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Bounds on Elastic and Inelastic Form Factors of the Nucleon and the Pion

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In this paper we derive bounds on the nucleon form factor, the anomalous magnetic moment of nucleons, and the pion form factor. By using sidewise dispersion relations and the Schwarz inequality, we are able to bound the elastic nucleon form factors $F_1(q^2)$ and $F_2(q^2)$ by integrals over the structure functions for inelastic electron-nucleon scattering, $W_{1,2}(q^2,\nu)$. At $q^2=0$, we then use unitarity to bound the anomalous magnetic moment by an integral over the nucleon propagator spectral function. Finally, by dispersing in q^2, the photon virtual mass, we are able to bound the pion form factor $F_\pi(q^2)$ by an integral over the total electron-positron annihilation cross section.

I. INTRODUCTION

THE absolute bounds on strongly interacting collisions of hadrons are derived primarily from the nonlinear character of the unitarity condition on the amplitude along with the usual assumptions of Lorentz invariance and the causality requirement as embodied in analyticity properties of the scattering amplitude. Collision amplitudes of weakly interacting particles such as the photon or lepton are also subject to unitarity restrictions. However, if one considers these amplitudes as a perturbation series expanded in the weak coupling as is done in practice, then the unitarity conditions will connect different order terms in the series so that order by order there are no useful bounds. For example, to first order in the perturbation parameter the unitarity condition is linear (with the exception of purely weak processes) and hence provides no restrictions on the magnitude of the amplitude; only the summed series is bounded. Although unitarity provides no absolute bound to first-order weak processes, it is possible to establish relative bounds of one weak process to first order in the coupling by another. It is this latter kind of bound that we consider in this article.

We first will study the nucleon form factors $F_{1,2}(q^2)$ by utilizing sidewise dispersion relations for the photon-nucleon vertex. This allows us to express the elastic form factors in terms of an integral of the absorptive part in the virtual nucleon mass W. The square of this absorptive part can be bounded from above by using the Schwarz inequality on the sum of states with positive norms in the Hilbert space, in which case it is bounded by $\rho_1(W^2)W_{1,2}^{(1/2^+)}(q^2,W^2)$. Here $\rho_1(W^2)$ is the first nucleon spectral function and $W_{1,2}^{(1/2^+)}(q^2,W^2)$ are the inelastic form factors with the final states restricted to the quantun numbers of the nucleon $J^P=\frac{1}{2}^+$. Using the Schwarz integral inequality, we then obtain for $q^2\leq 0$

$$-q^2|F_2(q^2)|^2\leq 32e^2(Z_2^{-1}-1)\times\int_{(2\mu M+\mu^2-q^2)/2M}^{\infty}d\nu\, W_2^{(1/2^+)}(q^2,\nu)\,, \quad (1.1)$$

$$|F_1(q^2)-e|^2\leq 32e^2(Z_2^{-1}-1)\times\int_{(2\mu M+\mu^2-q^2)/2M}^{\infty}d\nu\, W_2^{(1/2^+)}(q^2,\nu)\,,$$

$\nu=(W^2-M^2-q^2)/2M$, $Z_2^{-1}-1=\int\rho_1(W^2)dW^2$, and μ is the pion mass. The inequalities are valid for all spacelike $q^2\leq 0$. One may further use the inequality $W_{1,2}^{(1/2^+)}(q^2,\nu)\leq W_{1,2}(q^2,\nu)$, where $W_{1,2}(q^2,\nu)$ are the inelastic form factors into any hadronic final state. However, this procedure considerably weakens the bounds (1.1).

This method of the Schwarz inequality also allows us to place an absolute upper bound on the anomalous moment of a spin-$\frac{1}{2}$ particle. If we assume that the magnetic form factor $F_2^+(W^2)$ obeys $W^2\to\infty$, $F_2^+(W^2)\to 0$ [this assumption can be proven if $\rho_1(W^2)\to 0$, $W^2\to\infty$, where $\rho_1(W^2)$ is the spectral function for the propagator computed in the radiation gauge], then the anomalous moment is bounded by

$$\left|\kappa\frac{e}{2M}\right|\leqslant 8\int_{(M+\mu)^2}^{\infty}\frac{dW^2}{W^2-M^2}\left(\frac{8\rho_1(W^2)W^2}{W^2-M^2}\right)^{1/2}. \quad (1.2)$$

