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Particle Size and Contraction at High Velocity*

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The question of size and contraction of size at high velocity is considered in the context of particle physics. Size is defined through a simultaneous interaction with an external potential. To second order in the external potential, one is led to consider matrix elements of the form $\langle p|j_0(\mathbf{x},x_3,0)j_0(0)|p\rangle$. For large p such matrix elements are found to approach $p\delta(x_3)F(\mathbf{x})$ if there are no Regge singularities at $J=1$ when $t=0$. If there are such singularities at $J=1$ when $t=0$ and if they recede below $J=1$ for negative t, then matrix elements analogous to the one above, but for $t<0$, approach $\delta(x_3)$ at high velocity. $F(\mathbf{x})$ is related to the residue of a wrong-signature fixed pole at $J=1$ in a virtual Compton amplitude. $F(\mathbf{x})$ is also shown to be equal to the second-order impact factor in the operator droplet model. These results are then generalized to an arbitrary number of interactions with the external potential. More singular interactions, where the above analysis breaks down, are considered. It is found that for a certain strength of singularity on the light cone, the particle size may shrink to zero at high velocities. In a large class of models which give the scaling law for deeply inelastic electroproduction and have a constant asymptotic total cross section for electroproduction, it is found that the particle size does not shrink. A converse statement is also found, in that a simple argument shows that if electromagnetic particle size at $t=0$ shrinks, then the total asymptotic electroproduction cross section vanishes at high energy.

I. INTRODUCTION

THE idea that the "shape" of a particle may contract into a thin disk at high energies[1] can be based on a classical analogy. For suppose two, or more, simultaneous localized measurements of a static classical object are made in a fixed reference frame. (We take the direction of motion of the static object, O, to be the z direction.) Then as the velocity of O approaches the speed of light, these simultaneous measurements will give a null result unless they are made inside a thin disk whose width is the static size of O along the z direction in its rest system times $(1-v^2/c^2)^{1/2}=\gamma^{-1}$. This is easily seen. Suppose the two measurements in the laboratory system occur at points in (t,z) at $l_0=(0,0)$ and $l_1=(0,z_1)$. Then in the rest system of O the measurements take place at $l_0'=(0,0)$ and $l_1'=(-\gamma v z_1,\gamma z_1)$, where the velocity of light is taken to be unity. Now as v approaches 1 we see that the two measurements occur farther and farther apart in the rest system of O and thus must give a null result unless z_1 is less than the size of the object times γ^{-1}.

If O is not static, the situation is somewhat more complicated. If the motions of O are bounded, that is, if O can be put inside a static sphere of any finite size, then the arguments used above apply, and again a contraction in size occurs at high velocity. If, however, the motions are unbounded, no general statements about contraction can be made. In the following, when we deal with size in particle physics, we shall say that size contracts into a thin disk at high velocity if that size behaves essentially as the size of a static classical object of finite dimensions at high velocity.

For the case of particle physics we shall discuss the question of size in terms of an external potential. Thus when asking a question about the electromagnetic size, an external electromagnetic potential A_μ is introduced. Simultaneous interactions are obtained when $A_\mu(x)\to\delta(x_0)a_\mu(x)$ for two or more interactions with the external potential. As an example, take the scattering of a charged scalar meson of momentum p off an external potential. To second order,

$$f(p)=-i\int d^4x d^4y\, M_{\mu\nu}(p,\,x-y)A_\nu(x)A_\mu(y)\,,$$

where

$$M_{\mu\nu}(p,x)=\langle p|T(j_\nu(x)j_\mu(0))|p\rangle\,,$$

and p is taken along the z direction. If $A_0(x)\to\delta(x_0)\times a_0(x)$, then $\langle p|j_0(x_1,x_2,x_3,0)j_0(0)|p\rangle$, which appears in the integral for $f(p)$, is the amplitude for simultaneously finding charge density at the points (x_1,x_2,x_3) and $(0,0,0)$. If that equal-time matrix element does not exist, then the interaction is too singular to permit simultaneous interaction with a potential which is not smeared in x_0. We shall not discuss this case in detail; however, examples where this singular type of interaction occurs will be given in Secs. IV and V. Note, also, that when we say an equal-time matrix element is divergent or does not exist, we mean that its singularities in (x_1,x_2,x_3) are too severe to allow an integration with a test function without regularization.

As will be shown in Sec. II, the above equal-time matrix element decreases exponentially with $(x_1^2+x_2^2+x_3^2)^{1/2}$, showing that the particle labeled by $|p\rangle$ does have a finite interaction size in any reference frame. Furthermore, in that section it will be argued that if high-energy off-shell Compton scattering at $t=0$ is governed by Regge poles or other singularities in the angular momentum plane where the leading singularity is below $J=1$ at $t=0$, then particle size will contract at large p. That is, the matrix element $\langle p|j_0(x_1,x_2,x_3,0)j_0(0)|p\rangle$ will give a negligible contribution to $f(p)$ unless

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[1] T. T. Chou and C. N. Yang, Phys. Rev. 170, 1591 (1968).

$|x_3| \lesssim 1/p$ for large p. In fact, that matrix element will approach $p\delta(x_3)F(\mathbf{x})$, where $\mathbf{x}=(x_1,x_2)$. Even if there are singularities at $J=1$ when $t=0$, if these singularities recede below $J=1$ at negative t then matrix elements like $\langle p|j_0(\mathbf{x},x_3,0)j_0(0)|p'\rangle$ will approach a δ function in x_3 when p_3 and p_3' both go to infinity with $(p-p')^2=t$ fixed. In subsequent sections we shall only deal with $t=0$ explicitly. But whatever we obtain for $t=0$ when $\alpha(t=0)<1$ will, in general, be true for negative t, and if the Regge singularities move with t we can answer the $t<0$ question in general even if $\alpha(0)=1$.

Also, in Sec. II, we show that the $F(\mathbf{x})$ mentioned above is the two-dimensional Fourier transform of the residue of a wrong-signature nonsense fixed pole at $J=1$ in a virtual Compton amplitude.[2,3] Further $F(\mathbf{x})$, will be seen to be equal to the impact factor which occurs in the operator droplet model.[4–6] In fact, we shall show that the scattering of our scalar meson off a static external potential occurs according to the S matrix of the operator droplet model to second order in the external potential but to all orders of the hadronic couplings.

The connection of $F(\mathbf{x})$ with fixed poles shows the necessity of coupling the external potential to a vector current.[5] It is possible for a matrix element $\langle p|j(\mathbf{x},x_3,0)j(0)|p\rangle$ of scalar currents to contract to a δ function of x_3 for large p; however, in that case the scattering off an external potential to second order will decrease with p rapidly for large p whether or not the potentials emits only at $x_0=0$. It is possible to understand this in a very simple way. In the scattering off an external potential by the exchange of two spin-1 particles, the matrix element of the two currents between the initial and final scalar meson states has a wrong-signature fixed pole at $J=1$, as we have mentioned. The two orders of the external potential also have a wrong-signature fixed pole at $J=1$ which gives a fixed pole in a sense scattering amplitude of the meson off the potential. This fixed pole in J exactly accounts for the constant asymptotic cross section. If the currents are not of a vector type, the fixed poles will be at $J=0$ or lower depending on the nature of the currents.

In Sec. III the results of Sec. II are extended to an arbitrary number of interactions with the external potential. In order to achieve this, it is necessary to have a formalism for multiparticle amplitudes on which it is convenient to do an $O(2,1)$ analysis. Such a formula is developed in Appendix A. Matrix elements such as $\langle p|T(j_0(x_1)j_0(x_2)j_0(0))|p\rangle$ are considered at large values of p and found to approach $p\delta(x_{10}-x_{13})\delta(x_{20}-x_{23})\times F(\mathbf{x}_1,\mathbf{x}_2)$. By going to $x_{10}=x_{20}=0$, one again sees that particle size contracts under similar assumptions as in Sec. II. $F(\mathbf{x}_1,\mathbf{x}_2)$ is again related to a fixed pole at $J=1$ and is also seen to be the third-order term in the expansion of the S matrix in the operator droplet model. The third-order term for the scattering off a static external potential is thus shown to obey the operator droplet model.

Also in Sec. III it is observed that the complexity of matrix elements of the type

$$\langle p|T(j_{\mu_1}(x_1)j_{\mu_2}(x_2)\cdots j_{\mu_n}(x_n)j_{\mu_{n+1}}(0))|p\rangle$$

is no more than that for the three current case. Thus one obtains

$$\langle p|T(j_{\mu_1}(x_1)j_{\mu_2}(x_2)\cdots j_{\mu_n}(x_n)j_{\mu_{n+1}}(0))|p\rangle \to p\delta(x_{10}-x_{13})\times\delta(x_{20}-x_{23})\cdots \times\delta(x_{n0}-x_{n3})F(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$$

for large p. We see that electromagnetic size contracts for $n+1$ simultaneous interactions. Again the coefficient $F(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$ is related to a fixed pole at $J=1$ in a multiparticle amplitude. The operator droplet model is derived to the $n+1$ order for scattering off a static external potential, with F being the impact factor to this order.

In Sec. V more singular interactions are considered for the case of scalar currents. It is found that if the singularities of $\langle p|T(j(x)j(0))|p\rangle$ on the light cone have a strength $(x^2)^{-\gamma-2}$, then for $\gamma\geq -\frac{1}{2}$ the interaction is too singular to define scattering by a scalar external potential which approaches a δ function in time. However, for $-1<\gamma<-\frac{1}{2}$ it is found that $\langle p|j(\mathbf{x},x_3,0)j(0)|p\rangle$ approaches $\delta^2(\mathbf{x})\delta(x_3)$ for large p. That is, the particle shrinks to a point at large p. It is then shown that the rate of decrease in p of $f(p)$, the amplitude for scattering off an arbitrary external potential, and the rate of decrease with p of $\int d^4x\, e^{iqx}\langle p|T(j(x)j(0))|p\rangle$ for fixed q are related to the strength of the singularities in x of the operator product $j(x)j(0)$ as $x_\mu\to 0$. One result of this last result is that the rate of decrease in p should be independent of the momentum transfer of the scattering.

In Sec. V the question of the behavior of the matrix element of the time-ordered product of two electromagnetic currents is again considered for the case that the integrals in (2.10) and (2.13) do not exist and hence the arguments leading to contraction of particle size are no longer valid.

For a large class of models which obey the scaling law for deeply inelastic electroproduction and have a constant asymptotic total cross section for electroproduction,[7] it is explicitly shown that particle size does not contract with the velocity of the proton. That is, at high velocity, regions of $|x_3|>N/p$ for any fixed N in $\langle p|j_0(\mathbf{x},x_3,0)j_0(0)|p\rangle$ are dominant in the scattering off an external potential $A_\mu(x)=\delta(x_0)a_\mu(\mathbf{x}_1x_3)$ at large p.

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[2] A. H. Mueller and T. L. Trueman, Phys. Rev. **160**, 1306 (1967).

