Calculus for Functions of Noncommuting Operators and General Phase-Space Methods in Quantum Mechanics. III. A Generalized Wick Theorem and Multitime Mapping*

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(Received 8 May 1970)

The new c-number calculus for functions of noncommuting operators, developed in Paper I and employed in Paper II to formulate a general phase-space description of boson systems, deals with situations involving equal-time operators only. In the present paper extensions are presented for the treatment of problems involving boson operators at two or more instants of time. The mapping of time-ordered products onto c-number functions is studied in detail. The results make it possible to evaluate time-ordered products of boson operators by phase-space techniques. The usual Wick theorem for boson systems is obtained as a special case of a much more general theorem on time ordering. Our method of derivation appears to provide the first direct proof of Wick's theorem as well as a clear insight into its true meaning. A closed expression is also obtained for the time-evolution operator in terms of the solution of the c-number differential equation for the phase-space equivalent of this operator. The new calculus is also applied to the problem of evaluating normally ordered time-ordered, and also the antinormally ordered time-ordered, correlation functions.

I. INTRODUCTION

IN Paper I and Paper II of this series,[1,2] we developed a new calculus for functions of noncommuting operators, based on the concept of mapping a function $G(\hat{a},\hat{a}^\dagger)$ of noncommuting boson operators $\hat{a}$ and $\hat{a}^\dagger$ onto a c-number function $F(z,z^*)$ of complex variables z and z^*. We showed that this calculus leads to a general phase-space description of boson systems and provides a systematic method for solving a great variety of quantum-mechanical problems by c-number techniques. In these papers only problems involving operators that satisfy the equal-time commutation relations $[\hat{a}(t),\hat{a}^\dagger(t)]=1$ were considered.

In the present paper, we extend the theory to situations involving noncommuting boson operators at two or more instants of time. We study in detail the mapping of the time-ordered product of a set of operators onto c-number functions. In Sec. II we derive a general formula which makes it possible to evaluate time-ordered products in terms of products ordered according to some prescribed rule. The well-known Wick theorem[3,4] for boson systems, usually established by induction, is shown to follow readily from this new theorem as a special case. Our method of derivation seems to provide the first direct proof of Wick's theorem and gives a new insight into its true meaning. In Sec. III we obtain a new identity which makes it possible to express time-ordered products of a set of operators, which are functionals linear in the positive- and the negative-frequency parts of the field operators, in normally ordered forms. This identity is essentially a generalization of a formula given by Anderson,[5,6] which like Wick's theorem is frequently used in field theory. In Sec. IV, we present a closed-form expression for the unitary time-evolution operator of a boson system in terms of the solution of the c-number differential equation satisfied by the phase-space equivalent of this operator. We illustrate this result by deriving the explicit expression for the time-evolution operator for a forced harmonic oscillator. In Sec. V, we introduce the concept of multitime mapping of unequal-time boson operators onto c-number variables; we then show how this correspondence may be used to evaluate the normally ordered time-ordered, and also the antinormally ordered time-ordered, correlation functions. Some of the results of this section are analogous to those obtained recently by Lax[7] in connection with c-number techniques for the solution of problems in areas such as the theory of the laser and the statistics of photoelectrons. In Sec. VI, we present a brief summary of the main results obtained in these three papers and for comparison we display in a table the main quantum-mechanical equations, both in their conventional operator form and in our phase-space representation.

II. EVALUATION OF TIME-ORDERED PRODUCTS OF HEISENBERG OPERATORS BY PHASE-SPACE TECHNIQUES AND GENERALIZED WICK THEOREM

A well-known theorem of Wick[3,4] allows the evaluation of time-ordered products of operators in terms of

* Research supported jointly by the U. S. Army Research Office (Durham) and by the U. S. Air Force Office of Scientific Research. A preliminary account of some of the results obtained in this paper was presented at the New York meeting of the American Physical Society in February 1969 [Abstract GD6, Bull. Am. Phys. Soc. 14, 69 (1969)] and was summarized in notes published in Phys. Rev. Letters 21, 180 (1968), and in Nuovo Cimento Letters 1, 140 (1969).

[1] G. S. Agarwal and E. Wolf, second preceding paper, Phys. Rev. D 2, 2161 (1970), to be referred to as I.

[2] G. S. Agarwal and E. Wolf, first preceding paper, Phys. Rev. D 2, 2187 (1970), to be referred to as II.

[3] G. C. Wick, Phys. Rev. 80, 268 (1950).

[4] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1962); N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).

[5] J. L. Anderson, Phys. Rev. 94, 703 (1954).

[6] T. Matsubara, Progr. Theoret. Phys. (Kyoto) 14, 351 (1955).

[7] M. Lax, Phys. Rev. 172, 350 (1968).

normally ordered ones. The theorem is of basic importance in calculations, based on perturbation techniques, relating to the behavior of the unitary time-evolution operator of a quantum-mechanical system.

In this section we establish, with the help of the c-number techniques developed in the earlier parts of this investigation, an interesting generalization of Wick's theorem for boson systems. This generalized theorem allows the evaluation of time-ordered products of Heisenberg operators—whether linear or nonlinear in the annihilation and the creation operators—as products arranged according to a prescribed rule of ordering. Normal ordering plays, of course, a preferential role in field theory because of the significance of vacuum expectation values. However, as already pointed out in I, other rules of ordering are occasionally employed and some arise naturally in other branches of physics, e.g., Weyl ordering in quantum statistics[8] and antinormal ordering in quantum optics.[9] In any case, the generalized Wick theorem that we will now establish, and the considerations of Secs. III and IV of the present paper, bring into evidence a fact not previously explicitly recognized, namely, that the phase-space representation of operators plays a basic role in time-ordering problems.

Let T denote the time-ordering operator and let us consider the time-ordered product $T\{\hat{G}_1(t_1)\hat{G}_2(t_2)\cdots\hat{G}_M(t_M)\}$ of M Heisenberg operators $\hat{G}_m$ $(m=1,2,\ldots,M)$. The operators $\hat{G}_m$ will also depend on $\hat{a}$ and $\hat{a}^\dagger$ (i.e., $\hat{G}_m=G_m(\hat{a},\hat{a}^\dagger;t_m)$; $[\hat{a},\hat{a}^\dagger]=1$), but as a rule we will not exhibit this dependence explicitly. We may express the time-ordered product in the form

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=\sum_{\Pi}\theta(t_{i_1}-t_{i_2})\cdots\theta(t_{i_{M-1}}-t_{i_M})\hat{G}_{i_1}(t_{i_1})\cdots\hat{G}_{i_M}(t_{i_M}), \quad (2.1)$$

where

$$\begin{aligned}\theta(\tau)&=1 \quad \text{if } \tau>0\\ &=0 \quad \text{if } \tau<0,\end{aligned} \quad (2.2)$$

and $\sum_\Pi$ denotes the summation over all the permutations of the indices $1,2,\ldots,M$. Let $F_j^{(\Omega)}(z,z^*;t_j)$ be the Ω equivalent of the operator $\hat{G}_j$, i.e.,

$$\begin{aligned}G_j(\hat{a},\hat{a}^\dagger;t_j)&=\Omega\{F_j^{(\Omega)}(z,z^*;t_j)\},\\ F_j^{(\Omega)}(z,z^*;t_j)&=\Theta\{G_j(\hat{a},\hat{a}^\dagger;t_j)\},\end{aligned} \quad (2.3)$$

where Ω is an arbitrary linear mapping operator defined in Sec. II of I and Θ is its inverse. Then the Ω equivalent $F_{12\cdots M}^{(\Omega)}(z,z^*;t_1\cdots t_M)$ of the product $G_1(\hat{a},\hat{a}^\dagger;t_1)\cdots G_M(\hat{a},\hat{a}^\dagger;t_M)$, i.e., the c-number function such that

$$G_1(\hat{a},\hat{a}^\dagger;t_1)\cdots G_M(\hat{a},\hat{a}^\dagger;t_M)=\Omega\{F_{12\cdots M}^{(\Omega)}(z,z^*;t_1\cdots t_M)\}, \quad (2.4a)$$

$$F_{12\cdots M}^{(\Omega)}(z,z^*;t_1,\ldots,t_M)=\Theta\{G_1(\hat{a},\hat{a}^\dagger;t_1)\cdots G_M(\hat{a},\hat{a}^\dagger;t_M)\}, \quad (2.4b)$$

is given by the following formula, which is a generalization, for a product of an arbitrary number of operators, of Theorem V (Product Theorem) given in Sec. III of II, and which is derived in Appendix A of the present paper:

$$F_{12\cdots M}^{(\Omega)}(z,z^*;t_1,\ldots,t_M)=\exp\{\sum_j\sum_{i<j}\Lambda_{ij}\}\times\mathcal{U}_{12\cdots M}^{(\Omega)}\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}, \quad (2.5)$$

where

$$\Lambda_{ij}=\frac{1}{2}\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j^*}-\frac{\partial}{\partial z_i^*}\frac{\partial}{\partial z_j}\right), \quad (2.6)$$

$$\mathcal{U}_{12\cdots M}^{(\Omega)}=\prod_{m=1}^{M}\Omega\left(\frac{\partial}{\partial z_m^*},-\frac{\partial}{\partial z_m}\right)\times\bar{\Omega}\left(\sum_{m=1}^{M}\frac{\partial}{\partial z_m^*},-\sum_{m=1}^{M}\frac{\partial}{\partial z_m}\right). \quad (2.7)$$

In Eq. (2.7) the function $\Omega(\alpha,\beta)$ is the filter function for Ω mapping and $\bar{\Omega}(\alpha,\beta)=[\Omega(\alpha,\beta)]^{-1}$ is the filter function for the mapping that is reciprocal to Ω.