We also examine the pion form factor $F_\pi(q^2)$ by using dispersion relations directly in the q^2 variable. If we assume the Lehmann representation for the photon propagator exists, then $F_\pi(q^2)$ obeys a dispersion relation with no more than one subtraction required. We then establish from this dispersion relation the inequality

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valid to first order in $\alpha=e^2/4\pi$ for $q^2\leq 0$:

$$\left|\frac{F_\pi(q^2)-e}{q^2}\right| \leqslant \int_{4\mu^2}^{\infty} \frac{dq'^2}{q'^2(q'^2+q^2)} \times\left(\frac{4q'^2}{q'^2-4\mu^2}\right)^{3/4}\left(\frac{3q'^2\sigma_T{}^{e^+e^-}(q'^2)}{e^2\pi}\right)^{1/2}, \quad (1.3)$$

where $\sigma_T{}^{e^+e^-}(q'^2)$ is the electron-positron annihilation cross section for c.m. energy $(q'^2)^{1/2}$. In particular, this inequality requires for the pion charge radius

$$\langle r_\pi{}^2\rangle \leqslant 6\int_{4\mu^2}^{\infty}\frac{dq^2}{q^4}\left(\frac{4q^2}{q^2-4\mu^2}\right)^{3/4}\left(\frac{3q^2\sigma_T{}^{e^+e^-}(q^2)}{e^4\pi}\right)^{1/2}. \quad (1.4)$$

II. BOUNDS ON NUCLEON FORM FACTORS

In this section we derive our bounds on the nucleon form factors. Following Bincer,[1] we define the off-shell photon-nucleon vertex as follows (see Fig. 1):

$$\bar{u}(p's')\Gamma_\mu=\bar{u}(p's')\times\left\{e\gamma_\mu+\left[\left(q_\mu+\frac{\gamma_\mu q^2}{M-W}\right)F_3(q^2,W)-i\sigma_{\mu\nu}q^\nu F_2(q^2,W)\right]\times\left(\frac{W+p\!\!\!/}{2W}\right)+\left[\left(q_\mu+\frac{\gamma_\mu q^2}{M+W}\right)F_3(q^2.\,-W)-i\sigma_{\mu\nu}q^\nu F_2(q^2,\,-W)\right]\left(\frac{W-p\!\!\!/}{2W}\right)\right\}. \quad (2.1)$$

Here $p=p'+q$, $p^2=W^2$, $p'^2=M^2$, and q^2 is the virtual photon mass. In obtaining this general result, use has been made of the invariance of the theory under T, C, and P, and gauge transformations which implies the Ward-Takahashi identity $q^\mu\bar{u}(p's')\Gamma_\mu=e\bar{u}(p's')q\!\!\!/$. The usual charge and magnetic form factors can be identified as

$$F_1(q^2)=e+q^2F_3'(q^2,M)\,,\qquad F_2(q^2)=F_2(q^2,M),\quad F_2(0)=\kappa e/2M\,, \quad (2.2)$$

where the prime denotes differentiation with respect to W and κ is the anomalous moment.

One can rigorously prove that

$$F_i{}^+(q^2,W^2)=\tfrac{1}{2}[F_i(q^2,W)+F_i(q^2,\,-W)]$$

and

$$F_i{}^-(q^2,W^2)=(1/2W)[F_i(q^2,W)-F_i(q^2,\,-W)]$$

are analytic functions in the cut W^2 plane,[1] so that one can establish dispersion relations in W^2. These analyticity properties in W^2 can be transcribed to the W plane and if we assume that no subtractions are required for $F_2(q^2,W)$ and one subtraction for $F_3(q^2,W)$, then we have the following representation for the elastic form factors:

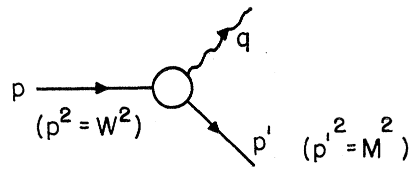

FIG. 1. Photon-nucleon vertex.

$$F_2(q^2)=\frac{1}{\pi}\int_{M+\mu}^{\infty}dW\left[\frac{\mathrm{Im}F_2(q^2,W)}{W-M}+\frac{\mathrm{Im}F_2(q^2,\,-W)}{W+M}\right],$$
$$\frac{F_1(q^2)-e}{q^2}=\frac{1}{\pi}\int_{M+\mu}^{\infty}dW\left[\frac{\mathrm{Im}F_3(q^2,W)}{(W-M)^2}+\frac{\mathrm{Im}F_3(q^2,\,-W)}{(W+M)^2}\right]. \quad (2.3)$$

