rev. D 2, 2298 (1970).
[12] V. Gribov and I. Pomeranchuk, Phys. Rev. Letters 9, 238 (1962).

Moreover, $F(A)$ satisfies the second-order ordinary differential equation

$$A(1-4A)F''(A)+[\lambda-\tau+2-2A(2\lambda+5)]\times F'(A)-(\lambda+1)(\lambda+2)F(A)=0. \quad \text{(A6)}$$

It therefore follows from (A5) and (A6) that $F(A)$ is a hypergeometric function [see Eq. (1) on p. 56 and Eq. (5) on p. 105 of Ref. 5]:

$$F(A)=2^{\lambda+1}\frac{\Gamma(1+\lambda)\Gamma(2\tau-\lambda-1)}{\Gamma(2\tau)}\times F(\tfrac{1}{2}+\tfrac{1}{2}\lambda;\,1+\tfrac{1}{2}\lambda;\,\tfrac{1}{2}+\tau;\,1-4A). \quad \text{(A7)}$$

Application of this result (A7) to (A3) gives

$$\bar{I}_1(\xi)=-4\pi\xi\csc\pi\xi\, e^{-i\pi\xi}[\Gamma(\xi)]^2[\Gamma(2+2\xi)]^{-1}\times\int_0^1 dx\, F(1,\tfrac{3}{2};\tfrac{3}{2}+\xi;4tx(1-x)-3+i\epsilon). \quad \text{(A8)}$$

Finally, by the Legendre duplication formula for the Γ function, (A8) can be put in the form

$$\bar{I}_1(\xi)=-2^{1-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{3}{2}+\xi)]^{-1}\times\int_0^1 dx\, F(1,\tfrac{3}{2};\tfrac{3}{2}+\xi;4tx(1-x)-3+i\epsilon)$$
$$=-2^{-\xi+3/2}\pi^{1/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)\int_0^1 dx \times P_{-\xi+1/2}{}^{-\xi-1/2}(z)(z^2-1)^{-(1+2\xi)/4}z^2, \quad \text{(A9)}$$

where

$$z=\tfrac{1}{2}[1-tx(1-x)-i\epsilon]^{-1/2}. \quad \text{(A10)}$$

In writing down (A9) we have used the relation between hypergeometric functions and associated Legendre functions [see Eq. (24) on pp. 128–129 of Ref. 5].

We see from (A9) that, for $t\neq 4$, $\bar{I}_1(\xi)$ has singularities at $\xi=0,-1,-2,\ldots$. For $t=4$, however, the argument of the hypergeometric function reaches its maximum at 1, and there are additional singularities to be discussed in detail in Sec. A 3 of this appendix. So far no approximation has been made and (A9) is exact.

2. Behavior away from Threshold

Let $t\neq 4$, and expand (A9) for ξ near zero. Since

$$F(1,\tfrac{3}{2};\tfrac{3}{2}+\xi;4tx(1-x)-3+i\epsilon)=\{4[1-tx(1-x)-i\epsilon]\}^{-1}\times F\left(1,\xi;\tfrac{3}{2}+\xi;\frac{4tx(1-x)-3+i\epsilon}{4tx(1-x)-4+i\epsilon}\right), \quad \text{(A11)}$$

the expansion of this hypergeometric function near $\xi=0$ is straightforward. For example, the leading term is given by

$$\bar{I}_1(\xi)\sim-\xi^{-2}\int_0^1 dx[1-tx(1-x)-i\epsilon]^{-1}, \quad \text{(A12)}$$

and hence, for $s\to\infty$ with fixed $t\neq 4$,

$$I_1\sim s^{-1}\ln s\int_0^1 dx\,[tx(1-x)-1+i\epsilon]^{-1}. \quad \text{(A13)}$$

This is the same as (2.5).