We note that under the interchange of the indices i and j, the operator Λ_{ij} changes sign, whereas the operator $\mathcal{U}_{12\cdots M}^{(\Omega)}$ remains unchanged. Hence it follows from (2.4a), (2.5), and (2.1) that

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=\Omega\{\exp[\sum_j\sum_{i<j}\Lambda_{ij}\epsilon(t_i-t_j)]\times\mathcal{U}_{12\cdots M}^{(\Omega)}\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}\}, \quad (2.8)$$

where

$$\begin{aligned}\epsilon(\tau)&=+1 \quad \text{if } \tau>0\\ &=-1 \quad \text{if } \tau<0.\end{aligned} \quad (2.9)$$

The right-hand side of (2.8) may be expressed in many different functional forms. In particular, it may be expressed as an Ω-ordered form. This form will be obtained on replacing the mapping operator Ω on the right-hand side of (2.8) by the substitution operator $S^{(\Omega)}$ for Ω mapping[10] [see Eq. (I.2.16)]. We then obtain the following formula:

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=S^{(\Omega)}\{\exp[\sum_j\sum_{i<j}\Lambda_{ij}\epsilon(t_i-t_j)]\times\mathcal{U}_{12\cdots M}^{(\Omega)}\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}\}. \quad (2.10)$$

This formula expresses the time-ordered product $T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}$ as an Ω-ordered form. We will refer

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[8] See, e.g., H. Mori, I. Oppenheim, and J. Ross, in *Studies in Statistical Mechanics*, edited by J. deBoer and G. E. Uhlenbeck (North-Holland, Amsterdam, 1962), Vol. I, p. 217.

[9] L. Mandel, Phys. Rev. 152, 438 (1966).

[10] Equations prefixed by I and II refer to equations in Refs. 1 and 2, respectively.

to (2.10) as a *generalized Wick theorem* for boson systems.

Let us now consider the special case when each of the operators $\hat{G}_m$ is a linear combination of the annihilation and the creation operators. We will take the $\hat{G}_m$'s in the interaction picture, so that their time dependence is the same as that of the free field operators, i.e.,

$$\hat{G}_m(t)=A_m\hat{a}e^{-i\omega t}+B_m\hat{a}^\dagger e^{i\omega t}\,, \tag{2.11}$$

where A_m and B_m are c numbers. Consider now the class of mappings whose filter functions are given by Eq. (I.3.38), i.e.,[11]

$$\Omega(\alpha,\beta)=\exp(\mu\alpha^2+\nu\beta^2+\lambda\alpha\beta)\,. \tag{2.12}$$

It then follows, according to the results expressed by Eqs. (I.3.34) and (I.3.36), that the Ω equivalent of the operator $\hat{G}_m(t)$ is given by

$$F_m{}^{(\Omega)}(z,z^*;t)=A_m z e^{-i\omega t}+B_m z^* e^{i\omega t}\,. \tag{2.13}$$

We note that for each operator $\hat{G}_m(t)$ of the form (2.11), the Ω equivalent [given by (2.13)], for any choice of Ω belonging to the class characterized by (2.12), is *independent* of the particular choice of Ω.

For the class of mappings characterized by (2.12), the operator $\mathfrak{U}_{1,2,\dots,M}{}^{(\Omega)}$ defined by (2.7) may be expressed in the form

$$\mathfrak{U}_{1,2,\dots,M}{}^{(\Omega)}=\prod_{i,j;\,i<j}\exp(u_{ij}{}^{(\Omega)})\,, \tag{2.14a}$$

where [cf. (II.3.10)]

$$u_{ij}{}^{(\Omega)}=-2\nu\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}-2\mu\frac{\partial}{\partial z_i^*}\frac{\partial}{\partial z_j^*}+\lambda\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j^*}+\frac{\partial}{\partial z_i^*}\frac{\partial}{\partial z_j}\right). \tag{2.14b}$$

Hence (2.10) may in such cases be written as

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=S^{(\Omega)}\{\exp[\sum_j\sum_{i<j}\Lambda_{ij}\epsilon(t_i-t_j)+u_{ij}{}^{(\Omega)}]\times\prod_{m=1}^{M}F_m{}^{(\Omega)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}\}\,. \tag{2.15}$$

Let us consider first the special case when $M=2$. On expanding the exponential in (2.15), and on using the fact that the c-number functions $F_1{}^{(\Omega)}(z,z^*;t)$ and $F_2{}^{(\Omega)}(z,z^*;t)$ are linear in z and z^*, we obtain from (2.15) the formula

$$\begin{aligned}T\{\hat{G}_1(t_1)\hat{G}_2(t_2)\}&=S^{(\Omega)}\{[1+\epsilon(t_1-t_2)\Lambda_{12}+u_{12}{}^{(\Omega)}]F_1{}^{(\Omega)}(z_1,z_1^*;t_1)\times F_2{}^{(\Omega)}(z_2,z_2^*;t_2)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}\}\\&=S^{(\Omega)}\{F_1{}^{(\Omega)}(z,z^*;t_1)F_2{}^{(\Omega)}(z,z^*;t_2)\}+[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(\Omega)}\,,\end{aligned} \tag{2.16}$$

where

$$[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(\Omega)}=S^{(\Omega)}\{[\Lambda_{12}\epsilon(t_1-t_2)+u_{12}{}^{(\Omega)}]F_1{}^{(\Omega)}(z_1,z_1^*;t_1)\times F_2{}^{(\Omega)}(z_2,z_2^*;t_2)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}\}\,. \tag{2.17}$$

We shall refer to $[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(\Omega)}$ as the *chronological contraction* of the operators $\hat{G}_1(t_1)$ and $\hat{G}_2(t_2)$ *for* Ω *mapping*. If we recall the definitions of the operators Λ_{12} and $u_{12}{}^{(\Omega)}$ [Eqs. (2.6) and (2.14b) with $i=1$, $j=2$] and use the explicit expressions for $F_1{}^{(\Omega)}$ and $F_2{}^{(\Omega)}$ [Eq. (2.13)], we readily find that

$$\begin{aligned}[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(\Omega)}&=-2\mu B_1B_2e^{i\omega(t_1+t_2)}-2\nu A_1A_2e^{-i\omega(t_1+t_2)}\\&\quad+A_1B_2[\lambda+\tfrac{1}{2}\epsilon(t_1-t_2)]e^{-i\omega(t_1-t_2)}\\&\quad+A_2B_1[\lambda+\tfrac{1}{2}\epsilon(t_2-t_1)]e^{-i\omega(t_2-t_1)}\,.\end{aligned} \tag{2.18}$$

More generally, if we expand the exponential on the right-hand side of (2.15) in a power series, we obtain the following expression for the time-ordered product of the M operators $\hat{G}_m(t_m)$:

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=S^{(\Omega)}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}[\sum_j\sum_{i<j}\Lambda_{ij}\epsilon(t_i-t_j)+u_{ij}{}^{(\Omega)}]^n\times\prod_{m=1}^{M}F_m{}^{(\Omega)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}\right\}. \tag{2.19}$$

A typical ij term for $n=1$ may be expressed in the form

$$\begin{aligned}&S^{(\Omega)}\{[\Lambda_{ij}\epsilon(t_i-t_j)+u_{ij}{}^{(\Omega)}]\times\prod_{m=1}^{M}F_m{}^{(\Omega)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}\}\\&=S^{(\Omega)}\{\prod_{m=1;\,m\neq i,j}^{M}F_m{}^{(\Omega)}(z,z^*;t_m)\times[\hat{G}_i{}^{\cdot}(t_i)\hat{G}_j{}^{\cdot}(t_j)]^{(\Omega)}\}\,.\end{aligned} \tag{2.20}$$

In a similar way we can simplify the contribution of a typical ij term for each value of n. It is obvious that each term in the expansion of

$$S^{(\Omega)}\{[\sum_j\sum_{i<j}u_{ij}{}^{(\Omega)}+\Lambda_{ij}\epsilon(t_i-t_j)]^n\times\prod_{m=1}^{M}F_m{}^{(\Omega)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}\}$$

will lead to n chronological contractions. Thus (2.19) finally leads to the identity

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=\mathfrak{F}_0{}^{(\Omega)}+\mathfrak{F}_1{}^{(\Omega)}+\cdots\,, \tag{2.21}$$

[11] The restriction to filter functions of the form (2.12) is not essential and is made here only for the sake of simplicity.

where[12]

$$\mathfrak{F}_0^{(\Omega)}=S^{(\Omega)}\{\prod_{m=1}^{M}F_m^{(\Omega)}(z,z^*;t_m)\},$$

$$\mathfrak{F}_1^{(\Omega)}=\sum_j\sum_{i<j}S^{(\Omega)}\{\prod_{m=1;m\neq i,j}^{M}F_m^{(\Omega)}(z,z^*;t_m)\times[\hat{G}_i{}^{\cdot}(t_i)\hat{G}_j{}^{\cdot}(t_j)]^{(\Omega)}\},\tag{2.22}$$

$$\mathfrak{F}_2^{(\Omega)}=\frac{1}{2!}\sum_j\sum_{i<j}\sum_l\sum_{k<l}S^{(\Omega)}\{\prod_{m=1;m\neq i,j,k,l}^{M}F_m^{(\Omega)}(z,z^*;t_m)\times[\hat{G}_i{}^{\cdot}(t_i)\hat{G}_j{}^{\cdot}(t_j)]^{(\Omega)}[\hat{G}_k{}^{\cdot}(t_k)\hat{G}_l{}^{\cdot}(t_l)]^{(\Omega)}\},$$

etc. Formula (2.21), together with (2.22), expresses the time-ordered product of a set of operators that are linear in the creation and the annihilation operators $\hat{a}$ and $\hat{a}^\dagger$, respectively, as the sum of all the Ω-ordered products of the $\hat{G}$'s, with all possible chronological contractions for the Ω mapping, including the term with no contraction.