It is important to remark that the subtraction assumptions we have made are crucial to the bounds we will derive. Although we can bound $\mathrm{Im}F_{2,3}(q^2,W)$ for $|W|\to\infty$ in terms of other functions and even if the asymptotic behavior of these functions imply the integrals in (2.3) exist, this is not sufficient to guarantee the $|W|\to\infty$ behavior of $F_{2,3}(q^2,W)$ required for the validity of (2.3). It is always possible that $\mathrm{Re}F_{2,3}(q^2,W)$ contain polynomials in W which are independent of the imaginary part and hence not specified by (2.3). In deriving our bounds, we explicitly assume the absence of such polynomial pieces in $F_{2,3}(q^2,W)$, so that if the integrals in (2.3) exist then they specify $F_{1,2}(q^2)$.

Our project is now to bound the imaginary parts appearing in these integrals using the method of the Schwarz inequality.[2] The absorptive amplitude is

$$\bar{u}(p's')\Gamma_\mu{}^A=\bar{u}(p',s')\times\left[\left(q_\mu+\frac{\gamma_\mu q^2}{M-W}\right)\mathrm{Im}F_3(q^2,W)-i\sigma_{\mu\nu}q^\nu\,\mathrm{Im}F_2(q^2,W)\right]\times\left(\frac{W+p\!\!\!/}{2M}\right)+\left[\left(q_\mu+\frac{\gamma_\mu q^2}{M+W}\right)\mathrm{Im}F_3(q^2,\,-W)-i\sigma_{\mu\nu}q^\nu\,\mathrm{Im}F_2(q^2,\,-W)\right]\left(\frac{W-p\!\!\!/}{2M}\right). \quad (2.4)$$

The unitarity condition on this amplitude is then

$$\bar{u}(p's')\Gamma_\mu{}^A=\tfrac{1}{2}\sum_n(2\pi)^4\delta^4(p_n-p)\times\langle p's'|j_\mu(0)|n\rangle\langle n|\bar{j}_N(0)|0\rangle\,, \quad (2.5)$$

[1] A. M. Bincer, Phys. Rev. **118**, 855 (1960).

[2] See S. D. Drell and F. Zachariasen, Phys. Rev. **119**, 463 (1960) for an earlier application of the technique.

where $j_\mu(x)$ is the electromagnetic current and $j_N(x) = (i\gamma_\mu\partial^\mu - M)\psi(x)$ is the nucleon source current.

Instead of (2.5) we consider the positive energy $W>0$ projection

$$\bar{u}(p's')\Gamma_\mu{}^A\left(\frac{W+\not p}{2W}\right)=\tfrac{1}{2}\sum_n a_n{}^*b_n{}^\mu,$$

$$a_n{}^*=[(2\pi)^4\delta^4(p_n-p)]^{1/2}\langle n|\bar{j}_N(0)|0\rangle\left(\frac{W+\not p}{2W}\right), \quad (2.6)$$

$$b_n{}^\mu=[(2\pi)^4\delta^4(p_n-p)]^{1/2}\langle p's'|j_\mu(0)|n\rangle.$$

Since we assume the states $|n\rangle$ have positive-definite norm, we may apply the Schwarz inequality to (2.6):

$$\sum_{s'}\left|\bar{u}(p's')\Gamma_\mu{}^A\left(\frac{W+\not p}{2W}\right)\right|^2\leq\tfrac{1}{4}\sum_n|a_n|^2\sum_{s'}\sum_n|b_n{}^\mu|^2. \quad (2.7)$$

It is straightforward to show that

$$\sum_n|a_n|^2=\frac{4\pi p_0(M-W)^2}{W}[\rho_1(W^2)W+\rho_2(W^2)], \quad (2.8)$$

where $\rho_{1,2}(W^2)$ are the nucleon spectral functions as defined in the representation

$$S_F(p)=\frac{\not p+M}{p^2-M^2}+\int\frac{dW^2}{W^2-M^2}\times[\rho_1(W^2)\not p+\rho_2(W^2)]. \quad (2.9)$$