[3] H. D. I. Abarbanel, F. E. Low, I. J. Muzinich, S. Nussinov, and J. H. Schwarz, Phys. Rev. **160**, 1329 (1967).

[4] T. T. Chou and C. N. Yang, Phys. Rev. **175**, 1832 (1968).

[5] B. W. Lee, Phys. Rev. D **1**, 2361 (1970).

[6] H. Cheng and T. T. Wu, Phys. Rev. **182**, 1852 (1969); **182**, 1868 (1969); **182**, 1873 (1969); **182**, 1899 (1969); **186**, 1611 (1969); Phys. Rev. Letters **23**, 670 (1969).

[7] R. A. Brandt, Phys. Rev. D **1**, 2808 (1970).

A converse statement is also possible. Suppose the equal-time matrix element contracts in x_3 to a disk whose size is $\sim 1/p$. By relativistic invariance this means that $\langle p|T(j_0(\mathbf{x},x_3,x_0)j_0(0))|p\rangle$ becomes nonzero only for $|x_0-x_3|\lesssim 1/p$. Consider the amplitude

$$M_{00}=-i\int d^4x\, e^{iqx}\langle p|T(j_0(\mathbf{x},x_3,x_0)j_0(0))|p\rangle$$

at large p. Write

$$\langle p|T(j_0(x)j_0(0))|p\rangle=p_0p_0d(p\cdot x,x^2)+\cdots.$$

Then $|x_0-x_3|\lesssim 1/p$ means that $d(p\cdot x,x^2)$ is independent of x_0+x_3 up to values $|x_0+x_3|\sim p$. But this means that $|q_0-q_3|\lesssim 1/p$, which requires that $q\cdot p$ not be large. Thus the leading contribution of $M_{00}(p,q)$ at large p, which grows as p, occurs only when $q\sim 1/p$, which requires that the total electroproduction cross section vanish at high energy. This heuristic argument makes it difficult to see how electromagnetic size can shrink if the total electroproduction cross section becomes asymptotically constant.[8,9] This is supported by the models of Sec. V.

Finally, in Sec. V a singular example is given where $\langle p|j_0(\mathbf{x},x_3,0)j_0(0)|p\rangle$ is divergent. In this case the integrals in (2.10) and (2.13) are convergent, but the interchange of limits leading to (2.10) and (2.13) is not valid.

II. PARTICLE SIZE TO SECOND ORDER IN EXTERNAL POTENTIAL

To begin our discussion of the interaction size of a particle, consider the scattering of a charged scalar particle in an external electromagnetic field. For simplicity, only forward scattering to second order in the external potential will be considered in this section. The general case of all orders in the external potential will be discussed subsequently. If the scalar particle has momentum p, the scattering amplitude is given by

$$f(p)=\int d^4q\, M_{\mu\nu}(p,q)A_\mu(q)A_\nu(-q)\,, \tag{2.1}$$

where A_μ is the external electromagnetic field, and $M_{\mu\nu}$ is given by

$$M_{\mu\nu}(p,q)=-i\int d^4x\, e^{iqx}\langle p|T(j_\nu(x)j_\mu(0))|p\rangle$$
$$=-i\int d^4x\, e^{iqx}M_{\mu\nu}(p,x)\,.$$

We can convert (2.1) to coordinate space, in which case

$$f(p)=-i\int d^4xd^4y\, M_{\mu\nu}(p,\,x-y)A_\nu(x)A_\mu(y)\,. \tag{2.2}$$

[8] J. D. Bjorken and E. A. Paschos, Phys. Rev. **185**, 1975 (1969).
[9] B. L. Ioffe, Phys. Letters **30B**, 123 (1969).

Further, at this point consider only the $\mu=0$ component of A_μ and choose $A_0(x)=\delta(x_0)a_0(\mathbf{x},x_3)$, where $\mathbf{x}=(x_1,x_2)$ is a two-dimensional spacelike vector. Then

$$f(p)=-i\int d^3xd^3y\, M_{00}(p,\,x-y)\bigg|_{x_0=0;\,y_0=0}\times a_0(\mathbf{x},x_3)a_0(\mathbf{y},y_3)\,. \tag{2.3}$$

By choosing the external potential in the above form, the expression inside the integral in (2.3) refers in effect to simultaneous measurements, as defined by interaction with the external potential, at the points $(\mathbf{x},x_3)$ and $(\mathbf{y},y_3)$. Thus

$$\langle p|T((\mathbf{x},x_3,0)j_0(0))|p\rangle=M_{00}(p,x)|_{x_0=0}$$
$$=\langle p|j_0(\mathbf{x},x_3,0)j_0(0)|p\rangle \tag{2.4}$$

is the amplitude for simultaneously finding an electromagnetic charge density at $(\mathbf{x},x_3)$ and $(\mathbf{y},y_3)$, and thus should provide a measure of the size of the scalar particle in the frame in which that particle moves with momentum p.

First it will be established that $M_{\mu\nu}(p,x)|_{x_0=0}$ has a finite size for any value of p. To show this, it is convenient to use the Jost-Lehmann-Dyson representation[10,11]

$$M_{\mu\nu}(p,x)=\int d\kappa^2\, \rho_{\mu\nu}(p,x,\kappa^2)\Delta_c(x,\kappa^2)\,. \tag{2.5}$$

$\rho_{\mu\nu}$ may contain a finite number of derivatives but the coefficients of these derivatives are bounded by polynomials in x. For spacelike x,

$$\Delta_c(x,\kappa^2)=-\frac{i\kappa}{4\pi^2(-x^2)^{1/2}}K_1(\kappa(-x^2)^{1/2})\,,$$

which decreases exponentially with $\kappa(-x^2)^{1/2}$. If there are no zero-mass particles in the theory, the κ^2 integration in (2.4) does not extend to zero [except for the pole term in which case $\Delta_c(x)=\delta(x^2)/4\pi$] so that $M_{\mu\nu}(p,x)$ decreases exponentially with $(-x^2)^{1/2}$ for large spacelike x^2. This result can also be obtained by a slight modification of a common proof of the cluster decomposition property.[12] The minimum value of κ^2, $\kappa_{\min}^2$, allowed in (2.4) does not depend on p, so the size determined by $\kappa_{\min}^{-1}$ does not contract in a rapidly moving reference frame. As we shall now show, $\kappa_{\min}^{-1}$ is a maximum size and the particle size which does contract in a rapidly moving reference system arises from a different source.

To investigate this question further, write $M_{\mu\nu}(p,x)$ as

$$M_{\mu\nu}(p,x)=\frac{i}{(2\pi)^4}\int d^4q\, e^{-iqx}M_{\mu\nu}(p,q) \tag{2.6}$$

[10] R. Jost and H. Lehmann, Nuovo Cimento **5**, 1598 (1957).
[11] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).
[12] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (Benjamin, New York, 1964).

and

$$M_{\mu\nu}(p,q)=ag_{\mu\nu}+bq_\mu q_\nu+c(q_\mu p_\nu+q_\nu p_\mu)+dp_\mu p_\nu\,, \quad (2.7)$$

where a, b, c, and d depend on $p\cdot q$ and q^2. Consider for the moment only the contribution of the last term in (2.7) to the integral in (2.6). Call this contribution $\tilde{M}_{\mu\nu}$. Then

$$\tilde{M}_{\mu\nu}(p,x)=p_\mu p_\nu\frac{i}{(2\pi)^4}\frac{1}{2}\int dq_+dq_-d^2q \times e^{-iq_+x_--iq_-x_++i\mathbf{q}\cdot\mathbf{x}}d(p\cdot q,q^2)\,, \quad (2.8)$$

where

$$q_\pm=(q_0\pm q_3)\,,\quad x_\pm=\tfrac{1}{2}(x_0\pm x_3)\,.$$

For large $p=p_3$, $p^\mu p_\mu=p_0^2-p_3^2$ remaining fixed, $p\cdot q\approx pq_-=\nu$, (2.8) can be written

$$\tilde{M}_{\mu\nu}(p,x)=\frac{p_\mu p_\nu}{2p}\frac{i}{(2\pi)^4}\int_{-\infty}^{\infty}d\nu\int dq_+d^2q \times e^{-iq_+x_--i\nu x_+/p+i\mathbf{q}\cdot\mathbf{x}}d(\nu,\,\nu q_+/p-\mathbf{q}^2)\,. \quad (2.9)$$

If the large-p limit can be taken inside the integral, then

$$\tilde{M}_{\mu\nu}(p,x)\to\frac{p_\mu p_\nu}{2p}\frac{i}{(2\pi)^3}\delta(x_-) \times\int_{-\infty}^{\infty}d\nu\int d^2q\; e^{i\mathbf{q}\cdot\mathbf{x}}d(\nu,\,-\mathbf{q}^2)\,. \quad (2.10)$$

Because of its importance to this section of the paper, I would like to present another derivation of (2.10).

Write $d(\nu,q^2)$ in a dispersion relation as

$$d(\nu,q^2)=\frac{1}{\pi}\int_0^\infty \mathrm{Im}d(\nu',q^2)\left(\frac{1}{\nu'-\nu-i\epsilon}+\frac{1}{\nu'+\nu-i\epsilon}\right).$$

Instead of considering d as a function of two invariants ν and q^2, consider it, for large p, as a function of pq_- and q^2. Then for large p,

$$d(pq_-,q^2)=\frac{1}{\pi p}\int_0^\infty \mathrm{Im}d(\nu',q^2)\mathcal{P}\left(\frac{1}{\nu'/p-q_-}+\frac{1}{\nu'/p+q_-}\right) +\frac{i}{p}\int_0^\infty \mathrm{Im}d(\nu',q^2)[\delta(\nu'/p-q_-)+\delta(\nu'/p+q_-)]\,, \quad (2.11)$$

where $\mathcal{P}$ indicates a principal value. The principal value vanishes as $p\to\infty$, so one obtains

$$d(pq_-,q^2)\to\frac{2i}{p}\delta(q_-)\int_0^\infty d\nu\;\mathrm{Im}d(\nu',\,-\mathbf{q}^2)\,. \quad (2.12)$$

Substituting (2.12) into (2.8), one obtains

$$\tilde{M}_{\mu\nu}(p,x)\to-\frac{p_\mu p_\nu}{p}\frac{\delta(x_-)}{(2\pi)^3} \times\int_0^\infty d\nu\int d^2q\; e^{i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{Im}d(\nu,\,-\mathbf{q}^2) \quad (2.13)$$

which, upon using

$$\int_{-\infty}^{\infty}d\nu d(\nu,\,-\mathbf{q}^2)=2i\int_0^\infty d\nu\;\mathrm{Im}d(\nu,\,-\mathbf{q}^2)\,,$$

is seen to be equal to (2.10).