We instead turn our attention to the case where the *fixed t* is near, but not equal to 4. Let

$$t=4(1-\delta); \quad \text{(A14)}$$

then, for small δ and ξ,

$$\bar{I}_1(\xi)=-2^{1-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{3}{2}+\xi)]^{-1}\times\int_0^1 dx\, F(1,\tfrac{3}{2};\tfrac{3}{2}+\xi;t(1-x^2)-3+i\epsilon)$$
$$\sim-2^{1-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{3}{2}+\xi)]^{-1}\times\int_0^1 dx\, F(1,\tfrac{3}{2};\tfrac{3}{2}+\xi;1-4\delta-4x^2+i\epsilon)$$
$$=-2^{1-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{3}{2}+\xi)]^{-1}\times\int_0^1 dx\Big[-\frac{1+2\xi}{2(1-\xi)}F(1,\tfrac{3}{2};2-\xi;4\delta+4x^2-i\epsilon)$$
$$+2\pi^{-1/2}\Gamma(\tfrac{3}{2}+\xi)\Gamma(1-\xi)(4\delta+4x^2-i\epsilon)^{-1+\xi}\times F(\tfrac{1}{2}+\xi;\xi;-1+\xi;4\delta+4x^2-i\epsilon)\Big]$$
$$\sim-\pi^2\csc^2\pi\xi\, e^{-i\pi\xi}\int_0^1 dx\,(\delta+x^2-i\epsilon)^{-1+\xi}$$
$$\sim-\tfrac{1}{2}\pi^{1/2}\xi^{-2}(\delta-i\epsilon)^{\xi-1/2}(1-i\pi\xi)\Gamma(\tfrac{1}{2}-\xi)/\Gamma(1-\xi)$$
$$\sim-\tfrac{1}{2}\pi\xi^{-2}(\delta-i\epsilon)^{-1/2}\times[1+\xi\ln(\delta-i\epsilon)+2\xi\ln 2-i\pi\xi]. \quad \text{(A15)}$$

Therefore, for $s\to\infty$ with a *fixed t* near, but not at, the threshold,

$$I_1\sim-\tfrac{1}{2}\pi(\delta-i\epsilon)^{-1/2}s^{-1}[\ln s+\ln 4(\delta-i\epsilon)-i\pi]$$
$$=-\pi s^{-1/2}[4s(\delta-i\epsilon)]^{-1/2}\{\ln[4s(\delta-i\epsilon)]-i\pi\}. \quad \text{(A16)}$$

This is the same as (2.11).

Note that the right-hand side of (A16) is of the form $s^{-1/2}$ multiplied by a function of $s\delta$. This is an indication that, if δ is zero or of the order $1/s$, I_1 is of the order of magnitude $s^{-1/2}$, not s^{-1}. This is indeed the case as already seen in Sec. 2.

3. Behavior at Threshold

Simplifications occur at the threshold

$$t=4. \tag{A17}$$

In this case, substitution into (A9) gives

$$\bar{I}_1(\xi)=-2^{1-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{3}{2}+\xi)]^{-1}\times\int_0^1 dx\, F(1,\tfrac{3}{2};\tfrac{3}{2}+\xi;1-4x^2)$$

$$=-2^{3-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{3}{2}+\xi)]^{-1}\times\int_0^1 dx\,(1+2x)^{-2}F\left(2,\tfrac{3}{2}-\xi;\tfrac{3}{2}+\xi;\frac{1-2x}{1+2x}\right)$$

$$=-2^{1-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{3}{2}+\xi)]^{-1}\frac{1+2\xi}{1-2\xi}\times[F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;1)-F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;-\tfrac{1}{3})]$$

$$=-2^{1-2\xi}\pi^{3/2}\csc\pi\xi\, e^{-i\pi\xi}\Gamma(\xi)[\Gamma(\tfrac{1}{2}+\xi)]^{-1}\times(1-2\xi)^{-1}[1-2F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;-\tfrac{1}{3})]. \tag{A18}$$

This result (A18) can also be obtained directly from (A3), as shown in Appendix B. In the present derivation, use has been made of the quadratic transformation of hypergeometric functions [see, for example, Eq. (26) on p. 65 of Ref. 5].

Equation (A18) shows explicitly that there is a singularity at $\xi=\frac{1}{2}$, which is absent when $t\neq 4$. We list some of the properties of $\bar{I}_1(\xi)$ that follow from this exact expression (2.22), as shown in Appendix C.