In the special case when we choose Ω to represent mapping according to the *normal* rule (superscript N), and use the fact that for the normal rule the parameters in (2.12) have the values $\mu=\nu=0$, $\lambda=\frac{1}{2}$ (cf. Table IV of I), the chronological contraction (2.18) reduces to

$$[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(N)}=\tfrac{1}{2}\epsilon(t_1-t_2)[A_1B_2e^{-i\omega(t_1-t_2)}-A_2B_1e^{i\omega(t_1-t_2)}]+\tfrac{1}{2}[A_1B_2e^{-i\omega(t_1-t_2)}+A_2B_1e^{i\omega(t_1-t_2)}].\tag{2.23}$$

The contraction $[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(N)}$ has a simple physical meaning. This is readily seen by taking the vacuum expectation value of (2.16) (with Ω again representing the normal rule). We then obtain

$$\langle 0|T\{\hat{G}_1(t_1)\hat{G}_2(t_2)\}|0\rangle=\langle 0|S^{(N)}\{F_1^{(N)}(z,z^*;t_1)F_2^{(N)}(z,z^*;t_2)\}|0\rangle+\langle 0|[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(N)}|0\rangle.\tag{2.24}$$

The first term on the right-hand side vanishes, since it is the expectation value of a normally ordered operator in the vacuum state. The second expression is equal to $[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(N)}$ since, being a c-number, it remains unchanged on taking the expectation value. Hence

$$[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(N)}=\langle 0|T\{\hat{G}_1(t_1)\hat{G}_2(t_2)\}|0\rangle,\tag{2.25}$$

showing that $[\hat{G}_1{}^{\cdot}(t_1)\hat{G}_2{}^{\cdot}(t_2)]^{(N)}$ is precisely the chronological product as usually defined.[3,4] It is now seen that in the special case when Ω is chosen to represent the normal rule of mapping formula (2.21) is (except for notation) nothing but Wick's theorem[3,4] for time-ordered products of a boson system.

[12] In more customary notation the expression $S^{(\Omega)}\{F_1^{(\Omega)}(z,z^*;t_1)\times\cdots F_M^{(\Omega)}(z,z^*;t_M)\}$ with Ω representing the normal rule of mapping would be written as $:\hat{G}_1(t_1)\cdots\hat{G}_M(t_M):$, where the colons indicate normal ordering. However, the customary notation disguises the important role that the phase-space representation plays in the ordering problem.

It is evident that our generalized Wick theorem, expressed by (2.10), from which we have just derived as a special case the usual form of Wick's theorem for boson systems, is of considerable generality. It allows us to express time-ordered products of Heisenberg operators $\hat{G}_1,\ldots,\hat{G}_M$ (*not* necessarily linear in $\hat{a}$ and $\hat{a}^\dagger$) of a boson system as a Ω-ordered form. In general, the use of our generalized Wick theorem requires the solution of the dynamical equation for the Ω equivalent $F_m^{(\Omega)}$ of each of the Heisenberg operators $\hat{G}_m$. In the special case when all the operators $\hat{G}_m$ are in the interaction picture, the solution of the dynamical equation is given by the very simple expressions (2.13), which are seen to be of the same mathematical form as the $\hat{G}_m$'s themselves [Eq. (2.11)]. It is presumably for this reason that the role of the phase-space representation of the Wick theorem has not been previously recognized.

Finally, we recall that there is also a Wick theorem for the *ordinary product* of operators. It is shown in Appendix A that a generalization of that theorem may also readily be obtained and that it bears the same relation to Wick's theorem for the ordinary product as (2.10) bears to Wick's theorem for the time-ordered product. This generalization is

$$\{G_1(\hat{a},\hat{a}^\dagger)\cdots G_M(\hat{a},\hat{a}^\dagger)\}=S^{(\Omega)}\{\exp[\sum_j\sum_{i<j}\Lambda_{ij}]\times\mathfrak{U}_{12\cdots M}{}^{(\Omega)}\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*)|_{z_m=z;\,z_m^*=z^*}\},\tag{2.26}$$

where the differential operators Λ_{ij} and $\mathfrak{U}_{12\cdots M}{}^{(\Omega)}$ have the same meaning as before. It is also shown in Appendix A that (2.26) leads to the following theorem.

Theorem. The ordinary product of a set of boson operators that are linear in the creation and the annihilation operators is equal to the sum of all Ω-ordered products of the G's, with all possible pairings for Ω mapping [defined by (A24)], including the term with no pairing. In particular, when Ω represents the normal rule of mapping, this theorem reduces to the usual Wick theorem for ordinary products.

III. GENERALIZATION OF ANDERSON'S THEOREM ON TIME-ORDERED PRODUCT OF FUNCTIONALS OF FIELD OPERATORS

We will now derive an interesting generalization of a result of Anderson[5,6] which expresses the time-ordered product of functions of field operators in terms of normally ordered ones. For this purpose we choose Ω to represent again the mapping according to the normal rule of association. Then if we substitute in the right-hand side of (2.10) from (2.6) and (2.14) (with $\mu=\nu=0$, $\lambda=\frac{1}{2}$, appropriate to the normal rule), and if we also use the identity $\theta(\tau)=\frac{1}{2}[1+\epsilon(\tau)]$ between the functions $\theta(\tau)$ and $\epsilon(\tau)$ defined by (2.2) and (2.9), respectively, we obtain the relation

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=S^{(N)}\left\{\exp\left[\sum_i\sum_j\theta(t_i-t_j)\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j^*}\right]\times\prod_{m=1}^{M}F_m^{(N)}(z_m,z_m^*;t_m)\,|_{z_m=z;\,z_m^*=z^*}\right\}. \quad (3.1)$$

Let us again assume that each of the operators $\hat{G}_m$ is of the form (2.11), so that its phase-space equivalent is given by (2.13). We will write this equivalent as

$$F_m^{(N)}\{z(t),z^*(t)\}=A_mz(t)+B_mz^*(t), \quad (3.2)$$

where $z(t)$ and its complex conjugate $z^*(t)$ are of the form

$$z(t)=ze^{-i\omega t},\quad z^*(t)=z^*e^{i\omega t}. \quad (3.3)$$

Identity (3.1) then becomes

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=S^{(N)}\left\{\exp\left[\sum_i\sum_j\theta(t_i-t_j)\times\exp[-i\omega(t_i-t_j)]\frac{\partial}{\partial z_i(t_i)}\frac{\partial}{\partial z_j^*(t_j)}\right]\times\prod_{m=1}^{M}F_m^{(N)}\{z_m(t_m),z_m^*(t_m)\}\Bigg|_{z_m(t_m)=z(t_m);\,z_m^*(t_m)=z^*(t_m)}\right\}. \quad (3.4)$$

Now in view of the linearity of $F_m^{(N)}$, (3.4) may be rewritten as

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=S^{(N)}\left\{\exp\left[\int dt'\int dt\,D(t'|t)\frac{\delta}{\delta z(t')}\frac{\delta}{\delta z^*(t)}\right]\times\prod_{m=1}^{M}F_m^{(N)}\{z(t_m),z^*(t_m)\}\right\}, \quad (3.5)$$

where

$$D(t'|t)=\theta(t'-t)\exp[-i\omega(t'-t)], \quad (3.6)$$

and $\delta/\delta z(t)$ denotes the functional derivative.

Identity (3.5) may readily be generalized to systems with an arbitrary number of degrees of freedom, in which the operators $\hat{G}_m(t)$ are of the form

$$\hat{G}_m(t)=\int[A_m(\mathbf{p})\hat{a}(\mathbf{p})e^{-i\omega_pt}+B_m(\mathbf{p})\hat{a}^\dagger(\mathbf{p})e^{i\omega_pt}]d^3p, \quad (3.7)$$

where the operators $\hat{a}(\mathbf{p})$ and $\hat{a}^\dagger(\mathbf{p})$ satisfy the commutation relations

$$[\hat{a}(\mathbf{p}),\hat{a}^\dagger(\mathbf{p}')]=\delta^{(3)}(\mathbf{p}-\mathbf{p}'). \quad (3.8)$$

The phase-space equivalent of $\hat{G}_m(t)$ for the normal rule of association is

$$F_m^{(N)}[z(\mathbf{p},t),z^*(\mathbf{p},t)]=\int[A_m(\mathbf{p})z(\mathbf{p},t)+B_m(\mathbf{p})z^*(\mathbf{p},t)]d^3p, \quad (3.9)$$

where $z(\mathbf{p},t)$ and $z^*(\mathbf{p},t)$ have the form

$$z(\mathbf{p},t)=z(\mathbf{p})e^{-i\omega_pt},\quad z^*(\mathbf{p},t)=z^*(\mathbf{p})e^{i\omega_pt}. \quad (3.10)$$

In place of (3.5), we then obtain the formula

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}=S^{(N)}\left\{\exp\left[\int dt'd^3p'\int dtd^3p\times D(t',\mathbf{p}'|t,\mathbf{p})\frac{\delta}{\delta z(\mathbf{p}',t')}\frac{\delta}{\delta z^*(\mathbf{p},t)}\right]\times\prod_{m=1}^{M}F_m^{(N)}[z(\{\mathbf{p}\},t_m),z^*(\{\mathbf{p}\},t_m)]\right\}, \quad (3.11)$$

with

$$D(t',\mathbf{p}'|t,\mathbf{p})=\delta^{(3)}(\mathbf{p}-\mathbf{p}')D(t'|t). \quad (3.12)$$

We are now in a position to obtain the generalization of the result of Anderson. Let $\hat{\varphi}^{(+)}$ and $\hat{\varphi}^{(-)}$ be the positive- and negative-frequency parts of a boson field operator $\hat{\varphi}$. In the interaction picture, the commutator $[\hat{\varphi}^{(+)}(\mathbf{x},t'),\hat{\varphi}^{(-)}(\mathbf{y},t)]$ is a c-number. Let us make the association[13] (analogous to the association $\hat{a}\to z$, $\hat{a}^\dagger\to z^*$)

$$\hat{\varphi}^{(+)}\to J,\quad \hat{\varphi}^{(-)}\to J^*, \quad (3.13)$$

where J is a c-number and J^* is its complex conjugate. In place of the operators $\hat{G}_m(t)$ defined by (2.11), we now have the operators

$$\hat{G}_m(t)=\int[A_m(\mathbf{x})\hat{\varphi}^{(+)}(\mathbf{x},t)+B_m(\mathbf{x})\hat{\varphi}^{(-)}(\mathbf{x},t)]d^3x, \quad (3.14)$$

i.e., linear functionals of the operators $\hat{\varphi}^{(+)}(\mathbf{x},t)$ and $\hat{\varphi}^{(-)}(\mathbf{x},t)$. The c-number equivalents of the operators (3.14) for the normal rule of association are

$$F_m^{(N)}[J(\{\mathbf{x}\},t),J^*(\{\mathbf{x}\},t)]=\int[A_m(\mathbf{x})J(\mathbf{x},t)+B_m(\mathbf{x})J^*(\mathbf{x},t)]d^3x. \quad (3.15)$$