Clearly the condition for the convergence of the integrals in (2.10) or (2.13) is that there be no Regge singularities greater than or equal to 1 at $t=0$. Assuming convergence we see that $\tilde{M}_{\mu\nu}(p,x)$ approaches zero except for $x_0\approx x_3$, that is, except for the region in a thin disk $|x_0-x_3|\sim 1/p$. In particular,

$$\tilde{M}_{33}(p,x)\approx\tilde{M}_{00}(p,x)\approx\tilde{M}_{03}(p,x)\approx\tilde{M}_{30}(p,x) \approx\frac{ip}{2(2\pi)^3}\delta(x_-)\int d\nu d^2q d(\nu,\,-\mathbf{q}^2)e^{+i\mathbf{q}\cdot\mathbf{x}}$$

for large p, while all other components of $\tilde{M}_{\mu\nu}$ are smaller by at least a single factor of p^{-1}. The reader can readily verify that the other terms of (2.7) do not give contributions to $M_{\mu\nu}$ which increase linearly with p for large p. Thus

$$M_{\mu\nu}(p,x)\approx\tilde{M}_{\mu\nu}(p,x)\,,\quad \mu=0,\,3\,,\quad \nu=0,\,3$$

for large p. If $x_0=0$, that is, the condition specified in (2.4), then

$$M_{00}(p,x)|_{x_0=0}\approx ip\frac{\delta(x_3)}{2(2\pi^3)}\int d\nu d^2q d(\nu,\,-\mathbf{q}^2)e^{i\mathbf{q}\cdot\mathbf{x}}. \quad (2.14)$$

Equation (2.12) shows that so long as there are no Regge singularities with $\alpha(t=0)\geq 1$ the electromagnetic size as defined by (2.4) contracts to a disk with width $\propto 1/p$.

Now, returning to (2.13), we note that $\int_0^\infty \mathrm{Im}d(\nu,-q^2)d\nu$ is the residue of a nonsense wrong-signature fixed pole at $J=1$ and $t=0$,[2,3] as is easily verified from a Froissart-Gribov representation for the continued partial-wave amplitude $F_{+-}{}^J(t=0)$ in the t channel.

We can very easily derive another representation for $M_{\mu\nu}(p,x)$ as p becomes large. To achieve this, use (2.13) to write

$$\langle p|T(j_\nu(\mathbf{x},x_3,x_0)j_\mu(0))|p\rangle\to -\frac{p_\mu p_\nu}{p}\frac{\delta(x_-)}{(2\pi)^3}\int d\nu d^2q\; e^{i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{Im}d(\nu,\,-\mathbf{q}^2)\,.$$

Integrating the above expression over x_3 and setting

$x_0=0$, one obtains

$$\int dx_3\langle p|\,j_\nu(\mathbf{x},x_3,0)j_\mu(0)\,|p\rangle\rightarrow -\frac{p_\mu p_\nu}{p}\frac{2}{(2\pi)^3}\int d\nu d^2q\, e^{i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{Im}d(\nu,\,-q^2)$$

or

$$M_{\mu\nu}(p,x)\rightarrow\tfrac{1}{2}\delta(x_-)\int dx_3\langle p|\,j_\nu(\mathbf{x},x_3,0)j_\mu(0)\,|p\rangle\,,\quad \mu=0,3\,,\quad \nu=0,3\,. \tag{2.15}$$

Using (2.15), consider the scattering of the charged scalar boson off a static potential. Then the scattering amplitude is

$$g(p)=-i\int d^4xd^3y\,M_{00}(p,\,x-y)\times\mathfrak{A}_0(\mathbf{x},y_3)\mathfrak{A}_0(\mathbf{y},y_2)\,,\quad y_0=0\,. \tag{2.16}$$

Substituting (2.15) into (2.16), one obtains

$$g(p)=-i\int d^2xd^2y\,F(\mathbf{x}-\mathbf{y})\bar{\mathfrak{A}}_0(\mathbf{x})\bar{\mathfrak{A}}_0(\mathbf{y})\,, \tag{2.17}$$

where

$$\bar{\mathfrak{A}}_0(\mathbf{x})=\int dx_3\,\mathfrak{A}_0(\mathbf{x},x_3) \tag{2.18}$$

and

$$F(\mathbf{x})=\int dx_3\langle p|\,j_0(\mathbf{x},x_3,0)j_0(0)\,|p\rangle\,. \tag{2.19}$$

Equations (2.17)–(2.19) are just the expressions given by the operator droplet model in second order.[5] Such expressions will not be obtained, however, if the potential is time dependent.

We have obtained the result, then, that the second-order term $\int\langle p|\,j_0(\mathbf{x},x_3,0)j_0(0)\,|p\rangle dx_3$ in the expansion of the operator droplet model is directly related to the residue of a nonsense wrong-signature fixed pole at $J=1$, and on the way to deriving this result we have actually obtained the operator droplet model for the scattering off an external static potential to second order assuming the absence of Regge trajectories, or cuts, with $\alpha(0)\geq 1$. In Sec. III we shall obtain an extension of this result to all orders in the external potential.

Before we conclude this section, let us note what the physical results are to second order in the external potential. In the first place, we have argued that a particle has a size, determined by the interaction with an external electromagnetic source, which contracts at high velocity if there are no Regge singularities greater or equal to one at $t=0$. This same assumption then allows one to derive the operator droplet model for the scattering of a charged particle off a static external potential. The impact factor which occurs in this model is then simply related to the residue of a fixed pole at $J=1$ in a nonsense wrong-signature partial-wave amplitude.

III. HIGHER-ORDER TERMS IN EXTERNAL POTENTIAL

In this section higher-order terms in the scattering of a scalar meson off an external electromagnetic potential will be considered. The result of forward scattering for three interactions is, for example,

$$f(p)=\int d^4x_1d^4x_2d^4x_3M_{\mu\nu\lambda}(p,x_1,x_2,x_3)\times A_\mu(x_1)A_\nu(x_2)A_\lambda(x_3)\,, \tag{3.1}$$

where

$$M_{\mu\nu\lambda}(p,x_1,x_2)=\langle p|\,T(j_\mu(x_1)j_\nu(x_2)j_\lambda(x_3))\,|p\rangle\,. \tag{3.2}$$

The extension of (3.1) to an arbitrary number of interactions is obvious. As in Sec. II, we wish to consider $M_{\mu\nu\lambda}$ when p is very large. To begin, let us consider the kinematics of the five-point amplitude

$$M_{\mu\nu\lambda}(p,q_1,q_2)=\int d^4x_1d^4x_2\times e^{iq_1x_1+iq_2x_2}M_{\mu\nu\lambda}(p,x_1,x_2,0)\,. \tag{3.3}$$

The invariant amplitudes appearing in an expansion of $M_{\mu\nu\lambda}$ will depend on five variables which can be chosen to be q_1^2, q_2^2, $(q_1+q_2)^2$, $p\cdot q_1$, and $p\cdot q_2$. One of the invariants appearing in (3.3) will occur as a coefficient of $p_\mu p_\nu p_\lambda$. We call this contribution

$$\tilde{M}_{\mu\nu\lambda}=d(p\cdot q_1,\,p\cdot q_2\cdot q_1^2,\,q_2^2,\,(q_1+q_2)^2)p_\mu p_\nu p_\lambda\,,$$

by analogy with the notation of Sec. II. Then

$$\tilde{M}_{\mu\nu\lambda}(p,x_1,x_2)=\frac{1}{(2\pi)^8}\int d^4q_1d^4q_2\times e^{-iq_1x_1-iq_2x_2}d(p,q_1,q_2)p_\mu p_\nu p_\lambda\,.$$

This equation can be written as

$$\tilde{M}_{\mu\nu\lambda}(p,x_1,x_2)=\frac{p_\mu p_\nu p_\lambda}{4(2\pi)^8}\int dq_{1+}dq_{1-}dq_{2+}dq_{2-}d^2q_1d^2q_2\times e^{-iq_{1+}x_{1-}-iq_{1-}x_{1+}-iq_{2+}x_{2-}-iq_{2-}x_{2+}+i\mathbf{q}_1\cdot\mathbf{x}_1+i\mathbf{q}_2\cdot\mathbf{x}_2}\times d(pq_{1-},\,pq_{2-},\,q_1^2,\,q_2^2,\,(q_1+q_2)^2)$$

for large p. Let $\nu_1=pq_{1-}$ and $\nu_2=pq_{2-}$. Assuming the interchange of the $p\rightarrow\infty$ limit and the integration over q for $\mu,\nu,\lambda=0,3$, one obtains

$$\tilde{M}_{\mu\nu\lambda}(p,x_1,x_2)\rightarrow\frac{p_\mu p_\nu p_\lambda}{p^24(2\pi)^6}\delta(x_{1-})\delta(x_{2-})\times\int d\nu_1d\nu_2d^2q_1d^2q_2e^{i\mathbf{q}_1\cdot\mathbf{x}_1+i\mathbf{q}_2\cdot\mathbf{x}_2}\times d(\nu_1,\,\nu_2,\,-\mathbf{q}_1^2,\,-\mathbf{q}_2^2,\,-(\mathbf{q}_1+\mathbf{q}_2)^2)\,. \tag{3.4}$$

Equation (3.4) is the three-current analog of (2.10) and similar assumptions are required for its validity. If

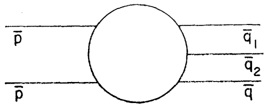

FIG. 1. Kinematics for the five-point function.

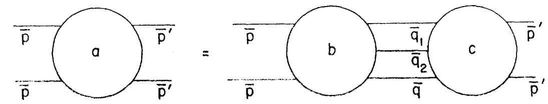

FIG. 2. Schematic diagram of (A12).

the time components of x_1 and x_2 are set equal to zero, $\tilde{M}$ vanishes identically unless the z components of x_1 and x_2 are also equal to zero, up to order $1/p$. Thus three simultaneous measurements by an external potential show that the fast-moving particle again contracts into a thin disk of width proportional to $1/p$. In the following it will be argued that (3.4) is related to a fixed pole at $J=1$ in a nonsense right-signature production amplitude. In completing this argument we shall also show that for large p, $M_{\mu\nu\lambda}$ approaches $\tilde{M}_{\mu\nu\lambda}$ for $\mu,\nu,\lambda=0,3$ so that the restriction to $dp_\mu p_\nu p_\lambda$ in M was, in fact, not a necessary assumption.