(a) For ξ near $\frac{1}{2}$, let $\xi=\frac{1}{2}+\zeta$ and

$$\bar{I}_1(\xi)=\tfrac{1}{2}i\pi^2\zeta^{-1}+O(1). \tag{A19}$$

(b) For ξ near 0,

$$\bar{I}_1(\xi)=\xi^{-2}+2\xi^{-1}(1+3^{-1/2}\pi-i\tfrac{1}{2}\pi)+O(1). \tag{A20}$$

(c) For ξ near -1, let $\xi=-1+\zeta$ and

$$\bar{I}_1(\xi)=(5/3)\zeta^{-2}+\zeta^{-1}[4/9+2\pi/\sqrt{3}-(5/3)i\pi]+O(1). \tag{A21}$$

Equation (A21) can be generalized to other negative integers and this generalization is also discussed in Appendix C.

It follows from (A19)–(A21) that, for $s\to\infty$ with $t=4$,

$$I_1=\tfrac{1}{2}i\pi^2s^{-1/2}+s^{-1}(\ln s-i\pi+2+2\pi/\sqrt{3})+s^{-2}[(5/3)(\ln s-i\pi)+4/9+2\pi/\sqrt{3}]+O(s^{-3}\ln s). \tag{A22}$$

Note that this behavior is quite different from that with $t\neq 4$ [(A16), for example].

The asymptotic behavior for $s\to\infty$ of I_1 is given by (2.5), (A22), and (2.9).

APPENDIX B

In this appendix, we shall derive, for $t=4$, (A18) more directly from (A3). By setting $t=4$ in (A3), we get

$$\bar{I}_1(\xi)=\pi\xi\csc\pi\xi[\Gamma(\xi)]^2[\Gamma(2\xi)]^{-1}e^{-i\pi\xi}\times\int_0^1 dx\int_0^\infty dy'y'(x^2y'^2+y'+1)^{-1-\xi}. \tag{B1}$$

We evaluate the integral on the right-hand side of (B1) by the change of variable

$$x^2=y'^{-2}(y'+1)x'^2 \tag{B2}$$

and

$$x'^2=y''^2/(y''+1), \tag{B3}$$

so that

$$\int_0^1 dx\int_0^\infty dy'y'(x^2y'^2+y'+1)^{-1-\xi}$$

$$=\int_0^\infty dx'\int_{[x'^2+(x'^4+4x'^2)^{1/2}]/2}^{\infty} dy'\,(y'+1)^{-\xi-1/2}\times(x'^2+1)^{-\xi-1}$$

$$=2(1-2\xi)^{-1}\int_0^\infty dx'\,\{1+\tfrac{1}{2}[x'^2+(x'^4+4x'^2)^{1/2}]\}^{-\xi+1/2}\times(x'^2+1)^{-\xi-1}$$

$$=(1-2\xi)^{-1}\int_0^\infty dy''\,(y''+2)(y''^2+y''+1)^{-1-\xi}$$

$$=\tfrac{1}{2}(1-2\xi)^{-1}[\xi^{-1}+6(1+2\xi)^{-1}F(\tfrac{1}{2},1;\tfrac{3}{2}+\xi;-3)]$$

$$=\tfrac{1}{2}(1-2\xi)^{-1}[\xi^{-1}+4(1+2\xi)^{-1}F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;-\tfrac{1}{3})]$$

$$=\tfrac{1}{2}(1-2\xi)^{-1}\xi^{-1}\{1+3[-1+\tfrac{4}{3}F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;-\tfrac{1}{3})]\}$$

$$=-(1-2\xi)^{-1}\xi^{-1}[1-2F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;-\tfrac{1}{3})]. \tag{B4}$$

In the above manipulation, we have used (A7) and also Eq. (6) on p. 111 and Eq. (39) on p. 103 of Ref. 5. The desired result (A18) then follows from (B1) and (B4) with the help of the Legendre duplication formula.

APPENDIX C

In this appendix, we derive (A19)–(A21) from (A18).