For positive-norm states, $\rho_1(W^2)\geq0$, $W\rho_1(W^2)\geq\rho_2(W^2)$, and so

$$\sum_n|a_n|^2\leq8\pi p_0(W-M)^2\rho_1(W^2). \quad (2.10)$$

The inelastic structure functions[3] $W_{1,2}(q^2,p'\cdot q)$ are defined by $(q^2\leq0)$

$$e^2W_{\mu\nu}=(2\pi)^3\tfrac{1}{2}\sum_{n,s'}\delta^4(p_n-p)\langle p's'|j_\mu(0)|n\rangle\times\langle n|j_\nu(0)|p's'\rangle,$$

$$e^2/4\pi=\alpha=1/137,$$

where $W_{\mu\nu}$ has the decomposition

$$W_{\mu\nu}=d_{\mu\nu}{}^1W_1(q^2,p'\cdot q)+d_{\mu\nu}{}^2W_2(q^2,p'\cdot q)/M^2,$$

$$d_{\mu\nu}{}^1=q_\mu q_\nu/q^2-g_{\mu\nu},\quad d_{\mu\nu}{}^2=\left(p_\mu'-\frac{p'\cdot q}{q^2}q_\mu\right)\left(p_\nu'-\frac{p'\cdot q}{q^2}q_\nu\right).$$

Hence

$$\sum_{\substack{s'n\\(\mu\text{ not summed})}}|b_n{}^\mu|^2=4\pi e^2[d_{\mu\mu}{}^1W_1(q^2,p'\cdot q)+d_{\mu\mu}{}^2W_2(q^2,p'\cdot q)/M^2]. \quad (2.11)$$

[3] See, e.g., S. D. Drell and D. Walecka, Ann. Phys. (N. Y.) 28, 18 (1964); F. J. Gilman, SLAC-PUB 674, 1969 (unpublished).

At this point we remark that in the original unitarity sum (2.5) the state $|n\rangle$ must have the quantum numbers of the nucleon (although the nucleon state itself is not in this sum) in virtue of the projection performed by $\langle n|\bar{j}_N(0)|0\rangle$. Consequently, the sum (2.11) is also over this restricted set of states with $J^P=\frac{1}{2}^+$ and we can write instead

$$\sum_{\substack{s'n\\(\mu\text{ not summed})}}|b_\mu{}^n|^2=e^24\pi d_{\mu\mu}{}^1W_1{}^{(1/2+)}(q^2,p'\cdot q)+d_{\mu\mu}{}^2W_2{}^{(1/2+)}(q^2,p'\cdot q)/M^2, \quad (2.12)$$

where $W_{1,2}{}^{(1/2^+)}(q^2,p'\cdot q)$ are the inelastic structure functions for final states with $J^P=\frac{1}{2}^+$ and hence represent the contribution from a few partial waves only.

For $q^2\leq0$, $W_{1,2}(q^2,p'\cdot q)$ and $W_{1/2}{}^{(1/2^+)}(q^2,p'\cdot q)$ can be shown to be positive definite, related as they are to the transverse and longitudinal cross sections[3]

$$W_1(q^2,\nu)=(k/4\pi^2\alpha)\,\sigma_T(\nu,q^2),$$
$$W_2(q^2,\nu)=\frac{k}{4\pi^2\alpha}\left(\frac{-q^2}{\nu^2-q^2}\right)[\sigma_T(\nu,q^2)+\sigma_L(\nu,q^2)], \quad (2.13)$$

with $\nu=p'\cdot q/2M=(W^2-M^2-q^2)/2M$, $k=\nu+q^2/2M=(W^2-M^2)/2M$. Of course, the partial cross sections are bounded by the total cross sections

$$W_{1,2}{}^{(1/2^+)}(q^2,\nu)\leq W_{1,2}(q^2,\nu). \quad (2.14)$$

We will now look in the rest frame of the nucleon, i.e.,

$$p'=(M,0,0,0),\quad q=(q_0,0,0,q_3)$$

and evaluate (2.7) for the cases $\mu=3$ and $\mu=1$, which are related to longitudinal and transverse cross sections.