Consider now the five-point function illustrated in Fig. 1 and whose matrix element is given by (3.2). For a particle with spacelike momentum $\bar{q}$, we can choose a frame where $q=(\bar{q},q_3)=(0,|\bar{q}|,0,q_3)=(q_0,q_1,q_2,q_3)$. In this frame, define the polarization vectors

$$\begin{aligned} \epsilon_\mu^+ &= \tfrac{1}{2}\sqrt{2}(1,0,1,0)\,,\\ \epsilon_\mu^- &= \tfrac{1}{2}\sqrt{2}(1,\,0,\,-1,\,0)\,,\\ \epsilon_\mu^0 &= (0,1,0,0)\,,\\ \epsilon_\mu^s &= (0,0,0,1)\,. \end{aligned} \tag{3.5}$$

The polarization vectors have the property

$$\sum_a \epsilon_\mu{}^a \epsilon_\nu{}^a = g_{\mu\nu}\,.$$

The amplitude given by (3.2) can be expressed in an helicity notation

$$M_{abc}=\sum_{\alpha\beta\gamma} M_{\alpha\beta\gamma}(p,q,q_1)\epsilon_\alpha{}^a(q_1)\epsilon_\beta{}^b(q_2)\epsilon_\gamma{}^c(q)\,. \tag{3.6}$$

This amplitude can be expanded in $O(2,1)$ harmonics according to[13–15]

$$M_{abc}(u^{-1}g,u^{-1}gg_1,g_1) =\sum_\rho \int d\rho(\Lambda)d\sigma(\mu)D_{0,\rho\mu}{}^\Lambda(u^{-1}g)M_{abc}{}^{\Lambda\rho\mu}(g_1)\,, \tag{3.7}$$

with the inversion

$$M_{abc}{}^{\Lambda\rho\mu}(g_1)=\int dv\, M_{abc}(v,vg_1,g_1)\bar{D}_{0,\rho\mu}{}^\Lambda(v)\,, \tag{3.8}$$

and where

$$\bar{p}=u\hat{p}\,,\qquad \hat{p}=(M,00)\,,$$
$$\bar{q}=g\hat{q}\,,\qquad \hat{q}=(0,|\bar{q}|,0)\,,$$
$$\bar{q}_1=gg_1\hat{q}_1\,,\qquad \hat{q}_1=(0,|\bar{q}_1|,0)\,.$$

[13] M. Toller, Nuovo Cimento **37**, 631 (1965).

[14] A. H. Mueller and I. J. Muzinich, Ann. Phys. (N. Y.) **57**, 20 (1970); **57**, 500 (1970).

[15] M. Ciafaloni, C. DeTar, and M. Misheloff, Phys. Rev. **188**, 2522 (1969).

u, g, and g_1 are $O(2,1)$ transformations, and we have for the moment dropped variables which do not enter directly into the $O(2,1)$ expansion. Now consider only the $dp_\mu p_\nu p_\lambda$ term. Then

$$\tilde{M}_{abc}=dp\cdot\epsilon^a p\cdot\epsilon^b p\cdot\epsilon^c$$

and

$$\tilde{M}_{abc}{}^{\Lambda\rho\mu}(g_1)=\int dvd(v,vg_1,g_1) \times p\cdot\epsilon^a p\cdot\epsilon^b p\cdot\epsilon^c \bar{D}_{0,\rho\mu}{}^\Lambda(v)\,. \tag{3.9}$$

Choose $\bar{q}=\hat{q}$ and $\bar{p}=v^{-1}\hat{p}=M(\cosh\xi\,\cosh\zeta,\,-\sinh\xi,\,\cosh\xi\,\sinh\zeta)$. Then $dv=2\pi d\,\sinh\xi d\zeta$.

The leading fixed poles in Λ will come from the leading pole in Λ of $D_{0,\rho\mu}{}^\Lambda$ which depends on μ. When the five-point function is used to calculate a scattering amplitude as illustrated in Fig. 2, for example, an integration over μ will occur. This integral over μ can be distorted in the μ plane until singularities appear. Suppose this leading singularity in μ is a pole at α. Then the leading pole in Λ will be given by the leading pole in $\bar{D}_{0,\rho\alpha}{}^\Lambda$, assuming that this pole can be reached before the integral in (3.9) diverges. Thus, the first step is to calculate the leading poles in μ which occur owing to the ζ integration in (3.9). For large ζ,

$$p\cdot\epsilon \xrightarrow[\zeta\to+\infty]{} M\,\cosh\xi\,\cosh\zeta(\epsilon_0-\epsilon_2)\,,$$
$$p\cdot\epsilon \xrightarrow[\zeta\to-\infty]{} M\,\cosh\xi\,\cosh\zeta(\epsilon_0+\epsilon_2)\,.$$

Substituting this into (3.9),

$$\begin{aligned} \tilde{M}_{abc}{}^{\Lambda\rho\mu}(g_1) \propto{}& M^3(\epsilon_0{}^a-\epsilon_2{}^a)(\epsilon_0{}^b-\epsilon_2{}^b)(\epsilon_0{}^c-\epsilon_2{}^c)\\ &\times\int d\,\sinh\xi\int_\Lambda^\infty d\zeta(\cosh\xi)^3(\cosh\zeta)^3 e^{\mu\zeta}\bar{d}_{0,\rho\mu}{}^\Lambda(\xi)d(\xi,\zeta,g_1)\\ &+M^3(\epsilon_0{}^a+\epsilon_2{}^a)(\epsilon_0{}^b+\epsilon_2{}^b)(\epsilon_0{}^c+\epsilon_2{}^c)\int d\,\sinh\xi\\ &\times\int_{-\infty}^{-\Lambda} d\zeta(\cosh\xi)^3(\cosh\zeta)^3 e^{\mu\zeta}\bar{d}_{0,\rho\mu}{}^\Lambda(\xi)d(\xi,\zeta,g_1) \end{aligned} \tag{3.10}$$

near the leading singularity in μ. Consider, for the present, the integral

$$I=\int_{-\infty}^\infty d\,\sinh\xi(\cosh\xi)^3\bar{d}_{0,\rho\mu}{}^\Lambda(\xi) \times\int_{-\infty}^{-\Lambda} d\zeta(\cosh\zeta)^3 e^{\mu\zeta}d(\xi,\zeta,g_1)\,.$$

Now use (3.4), which states that as $\zeta \to -\infty$

$$d(\bar{p}\cdot\hat{q},\bar{p}\cdot\bar{q}_1,\bar{q},\bar{q}_1) \xrightarrow[\bar{p}\cdot\bar{q}_1\to\infty]{} \frac{\delta(q_{10}-q_{12})}{M\cosh\xi\cosh\zeta} \times\int d\nu_1 d(\bar{p}\cdot\hat{q},\nu_1,\hat{q},\bar{q}_1).$$

Thus

$$I=\frac{1}{M}\int_{-\infty}^{\infty} d\sinh\xi(\cosh\xi)^2\bar{d}_{0,\rho\mu}{}^{\Lambda}(\xi)\int_{-\infty}^{-\Lambda} d\zeta(\cosh\zeta)^2 e^{\mu\zeta} \times\int d\nu_1 d(\bar{p}\cdot\hat{q},\nu_1,\hat{q},\bar{q}_1)\delta(q_{10}-q_{12}).$$

The ζ dependence has now been isolated into the $(\cosh\zeta)^2$ factor. The leading pole in μ occurs at $\mu=2$, that is,

$$\int_{-\infty}^{-\Lambda} d\zeta(\cosh\zeta)^2 e^{\mu\zeta}\approx(\tfrac{1}{2})^{\mu}\int_{\Lambda'}^{\infty} d\cosh\zeta(\cosh\zeta)^{1-\mu}\approx\frac{1}{4}\frac{1}{\mu-2}$$

near $\mu=2$. Now, using $\bar{d}_{0,\rho\mu}{}^{\Lambda}(\xi)=d_{0,-\rho-\mu}{}^{-\Lambda-1}(-\xi)$,[14,15] and keeping only the leading μ pole, I can be written as

$$I\approx\frac{\delta(q_{10}-q_{12})}{4M(\mu-2)}\int d\sinh\xi(\cosh\xi)^2 d_{0,-\rho-2}{}^{-\Lambda-1}(-\xi) \times\int d\nu_1 d(\bar{p}\cdot\hat{q},\nu_1,\hat{q},\bar{q}_1).$$

Noting that $d_{0,-\rho-2}{}^{-\Lambda-1}(-\xi)$ has a pole in Λ at $\Lambda=-2$ whose residue is a constant times $(\cosh\xi)^{-2}$, we can write

$$I\propto\frac{\delta(q_{10}-q_{12})}{M(\mu-2)(\Lambda+2)}\int d\sinh\xi\int d\nu_1 d(\bar{p}\cdot\hat{q},\nu_1,\hat{q},\bar{q}_1)$$

or

$$I\propto\frac{\delta(q_{10}-q_{12})}{M^2(\mu-2)(\Lambda+2)}\int d\nu_1 d\nu d(\nu,\nu_1,\hat{q},\bar{q}_1).$$

Thus near $\mu=2$ and $\Lambda=-2$ the first term of (3.10) is

$$\tilde{M}_{abc}{}^{\Lambda\rho\mu}(g_1)\propto\frac{M\delta(q_{10}-q_{12})}{(\mu-2)(\Lambda+2)}(\epsilon_0{}^a-\epsilon_2{}^a)(\epsilon_0{}^b-\epsilon_2{}^b) \times(\epsilon_0{}^c-\epsilon_2{}^c)\int d\nu d\nu_1 d(\nu,\nu_1,\hat{q},\bar{q}_1). \quad (3.11)$$

The second term in (3.10) is similar, where again the integral

$$\int d\nu d\nu_1 d(\nu,\nu_1,\hat{q},\bar{q}_1)$$

appears as a factor. This integral is the same one which appears in (3.4), and the fact that it occurs as the coefficient of $1/(\Lambda+2)$ means that it is the residue of an $O(2,1)$ fixed pole.

Furthermore, this fixed pole at $\Lambda=-2$ is the leading Λ pole in $\tilde{M}_{abc}{}^{\Lambda\rho\mu}(g_1)$ so long as there are no Regge trajectories α which have $\alpha(0)\geq 1$. It now becomes clear why only the $p_\mu p_\nu p_\lambda$ term of $M_{\mu\nu\lambda}$ contributes to the leading right-signature fixed pole. In the calculation just completed, it was critical that there be a pole at $\mu=2$ in order to get a pole at $\Lambda=-2$. A pole at $\mu=1$, for example, would give a pole at $\Lambda=-1$ which corresponds to a pole at $J=0$ in a Sommerfeld-Watson amplitude, just as a pole at $\Lambda=-2$ corresponds to a pole at $J=1$. The pole at $\mu=2$ came from the $(\cosh\zeta)^3$ factor, which in turn came from the three factors of p in $p_\mu p_\nu p_\lambda$. A term whose tensor indices were $p_\mu p_\nu q_\lambda$, for example, would have only a $(\cosh\zeta)^2$ factor which would give a pole at $\mu=1$ and hence at $\Lambda=-1$.

At this point it may also be in order to comment on the reason why only spacelike $\bar{q}$, $\bar{q}_1$, and $(\bar{q}-\bar{q}_1)$ were considered. We have been interested in the large-p behavior of $M_{\mu\nu\lambda}$. If any of the variables $\bar{q}$, $\bar{q}_1$, or $(\bar{q}-\bar{q}_1)$ is timelike then $\bar{p}\cdot\bar{q}$, if $\bar{q}$ is timelike, must go to infinity with large p. Thus in that case the behavior of $M_{\mu\nu\lambda}$ would be governed by the leading Regge singularities in the t channel. If there are no $\alpha(0)\geq 1$, the singularity which we have found above at $\Lambda=-2$ is greater than the singularities derived from Regge asymptotic behavior, so that the spacelike values of $\bar{q}$, $\bar{q}_1$, and $(\bar{q}-\bar{q}_1)$ are the only important ones for our considerations.