It should be noted that (3.15) is of the same form as (3.9) when transformation to the momentum space is made. One may also show by straightforward functional

[13] This representation of the field $\hat{\varphi}+(\mathbf{x})$ by the c-number $J(\mathbf{x})$ is very similar to one employed by J. Schwinger [Proc. Natl. Acad. Sci. **37**, 452 (1951); **37**, 455 (1951)] in his external source representation of the boson field.

differentiation that

$$\int dt' d^3x \int dt d^3y \, D(t',\mathbf{x}|t,\mathbf{y}) \frac{\delta}{\delta J(\mathbf{x},t')} \frac{\delta}{\delta J^*(\mathbf{y},t)}$$

$$= \int dt' d^3p' \int dt d^3p \, D(t',\mathbf{p}'|t,\mathbf{p}) \frac{\delta}{\delta z(\mathbf{p}',t')} \frac{\delta}{\delta z^*(\mathbf{p},t)}, \quad (3.16)$$

where

$$D(t',\mathbf{x}|t,\mathbf{y}) = \theta(t'-t)[\hat{\varphi}^{(+)}(\mathbf{x},t'),\hat{\varphi}^{(-)}(\mathbf{y},t)]. \quad (3.17)$$

Hence (3.11) leads to the following identity:

$$T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\} = S^{(N)}\Big\{\exp\Big[\int dt' d^3x \int dt d^3y \times D(t',\mathbf{x}|t,\mathbf{y}) \frac{\delta}{\delta J(\mathbf{x},t')} \frac{\delta}{\delta J^*(\mathbf{y},t)}\Big] \times \prod_{m=1}^{M} F_m^{(N)}[J(\{\mathbf{x}\},t_m),J^*(\{\mathbf{x}\},t_m)]\Big\}. \quad (3.18)$$

An identity that expresses the time-ordered product of M operators which are linear functionals of the boson field operator $\hat{\varphi}(\mathbf{x},t)$ in terms of normally ordered products was derived by Anderson[5] [his formulas (8), (12), and (13)]. Our formula (3.18) may be regarded as a generalization of Anderson's identity when the operators are linear functionals of the positive- and the negative-frequency parts $\hat{\varphi}^{(+)}(\mathbf{x},t)$ and $\hat{\varphi}^{(-)}(\mathbf{x},t)$ of the field operator $\hat{\varphi}(\mathbf{x},t)$. Moreover, as in our formulation of Wick's theorem given in Sec. II, formula (3.18) shows explicitly that phase-space representation of the operators plays a key role for a clear understanding of this identity.

IV. PHASE-SPACE EQUIVALENTS AND CLOSED-FORM EXPRESSION FOR TIME-EVOLUTION OPERATOR

In this section we show that our phase-space techniques provide a new systematic way of evaluating the unitary time-evolution operator of a boson system. For the sake of simplicity we will restrict ourselves to a system with one degree of freedom only. The generalization to systems with an arbitrary number of degrees of freedom is straightforward.

The time-evolution operator $U(\hat{a},\hat{a}^\dagger;t,t_0)$ satisfies the Schrödinger equation

$$i\hbar \frac{\partial U(\hat{a},\hat{a}^\dagger;t,t_0)}{\partial t} = H(\hat{a},\hat{a}^\dagger;t)\,U(\hat{a},\hat{a}^\dagger;t,t_0), \quad (4.1)$$

where $H(\hat{a},\hat{a}^\dagger;t,t_0)$ is the Hamiltonian of the system. $\hat{U}$ satisfies the initial condition

$$U(\hat{a},\hat{a}^\dagger;t_0,t_0) = 1. \quad (4.2)$$

As is well known, the formal solution of (4.1) subject to (4.2) is[4]

$$U(\hat{a},\hat{a}^\dagger;t,t_0) = T\Big\{\exp\Big[-\frac{i}{\hbar}\int_{t_0}^{t} H(\hat{a},\hat{a}^\dagger;t')dt'\Big]\Big\}. \quad (4.3)$$

Let $F_U^{(\Omega)}(z,z^*;t,t_0)$ be the Ω equivalent of the time-evolution operator $\hat{U}$, i.e.,

$$F_U^{(\Omega)}(z,z^*;t,t_0) = \Theta\{U(\hat{a},\hat{a}^\dagger;t,t_0)\}. \quad (4.4)$$

We have shown in Sec. IV A of II that $F_U^{(\Omega)}$ satisfies the equation

$$i\hbar \frac{\partial F_U^{(\Omega)}}{\partial t} = \mathfrak{L}_+ F_U^{(\Omega)}, \quad (4.5)$$

where the "Liouville operator" $\mathfrak{L}_+$ is defined by Eq. (II.3.16a), viz.,

$$\mathfrak{L}_+ F_U^{(\Omega)} = \exp(\Lambda_{12})\mathfrak{U}_{12}^{(\Omega)} F_H^{(\Omega)}(z_1,z_1^*;t) \times F_U^{(\Omega)}(z_2,z_2^*;t,t_0)\big|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}. \quad (4.6)$$

Here Λ_{12} and $\mathfrak{U}_{12}^{(\Omega)}$ are, respectively, the differential operators defined by Eqs. (II.3.4) and (II.3.5) and $F_H^{(\Omega)}$ is the Ω equivalent of the Hamiltonian operator $\hat{H}$. Equation (4.5) is to be solved subject to the initial condition

$$F_U^{(\Omega)}(z,z^*;t_0,t_0) = 1 \quad \text{for all } z \text{ and } z^*. \quad (4.7)$$

Equations (4.5) and (4.7) are just the Ω equivalents of Eqs. (4.1) and (4.2), respectively.

Once the partial differential equation (4.5) is solved, subject to the initial condition (4.7), one may readily determine the time-evolution operator. For, according to (4.4) and our mapping theorem I [Eq. (I.2.22)], $\hat{U}$ is then evidently given by

$$U(\hat{a},\hat{a}^\dagger;t,t_0) = S^{(\Omega)}\{F_U^{(\Omega)}(z,z^*;t,t_0)\}, \quad (4.8)$$

where $S^{(\Omega)}$ is the substitution operator for Ω mapping. Let us rewrite (4.8) in a more explicit form:

$$T\Big\{\exp\Big[-\frac{i}{\hbar}\int_{t_0}^{t} H(\hat{a},\hat{a}^\dagger;t')dt'\Big]\Big\} = S^{(\Omega)}\{F_U^{(\Omega)}(z,z^*;t,t_0)\}. \quad (4.9)$$

This formula shows the following: The time-evolution operator of a boson system may be determined by first solving the phase-space equation of motion (4.5) for the Ω equivalent of the evolution operator [subject to the initial condition (4.7)] and then applying to the solution the substitution operator $S^{(\Omega)}$ for Ω mapping.

In the special case when Ω represents the mapping according to the normal rule of association, identity (4.9) is at the root of the normal-ordering techniques developed in recent years for solving quantum-dynamical problems with time-dependent Hamiltonians.[14] We will see shortly that (4.9) is, in fact, intimately related to our generalized Wick theorem.

We will illustrate the use of the new identity (4.9) by determining the time-evolution operator for a forced harmonic oscillator, with the Hamiltonian

$$\hat{H}(t)=\hbar\omega\hat{a}^\dagger\hat{a}+\hbar f(t)(\hat{a}+\hat{a}^\dagger)\,, \qquad (4.10)$$

where $f(t)$ is real and represents the driving force. The interaction Hamiltonian in the interaction picture is, in this case, given by

$$\hat{H}_I(t)=\hbar f(t)(\hat{a}e^{-i\omega t}+\hat{a}^\dagger e^{i\omega t})\,. \qquad (4.11)$$

The formal expression for the time-evolution operator is, according to (4.11) and (4.3), now given by

$$U_I(t)=T\times\left\{\exp\left[-i\int_0^t f(t')(\hat{a}e^{-i\omega t'}+\hat{a}^\dagger e^{i\omega t'})dt'\right]\right\}\,, \qquad (4.12)$$

where we have chosen $t_0=0$.

Let us now apply our technique to evaluate (4.12). We will restrict ourselves to the class of Ω mappings for which the filter function is again of the form (2.12). The Ω equivalent of the Hamiltonian (4.11) is given by

$$F_H^{(\Omega)}(z,z^*;t)=\hbar f(t)(ze^{-i\omega t}+z^*e^{i\omega t})\,. \qquad (4.13)$$

If we substitute from (4.13) into (4.6), we readily obtain the explicit form of $\mathcal{L}_+F_U^{(\Omega)}$ [cf. Eq. (II.3.18a)], and from (4.5) we then obtain the following equation for the Ω equivalent of the time-evolution operator for the present problem:

$$\frac{i\partial F_U^{(\Omega)}}{\partial t}=\left[f(t)e^{-i\omega t}z+f^*(t)e^{i\omega t}z^*-2\mu f(t)e^{i\omega t}\frac{\partial}{\partial z^*}-2\nu f(t)e^{-i\omega t}\frac{\partial}{\partial z}+(\lambda+\tfrac{1}{2})f(t)e^{-i\omega t}\frac{\partial}{\partial z^*}+(\lambda-\tfrac{1}{2})f(t)e^{i\omega t}\frac{\partial}{\partial z}\right]F_U^{(\Omega)}\,. \qquad (4.14)$$

The parameters λ, μ, and ν characterize, of course, a particular mapping.