Using (2.7), (2.10), and (2.12), we obtain for $(q^2\leq0)$, $\mu=3$,

$$p^0\nu^2\frac{(W-M)^2-q^2}{2MW^2}\times\left[\operatorname{Im}F_2(q^2,W)-\left(\frac{W+M}{W-M}\right)\operatorname{Im}F_3(q^2,W)\right]^2$$
$$\leq p^0(W-M)^2e^2(4\pi)^2\rho_1(W^2)\left(\frac{\nu^2}{-q^2}\right)\times\left[\frac{\nu^2-q^2}{-q^2}W_2{}^{(1/2^+)}(q^2,\nu)-W_1{}^{(1/2^+)}(q^2,\nu)\right], \quad (2.15a)$$

and for $\mu=1$,

$$p^0\frac{(W-M)^2-q^2}{2MW^2}\times\left[(W+M)\operatorname{Im}F_2(q^2,W)-\frac{q^2}{W-M}\operatorname{Im}F_3(q^2,W)\right]^2$$
$$\leq p^0(W-M)^2e^2(4\pi)^2\rho_1(W^2)W_1{}^{(1/2^+)}(q^2,\nu). \quad (2.15b)$$

Multiplying (2.15a) by $-q^2/\nu^2 \geq 0$ and adding the result to (2.15b), we get

$$|\mathrm{Im}F_2(q^2,W)|^2 - \frac{q^2}{(W-M)^2}|\mathrm{Im}F_3(q^2,W)|^2 \leq (e^2)(4\pi)^2 \frac{(W-M)^2}{2M}\left(\frac{\rho_1(W^2)W^2W_2{}^{(1/2^+)}(q^2,\nu)}{-q^2}\right).$$

Since each term on the left is positive,

$$\left|\frac{\mathrm{Im}F_2(q^2,W)}{W-M}\right| \leq 4\pi e|W|\left[\frac{\rho_1(W^2)W_2{}^{(1/2^+)}(q^2,\nu)}{-2q^2M}\right]^{1/2}, \quad (2.16\mathrm{a})$$

$$\left|\frac{\mathrm{Im}F_3(q^2,W)}{(W-M)^2}\right| \leq \frac{4\pi e|W|}{-q^2}\left[\frac{\rho_1(W^2)W_2{}^{(1/2^+)}(q^2,\nu)}{2M}\right]^{1/2}. \quad (2.16\mathrm{b})$$

These two inequalities do not exhaust the content of our original inequalities (2.15) but they will suffice for our purposes here.

In order for the integrals (2.3) to converge, we require from (2.16)

$$W^4\rho_1(W^2)W_2{}^{(1/2^+)}(q^2,\nu) \to W^{-\epsilon}, \quad W^2\to\infty, \quad \epsilon>0. \quad (2.17)$$

We can consider under what conditions this requirement is satisfied. If we use $W_2{}^{(1/2^+)}(q^2,\nu) \leq W_2(q^2,\nu)$ and assume the total cross sections in (2.13) are bounded by constants, then (2.17) implies $W^2\rho_1(W^2) \to W^{-\epsilon}$ as $W^2 \to\infty$. Hence under these assumptions $Z_2{}^{-1}-1=\int\rho_1(W^2)dW<\infty$ exists.

We view this requirement that $Z_2{}^{-1}<\infty$ as a rather strong one on $\rho_1(W^2)$. However, use of the inequality $W_2{}^{(1/2)+}(q^2,\nu)\leq W_2(q^2,\nu)$ is probably very inefficient. If the high-energy behavior of partial-wave amplitudes for photon processes is anything similar to the bound forced by unitarity on hadronic partial-wave amplitudes, then we would expect $W_2{}^{(1/2^+)}(q^2,\nu) \to 1/W^4$, $W^2\to\infty$. Then we need only require $\rho_1(W^2)\to W^{-\epsilon}$ to satisfy (2.17). This assumption on the nucleon spectral function is a much weaker assumption, the existence of a Lehmann representation (2.9) for the nucleon propagator.

We have not been able to establish any good relative bound for $W_{1,2}{}^{(1/2^+)}(q^2,\nu)$ in terms of the experimentally measured $W_{1,2}(q^2,\nu)$ other than the trivial observation that $W_{1,2}{}^{(1/2^+)}(q^2,\nu)\leq W_{1,2}(q^2,\nu)$ which, as has already been remarked, is inefficient. It would be generally desirable to establish such a bound.