In concluding this section let us again go back to (3.4). For $\mu,\nu,\lambda=0,3$, we can write

$$M_{\mu\nu\lambda}(p,x_1,x_2)\xrightarrow[p\to\infty]{}\tfrac{1}{4}\delta(x_{1-})\delta(x_{2-})\int dx_{13}dx_{23} \times\langle p|j_\mu(\mathbf{x}_1,x_{13},0)j_\nu(\mathbf{x}_2,x_{23},0)j_\lambda(0)|p\rangle, \quad (3.12)$$

and $M_{\mu\nu\lambda}$ is independent of μ, ν, λ so long as these indices take on the values 0 and 3. Thus, again the operator droplet expansion is obtained, this time to third order in the external potential. However, the higher orders are trivial once the three-current problem has been dealt with, so we reach the conclusion that to any order in the external potential the scattering of a scalar meson off a static electromagnetic potential is given by the operator droplet model. We note, however, that this will not be true for external potentials which are not static, nor will it be true in general for particle-particle scattering to higher orders in electromagnetism.

IV. SINGULAR EXAMPLE

There is a different and rather singular way in which particle size can contract at high velocity. This case will first be analyzed for scalar currents and then the extension to vector currents will be discussed.

It will be convenient to use a formalism which expresses high-energy behavior of scattering amplitudes in terms of a Mellin transform. Such a formalism is developed in Appendix B. We restate a few relevant results for completeness. If

$$f(p,q)=-i\int d^4x\, e^{iqx}\langle p|T(j(x)j(0))|p\rangle \quad (4.1)$$

and $p=(M,0,0,0)$, then $f(p,q)=f(Q,\mu^2)$, where $Q^2=q_1^2+q_2^2+q_3^2$ and $\mu^2=q^2$. Furthermore,

$$f(Q,\mu^2)=Q^{-1}[g(Q,\mu^2)+g(-Q,\mu^2)], \quad (4.2)$$

with

$$g(Q,\mu^2)=-i\int rdrdx_0 f(x_0,r)e^{iq_0x_0-iQr}$$

and $f(x_0,r)=\langle p|T(j(x)j(0))|p\rangle$. Also,

$$g(Q,\mu^2)=\int dl\, Q^l g_l(\mu^2),$$

and the leading singularities in the left half l plane are determined by

$$g_l(\mu^2)=2\pi^2 ie^{i(\pi/2)l}\int rdrdx_0\, f(x_0,r) \times\left[\frac{\mu^2x_0}{2(x_0-r)}\right]^{-l/2} H_l^{(1)}([2\mu^2x_0(x_0-r)]^{1/2}). \quad (4.3)$$

If the leading singularities in l of $g_l(\mu^2)$ occur at $x_0=r$ but finite x_0 in (4.3), then again we shall have an instance where particle size will contract at high velocity. This is most easily seen by use of an example.

Suppose $f(p\cdot x,x^2)=\langle p|T(j(x)j(0))|p\rangle$ has a singularity near $x^2=0$ of the type

$$f(p\cdot x,x^2)\underset{x^2\to 0}{\longrightarrow}(x^2)^{-\gamma-2}a(p\cdot x), \quad (4.4)$$

where $a(p\cdot x)$ decreases rapidly in $p\cdot x$. Then from (4.1)

$$f(p\cdot q,q^2)=-i\int d^4x\, e^{iqx}(x^2)^{-\gamma-2}a(p\cdot x)+\cdots. \quad (4.5)$$

For large $p\cdot q$ and fixed q^2, $f(p\cdot q,q^2)$ goes as $(p\cdot q)^\gamma$. This can be seen by substituting (4.4) into (4.3) and calculating the leading l-plane singularity at $l=\alpha+1$. Now consider scattering off an external potential $\varphi(x)$. Then the scattering amplitude is

$$f(p)=\int d^4xd^4y\langle p|T(j(x)j(y))|p\rangle\varphi(x)\varphi(y),$$

in complete analogy with (2.1). Suppose now that $\varphi(x)\to\varphi(\mathbf{x},x_3)\delta(x_0)$. Then

$$f(p)\to\int d^2xd^2ydx_3dy_3 \times f(-p(x_3-y_3),\,-(\mathbf{x}-\mathbf{y})^2-(x_3-y_3)^2) \times\varphi(\mathbf{x},x_3)\varphi(\mathbf{y},y_3),$$

or

$$f(p)\to\int d^2xd^2zdx_3dz_3\, f(-pz_3,\,-\mathbf{z}^2-z_3^2) \times\varphi(\mathbf{x},x_3)\varphi(\mathbf{x}-\mathbf{z},\,x_3-z_3). \quad (4.6)$$

However, if $\gamma\geq-\frac{1}{2}$, the above expression diverges, and the interaction is too singular to give a meaning to simultaneous measurements. This is a statement that $\langle p|j(\mathbf{x},x_3,0)j(0)|p\rangle$ has a very strong singularity at $\mathbf{x}=0$, $x_3=0$. This singularity is too strong to be integrated with a test function over $\mathbf{x}$ and x_3 unless the test function vanishes at $\mathbf{x}=0$, $x_3=0$. Of course, $\langle p|j(\mathbf{x},x_3,0)j(0)|p\rangle$ does correspond to a simultaneous measurement, and it is finite, but the relative probability of making such a measurement to the ∞ probability indicated at $\mathbf{x}=x_3=0$ by the divergence of (4.6) makes it difficult to interpret the $\gamma\geq-\frac{1}{2}$ case.

In the range $-1<\gamma<-\frac{1}{2}$, the integral in (4.6) exists and can be rewritten as

$$f(p)=\frac{1}{p}\int d^2xdx_3d^2zd\nu\, f(-\nu,\,-\mathbf{z}^2-\nu^2/p^2) \times\varphi(\mathbf{x},x_2)\varphi(\mathbf{x}-\mathbf{z},\,x_3-\nu/p) \quad (4.7)$$

for large p, where $\nu=pz_3$. The large-p limit cannot be interchanged with the integration because of the singularity on the light cone. Using (4.4) and observing that only the region near $\mathbf{z}=0$ and $z_3=0$ contributes to the large-p limit, one obtains

$$f(p)\to p^{2\gamma+1}\int d^2xdx_3d^2\lambda d\nu\, a(\nu) \times(\lambda^2+\nu^2)^{-\gamma-2}\varphi(\mathbf{x},x_3)\varphi(\mathbf{x},x_3). \quad (4.8)$$

Equation (4.8) makes it clear that in fact the particle has shrunk to a point for simultaneous interactions at high energy.

Thus if the interaction is singular (in the sense that the high-energy behavior is determined by singularities on the finite regions of the light cone), then the particle shrinks to a point at high energies if the interaction is not too singular: if $-1<\gamma<-\frac{1}{2}$, in the above example. If the interaction is very singular ($\gamma>-\frac{1}{2}$ in the above example), it appears difficult to give meaning to particle size without a more detailed treatment of smearing in time. The question of more than two simultaneous interactions has not been investigated.

A possibly interesting feature arises in the present context. To illustrate this feature, we shall display $\langle p|j(x)j(0)|p\rangle$ by means of a Jost-Lehmann-Dyson representation[10,11]

$$\langle p|j(x)j(0)|p\rangle=\int_0^\infty d\kappa^2\,\rho(x,\kappa^2)\Delta^-(x,\kappa^2) =f(p\cdot x,x^2), \quad (4.9)$$

where

$$\rho(x,\kappa^2)=\int d^4u\, e^{-iu\cdot x}\rho(u,\kappa^2)$$

and where $\rho(u,\kappa^2)$ has bounded support in u. Now from (4.9) one sees that singularities on the light cone arise from the large-κ^2 behavior of $\rho(x,\kappa^2)$, apart of course from the usual $\delta(x^2)$ and possible derivatives of $\delta(x^2)$. If, for example, $\rho(x,\kappa^2) \to (\kappa^2)^\gamma \rho(x)$, with $\gamma > -1$, then

$$f(p\cdot x,x^2) \underset{x^2\to 0}{\longrightarrow} (x^2)^{-\gamma-2}\rho(x)\,,$$

as in (4.4). [For the moment we assume that $\rho(x,\kappa^2)$ is simply a function.] However, such behavior means that

$$f(p\cdot x,x^2) \underset{x_\mu\to 0}{\longrightarrow} (x^2)^{-\gamma-2}\rho(0)\,.$$

This indicates that the *strength* of the singularities on the finite light cone and the *strength* of the singularity at $x=0$ are derived from a common source. These strengths are not necessarily equal, but could differ by integer powers. Now[16–18]

$$j(x)j(0) \underset{x_\mu\to 0}{\longrightarrow} E_0(x^2)+E_1(x^2)\Phi(0)+\cdots\,,$$

where E_0 and E_1 are numerical functions and $\Phi(x)$ is a local field. Thus we could move away from $t=0$ and consider matrix elements like $\langle p|j(x)j(0)|p'\rangle$ with values of $t=(p-p')^2$ different from 0. Of course, the leading singularities of $\langle p|j(x)j(0)|p'\rangle$ near $x_\mu=0$ would be the same as those of $\langle p|j(x)j(0)|p'\rangle$ by the fact that $\Phi(x)$ is local. The arguments following Eq. (4.9) suggest that the strengths of the singularities on the finite light cone are then also independent of t. This has two immediate consequences.

(i) The size which depends on singularities on the finite light cone is independent of t. That is, this size, defined by interaction with an external potential, shrinks to zero at large velocities for any finite value of t.

(ii) If high-energy scattering is actually determined from singularities on the finite light cone, then the asymptotic behavior with energy is independent of t as far as the rate of growth with energy is concerned.

These results do not appear to depend on the assumption, used in the above example, that $\rho(x,u)$ involves no derivatives. The central point of the argument is simply that it is the large-κ^2 region in (4.9) which is important, and that this region governs both the short-distance behavior and light-cone singularities.

V. SOME EXAMPLES IN VIRTUAL COMPTON SCATTERING

We shall now consider again the case of the large-momentum limit of the product of two electromagnetic currents for the case where the integrals in (2.10) and (2.13) do not exist. We shall treat two cases in detail and then comment on general features of the problem. The first case to be considered will have a nonsingular light-cone commutator, will obey the scaling law in deeply inelastic electroproduction,[19] and will give a constant total inelastic electroproduction cross section at high energies.[7] In the second example the light-cone commutator will be marginally singular but the scaling law will be violated, while constant total cross sections will result.