(a) When ξ is near $\frac{1}{2}$, let $\xi=\frac{1}{2}+\zeta$ and

$$\bar{I}_1(\xi)=-\pi^{3/2}(-i)\pi^{1/2}(-2\zeta)^{-1}\times[1-2F(1,0;1;-\tfrac{1}{3})]+O(1)=\tfrac{1}{2}i\pi^2\zeta^{-1}+O(1). \tag{C1}$$

When ξ is near $-\frac{1}{2}$, $\bar{I}_1(\xi)$ is bounded because the singularity due to the hypergeometric function in (A18) is canceled by the $\Gamma(\frac{1}{2}+\xi)$ in the denominator. More generally, this holds for ξ near $-n-\frac{1}{2}$, $n=0, 1, 2, \ldots$.

(b) When ξ is small, we need the relations

$$\psi(1)=-\gamma \tag{C2}$$

and

$$\psi(\tfrac{1}{2})=-\gamma-2\ln 2, \tag{C3}$$

where

$$\psi(z)=\Gamma'(z)/\Gamma(z) \tag{C4}$$

and γ is Euler's constant. Moreover, for small ξ,

$$\begin{aligned}F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;-\tfrac{1}{3})&=\tfrac{3}{4}F(1,2\xi;\tfrac{1}{2}+\xi;\tfrac{1}{4})\\&\sim\tfrac{3}{4}[1+\xi F(1,1;\tfrac{3}{2};\tfrac{1}{4})]\\&=\tfrac{3}{4}[1+2\times 3^{-3/2}\pi\xi].\end{aligned} \tag{C5}$$

The substitution of (C2), (C3), and (C5) into (A18) gives

$$\begin{aligned}\bar{I}_1(\xi)&\sim-2(1-2\xi\ln 2)(1-i\pi\xi)\\&\quad\times\xi^{-2}(1+2\xi\ln 2)(1+2\xi)(-\tfrac{1}{2}-3^{-1/2}\pi\xi)\\&\sim\xi^{-2}[1+\xi(2-i\pi+2\times 3^{-1/2}\pi)].\end{aligned} \tag{C6}$$

(c) For ξ near -1, let $\xi=-1+\zeta$ and

$$\begin{aligned}&F(1,\tfrac{1}{2}-\xi;\tfrac{1}{2}+\xi;-\tfrac{1}{3})\\&=F(1,\tfrac{3}{2}-\zeta;-\tfrac{1}{2}+\zeta;-\tfrac{1}{3})\\&=\tfrac{3}{4}(\tfrac{1}{2}-\zeta)^{-1}[(-\tfrac{3}{2}+\zeta)-(-2+2\zeta)\\&\qquad\times F(1,\tfrac{1}{2}-\zeta;-\tfrac{1}{2}+\zeta;-\tfrac{1}{3})]\\&=\tfrac{3}{4}(\tfrac{1}{2}-\zeta)^{-1}\{(-\tfrac{3}{2}+\zeta)+\tfrac{3}{2}(1-\zeta)\\&\qquad\times[1+\tfrac{2}{3}F(1,\tfrac{1}{2}-\zeta;\tfrac{1}{2}+\zeta;-\tfrac{1}{3})]\}\\&\sim\tfrac{3}{2}\{-\tfrac{1}{2}\zeta+\tfrac{3}{4}[1+\zeta+2\times 3^{-3/2}\pi\zeta]\}\\&=(9/8)[1+\tfrac{1}{3}\zeta(1+2\times 3^{-1/2}\pi)].\end{aligned} \tag{C7}$$

The substitution of (C2), (C3), and (C7) into (A18) then gives

$$\begin{aligned}\bar{I}_1(\xi)&\sim-8(1-2\zeta\ln 2)\pi^{1/2}\zeta^{-2}(1-i\pi\zeta)\pi^{-1/2}(1+2\zeta\ln 2)\\&\quad\times\tfrac{1}{6}(1-\tfrac{1}{3}\zeta)\{1-(9/4)[1+\tfrac{1}{3}\zeta(1+2\times 3^{-1/2}\pi)]\}\\&\sim(5/3)[\zeta^{-2}+\zeta^{-1}(4/15+2\sqrt{3}\pi/5-i\pi)].\end{aligned} \tag{C8}$$