Let us take as a trial solution of (4.14) an expression of the form[15]

$$F_U^{(\Omega)}(z,z^*;t)=\exp[A(t)+B(t)z+C(t)z^*]\,. \qquad (4.15)$$

The initial condition (4.7) requires that

$$A(0)=B(0)=C(0)=0\,. \qquad (4.16)$$

We next substitute from (4.15) into (4.14). Each side is then of the form of a product of $F_U^{(\Omega)}$ and a polynomial in z and z^*. By equating the coefficients of equal powers of z and z^* on the two sides, we obtain the following set of equations:

$$i\partial B/\partial t=f(t)e^{-i\omega t}\,, \qquad (4.17a)$$

$$i\partial C/\partial t=f(t)e^{i\omega t}\,, \qquad (4.17b)$$

$$i\partial A/\partial t=-2\mu f(t)e^{i\omega t}C(t)-2\nu f(t)e^{-i\omega t}B(t)+(\lambda+\tfrac{1}{2})f(t)e^{-i\omega t}C(t)+(\lambda-\tfrac{1}{2})f(t)e^{i\omega t}B(t)\,. \qquad (4.17c)$$

This coupled set of differential equations, subject to the initial conditions (4.16), is readily solved and the result is

$$B(t)=-i\int_0^t f(t')e^{-i\omega t'}dt'\,, \qquad (4.18a)$$

$$C(t)=-i\int_0^t f(t')e^{i\omega t'}dt'=-B^*(t)\,, \qquad (4.18b)$$

$$A(t)=-\mu C^2(t)-\nu B^2(t)+(\lambda-\tfrac{1}{2})B(t)C(t)-i\int_0^t f(t')e^{-i\omega t'}C(t')dt'\,. \qquad (4.18c)$$

It follows from (4.9), (4.12), and (4.15) that

$$T\left\{\exp\left[-i\int_0^t f(t')(\hat{a}e^{-i\omega t'}+\hat{a}^\dagger e^{i\omega t'})dt'\right]\right\}=S^{(\Omega)}\{\exp[A(t)+B(t)z+C(t)z^*]\}\,, \qquad (4.19)$$

where $A(t)$, $B(t)$, and $C(t)$ are given by Eqs. (4.18).

In particular, let us consider the special case when Ω represents the normal rule of association ($\mu=\nu=0$, $\lambda=\frac{1}{2}$; cf. Table IV of I). We note that of the three coefficients given by (4.18) only the A coefficient depends on the particular choice of association and for the normal rule (suffix N) it becomes

$$A_N=-\int_0^t f(t')e^{-i\omega t'}dt'\int_0^{t'} f(t'')e^{i\omega t''}dt''=-\frac{1}{2}\left|\int_0^t f(t')e^{-i\omega t'}dt'\right|^2=-\tfrac{1}{2}|B(t)|^2\,. \qquad (4.20)$$

Noting also that, according to (4.18a) and (4.18b), $C(t)=-B^*(t)$ [since $f(t)$ is real], (4.19) now reduces to

$$T\left\{\exp\left[-i\int_0^t f(t')(\hat{a}e^{-i\omega t'}+\hat{a}^\dagger e^{i\omega t'})dt'\right]\right\}=S^{(N)}\{\exp[-\tfrac{1}{2}|B(t)|^2]\exp[B(t)z]\exp[-B^*(t)z^*]\}=\exp[-\tfrac{1}{2}|B(t)|^2]\exp[-B^*(t)\hat{a}^\dagger]\times\exp[B(t)\hat{a}]\,, \qquad (4.21)$$

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[14] See, e.g., W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964).

[15] This form of the trial solution is suggested by a theorem of J. H. Marburger [J. Math. Phys. **7**, 829 (1966)] relating to solutions of c-number equations for systems whose Hamiltonians are quadratic in $\hat{a}$ and $\hat{a}^\dagger$.

where $B(t)$ is given by (4.18a). The identity (4.21) has been known for a long time.[16] More recently, it was derived by c-number techniques by Heffner and Louisell.[17]

Returning to the general case, we see that the possibility of obtaining a closed-form solution for the time-evolution operator by phase-space techniques depends strongly on whether a closed-form solution of the differential equation (4.5) can be found. However, Eq. (4.5) may always be solved by a standard perturbation procedure. For this purpose let us write the perturbation-series solution of (4.5) in the form

$$F_U^{(\Omega)}=\sum_{n=0}^{\infty}[F_U^{(\Omega)}]_n, \qquad (4.22)$$

where

$$[F_U^{(\Omega)}]_n=\left(-\frac{i}{\hbar}\right)^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n \times\mathfrak{L}_+(t_1)\cdots\mathfrak{L}_+(t_n)F_U^{(\Omega)}(z,z^*;0). \qquad (4.23)$$

Now according to (4.6),

$$\mathfrak{L}_+(t_n)F_U^{(\Omega)}(z,z^*;0) =\exp(\Lambda_{12})\mathfrak{U}_{12}^{(\Omega)}F_H^{(\Omega)}(z_1,z_1^*;t_n) \times F_U^{(\Omega)}(z_2,z_2^*;0)\big|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}. \qquad (4.24)$$

The operator $\mathfrak{U}_{12}^{(\Omega)}$ is, in view of Eqs. (I.3.30) and (II.3.5), of the form $\exp(u_{12}^{(\Omega)})$, where $u_{12}^{(\Omega)}$ is a power series in $\partial/\partial z_i$ and $\partial/\partial z_i^*$ ($i=1,2$), and Λ_{12} is a quadratic function in these differentials. Since, according to (4.7), $F_U^{(\Omega)}(z,z^*;0)=1$, it follows at once that the operators $\exp(\Lambda_{12})$ and $\mathfrak{U}_{12}^{(\Omega)}$ in (4.24) may be replaced by unity, and (4.24) then reduces to

$$\mathfrak{L}_+(t_n)F_U^{(\Omega)}(z,z^*;0)=F_H^{(\Omega)}(z,z^*;t_n). \qquad (4.25)$$

If we use (4.25), (4.23) may be rewritten as

$$[F_U^{(\Omega)}]_n=\left(-\frac{i}{\hbar}\right)^n\int_0^t dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n \times\mathfrak{L}_+(t_1)\cdots\mathfrak{L}_+(t_{n-1})F_H^{(\Omega)}(z,z^*;t_n). \qquad (4.26)$$

It is shown in Appendix B that the integrand of (4.26) may be put in the following form:

$$\mathfrak{L}_+(t_1)\cdots\mathfrak{L}_+(t_{n-1})F_H^{(\Omega)}(z,z^*;t_n)=\exp\{\sum_j\sum_{i<j}\Lambda_{ij}\} \times\mathfrak{U}_{12\cdots n}^{(\Omega)}\prod_{m=1}^{n}F_H^{(\Omega)}(z_m,z_m^*;t_m)\bigg|_{z_m=z;\,z_m^*=z^*}. \qquad (4.27)$$

On substituting from (4.27) into (4.26) and on using the symmetry properties of the operators Λ_{ij} and $\mathfrak{U}_{1\cdots n}^{(\Omega)}$ under permutation of the indices [see the remarks that precede Eq. (2.8)], it follows that the nth-order term $[F_U^{(\Omega)}]_n$ in the perturbation expansion (4.22) of $F_U^{(\Omega)}$ may be expressed in the form

$$[F_U^{(\Omega)}]_n=\left(-\frac{i}{\hbar}\right)^n\frac{1}{n!}\int_0^t\cdots\int_0^t dt_1\cdots dt_n \times\exp[\sum_j\sum_{i<j}\Lambda_{ij}\epsilon(t_i-t_j)] \times\mathfrak{U}_{12\cdots n}^{(\Omega)}\prod_{m=1}^{n}F_H^{(\Omega)}(z_m,z_m^*;t_m)\bigg|_{z_m=z;\,z_m^*=z^*}, \qquad (4.28)$$

where the function $\epsilon(\tau)$ is defined by (2.9).

From (4.8) and (4.22), we finally obtain the following expression for the time-evolution operator:

$$U(\hat{a},\hat{a}^\dagger;t,0)=\sum_{n=0}^{\infty}[U(\hat{a},\hat{a}^\dagger;t,0)]_n, \qquad (4.29)$$

where

$$[U(\hat{a},\hat{a}^\dagger;t,0)]_n=S^{(\Omega)}\{[F_U^{(\Omega)}(z,z^*;t,0)]_n\}. \qquad (4.30)$$

The significance of the expression on the right-hand side of (4.30) can be seen at once by observing that the integrand in expression (4.28) for $[F_U^{(\Omega)}]_n$ has precisely the same form as the expression on the right-hand side of our generalized Wick's theorem expressed by Eq. (2.10). It therefore follows from the generalized Wick theorem and from (4.28) that

$$S^{(\Omega)}\{[F_U^{(\Omega)}(z,z^*;t,0)]_n\} =\left(-\frac{i}{\hbar}\right)^n\frac{1}{n!}\int_0^t\cdots\int_0^t dt_1\cdots dt_n \times T\{\hat{H}(t_1)\cdots\hat{H}(t_n)\}. \qquad (4.31)$$

But the expression on the right-hand side of (4.31) is precisely the nth-order term in the usual perturbation expansion of the evolution operator (4.3). Thus Eq. (4.30) implies that *the nth-order term in the perturbation expansion of the evolution operator* $\hat{U}$ *may be obtained by applying the substitution operator* $S^{(\Omega)}$ *for* Ω *mapping to the nth-order perturbation solution* $[F_U^{(\Omega)}]_n$ *of Eq. (4.5) for the* Ω *equivalent* $F_U^{(\Omega)}=\Theta\{\hat{U}\}$ *of the time-evolution operator.*

V. MULTITIME CORRELATION FUNCTIONS AS PHASE-SPACE AVERAGES

In Sec. II of Paper II of this series, we considered the problem of calculating by phase-space techniques the trace of the product of two operators and we showed, in particular, how the phase-space methods may be used to determine the expectation value of an observable when the system is in a state characterized by the density operator $\hat{\rho}$. In that paper, operators that were functions of annihilation and creation operators at one time only were considered.

[16] R. P. Feynman, Phys. Rev. **80**, 440 (1950); **84**, 108 (1951); R. J. Glauber, *ibid.* **84**, 395 (1951).
[17] H. Heffner and W. H. Louisell, J. Math. Phys. **6**, 474 (1965).