For the present let us assume $Z_2{}^{-1}<\infty$. Then the integrals (2.3) and bounds (2.16) imply ($q^2\leq 0$)

$$|F_2(q^2)|\leq 4e\int_{(M+\mu)^2}^{\infty} dW^2\left[\frac{\rho_1(W^2)W_2{}^{(1/2^+)}(q^2,\nu)}{-2Mq^2}\right]^{1/2},$$
$$|F_1(q^2)-e|\leq 4e\int_{(M+\mu)^2}^{\infty} dW^2\left[\frac{\rho_1(W^2)W_2{}^{(1/2^+)}(q^2,\nu)}{2M}\right]^{1/2}. \quad (2.18)$$

Using the Schwarz integral inequality,

$$-q^2|F_2(q^2)|^2\leq 32e^2(Z_2{}^{-1}-1)\times\int_{(+2M\mu+\mu^2-q^2)/2M}^{\infty} d\nu\, W_2{}^{(1/2^+)}(q^2,\nu), \quad (2.19)$$
$$|F_1(q^2)-e|^2\leq 32e^2(Z_2{}^{-1}-1)\times\int_{(+2M\mu+\mu^2-q^2)/2M}^{\infty} d\nu\, W_2{}^{(1/2^+)}(q^2,\nu).$$

In using the integral inequality on (2.18), we should comment that if the functions $W_2{}^{(1/2^+)}(q^2,\nu)$ decrease very rapidly in W^2, then we can exchange a higher moment of $\rho(W^2)$ for a higher moment of $W_2{}^{(1/2^+)}(q^2,\nu)$.

It is interesting to examine the consequences of the scaling hypothesis that $\nu W_2(q^2,\nu)\to F_2(\omega)$ as $\nu\to\infty$, $\omega=-q^2/\nu$ fixed. Then we obtain from (2.19) and (2.14) as $-q^2\to\infty$,

$$-q^2|F_2(q^2)|^2\leq 32e^2(Z_2{}^{-1}-1)\int_0^2\frac{d\omega}{\omega}F_2(\omega),$$
$$|F_1(q^2)-e|^2\leq 32e^2(Z_2{}^{-1}-1)\int_0^2\frac{d\omega}{\omega}F_2(\omega). \quad (2.20)$$

Needless to say, with the measured nucleon elastic form factors falling like q^{-4} there is little danger that (2.20) is violated.

III. ABSOLUTE BOUND ON ANOMALOUS MAGNETIC MOMENT

From Sec. II we can establish the following theorem: If $F_2(0,W)\to 0$ as $|W|\to\infty$, so that this function satisfies an unsubtracted dispersion relation, then the anomalous magnetic moment is bounded by

$$\left|\kappa\frac{e}{2M}\right|\leq 8\int_{(M+\mu)^2}^{\infty}\frac{dW^2}{W^2-M^2}\left[\frac{8\rho_1(W^2)W^2}{W^2-M^2}\right]^{1/2}, \quad (3.1)$$

where $\rho_1(W^2)$ is the spectral function of the hadron.

The result is established in the following way. If $F_2(0,W) \to 0$ as $|W| \to \infty$, then $F_2(0,M) = \kappa e/2M$ is specified in terms of the unsubtracted integral over the imaginary part (2.3). Using the expression (2.13) for $W_2(q^2,\nu)$ in terms of the total cross section, we find

$$\lim_{q^2\to 0} \frac{e^2 W_2^{(1/2^+)}(q^2,\nu)}{-2Mq^2} = \frac{\sigma_T^{(1/2^+)}(W^2)}{\pi(W^2-M^2)}.$$

Then (2.18) implies with $q^2=0$,

$$\left|\kappa\frac{e}{2m}\right| \leq 4\int_{(M+\mu)^2}^{\infty} dW^2 \left[\frac{\rho_1(\omega^2)\sigma_T^{(1/2^+)}(W^2)}{\pi(W^2-M^2)}\right]^{1/2}.$$

In this case we are dealing with physical photons $q^2=0$, so $\sigma_T^{(1/2^+)}(W^2)$ represents partial-wave projections of a physical S-matrix element (no such statement can be made for $q^2\neq 0$). Consequently, $\sigma_T^{(1/2^+)}(W^2)$ has a unitarity bound given by

$$\sigma_T^{(1/2^+)}(W^2) \leq 8\pi\left(\frac{2W}{W^2-M^2}\right)^2.$$

However, as remarked in the Introduction, such a bound need not be satisfied order by order in perturbation theory. Using this bound, we have

$$\left|\kappa\frac{e}{2M}\right| \leq 8\int_{(M+\mu)^2}^{\infty} \frac{dW^2}{W^2-M^2}\left[\frac{8\rho_1(W^2)W^2}{W^2-M^2}\right]^{1/2}. \quad (3.2)$$

It is worth remarking that our method gives no such bounds for quantities like the charge radius $F_1'(0)$, since to obtain such a bound one must analytically continue the unitarity condition into the unphysical region $q^2\neq 0$ and we know of no absolute bounds on the amplitude in this region.