(d) The procedure used to derive (C7) and (C8) can be generalized to the vicinity of any negative integer. We indicate briefly how this may be carried out. If ξ is near $-n$, where $n=1, 2, \ldots$, define $\xi=-n+\zeta$. Also define

$$a_n=F(1,\tfrac{1}{2}+n-\zeta;\tfrac{1}{2}-n+\zeta;-\tfrac{1}{3}). \tag{C9}$$

We want to relate a_n to a_{n-1}. For this purpose, we use Gauss's relation between contiguous hypergeometric functions [see Eqs. (37) and (38) on p. 103 of Ref. 5]:

$$\begin{aligned}&(-\tfrac{1}{2}+n-\zeta)\tfrac{4}{3}a_n-(-\tfrac{1}{2}-n+\zeta)+(-2n-2\zeta)\\&\qquad\times F(1,-\tfrac{1}{2}+n-\zeta;\tfrac{1}{2}-n+\zeta;-\tfrac{1}{3})=0\end{aligned} \tag{C10}$$

and

$$\tfrac{4}{3}F(1,-\tfrac{1}{2}+n-\zeta;\tfrac{1}{2}-n+\zeta;-\tfrac{1}{3})-1-\tfrac{2}{3}a_{n-1}=0. \tag{C11}$$

It follows from (C10) and (C11) that

$$a_n=\tfrac{3}{4}(n-\tfrac{1}{2}-\zeta)^{-1}[\tfrac{1}{2}(n-1-\zeta)+(n-\zeta)a_{n-1}]. \tag{C12}$$

The required answer may be obtained by solving this linear difference equation of first order.

APPENDIX D

In this appendix, we derive (3.18) from (3.17). First the y_1 and y_2 integrals can be carried out trivially:

$$(3.17)=-\tfrac{1}{4}\pi\int_0^1 dx\times\int_0^\infty\frac{[x(1-x)]^{1/2}d\alpha_1 d\alpha_2 d\alpha_3}{[\alpha_1\alpha_2\alpha_3-(\Delta+\alpha_1+\alpha_2+\alpha_3)x(1-x)+i\epsilon]^2}, \tag{D1}$$

where the primes on the dummy variables α have been omitted. We next carry out explicitly the α_3 and x integrations to get

$$\begin{aligned}(3.17)&=\tfrac{1}{4}\pi\int_0^1 dx\times\int_0^\infty\frac{[x(1-x)]^{-1/2}d\alpha_1 d\alpha_2}{[\alpha_1\alpha_2-x(1-x)+i\epsilon](\Delta+\alpha_1+\alpha_2)}\\&=\tfrac{1}{4}\pi^2\int_0^\infty d\alpha_1 d\alpha_2\,[\alpha_1\alpha_2(\alpha_1\alpha_2-\tfrac{1}{4})+i\epsilon]^{-1/2}\\&\qquad\times(\Delta+\alpha_1+\alpha_2)^{-1}.\end{aligned} \tag{D2}$$

To get rid of the $i\epsilon$, it is convenient to shift the contours of integration to the positive imaginary axis:

$$(3.17)=\tfrac{1}{4}\pi^2\int_0^\infty d\alpha_1 d\alpha_2\,[\alpha_1\alpha_2(\alpha_1\alpha_2+\tfrac{1}{4})]^{-1/2}\times[\Delta+i(\alpha_1+\alpha_2)]^{-1}. \tag{D3}$$

Let

$$x_1=(\alpha_1+\alpha_2)/\Delta \tag{D4}$$

and

$$\sin\phi_1=(\alpha_1-\alpha_2)/(\alpha_1+\alpha_2); \tag{D5}$$

then

$$(3.17)=\pi^2\int_0^{\pi/2}d\phi_1\int_0^\infty dx_1\,(1+\Delta^2x_1^2\cos^2\phi_1)^{-1/2}\times(1+ix_1)^{-1}. \tag{D6}$$

Equation (3.18) follows from (D6).