We will now show that these techniques may be extended to calculations of the expectation value of an observable that is represented by an operator which is an arbitrary function of the annihilation and the creation operators at different times. Expectation values of operators of this type have recently become of importance in quantum optics, particularly in connection with the theory of the laser[18] and in problems concerning the coherence properties of light beams.[19] The calculation of such expectation values is, in general, a rather involved problem.

Let $\hat{a}(t)$ and $\hat{a}^\dagger(t)$ be the boson annihilation and creation operators at time t. These operators at different times do not commute in general, i.e.,

$$[\hat{a}(t_1),\hat{a}(t_2)]\neq 0 \quad \text{if } t_1\neq t_2. \tag{5.1}$$

Let $G[\hat{a}(t_1),\hat{a}^\dagger(t_1);\ldots;\hat{a}(t_n),\hat{a}^\dagger(t_n)]$ be an arbitrary function of the boson operators, considered at different times $t_1,t_2,\ldots,t_n$, where[20]

$$t_n\geqslant t_{n-1}\geqslant\cdots\geqslant t_1. \tag{5.2}$$

Further, let $z_1,z_2,\ldots,z_n$ be the c-numbers associated with the annihilation operators $\hat{a}(t_1),\ldots,\hat{a}(t_n)$ and $z_1^*,\ldots,z_n^*$ (where z_j^* is the complex conjugate of z_j) be the c-numbers associated with the creation operators $\hat{a}^\dagger(t_1),\ldots,\hat{a}^\dagger(t_n)$. We now introduce a linear mapping operator $\Omega_T^{(N)}$ defined by the formula[21]

$$\begin{aligned}\Omega_T^{(N)}&\{(z_1^*)^{i_1}\cdots(z_n^*)^{i_n}(z_1)^{j_1}\cdots(z_n)^{j_n}\}\\&=\tilde{T}\{[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}\}T\{[\hat{a}(t_1)]^{j_1}\cdots[\hat{a}(t_n)]^{j_n}\}\\&=[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}[\hat{a}(t_n)]^{j_n}\cdots[\hat{a}(t_1)]^{j_1}.\end{aligned} \tag{5.3}$$

Here $i_1\cdots i_n$, $j_1\cdots j_n$ are any non-negative integers and T and $\tilde{T}$ denote the chronological and the antichronological ordering, respectively. We also introduce the inverse operator $\Theta_T^{(N)}$, defined by the formula

$$\begin{aligned}\Theta_T^{(N)}\{\tilde{T}\{[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}\}&\\ \times T\{[\hat{a}(t_1)]^{j_1}\cdots[\hat{a}(t_n)]^{j_n}\}\}&\\ =(z_1^*)^{i_1}\cdots(z_n^*)^{i_n}(z_1)^{j_1}\cdots(z_n)^{j_n}.&\end{aligned} \tag{5.4}$$

It is seen from (5.3) that the operator $\Omega_T^{(N)}$ replaces all the c-numbers by the corresponding annihilation and creation operators according to the rule

$$z_j\to\hat{a}(t_j)\,,\quad z_j^*\to\hat{a}^\dagger(t_j)\,, \tag{5.5}$$

and it places all the creation operators to the left of the annihilation operators and puts all the annihilation operators into chronological order and all the creation operators into antichronological order. Following Lax,[7] we call the product occurring on the right-hand side of (5.3) a *normally ordered time-ordered product*. Such products arise naturally, for example, in the analysis of photoelectric detection of photons.[19] In such an analysis one is led to correlation functions of the form

$$\begin{aligned}\Gamma_T^{(N)}=\langle\tilde{T}\{[\hat{A}^{(-)}(t_1)]^{i_1}\cdots[\hat{A}^{(-)}(t_n)]^{i_n}\}&\\ \times T\{[\hat{A}^{(+)}(t_1)]^{j_1}\cdots[\hat{A}^{(+)}(t_n)]^{j_n}\}\rangle\,,&\end{aligned} \tag{5.6}$$

where $\hat{A}^{(+)}(t)$ and $\hat{A}^{(-)}(t)$ are the positive- and negative-frequency parts of the appropriate field operator $\hat{A}$. We will refer to (5.6) as a *normally ordered time-ordered correlation function*. When the field consists of a single mode, (5.6) reduces to

$$\begin{aligned}\Gamma_T^{(N)}=\langle[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}&\\ \times[\hat{a}(t_n)]^{j_n}\cdots[\hat{a}(t_1)]^{j_1}\rangle.&\end{aligned} \tag{5.7}$$

In a similar manner, one can introduce other linear mapping operators. For example, the operator $\Omega_T^{(A)}$ associated with the antinormal rule of association is defined by the property that

$$\begin{aligned}\Omega_T^{(A)}&\{(z_1^*)^{i_1}\cdots(z_n^*)^{i_n}(z_1)^{j_1}\cdots(z_n)^{j_n}\}\\&=[\hat{a}(t_1)]^{j_1}\cdots[\hat{a}(t_n)]^{j_n}[\hat{a}^\dagger(t_n)]^{i_n}\cdots[\hat{a}^\dagger(t_1)]^{i_1}.\end{aligned} \tag{5.8}$$

By analogy with (5.7), we define the *antinormally ordered time-ordered correlation function* $\Gamma_T^{(A)}$ as

$$\begin{aligned}\Gamma_T^{(A)}=\langle[\hat{a}(t_1)]^{j_1}\cdots[\hat{a}(t_n)]^{j_n}&\\ \times[\hat{a}^\dagger(t_n)]^{i_n}\cdots[\hat{a}^\dagger(t_1)]^{i_1}\rangle.&\end{aligned} \tag{5.9}$$

Correlation functions of this type occur in the theory of photon detectors which operate via emission rather than absorption of photons.[9]

In this section, we will consider in detail the mapping characterized by the operator $\Omega_T^{(N)}$. Let $F_T^{(N)}$ be the c-number equivalent of the operator $\hat{G}$ in this mapping:

$$\begin{aligned}G[\hat{a}(t_1),\hat{a}^\dagger(t_1);\cdots;\hat{a}(t_n),\hat{a}^\dagger(t_n)]&\\ =\Omega_T^{(N)}\{F_T^{(N)}(z_1,z_1^*;\ldots;z_n,z_n^*)\}.&\end{aligned} \tag{5.10}$$

Then the expectation value of $\hat{G}$ may be expressed as

$$\begin{aligned}\langle\hat{G}\rangle&=\langle G[\hat{a}(t_1),\hat{a}^\dagger(t_1);\ldots;\hat{a}(t_n),\hat{a}^\dagger(t_n)]\rangle\\&=\langle\Omega_T^{(N)}\{F_T^{(N)}(z_1,z_1^*;\ldots;z_n,z_n^*)\}\rangle.\end{aligned} \tag{5.11}$$

Equation (5.11) may be rewritten in the form

$$\langle\hat{G}\rangle=\left\langle\Omega_T^{(N)}\left\{\int\cdots\int d^2\{\bar{z}_n\}F_T^{(N)}(\bar{z}_1,\bar{z}_1^*;\ldots;\bar{z}_n,\bar{z}_n^*)\times\prod_{\lambda=1}^{n}\delta^{(2)}(z_\lambda-\bar{z}_\lambda)\right\}\right\rangle, \tag{5.12}$$

which, in view of the linearity of $\Omega_T^{(N)}$, can be expressed

[18] Such correlation functions, which occur in the theory of lasers, based on the van der Pohl oscillator model have been computed by several authors. See, e.g., H. Risken, Z. Physik **191**, 302 (1966); R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967); H. Risken, C. Schmidt, and W. Weidlich, Z. Physik **193**, 37 (1966); **194**, 337 (1966).

[19] P. L. Kelley and W. H. Kleiner, Phys. Rev. **136**, A316 (1964).

[20] We will retain the labeling implied by (5.2) throughout this section.

[21] The expression on the right-hand side of Eq. (5.3) can, of course, be written in many different functional forms by the use of the commutation relations. By analogy with Eq. (I.2.16), we may also introduce an associated substitution operator $S_T^{(N)}$ such that $S_T^{(N)}\{(z_1^*)^{i_1}\cdots(z_n^*)^{i_n}(z_1)^{j_1}\cdots(z_n)^{j_n}\}\equiv[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}\times[\hat{a}(t_n)]^{j_n}\cdots[\hat{a}(t_1)]^{j_1}$, where the identity sign is used in the same sense as before [cf. the discussion following Eq. (I.2.13)].

as

$$\langle\hat{G}\rangle=\int\cdots\int d^2\{\bar{z}_n\}F_T{}^{(N)}(\bar{z}_1,\bar{z}_1{}^*;\ldots;\bar{z}_n,\bar{z}_n{}^*)\times\Phi_T{}^{(A)}(\bar{z}_1,\bar{z}_1{}^*,t_1;\ldots;\bar{z}_n,\bar{z}_n{}^*,t_n),\quad(5.13)$$

where

$$\Phi_T{}^{(A)}(\bar{z}_1,\bar{z}_1{}^*,t_1;\ldots;\bar{z}_n,\bar{z}_n{}^*,t_n)=\langle\Omega_T{}^{(N)}\{\prod_{\lambda=1}^{n}\delta^{(2)}(z_\lambda-\bar{z}_\lambda)\}\rangle.\quad(5.14)$$

We see from Eq. (5.13) that the function $\Phi_T{}^{(A)}$ plays the role of a multitime phase-space distribution function.[22] We can rewrite (5.14) as follows:

$$\Phi_T{}^{(A)}(\bar{z}_1,\bar{z}_1{}^*,t_1;\ldots;\bar{z}_n,\bar{z}_n{}^*,t_n)=\langle\hat{\Delta}_T{}^{(N)}\rangle,\quad(5.15)$$

where $\hat{\Delta}_T{}^{(N)}$ is the mapping Δ operator for the rule of association characterized by $\Omega_T{}^{(N)}$, and it is given by

$$\hat{\Delta}_T{}^{(N)}=\Omega_T{}^{(N)}\{\prod_{\lambda=1}^{n}\delta^{(2)}(z_\lambda-\bar{z}_\lambda)\}.\quad(5.16)$$