A. Nonsingular Light-Cone Behavior

In this section we shall follow the formalism and ideas of Ref. 7. We restrict our discussion to $W_2(\nu,q^2)$ for brevity. Recall that

$$W_2(\nu,q^2)=q^2\int_0^\infty da\int_{-1}^1 db\,\sigma_2(a,b)\delta(q^2+2b\nu-a) \quad (5.1)$$

for $\nu=p\cdot q>0$, where

$$W_{\mu\nu}(p,q)=\frac{1}{2\pi}\int d^4x\,e^{iqx}\langle p|[j_\mu(x),j_\nu(0)]|p\rangle$$
$$=\left(p_\mu-\frac{\nu q_\mu}{q^2}\right)\left(p_\nu-\frac{\nu q_\nu}{q^2}\right)W_2(\nu,q^2)-\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)W_1(\nu,q^2)\,, \quad (5.2)$$

and where $p^2=1$. If

$$\int_0^\infty da\,\sigma_2(a,b)=0 \quad (5.3)$$

and

$$\bar\sigma_2(a,b)\to\sigma_2(a)b^{-1}\text{ as } b\to 0, \quad (5.4)$$

where $\bar\sigma_2(a,b)=\partial\sigma(a,b)/\partial b$, one obtains the usual scaling property $\nu W_2(\nu,q^2)\to F_2(-\nu/q^2)=F_2(\rho)$ as ν and q^2 become large with ρ fixed. Furthermore, (5.3) and (5.4) give $F_2(\infty)=w_2$, which in general is not zero. We shall now examine the question of the size of the proton at high velocity as determined by (5.1), (5.3), and (5.4).

It is convenient to consider not the Fourier transform

$$\hat W_2(p\cdot x,x^2)=\frac{1}{(2\pi)^4}\int d^4q\,e^{-iqx}W_2(\nu,q^2)\,,$$

but rather the function $\hat V_2(p\cdot x,x^2)$,[7,20] which is related to $\hat W_2$ by $\hat W_2(p\cdot x,x^2)=-\Box\hat V_2(p\cdot x,x^2)$. $\hat V_2$ can be written as

$$\hat V_2(p\cdot x,x^2)=\frac{i}{2\pi}\int da db\,\sigma_2(a,b)e^{+ibp\cdot x}\Delta(x,a+b^2)\,. \quad (5.5)$$

If we switch to the time-ordered product rather than

[16] K. Wilson, Cornell LNS Report No. 64-15, 1964 (unpublished); Phys. Rev. **179**, 1499 (1969).
[17] R. A. Brandt, Ann. Phys. (N. Y.) **44**, 221 (1967).
[18] W. Zimmermann, Commun. Math. Phys. **6**, 161 (1967).
[19] J. D. Bjorken, Phys. Rev. **179**, 1547 (1968).
[20] J. W. Meyer and H. Suura, Phys. Rev. **160**, 1366 (1967).

the commutator, we obtain

$$\hat{V}_{2c}(p\cdot x,x^2)=\frac{i}{2\pi}\int dadb\ \sigma_2(a,b)e^{ibp\cdot x}\Delta_c(x,\ a+b^2)\,, \quad (5.6)$$

with

$$(1/2\pi)\langle p|T(j_\mu(x)j_\nu(0))|p\rangle = -p_\mu p_\nu \Box\,\hat{V}_{2c}(p\cdot x,x^2)+\cdots. \quad (5.7)$$

The other terms in (5.7) will turn out to be comparable to the $p_\mu p_\nu$ term at large p, but for the essence of the argument (5.7) will suffice. Then

$$(1/2\pi)\langle p|j_\mu(\mathbf{x},x_3,0)j_\nu(0)|p\rangle = -p_\mu p_\nu[\Box\,\hat{V}_{2c}(p\cdot x,x^2)]|_{x_0=0}+\cdots. \quad (5.8)$$

We shall be interested in seeing in which regions of $(\mathbf{x},x_3)$ Eq. (5.8) is not small when $p\to\infty$. Now for large p, we can write (5.6) as

$$\hat{V}_{2c}(p(x_0-x_3),\ x^2) = \frac{i}{2\pi}\int dadb\ e^{+ibp(x_0-x_3)}\sigma_2(a,b)\Delta_c(x,\ a+b^2)\,. \quad (5.9)$$

Let us first look in the region where $|x_3|\gg 1/p$ while $x_0\ll|x_3|$. Then clearly only the small-b part of the integration in (5.9) is important. From (5.4) we see that $\sigma_2(a,b)\approx\sigma_2(a)\ln|b|$ for small b, so that (5.9) becomes

$$\hat{V}_{2c}(p(x_0-x_3),\ x^2) \approx\int dadb\ \sigma_2(a)\Delta_c(x,a)\ln|b|\ e^{ibp(x_0-x_3)}. \quad (5.10)$$

Now for large p

$$\int e^{ibp(x_0-x_3)}\ln|b|\,db=c\frac{1}{p|x_0-x_3|},$$

where c is a constant. Thus

$$\hat{V}_{2c}(p(x_0-x_3),\ x^2)\approx\frac{c}{p|x_0-x_3|}\int da\ \sigma_2(a)\Delta_c(x,a) \quad (5.11)$$

and

$$[-\Box\,\hat{V}_{2c}(p(x_0-x_3),\ x^2)]|_{x_0=0} =\frac{c}{p|x_3|}\int da\ \sigma_2(a)\bigg[\Delta^-(-\mathbf{x}^2-x_3{}^2,\ a)a +\frac{\partial}{\partial x_3{}^2}\Delta^-(-\mathbf{x}^2-x_3{}^2,\ a)\bigg]. \quad (5.12)$$

Using (5.12) and (5.8), we obtain

$$\frac{1}{2\pi}\langle p|j_\mu(\mathbf{x},x_3,0)j_\nu(0)|p\rangle =\frac{cp_\mu p_\nu}{p|x_3|}\int da\ \sigma_2(a)\bigg[\Delta^-(-\mathbf{x}^2-x_3{}^2,\ a)a +\frac{\partial}{\partial x_3{}^2}\Delta^-(-\mathbf{x}^2-x_3{}^2,\ a)\bigg] \quad (5.13)$$

for $|x_3|\gg 1/p$, and where p is very large.

Except for the finite-size effect due to $\Delta^-(-\mathbf{x}^2-x_3{}^2,\ a)$ which we encountered in the discussion following (2.5), the only decrease in size outside the disk $|x_3|>1/p$ is the factor $1/p|\mathbf{x}_3|$ in (5.13). Now the amplitude for an interaction inside the thin disk $|x_3|\lesssim 1/p$ will be proportional to $p_\mu p_\nu$ times $1/p$. The $1/p$ comes from the fact that the disk has a width $1/p$. However, the amplitude, given by (5.13) for interaction outside the disk, $|x_3|>1/p$, goes as $(p_\mu p_\nu/p)\ln p$. That is, the amplitude

$$f(p)=-i\int d^3xd^3y\langle p|j_0(\mathbf{x},x_3,0)j_0(\mathbf{y},y_3,0)|p\rangle \times a_0(\mathbf{x},x_3)a_0(\mathbf{y},y_3)$$

given by (2.3) goes as $p\ln p$, and this asymptotic contribution can be obtained without approximation by dropping the region $|x_3-y_3|<N/p$, where N is any finite number, as the region $|x_3-y_3|<N/p$ contributes a term proportional to p.

Thus particle size clearly does not contract in a rapidly moving frame in this large class of models which give scaling and a constant asymptotic total cross section for inelastic electroproduction.

B. Example of Singular Light-Cone Commutator

Now we shall drop the assumptions of Sec. V A and rather assume that the $\sigma_2(a,b)$ of (5.1) behaves as $\sigma_2(a,b)\to\sigma_2(b)/a$ for large a with $\sigma_2(0)=0$. Then for $\nu\to\infty$ and fixed q^2,

$$W_2(\nu,q^2)\approx\frac{q^2}{2\nu}\int_0^1 db\frac{\sigma_2(b)}{b}, \quad (5.14)$$

while for $\nu\to\infty$, $q^2\to-\infty$, and ν/q^2 fixed,

$$W_2(\nu,q^2)=\frac{q^2}{2\nu}\int da\ \sigma_2\bigg(a,\ \frac{a-q^2}{2\nu}\bigg) =(q^2/2\nu)\ln\nu\ \sigma_2(q^2/2\nu)\,. \quad (5.15)$$

Equation (5.15) violates the scaling law, as apparently do all singular commutators. The major difficulty is that one needs a condition like (5.3) to obtain scaling while any attempt to obtain a constant total cross section, as in (5.14), requires $\sigma_2(a,b)\sim 1/a$ if this constancy is to be achieved from a singularity on the finite light cone. It is a rather simple exercise to show that this interaction is too singular to permit simultaneous interaction, as in this example $\langle p|j_0(\mathbf{x},x_3,0)j_0(0)|p\rangle$ is a divergent quantity.

Suppose that we had taken $\sigma_2(a,b)\to\sigma_2(b)a^{-1-\epsilon}$ for large a where $0<\epsilon\leq 1$ and taken $\sigma_2(0)$ constant where $\sigma_2(a,0)=\partial\bar{\sigma}_2(a,0)/\partial a$. Now the integrals in (2.10) and (2.13) do not diverge but the steps leading to the equations represented by (2.10) and (2.13) are no longer valid, as again $\langle p|j_0(\mathbf{x},x_3,0)j_0(0)|p\rangle$ is a divergent quantity for large p. This shows up in the fact that the impact factor $F(\mathbf{x})$, which one would obtain in this example, goes as $(\mathbf{x}^2)^{-2+\epsilon}$ near $\mathbf{x}=0$. A singularity of this strength is too strong to be integrated over without regularization. High-energy behavior cannot be the only criterion for shrinkage of size to a thin disk.

APPENDIX A: O(2,1) FORMALISM FOR MULTIPARTICLE AMPLITUDES

In this Appendix we shall develop an $O(2,1)$ formalism for multiparticle amplitudes which exhibits nonsense fixed poles in an explicit way. We shall consider first the four-point function, and in the following subsection a formalism will be given for the five-point function. The other n-point functions which will be needed in this paper will be obvious extensions of the analysis carried out for the five-point function.

A. Four-Point Function

The relation between the $O(2,1)$ harmonic analysis and the Froissart-Gribov continuation of the partial-wave amplitude has been discussed by Olive[21] for the case of four-point functions where all the external momenta of the four legs are timelike. In the case considered by Olive, there are no fixed poles in the $O(2,1)$ amplitudes at wrong-signature nonsense points. However, if two of the external particles have negative mass, then this analysis must be modified. This will be done for the case of spinless particles in what follows.