APPENDIX E

In this appendix we obtain (3.19) from (3.17). It is convenient to take the Mellin transform of (D1) and define, for $i=1, 2, 3$,

$$\alpha_i=[x(1-x)]^{1/2}\bar{\alpha}_i. \tag{E1}$$

The result is then

$$\int_0^\infty d\Delta'\,\Delta'^{-\xi}\lim_{s\to\infty} s^{1/2}I_2$$

$$=-\tfrac{1}{4}\pi^2\xi\csc\pi\xi\int_0^1 dx\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3[x(1-x)]^{\xi-1/2}$$

$$\times[-\alpha_1\alpha_2\alpha_3+(\alpha_1+\alpha_2+\alpha_3)x(1-x)-i\epsilon]^{-1-\xi}$$

$$=-\tfrac{1}{4}\pi^2\xi\csc\pi\xi[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\xi)]^2[\Gamma(1-\xi)]^{-1}\int_0^\infty d\bar\alpha_1 d\bar\alpha_2 d\bar\alpha_3$$

$$\times[-\bar\alpha_1\bar\alpha_2\bar\alpha_3+(\alpha_1+\alpha_2+\alpha_3)-i\epsilon]^{-1-\xi}$$

$$=\tfrac{1}{4}\pi^2\csc\pi\xi[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\xi)]^2[\Gamma(1-\xi)]^{-1}e^{-i\pi\xi/2}\int_0^\infty d\bar\alpha_1 d\bar\alpha_2$$

$$\times(1+\bar\alpha_1\bar\alpha_2)^{-1}(\bar\alpha_1+\bar\alpha_2)^{-\xi}$$

$$=\tfrac{1}{8}\pi^2\csc\pi\xi[\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\xi)]^2[\Gamma(1-\xi)]^{-1}e^{-i\pi\xi/2}$$

$$\times[\Gamma(\tfrac{1}{2}\xi)]^3\Gamma(1-\tfrac{1}{2}\xi)[\Gamma(\xi)]^{-1}$$

$$=\tfrac{1}{8}\pi^2(\csc\tfrac{1}{2}\pi\xi)e^{-i\pi\xi/2}[\Gamma(\tfrac{1}{2}\xi)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\xi)]^2. \quad \text{(E2)}$$

APPENDIX F

We study further in this appendix the behavior of I_2, as given by (3.3), when $4-t$ is of the order of s^{-1}. More precisely, we consider

$$\frac{\partial I_2}{\partial t}=-6\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\beta_1 d\beta_2 d\gamma_1 d\gamma_2$$

$$\times\delta(1-\alpha_1-\alpha_2-\alpha_3-\beta_1-\beta_2-\gamma_1-\gamma_2)\Lambda$$

$$\times[\beta_1\gamma_1(\beta_2+\gamma_2+\alpha_2+\alpha_3)+\beta_2\gamma_2(\beta_1+\gamma_1+\alpha_1+\alpha_2)$$

$$+\alpha_2(\beta_1\gamma_2+\beta_2\gamma_1)]D_2^{-4} \quad \text{(F1)}$$

in the limit of large s with fixed T, where T is defined by (2.7). Note that this integral fails to converge when $T=0$. We shall therefore assume throughout this appendix

$$T\neq 0. \quad \text{(F2)}$$

The leading contribution to the right-hand side of (F1) comes from two *distinct* regions: (i) α_1, α_2, β_1, γ_1, and $\beta_2-\gamma_2$ are all small; and (ii) α_1, α_2, α_3, β_2, γ_2, and $\beta_1-\gamma_1$ are all small. By symmetry the contributions from these two regions are the same, and therefore it is sufficient to consider only the first region. Let

$$\alpha_1=\alpha_1'/s,\quad \alpha_2=\alpha_2'/s,\quad \alpha_3=\alpha_3'/s,$$
$$\beta_1=\beta_1'/s,\quad \gamma_1=\gamma_1'/s,$$

and

$$\beta_2=\tfrac{1}{2}(1+y/\sqrt{s}); \quad \text{(F3)}$$

then

$$\gamma_2\sim\tfrac{1}{2}(1-y/\sqrt{s}). \quad \text{(F4)}$$

Accordingly, for fixed $T\neq 0$,

$$\lim_{s\to\infty} s^{-1/2}\frac{\partial I_2}{\partial t}$$

$$=3\int_0^\infty d\alpha_1'd\alpha_2'd\alpha_3'd\beta_1'd\gamma_1'dy\,(\beta_1'+\gamma_1'+\alpha_1'+\alpha_2')^2$$