We further remark that caution must be applied if one desires to extend these results to all orders in the electromagnetic coupling. In the case of quantum electrodynamics, for example, for which the fermion in question is the electron, it is well known[4] that the electron spectral function is gauge dependent and (depending on the quantization procedure for the photon field) the norm of states containing photons need not be positive definite, so we lose the positivity condition on spectral functions. Also, the Schwarz inequality is not valid for these states. If, however, we stipulate the transverse gauge and quantize in this gauge, the norm of states with photons is positive definite and these results can be expected to apply. We also can show by considering the one-photon–one-fermion contribution to the spectral function $\rho_1(W^2)$ in the transverse gauge that as $W^2\to\infty$, $\rho_1^{(2)}(W^2) \sim |e+(W+M)F_2(0,W)|^2/W^2$. Hence, if the Lehmann representation exists in this gauge, then $F_2(0,W)\to 0$, $|W|\to\infty$, and our hypothesis of an unsubtracted dispersion relation for $F_2(0,W)$ is justified.

IV. BOUND ON PION FORM FACTOR

Here we present some simple relative bounds on the pion form factor in terms of the e^+e^- annihilation cross section which, to our knowledge, have not been reported in the literature. We will not follow the method of the previous sections since in the case of the pion, owing to the special circumstance that it is the lowest mass hadron, we have found it useful to consider dispersion relations in q^2, the photon mass.

We begin our discussion by considering the photon spectral function

$$J_{\mu\nu} = \sum_n (2\pi)^3\delta^4(p_n-q)\langle 0|j_\mu(0)|n\rangle\langle n|j_\nu(0)|0\rangle \quad (4.1)$$
$$= (q_\mu q_\nu/q^2 - g_{\mu\nu})J(q^2),$$

and assume that a Lehmann representation for the photon propagator $\pi(q^2)$ exists:

$$\pi(q^2) = \frac{1}{q^2} + \int_{4\mu^2}^{\infty} \frac{dq'^2\rho(q'^2)}{q^2-q'^2}, \quad \rho(q^2)\geq 0 \quad (4.2)$$

where $\rho(q^2) = J(q^2)/q^4$. The threshold $4\mu^2$ in (4.2) corresponding to production of pion pairs arises in the approximation of neglecting all but the hadronic contribution to the states $|n\rangle$ in (4.1), which is valid to order e^2.

It is straightforward to compute the two-pion contributions to $\rho(q^2)$, with the result

$$\rho_{2\pi}(q^2) = \frac{|F_\pi(q^2)|^2}{6\pi^2 q^2}\left(\frac{q^2-4\mu^2}{4q^2}\right)^{3/2}\theta(q^2-4\mu^2), \quad (4.3)$$

where $F_\pi(q^2)$ is the pion form factor. Since each state contributes a positive-definite amount to $\rho(q^2)$, we have $\rho(q^2)\geq\rho_{2\pi}(q^2)$. For the Lehmann representation (4.2) to exist, we must have $\rho(q^2)\to 0$, $q^2\to\infty$ which implies, from (4.3), that $|F_\pi(q^2)|^2/q^2\to 0$ as $q^2\to\infty$. If we postulate the usual analyticity properties of $F_\pi(q^2)$, this asymptotic behavior implies that we can write a dispersion relation for $F_\pi(q^2)$ with no more than one subtraction:

$$F_\pi(q^2) = e + \frac{q^2}{\pi}\int_{4\mu^2}^{\infty} \frac{\mathrm{Im}F_\pi(q'^2)dq'^2}{q'^2(q'^2-q^2)}. \quad (4.4)$$

Our interest is in establishing a relative bound to first order in e for $F_\pi(q^2)$ in the spacelike region $q^2\leq 0$. From (4.4),

$$\left|\frac{F_\pi(q^2)-e}{q^2}\right|_{q^2\leq 0} \leq \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{dq'^2\,\mathrm{Im}F_\pi(q'^2)}{q'^2(q'^2+|q|^2)}, \quad (4.5)$$