Consider the process shown in Fig. 3, with $p_1^2=p_2^2=m^2>0$ and $q_1^2=\mu_1^2<0$, $q_2^2=\mu_2^2<0$. Work in the Breit frame defined by

$$p_2=((M^2-\tfrac{1}{4}t)^{1/2},0,0,(-\tfrac{1}{4}t)^{1/2}),\qquad p_1=((M^2-\tfrac{1}{4}t)^{1/2},0,0,-(-\tfrac{1}{4}t)^{1/2}),\tag{A1}$$

where $t=(p_2-p_1)^2<0$. Furthermore, suppose that

$$\bar{q}_2^2=(q_2)_0^2-(q_2)_1^2-(q_2)_2^2<0$$

and

$$\bar{q}_1^2=(q_1)_0^2-(q_1)_1^2-(q_1)_2^2<0.$$

If these last conditions are not fulfilled, then again there will be no wrong-signature nonsense fixed poles in $O(2,1)$ amplitudes. This point should become clear in the following discussion. Label q_1 and q_2 by

$$q_1=\left(\bar{q},\frac{-\mu_1^2+\mu_2^2-t}{2(-t)^{1/2}}\right)=(q_{10},q_{11},q_{12},q_{13}),\qquad q_2=\left(\bar{q},\frac{-\mu_1^2+\mu_2^2+t}{2(-t)^{1/2}}\right),\tag{A2}$$

with

$$\bar{q}=|\bar{q}|(\sinh\xi,\cosh\xi\cosh\varphi,\cosh\xi\sin\varphi).\tag{A3}$$

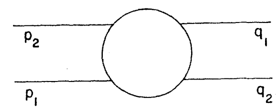

FIG. 3. Kinematics for the four-point function.

For the amplitude describing the process shown in Fig. 3, we can choose $\varphi=0$. This choice can be made since the amplitude can only depend on invariants which are independent of φ. Let f be the invariant transition amplitude describing this process. Then f can be expanded according to[14,15]

$$f(t,\xi)=\sum_{\rho=1,2}\int d\rho(\Lambda)d_{0,\rho 0}{}^{\Lambda}(\xi)f_\rho{}^{\Lambda}(t),\tag{A4}$$

with

$$f_\rho{}^{\Lambda}(t)=\int_{-\infty}^{\infty}d\sinh\xi\, f(t,\xi)\bar{d}_{0,\rho 0}{}^{\Lambda}(\xi).\tag{A5}$$

The $d_{0,\rho 0}{}^{\Lambda}$ are special cases of the representation functions of $O(2,1)$ in a mixed $O(2)$ and $O(1,1)$ basis, $D_{m,\rho\mu}{}^{\Lambda}(\varphi,\xi,\zeta)$. [$m$ is an $O(2)$ quantum number while μ is an $O(1,1)$ quantum number.] Explicit formulas for these functions in terms of hypergeometric functions are given in Refs. 14 and 15. A mixed basis is appropriate here, since $\bar{q}^2<0$.

For $m=0$, $\mu=0$ we have

$$\begin{aligned}d_{0,+0}{}^{\Lambda}(\xi)&=-\pi^{-1}e^{i\pi(\Lambda-1)/2}Q_\Lambda(i\sinh\xi),\quad \xi>0\\ d_{0,-0}{}^{\Lambda}(\xi)&=-\pi^{-1}e^{i\pi(\Lambda-1)/2}[Q_\Lambda(-i\sinh\xi)\\&\qquad -i\pi P_\Lambda(-i\sinh\xi)],\quad \xi>0\\ d_{0,+0}{}^{\Lambda}(-\xi)&=d_{0,-0}{}^{\Lambda}(\xi),\quad d_{0,-0}{}^{\Lambda}(-\xi)=d_{0,+0}{}^{\Lambda}(\xi),\\&\qquad\qquad \xi<0.\end{aligned}\tag{A6}$$

Thus, for example,

$$f_+{}^{\Lambda}(t)=\int_{-\infty}^{\infty}d\sinh\xi\, f(t,\xi)\bar{d}_{0,+0}{}^{\Lambda}(\xi).\tag{A7}$$

Using the above formulas for the representation functions, one finds that $f_+{}^{\Lambda}(t)$ has a pole at $\Lambda=0$. The fact that this pole occurs at $\Lambda=0$, rather than at $\Lambda=-1$ as might be expected, can be understood by the following argument. Suppose one considers the process described in Fig. 4, where $\bar{p}$ is the same as given in (A1), $\bar{q}$ is as given in (A2), and $\bar{p}'=u\bar{p}$, where u is an $O(2,1)$ transformation. We take all particles spinless. This

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[21] D. Olive, Nucl. Phys. B15, 617 (1970).

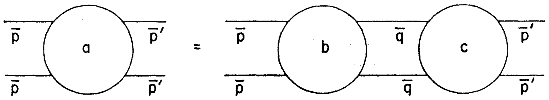

FIG. 4. Schematic diagram of (A8).

process can be written as

$$a(u)=\int d\tilde{g}\, b(g)c(g^{-1}u)\,,$$
$$g=R_z(\varphi)B_x(\xi)B_y(\zeta)\,, \tag{A8}$$
$$\tilde{g}=R_z(\varphi)B_x(\xi)\,,\quad d\tilde{g}=d\phi d\sinh\xi\,,$$

where integrations over dq_3 and $d|\bar{q}|$ are ignored since they do not enter into the group-theory arguments. If we carry out an $O(2,1)$ analysis

$$a^\Lambda=\int du\, a(u)\bar{D}_{00}{}^\Lambda(u)$$
$$=\int dud\tilde{g}\, b(g)c(g^{-1}u)\bar{D}_{00}{}^\Lambda(u)$$
$$=\sum_\rho \int d\sigma(u)dud\tilde{g}\, b(g)c(g^{-1}u)\bar{D}_{0,\rho\mu}{}^\Lambda(g)\bar{D}_{\rho\mu,0}{}^\Lambda(g^{-1}u)$$
$$=\sum_\rho \int d\sigma(\mu)\int d\tilde{g}\, b(g)\bar{D}_{0,\rho\mu}{}^\Lambda(g)\int du\, c(u)\bar{D}_{\rho\mu,0}{}^\Lambda(u)\,,$$

and use

$$\int du\, c(u)\bar{D}_{\rho\mu,0}{}^\Lambda(u)$$
$$=(2\pi)^2\delta(i\mu)\int d\sinh\xi\, c(\sinh\xi)\bar{d}_{\rho0,0}{}^\Lambda(\xi)$$
$$=(2\pi)^2\delta(i\mu)\int d\sinh\xi\, c(\sinh\xi)\bar{d}_{0,-\rho0}{}^{-\Lambda-1}(\xi)\,,$$

$$\int d\tilde{g}\, b(g)\bar{D}_{0,\rho0}{}^\Lambda(g)=(2\pi)\int d\sinh\xi\, b(\sinh\xi)\bar{d}_{0,\rho0}{}^\Lambda(\xi)\,,$$

then we obtain

$$a=(2\pi)^2\sum_\rho b_\rho{}^\Lambda c_{-\rho}{}^{-\Lambda-1}=a^{-\Lambda-1}\,, \tag{A9}$$

where $|\bar{q}|$ and q_3 integrations are understood. Now $b_\rho{}^\Lambda$ has a pole in Λ at $\Lambda=0$, while $c_{-\rho}{}^{-\Lambda-1}$ has a pole in Λ at $\Lambda=-1$. Thus a^Λ has first-order poles at $\Lambda=0$ and $\Lambda=-1$ as its leading fixed poles.

We can now see why the vector $\bar{q}$, as defined in (A2), must be spacelike for a nonsense wrong-signature fixed pole to appear in an $O(2,1)$ analysis. We have seen that such a fixed pole comes from a pole in the representation function. $O(2,1)$ representation functions described purely in an $O(2)$ bases have no such poles.[13] An $O(2)$ basis occurs for timelike $\bar{q}$ and an $O(1,1)$ basis occurs for spacelike q. Since $\bar{p}$ is fixed to be timelike, it is necessary for $\bar{q}$ to be spacelike in order to avoid having the $O(2,1)$ functions in a purely $O(2)$ basis. This is not the case for right-signature fixed poles since they arise from divergences in integrals over the group volume in an $O(2,1)$ analysis.

B. Five-Point Function

Consider the process whose momenta are described in Fig. 1. Again label the first three components of p_0 by $\bar{p}$, that is, $\bar{p}=(p_0,p_1,p_2)$, and similarly for q_i. As before, we take all particles to be scalar. We use the representation

$$\bar{p}=(M^2-\tfrac{1}{4}t)^{1/2}(\cosh\xi\cosh\zeta,\ -\sinh\xi,\ \sinh\xi\sinh\zeta)\,,$$
$$\bar{p}=u\hat{p}\,,\quad \hat{p}=(M^2-\tfrac{1}{4}t)^{1/2}(1,0,0)$$
$$u=B_y(\zeta)\ B_x(\xi)\,.$$

Also, choose $\bar{q}=|\bar{q}|(0,1,0)=\hat{q}$. We shall later argue that only spacelike $\bar{q}$ and $\bar{q}_i$ will be important for the question of fixed poles, as was the case in the four-point function.

The amplitude for the process shown in Fig. 1 can be expanded as

$$f(\bar{p}\cdot\bar{q},\bar{p}\cdot\bar{q}_1,\bar{q}\cdot\bar{q}_1)=f(u^{-1}g,u^{-1}gg_1,g_1)$$
$$=\sum_\rho\int d\rho(\Lambda)d\sigma(\mu)D_{0,\rho\mu}{}^\Lambda(u^{-1}g)f_{\rho\mu}{}^\Lambda(g_1)\,, \tag{A10}$$

with the inversion

$$f_{\rho\mu}{}^\Lambda(g_1)=\int dv\, f(v,vg_1,g_1)\bar{D}_{0,\rho\mu}{}^\Lambda(v)\,, \tag{A11}$$

where

$$v=R_z(\psi)B_x(\alpha)B_y(\beta)\,,$$
$$-\infty<\beta<\infty\,,\quad -\infty<\alpha<\infty\,,\quad 0<\psi<2\pi$$
$$dv=d\psi d\sinh\xi\, d\beta\,,$$

and

$$\bar{p}=u\hat{p}\,,\qquad \hat{p}=(M^2-\tfrac{1}{4}t)^{1/2}(1,0,0)$$
$$\bar{q}=g\hat{q}\,,\qquad \hat{q}=|\bar{q}|(0,1,0)$$
$$\bar{q}_1=gg_1\hat{q}_1\,,\quad \hat{q}_1=|\bar{q}_1|(0,1,0)\,.$$

To see that Λ is actually the label of the usual Casimir operator, we form the amplitude, shown in Fig. 2,

$$a(\bar{p}\cdot\bar{p}')=\int d^3\bar{q}d^3\bar{q}_1\, b(\bar{p},\bar{q},\bar{q}_1)c(\bar{p}',\bar{q},\bar{q}_1)\,. \tag{A12}$$