$$\times[-\alpha_1'\alpha_2'\alpha_3'+(T+\alpha_1'+\alpha_2'+\alpha_3')$$

$$\times(\beta_1'+\gamma_1'+\alpha_1'+\alpha_2')+(\beta_1'-\gamma_1')^2$$

$$+x^2(\beta_1'+\gamma_1'+\alpha_1'+\alpha_2')-i\epsilon]^4. \quad \text{(F5)}$$

For this sixfold integral, four of the integrations can be explicitly carried out: α_3', x, $\beta_1'-\gamma_1'$, and $\alpha_1'-\alpha_2'$. Let

$$x_1=\beta_1'+\gamma_1' \quad\text{and}\quad x_2=\alpha_1'+\alpha_2'; \quad \text{(F6)}$$

then[9]

$$\lim_{s\to\infty} s^{-1/2}\frac{\partial I_2}{\partial t}=\tfrac{1}{4}\pi\left\{\int_{R_1} dx_1dx_2x_1(x_1+x_2)^{-1/2}(T+x_2)^{-2}\right.$$

$$\times[2x_1^2+3(T+x_2)(x_1+x^2)][x_1^2+(T+x_2)(x_1+x_2)]^{-3/2}$$

$$\times[4(x_1+x_2)-x_2^2]^{-1/2}\sin^{-1}(\tfrac{1}{2}x_2(x_1+x_2)^{-1/2})$$

$$+\int_{R_2} dx_1dx_2x_1(x_1+x_2)^{-1/2}(T+x_2)^{-2}$$

$$\times[2x_1^2+3(T+x_2)(x_1+x_2)][x_1^2+(T+x_2)(x_1+x_2)]^{-3/2}$$

$$\times[x_2^2-4(x_1+x_2)]^{-1/2}$$

$$\left.\times[\tfrac{1}{2}i\pi-\cosh^{-1}(\tfrac{1}{2}x_2(x_1+x_2)^{-1/2})]\right\}, \quad \text{(F7)}$$

where R_1 and R_2 are the regions where $x_1>0$, $x_2>0$, and $4(x_1+x_2)-x_2^2$ is positive and negative, respectively. For small T, the right-hand side of (F7) behaves like $T^{-1/2}$.

The meaning of this result is as follows: When $4-t$ is of the order of s^{-1}, the leading term for I_2, of order $s^{-1/2}\ln s$, does not depend on T, but the next term, of order $s^{-1/2}$, depends on T in a complicated manner. It is this term that reflects the complicated singularity of Gribov type.[12]

APPENDIX G

In this appendix we derive (4.20) from (4.19). It is convenient to carry out the y integrations first:

$$\lim_{s\to\infty} s^{1/2}I_n=-(-\tfrac{1}{2}\sqrt{\pi})^n\Gamma(\tfrac{1}{2}n+1)\int_0^1 dx_1dx_2\cdots dx_n$$

$$\times\delta(1-\sum_{i=1}^n x_i)(\prod_{i=1}^n x_i)^{(n-1)/2}\int_0^\infty d\alpha_1'd\alpha_2'\cdots d\alpha_{n+1}'$$

$$\times[-(\prod_{i=1}^{n+1}\alpha_i')+(\Delta'+\sum_{n=1}^{n+1}\alpha_i')\prod_{i=1}^n x_i-i\epsilon]^{-(n+2)/2}. \quad \text{(G1)}$$