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[4] For a discussion, see L. Evans, P. Feldman, and P. T. Matthews, Ann. Phys. (N. Y.) **13**, 268 (1961).

and from (4.3) and the positivity $\rho(q^2)\geq\rho_{2\pi}(q^2)$,

$$|\mathrm{Im}F_\pi(q^2)|^2\leq 6\pi^2\left(\frac{4q^2}{q^2-4\mu^2}\right)^{3/2} q^2\rho(q^2)\,. \qquad (4.6)$$

If we further use the relation valid to first order in $\alpha=e^2/4\pi$, between the photon spectral function and the annihilation cross section $\rho(q^2)=\sigma_T{}^{e^+e^-}(q^2)/8\pi^2\alpha$, where q^2 is the c.m. energy of the pair, and combining (4.5) and (4.6), we have our relative bound

$$\left|\frac{F_\pi(q^2)-e}{q^2}\right|_{q^2\leq 0}\leq\int_{4\mu^2}^{\infty}\frac{dq'^2}{q'^2(q'^2+|q|^2)}\left(\frac{4q'^2}{q'^2-4\mu^2}\right)^{3/4}\times\left[\frac{3q'^2\sigma_T{}^{e^+e^-}(q'^2)}{\pi e^2}\right]^{1/2}. \qquad (4.7)$$

There immediately follows from this expression valid for $q^2\leq 0$ a bound on the charge radius of the pion:

$$\langle r_\pi{}^2\rangle=\frac{6}{F_\pi(0)}\frac{dF_\pi(q^2)}{dq^2}\bigg|_{q^2=0}\leq 6\int_{4\mu^2}^{\infty}\frac{dq^2}{q^4}\left(\frac{4q^2}{q^2-4\mu^2}\right)^{3/4}\times\left[\frac{3q^2\sigma_T{}^{e^+e^-}(q^2)}{\pi e^4}\right]^{1/2}. \qquad (4.8)$$

Also perhaps of interest is the observation that if $F_\pi(q^2)\to 0$, $-q^2\to\infty$, then (4.7) implies

$$\int_{4\mu^2}^{\infty}\frac{dq'^2}{q'^2}\left(\frac{4q'^2}{q'^2-4\mu^2}\right)^{3/4}\left[\frac{3q'^2\sigma_T{}^{e^+e^-}(q^2)}{\pi e^4}\right]^{1/2}\geq 1\,. \qquad (4.9)$$

The integral here probably diverges as is expected in the quark-algebra estimate of $\sigma_T{}^{e^+e^-}(q^2)$, $q^2\to\infty$, so the inequality is trivial.

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Relation Between the Multi-Regge Model and the Missing-Mass Spectrum*

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The integral equation approach to the multi-Regge peripheral model is applied to give the missing-mass spectrum with Regge behavior in s and M^2. A simple factorizable model for the double Regge coupling then gives the magnitude and t dependence of the cross section. This model is found to be in reasonable agreement with the backward $\pi^-+p\to p+X^-$ data.

THE integral equation approach to the multi-Regge production model for computing the contribution to the elastic absorptive part has recently been formulated.[1–5] The approach has been used to predict total cross sections at high energies with results that are encouraging.[6] Recently, this approach has also been applied by Caneschi and Pignotti[7] to studying the missing-mass spectrum at high energies. The importance of the missing-mass experiments as a test of the integral equation approach has motivated us to examine this relationship in more explicit detail. In this paper we show how the integral equations may be readily applied to give the missing-mass spectrum in high-energy inelastic collisions in terms of the forward Reggeon-particle absorptive amplitude $\mathcal{A}(t;s)$. The Reggeon-particle absorptive amplitude at general momentum transfer, forward or nonforward, can be obtained by solving a multi-Regge integral equation.[3–5] The predicted missing-mass spectrum has characteristic properties which can be tested experimentally. Furthermore, using the simplified model of a factorizable and ω-angle-independent double Regge coupling, we achieve an expression of the missing-mass cross section entirely in terms of two-body cross sections and coupling constants. This allows us to predict not only the Regge behavior in energy and missing mass, but also the magnitude of the missing-mass cross section. The result is applied with reasonable agreement to "backward" $\pi^-+p\to p+X^-$ reaction,[8] as production on the end of a multiperipheral chain. In a later paper,[9] we will examine more explicitly the formulation and results of the missing-mass contribution from the particles emitted from the central

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