Neglecting those variables invariant under $O(2,1)$ transformations and keeping only that part of the phase space relevant to the $O(2,1)$ group structure, we can write (A12) as

$$a(u^{-1}v)=\int d^2g d^2g_1 b(u^{-1}g,u^{-1}gg_1,g_1)\times c(v^{-1}g,v^{-1}gg_1,g_1)\,, \quad \text{(A13)}$$

where

$$\bar{p}'=vp'\,,\quad \hat{p}'=(M^2-\tfrac{1}{4}t)^{1/2}(1,0,0)\,,$$
$$g=R_z(\varphi)B_x(\xi)\,,\quad d^2g=d\varphi d\sinh\xi\,,\quad 0<\varphi<2\pi,\quad -\infty<\xi<\infty$$
$$g_1=R_z(\varphi_1)B_x(\xi_1)\,,\quad d^2g_1=d\varphi_1 d\sinh\xi_1\,,\quad 0<\varphi_1<2\pi\,,\quad -\infty<\xi_1<\infty\,.$$

Equation (A13) can be diagonalized as

$$a^\Lambda=\int d(u^{-1}v)a(u^{-1}v)\bar{D}_{00}{}^\Lambda(u^{-1}v)$$
$$=\sum_\rho\int d(u^{-1}v)d^2gd^2g_1d\sigma(\mu)b(u^{-1}g,u^{-1}gg_1,g_1)\times c(v^{-1}g,v^{-1}gg_1,g_1)\bar{D}_{0,\rho\mu}{}^\Lambda(u^{-1}g)D_{\rho\mu,0}{}^\Lambda(g^{-1}v)\,.$$

Using $D_{\rho\mu,0}{}^\Lambda(g^{-1}v)=D_{0,-\rho-\mu}{}^{-\Lambda-1}(v^{-1}g)$, we obtain

$$a^\Lambda=\sum_\rho\int d^2g_1 d\sigma(\mu)b_{\rho\mu}{}^\Lambda(g_1)c_{-\rho-\mu}{}^{-\Lambda-1}(g_1)\,. \quad \text{(A14)}$$

Equation (A14) is the three-particle analog of (A9) and shows that the Λ of (A10) and (A11) actually is the usual Λ appearing in expansions of elastic scattering amplitudes.

Nonsense fixed poles of $b_{\rho\mu}{}^\Lambda$ and $c_{-\rho-\mu}{}^{-\Lambda-1}$ in Λ again appear through the Λ poles of $\bar{D}_{0,\rho\mu}{}^\Lambda$ in (A11). As we have noted in Sec. III, $\bar{q}_1$, $\bar{q}_2$, and $\bar{q}$ must be spacelike in order for the poles in μ of $b_{\rho\mu}{}^\Lambda$ and $c_{-\rho-\mu}{}^{-\Lambda-1}$ in (A14) not to arise only from Regge asymptotic behavior. In general, the fixed poles in μ will involve δ functions of part of the group volume g_1 in (A14), as illustrated in (3.11) and the three equations preceding (3.11). However, the δ functions in g_2 will be integrated over when one considers a physical amplitude such as the one given in (A14).

The extension to the $(n+4)$-point function is now obvious. The analog of (A11) is

$$f_{\rho\mu}{}^\Lambda(g_1,g_2,\ldots,g_n)=\int dv\, f(v,vg_1,vg_2,\ldots,vg_n,g_1,g_2,\ldots,g_n)\,,$$

while the analog of (A14) is

$$a^\Lambda=\sum\int d\sigma(\mu)d^2g_1d^2g_2\cdots d^2g_n b_{\rho\mu}{}^\Lambda(g_1,g_2,\ldots,g_n)\times c_{-\rho-\mu}{}^{-\Lambda-1}(g_1,g_2,\ldots,g_n)\,.$$

APPENDIX B

Consider the amplitude

$$f(q)=-i\int d^4x\, e^{iqx}\langle p|R(j(x)j(0))|p\rangle\,,$$

where both the particle whose momentum is p and the current j are scalar quantities. If $p=(M,0,0,0)$, then $\langle p|R(j(x)j(0))|p\rangle=f(x_0,r)$ and

$$f(q)=f(Q,\mu^2)=-i\int d^4x\, e^{iqx}f(x_0,r)\,, \quad \text{(B1)}$$

where $Q^2=q_1^2+q_2^2+q_3^2$ and $\mu^2=q^2$. The angular integrations can be completed to give

$$f(Q,\mu^2)=\frac{2\pi}{Q}\int rdrdx_0f(x_0,r)e^{iq_0x_0}(e^{-iQr}-e^{iQr})$$
$$=[g(Q,\mu^2)+g(-Q,\mu^2)]Q^{-1}\,, \quad \text{(B2)}$$

where

$$g(Q,\mu^2)=2\pi\int rdr\, x_0f(x_0,r)e^{iq_0x_0-iQr}\,. \quad \text{(B3)}$$

g can be represented in terms of a Mellin transform as

$$g(Q,\mu^2)=\frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty}dl\, Q^lg_l(\mu^2)\,,\quad a<0 \quad \text{(B4)}$$

with the inversion

$$g_l(\mu^2)=\int_0^\infty dQ\, Q^{-l-1}g(Q,\mu^2)\,. \quad \text{(B5)}$$

Substituting (B3) into (B5), one obtains

$$g_l(\mu^2)=(2\pi)\int rdrdx_0\, f(x_0,r)\int dQ\, Q^{-l-1}e^{iq_0x_0-iQr}\,. \quad \text{(B6)}$$

The poles in the left half l plane are determined by the asymptotic behavior in Q, so one may use $q_0=(Q^2+\mu^2)^{1/2}\approx Q+\mu^2/2Q$ for the leading poles. In this approximation, (B6) becomes

$$g_l(\mu^2)\approx(2\pi)\int rdrdx_0\, f(x_0,r)\int dQ\, Q^{-l-1}\times\exp\left[iQ(x_0-r)+i\frac{\mu^2x_0}{2Q}\right]. \quad \text{(B7)}$$

Now[22]

$$\int dQ\, Q^{-l-1}\exp\left[iQ(x_0-r)+i\frac{\mu^2}{2Q}x_0\right]=i\pi e^{i\pi l/2}\left[\frac{\mu^2x_0}{2(x_0-r)}\right]^{-l/2}H_l^{(1)}([2\mu^2x_0(x_0-r)]^{1/2})\,, \quad \text{(B8)}$$

[22] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1965), p. 340.

so that (B7) becomes

$$g_l(\mu^2)=2\pi^2 i e^{i\pi l/2}\int r dr dx_0 f(x_0,r) \times\left[\frac{\mu^2 x_0}{2(x_0-r)}\right]^{-l/2} H_l^{(1)}([2\mu^2 x_0(x_0-r)]^{1/2}). \quad \text{(B9)}$$

From (B9), one can determine which regions in coordinate space give the leading singularities in l and hence are responsible for the leading terms at high energy in q. Using

$$H_l^{(1)}(z) \underset{z\to 0}{\longrightarrow} \frac{-i}{\sin\pi l}\left[\tfrac{1}{2}z^{-l}\frac{1}{\Gamma(1-l)}-\tfrac{1}{2}z^{l}\frac{1}{\Gamma(l+1)}e^{-i\pi l}\right],$$

we note the following properties of

$$\left[\frac{\mu^2 x_0}{2(x_0-r)}\right]^{-l/2} H_l^{(1)}([2\mu^2 x_0(x_0-r)]^{1/2})=h(x_0,r,\mu^2):$$

(i) As $x_0\to\infty$ with $x_0(x_0-r)$ finite,

$$h(x_0,r)\to (x_0\mu^2)^{-l}[2\mu^2 x_0(x_0-r)]^{l/2}H_l^{(1)} \times([2\mu^2 x_0(x_0-r)]^{1/2}),$$

or equivalently,

$$h(x_0,r)\to (x_0-r)^{l}[\tfrac{1}{2}\mu^2 x_0(x_0-r)]^{-l/2}H_l^{(1)} \times([2\mu^2 x_0(x_0-r)]^{1/2}).$$

(ii) As $x_0(x_0-r)\to 0$,

$$h(x_0,r)\to\frac{-i}{\sin\pi l}\left[(\tfrac{1}{2}x_0\mu^2)^{-l}\frac{1}{\Gamma(1-l)} -(x_0-r)^{l}\frac{1}{\Gamma(l+1)}e^{-i\pi l}\right].$$

(iii) As $x_0\to\infty$ and $x_0(x_0-r)\to\infty$,

$$h(x_0,r)\to\frac{-i}{\sin\pi l}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{\mu^2 x_0}{2(x_0-r)}\right)^{-l/2}[2\mu^2 x_0(x_0-r)]^{-1/4} \times\{\cos([2\mu^2 x_0(x_0-r)]^{1/2}+\tfrac{1}{2}l\pi-\tfrac{1}{4}\pi) -\cos([2\mu^2 x_0(x_0-r)]^{1/2}-\tfrac{1}{2}l\pi-\tfrac{1}{4}\pi)e^{-i\pi l}\}.$$

From these expressions and (B9), one can see that singularities in $g_l(\mu^2)$ can arise from regions in coordinate space which are a finite distance from $x_0=r=0$ as indicated by case (ii), or from regions which are infinitely far from the origin as in case (i). For example, suppose $f(x_0,r)\to x_0^{\alpha}f(x^2)$ as $x_0\to\infty$ for fixed x^2. Then from (i) and (B9),

$$g_l(\mu^2)\approx 2\pi^2 i e^{i\pi/2}\left(\frac{2}{\mu^2}\right)^{l/2}\int_0^\infty dy\, y^{l/2}f(2y)H_l^{(1)} \times((2\mu^2 y)^{1/2})\int_\Lambda^\infty dr\, r^{\alpha-l}$$

$$\approx\frac{2\pi^2 i e^{i\pi(\alpha+1)/2}}{l-(\alpha+1)}\left(\frac{2}{\mu^2}\right)^{(\alpha+1)/2} \times\int_0^\infty dy\, y^{(\alpha+1)/2}f(2y)H_{\alpha+1}^{(1)}((2\mu^2 y)^{1/2})$$

near $l=\alpha+1$. This pole in l at $\alpha+1$ corresponds to an asymptotic behavior of $f(Q)$ in (B2) which is $f(q)\to Q^{\alpha}$. This is the typical way in which a Regge pole arises in coordinate space, as can be verified explicitly in a ϕ^3-type theory. If there are also strong singularities on the light cone, $x_0=r$, then the singularities in l from $x_0=\infty$ and from $x_0=r$ are additive as indicated in case (ii).

The above formalism can be used for operator products other than the retarded product with a minor modification. Other operator products will, in general, not vanish outside the interior of the forward light cone, in which case one must go around the branch points at $x_0=r$ and at $x_0=0$ in

$$\left[\frac{x_0\mu^2}{2(x_0-r)}\right]^{-l/2} H_l^{(1)}([2\mu^2 x_0(x_0-r)]^{1/2}).$$

In the first instance one should take $x_0-r\to x_0-r-i\epsilon$, and in the latter instance, $\mu^2 x_0\to\mu^2 x_0-i\epsilon$. These prescriptions follow immediately from the integral representation (B8).