Mellin transformation with respect to Δ' then gives

$$\int_0^\infty d\Delta'\,\Delta'^{-\xi}\lim_{s\to\infty}s^{1/2}I_n$$

$$=-(-\tfrac{1}{2}\sqrt{\pi})^n\Gamma(1-\xi)\Gamma(\tfrac{1}{2}n+\xi)\int_0^1 dx_1dx_2\cdots dx_n$$

$$\times\delta(1-\sum_{i=1}^n x_i)(\prod_{i=1}^n x_i)^{\xi+(n-3)/2}\int_0^\infty d\alpha_1'd\alpha_2'\cdots d\alpha_{n+1}'$$

$$\times[-(\prod_{i=1}^{n+1}\alpha_i')+(\sum_{n=1}^{n+1}\alpha_i')\prod_{i=1}^n x_i-i\epsilon]^{-\xi-n/2}$$

$$=-(-\tfrac{1}{2}\sqrt{\pi})^n\Gamma(1-\xi)\Gamma(\tfrac{1}{2}n+\xi)\int_0^1 dx_1dx_2\cdots dx_n$$

$$\times\delta(1-\sum_{i=1}^n x_i)(\prod_{i=1}^n x_i)^{(1-\xi)/n-1}\int_0^\infty d\alpha_1''d\alpha_2''\cdots d\alpha_{n+1}''$$

$$\times[-(\prod_{i=1}^{n+1}\alpha_i'')+(\sum_{n=1}^{n+1}\alpha_i'')-i\epsilon]^{-\xi-n/2}$$

$$=-(-\tfrac{1}{2}\sqrt{\pi})^n\left[\Gamma\left(\frac{1-\xi}{n}\right)\right]^n\Gamma(\tfrac{1}{2}n+\xi)$$

$$\times\int_0^\infty d\alpha_1''d\alpha_2''\cdots d\alpha_{n+1}''$$

$$\times[-(\prod_{i=1}^{n+1}\alpha_i'')+(\sum_{i=1}^{n+1}\alpha_i'')-i\epsilon]^{-\xi-n/2}. \quad \text{(G2)}$$

In the above, the variables α_i'', $i=1, 2, \ldots, n$, are defined by

$$\alpha_i''=(\prod_{j=1}^n x_i)^{-1/n}\alpha_i'. \quad \text{(G3)}$$

In (G2), we change the variables further by

$$\rho''=\sum_{i=1}^{n+1}\alpha_i'', \quad \text{(G4)}$$

$$\bar{\alpha}_i=\alpha_i''/\rho'', \quad \text{(G5)}$$

and

$$\rho'=(\prod_{i=1}^{n+1}\alpha_i)^{1/n}\rho''; \quad \text{(G6)}$$

then

$$\int_0^\infty d\Delta'\,\Delta'^{-\xi}\lim_{s\to\infty}s^{1/2}I_n$$

$$=-(-\tfrac{1}{2}\sqrt{\pi})^n\left[\Gamma\left(\frac{1-\xi}{n}\right)\right]^n\Gamma(\tfrac{1}{2}n+\xi)$$

$$\times\int_0^\infty d\rho''\int_0^1 d\bar{\alpha}_1d\bar{\alpha}_2\cdots d\bar{\alpha}_{n+1}\rho''^n$$

$$\times\delta(1-\sum_{i=1}^{n+1}\bar{\alpha}_i)\{\rho''[1-\rho''^n(\prod_{i=1}^{n+1}\bar{\alpha}_i)-i\epsilon]\}^{-\xi-n/2}$$

$$=-(-\tfrac{1}{2}\sqrt{\pi})^n\left[\Gamma\left(\frac{1-\xi}{n}\right)\right]^n\Gamma(\tfrac{1}{2}n+\xi)$$

$$\times\int_0^\infty d\rho'\,\rho'^{-\xi+n/2}\int_0^1 d\bar{\alpha}_1d\bar{\alpha}_2\cdots d\bar{\alpha}_{n+1}$$

$$\times\xi(1-\sum_{i=1}^{n+1}\bar{\alpha}_i)(\prod_{i=1}^{n+1}\bar{\alpha}_i)^{-(-\xi+1+n/2)/n}$$

$$\times(1-\rho'^n-i\epsilon)^{-\xi-n/2}$$

$$=-i(-\tfrac{1}{2}\sqrt{\pi})^n\frac{1}{n}e^{i\pi(1-\xi)/n}\pi$$

$$\times\sec\left[\frac{(1-\xi)\pi}{n}\right]\left[\Gamma\left(\frac{1-\xi}{n}\right)\Gamma\left(\tfrac{1}{2}-\frac{1-\xi}{n}\right)\right]^n. \quad \text{(G7)}$$