On using the integral representation for the δ function, relation (5.16) reduces to

$$\begin{aligned}\hat{\Delta}_T{}^{(N)}&=\frac{1}{\pi^{2n}}\int\cdots\int d^2\{\alpha\}\prod_{\lambda=1}^{n}\exp[-(\alpha_\lambda\bar{z}_\lambda{}^*-\alpha_\lambda{}^*\bar{z}_\lambda)]\\&\quad\times\Omega_T{}^{(N)}\{\prod_{\lambda=1}^{n}\exp(\alpha_\lambda z_\lambda{}^*-\alpha_\lambda{}^*z_\lambda)\}\\&=\frac{1}{\pi^{2n}}\int\cdots\int d^2\{\alpha\}\prod_{\lambda=1}^{n}\exp[-(\alpha_\lambda\bar{z}_\lambda{}^*-\alpha_\lambda{}^*\bar{z}_\lambda)]\\&\quad\times\exp[\alpha_1\hat{a}^\dagger(t_1)]\cdots\exp[\alpha_n\hat{a}^\dagger(t_n)]\\&\quad\times\exp[-\alpha_n{}^*\hat{a}(t_n)]\cdots\exp[-\alpha_1{}^*\hat{a}(t_1)],\end{aligned}\quad(5.17)$$

where (5.2) and (5.3) have been used. Relations (5.13), (5.15), and (5.17) are natural generalizations to the case of multitime operators of our results expressed by Eqs. (II.2.9) and (II.2.8) and (I.3.14) (with $\Omega=\Omega^{(N)}$). From (5.13) we obtain, in particular, the following expression for the normally ordered time-ordered correlation function (5.7):

$$\Gamma_T{}^{(N)}=\int\cdots\int d^2\{z_n\}\Phi_T{}^{(A)}(z_1,z_1{}^*,t_1;\ldots;z_n,z_n{}^*,t_n)\times\prod_{\lambda=1}^{n}\{(z_\lambda{}^*)^{i_\lambda}(z_\lambda)^{j_\lambda}\}.\quad(5.18)$$

This formula is a generalization of Eq. (II.2.19).

We will now derive an explicit expression for the multitime phase-space distribution function $\Phi_T{}^{(A)}$. We will show that $\Phi_T{}^{(A)}$ can be expressed in terms of the Green's function of the c-number equation of motion for the phase-space equivalent of the density operator. The main properties of this Green's function and its application to calculations of multitime correlation functions has been discussed in another publication.[23] We will first recall some of the results derived in Ref. 23.

Let $K^{(\Omega)}(z,z^*,t|z_0,z_0{}^*,t_0)$ be the Green's function associated with the equation of motion for the Ω equivalent $F_\rho{}^{(\Omega)}(z,z^*,t)$ of the density operator $\hat{\rho}$. This Green's function satisfies the equation of motion (II.4.7), viz.,

$$i\hbar\partial K^{(\Omega)}/\partial t=(\mathcal{L}_+-\mathcal{L}_-)K^{(\Omega)}\quad(5.19)$$

[with the operators $\mathcal{L}_+$ and $\mathcal{L}_-$ being defined by Eqs. (II.3.16)], subject to the initial condition

$$K^{(\Omega)}(z,z^*,t_0|z_0,z_0{}^*,t_0)=\delta^{(2)}(z-z_0).\quad(5.20)$$

It was shown in Ref. 23 that $K^{(\Omega)}$ may be expressed in the form

$$\begin{aligned}K^{(\Omega)}(z,z^*,t|z_0,z_0{}^*,t_0)&=\pi\,\mathrm{Tr}\{\Delta^{(\Omega)}(z_0-\hat{a},z_0{}^*-\hat{a}^\dagger)\hat{U}^\dagger(t,t_0)\\&\quad\times\Delta^{(\tilde{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger)\hat{U}(t,t_0)\}.\end{aligned}\quad(5.21)$$

We now introduce the mapping Δ operator $\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger;t)$ in the Heisenberg picture. It is related to the mapping Δ operator in the Schrödinger picture by the unitary transformation:

$$\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger;t)=\hat{U}^\dagger(t,t_0)\times\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger)\hat{U}(t,t_0).\quad(5.22)$$

On combining (5.21) and (5.22), we find that

$$K^{(\Omega)}(z,z^*,t|z_0,z_0{}^*,t_0)=\pi\,\mathrm{Tr}\{\Delta^{(\Omega)}(z_0-\hat{a},z_0{}^*-\hat{a}^\dagger)\times\Delta^{(\tilde{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger;t)\}.\quad(5.23)$$

This relation has a simple meaning. For in view of Theorem III [Eq. (I.3.25)], it follows that

$$K^{(\Omega)}(z_1,z_1{}^*,t|z,z^*,t_0)=\tilde{\Theta}\{\Delta^{(\tilde{\Omega})}(z_1-\hat{a},z_1{}^*-\hat{a}^\dagger;t)\},\quad(5.24)$$

or, on inverting,

$$\Delta^{(\tilde{\Omega})}(z_1-\hat{a},z_1{}^*-\hat{a}^\dagger;t)=\tilde{\Omega}\{K^{(\Omega)}(z_1,z_1{}^*,t|z,z^*,t_0)\}.\quad(5.25)$$

Hence $K^{(\Omega)}$ is the $\tilde{\Omega}$ equivalent of the Δ operator for $\tilde{\Omega}$ mapping in the Heisenberg picture. It is shown in Appendix C that the Green's function $K^{(\Omega)}$ satisfies a

[22] This relation should be distinguished from one introduced not long ago by M. Lax [Ref. 7, Eq. (5.3)]. Lax defines a multitime phase-space distribution function P_n by the formula which in his notation is $P_n=\langle\delta(\beta_1{}^*-b^\dagger(t_1))\cdots\delta(\beta_n{}^*-b^\dagger(t_n))\delta(\beta_n-b(t_n))\times\cdots\delta(\beta_1-b(t_1))\rangle$. This definition is seen to involve δ functions of arguments that are not real c-numbers; their use is not always without ambiguities.

[23] G. S. Agarwal, Phys. Rev. **177**, 400 (1969). The results in this reference were established under the assumption that $\Omega(\alpha,\beta)$ was a symmetric function of α and β and that it was of the form (2.12). These restrictions are easily relaxed (cf. Agarwal, Ref. 25).

relation of the Chapman-Kolmogorov type, viz.,

$$K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)=\int K^{(\Omega)}(z,z^*,t|z_1,z_1^*,t_1)\times K^{(\Omega)}(z_1,z_1^*,t_1|z_0,z_0^*,t_0)d^2z_1, \quad (5.26)$$

for all values of t_1, such that $t\geqslant t_1\geqslant t_0$.

We will now write the operator $G[\hat{a}(t),\hat{a}^\dagger(t)]$ in terms of a complete set of operators at some earlier instant of time, say t_1. We have, according to Theorem II [Eq. (I.3.13)],

$$G[\hat{a}(t),\hat{a}^\dagger(t)]=\int F_G{}^{(\Omega)}(z,z^*)\times\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger;t)d^2z, \quad (5.27)$$

where $F_G{}^{(\Omega)}(z,z^*)$ is the Ω equivalent of $G[\hat{a},\hat{a}^\dagger]$. Now, from (5.25) and Theorem II [Eq. (I.3.13)], we have

$$\Delta^{(\Omega)}(z_1-\hat{a},z_1^*-\hat{a}^\dagger;t)=\int K^{(\tilde{\Omega})}(z_1,z_1^*,t|z,z^*,t_1)\times\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger;t_1)d^2z. \quad (5.28)$$

Hence, on substituting from (5.28) in (5.27), we obtain the required expression for $G[\hat{a}(t),\hat{a}^\dagger(t)]$:

$$G[\hat{a}(t),\hat{a}^\dagger(t)]=\int\int F_G{}^{(\Omega)}(z,z^*)K^{(\tilde{\Omega})}(z,z^*,t|z_1,z_1^*,t_1)\times\Delta^{(\Omega)}(z_1-\hat{a},z_1^*-\hat{a}^\dagger;t_1)d^2zd^2z_1, \quad (5.29)$$

where t_1 ($t_0\leqslant t_1\leqslant t$) is an arbitrary instant of time.[24]

We will now show that by the repeated use of (5.29), specialized to the case of the normal rule of mapping, we may obtain an explicit expression for the normally ordered time-ordered correlation function $\Gamma_T{}^{(N)}$ defined by (5.7). We first make use of Eqs. (II.2.8) and (II.2.9) to express $\Gamma_T{}^{(N)}$ in the form

$$\Gamma_T{}^{(N)}=\int d^2z_0\Phi^{(A)}(z_0,z_0^*,t_0)\times F_G{}^{(N)}(z_0,z_0^*;t_0,t_1,\ldots,t_n), \quad (5.30)$$

[24] Result (5.29) leads to the following two interesting formulas:

$$\langle\hat{G}_2(t_2)\rangle=\int\int d^2zd^2z_1F_2{}^{(\Omega)}(z,z^*)K^{(\tilde{\Omega})}(z,z^*,t|z_1,z_1^*,t_1)\times\langle\Delta^{(\Omega)}(z_1-\hat{a},z_1^*-\hat{a}^\dagger;t_1)\rangle\quad(t\geqslant t_1),$$

and

$$\langle\hat{G}_2(t_2)\hat{G}_1(t_1)\rangle=\int\int d^2zd^2z_1F_2{}^{(\Omega)}(z,z^*)K^{(\tilde{\Omega})}(z,z^*,t_2|z_1,z_1^*,t_1)\times\langle\Delta^{(\Omega)}(z_1-\hat{a},z_1^*-\hat{a}^\dagger;t_1)\hat{G}_1(t_1)\rangle\quad(t_2\geqslant t_1).$$

These formulas imply that if the average of an operator at time t is expressed in terms of the averages of our mapping Δ operators at an earlier time t_1 (which, of course, form a complete set), then the average of the product of two operators is simply obtained by replacing $\langle\hat{\Delta}^{(\Omega)}\rangle$ by $\langle\hat{\Delta}^{(\Omega)}\hat{G}_1\rangle$. This statement is essentially the content of the important "quantum regression theorem" formulated by Lax (Ref. 7).

where $F_G{}^{(N)}=\Theta^{(N)}\{\hat{G}\}$ is the Ω equivalent, for the normal rule of association, of the operator

$$\hat{G}=[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}[\hat{a}(t_n)]^{j_n}\cdots[\hat{a}(t_1)]^{j_1}, \quad (5.31)$$

and

$$\Phi^{(A)}(z_0,z_0^*,t_0)=(1/\pi)\Theta^{(A)}\{\hat{\rho}\}.$$

Consider now the product of the two operators in Eq. (5.31) with the latest time argument t_n, i.e., the product $[\hat{a}^\dagger(t_n)]^{i_n}[\hat{a}(t_n)]^{j_n}$. Since this product is normally ordered, its equivalent for the normal rule of mapping is clearly $(z^*)^{i_n}(z)^{j_n}$. Hence (5.29) gives

$$[\hat{a}^\dagger(t_n)]^{i_n}[\hat{a}(t_n)]^{j_n}=\int\int d^2z_nd^2z'(z_n^*)^{i_n}(z_n)^{j_n}\times K^{(A)}(z_n,z_n^*,t_n|z',z'^*,t_{n-1})\times\Delta^{(N)}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-1}). \quad (5.32)$$

It follows that the operator $\hat{G}$ may be expressed in the form

$$\hat{G}=\int\int d^2z_nd^2z'(z_n^*)^{i_n}(z_n)^{j_n}\times K^{(A)}(z_n,z_n^*,t_n|z',z'^*,t_{n-1})\times[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_{n-1})]^{i_{n-1}}\times\Delta^{(N)}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-1})\times[\hat{a}(t_{n-1})]^{j_{n-1}}\cdots[\hat{a}(t_1)]^{j_1}. \quad (5.33)$$

Next we make use of the fact that the c-number equivalent, for the normal rule of mapping, of the operator

$$[\hat{a}^\dagger(t_{n-1})]^{i_{n-1}}\Delta^{(N)}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-1})[\hat{a}(t_{n-1})]^{j_{n-1}}$$

is the function $(z^*)^{i_{n-1}}(z)^{j_{n-1}}\delta^{(2)}(z'-z)$, so that in view of (5.29) we have

$$[\hat{a}^\dagger(t_{n-1})]^{i_{n-1}}\Delta^{(N)}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-1})[\hat{a}(t_{n-1})]^{j_{n-1}}=\int\int d^2z_{n-1}d^2z''(z_{n-1}^*)^{i_{n-1}}(z_{n-1})^{j_{n-1}}\delta^{(2)}(z'-z_{n-1})\times K^{(A)}(z_{n-1},z_{n-1}^*,t_{n-1}|z'',z''^*,t_{n-2})\times\Delta^{(N)}(z''-\hat{a},z''^*-\hat{a}^\dagger;t_{n-2}). \quad (5.34)$$

On substituting (5.34) in (5.33) and on integrating, we obtain the formula

$$\hat{G}=\int\int\int d^2z_nd^2z_{n-1}d^2z'(z_n^*)^{i_n}(z_n)^{j_n}(z_{n-1}^*)^{i_{n-1}}(z_{n-1})^{j_{n-1}}\times K^{(A)}(z_n,z_n^*,t_n|z_{n-1},z_{n-1}^*,t_{n-1})\times K^{(A)}(z_{n-1},z_{n-1}^*,t_{n-1}|z',z'^*,t_{n-2})\times[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_{n-2})]^{i_{n-2}}\times\Delta^{(N)}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-2})\times[\hat{a}(t_{n-2})]^{j_{n-2}}\cdots[\hat{a}(t_1)]^{j_1}. \quad (5.35)$$

On repeating this procedure again and again we finally arrive at the following expression for $\hat{G}$:

$$\hat{G}=\int\cdots\int d^2\{z_n\}d^2z_0$$

$$\times\prod_{\lambda=1}^{n}\{(z_\lambda^*)^{i_\lambda}(z_\lambda)^{j_\lambda}K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})\}$$

$$\times\Delta^{(N)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger;t_0).\quad(5.36)$$

From (5.36) and Theorem II [Eq. (I.3.13)], it follows at once that the c-number equivalent of the operator $\hat{G}$, for the normal rule of mapping, is given by

$$F_G^{(N)}=\int\cdots\int\prod_{\lambda=1}^{n}\{(z_\lambda^*)^{i_\lambda}(z_\lambda)^{j_\lambda}$$

$$\times K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})\}d^2\{z_\lambda\}.\quad(5.37)$$

Finally on substituting from (5.37) into (5.30), we obtain the following expression for the normally ordered time-ordered correlation function $\Gamma_T^{(N)}$ in terms of the phase-space distribution function $\Phi^{(A)}$ and the associated Green's function $K^{(A)}$:

$$\Gamma_T^{(N)}=\int\cdots\int\Phi^{(A)}(z_0,z_0^*,t_0)\prod_{\lambda=1}^{n}\{(z_\lambda^*)^{i_\lambda}(z_\lambda)^{j_\lambda}$$

$$\times K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})\}d^2\{z_\lambda\}d^2z_0.\quad(5.38)$$

In a similar way, we may derive an expression for the *antinormally ordered time-ordered correlation function* $\Gamma_T^{(A)}$ defined by Eq. (5.9), and we find that

$$\Gamma_T^{(A)}=\int\cdots\int\Phi^{(N)}(z_0,z_0^*,t_0)\prod_{\lambda=1}^{n}\{(z_\lambda^*)^{i_\lambda}(z_\lambda)^{j_\lambda}$$

$$\times K^{(N)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})\}d^2\{z_\lambda\}d^2z_0.\quad(5.39)$$

We now consider the multitime phase-space distribution function $\Phi_T^{(A)}(z_1,z_1^*,t_1;\ldots;z_n,z_n^*,t_n)$ defined by (5.14). We substitute (5.17) in (5.15) and expand each of the operators $\exp[\alpha_i\hat{a}^\dagger(t_i)]$ and $\exp[-\alpha_i^*\hat{a}(t_i)]$ in a power series. Then on using (5.38), we find that

$$\Phi_T^{(A)}=\int\Phi^{(A)}(z_0,z_0^*,t_0)$$

$$\times\prod_{\lambda=1}^{n}K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})d^2z_0.\quad(5.40)$$

The integration over z_0 can be carried out by noting that

$$\int\Phi^{(A)}(z_0,z_0^*,t_0)K^{(A)}(z,z^*,t|z_0,z_0^*,t_0)d^2z_0$$

$$=\Phi^{(A)}(z,z^*,t).\quad(5.41)$$

Hence *the multitime phase-space distribution function* $\Phi_T^{(A)}$ *can be calculated in terms of the Green's function* $K^{(A)}$ *and the single-time distribution function* $\Phi^{(A)}$ *by means of the formula*

$$\Phi_T^{(A)}(z_1,z_1^*,t_1;\ldots;z_n,z_n^*,t_n)=\Phi^{(A)}(z_1,z_1^*,t_1)$$

$$\times\prod_{\lambda=2}^{n}K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1}).\quad(5.42)$$

This remarkable result was first obtained by Lax[7] in quite a different manner on the basis of his "quantum regression theorem."

It is also possible to give an expression for $\Phi_T^{(A)}$ in terms of $\Phi^{(\Omega)}$ and $K^{(\Omega)}$, for any rule of mapping Ω, characterized by a filter function of the form (2.12). The result and the proof[25] are given in Appendix D.

VI. SUMMARY AND COMPARISON OF QUANTUM EQUATIONS IN GENERALIZED PHASE SPACE WITH THEIR CONVENTIONAL OPERATOR FORM

In view of the considerable length of the three papers of this series and because of the great generality of our analysis, it might be helpful to summarize our main results. This will now be done.

Basic in the present theory is the concept of the mapping of a function $G(\hat{a},\hat{a}^\dagger)$ of the noncommuting boson operators $\hat{a}$ and $\hat{a}^\dagger$ onto functions of c-numbers $F(z,z^*)$. In Sec. II of I we introduced a class of linear mappings, each characterized by a mapping operator Ω [$\hat{G}=\Omega\{F\}$]. We also introduced the inverse mapping operator Θ [$F=\Theta\{\hat{G}\}$]. We have shown that such mappings are intimately related to the problem of ordering of functions of the noncommuting boson operators according to some prescribed rule. In fact, the mapping and the ordering problem were found to be essentially equivalent to each other. In Sec. III of I we showed that each mapping Ω (satisfying some obvious regularity conditions) is characterized by an entire analytic function $\Omega(\alpha,\beta)$ of two complex variables α and β. We also derived in Sec. III of I closed expressions for the c-number function corresponding to a given operator and for the operator corresponding to a given c-number function, for any prescribed mapping of this class. The solution to the mapping problem is expressed with the help of the operator $\hat{\Delta}^{(\Omega)}$, which we called the mapping Δ operator and which acts as a transformation kernel. This mapping Δ operator is the operator onto which the two-dimensional Dirac δ function is mapped by Ω.

In Paper II we showed how this new calculus may be used to calculate systematically quantum-mechanical

[25] Alternative proofs of Eqs. (5.38) and (D17) based on the equation of motion for the Ω equivalent of the generating functional for normally ordered time-ordered correlation functions were given by G. S. Agarwal, Ph.D. thesis, University of Rochester, 1969 (unpublished).

TABLE I. The main quantum-mechanical formulas in phase-space form and in conventional form.

Conventional operator theory	Present phase-space theory
Arbitrary operator $\hat{G}$.	Phase-space equivalent of $\hat{G}$: $F_G^{(\Omega)}(z,z^*)=\Theta\{G(\hat{a},\hat{a}^\dagger)\}=\pi\,\mathrm{Tr}\{\hat{G}\Delta^{(\bar{\Omega})}(z-\hat{a},\,z^*-\hat{a}^\dagger)\}$.
State of system characterized by density operator $\hat{\rho}$.	State of system characterized by generalized distribution function: $\Phi^{(\Omega)}(z,z^*)=(1/\pi)\Theta\{\rho(\hat{a},\hat{a}^\dagger)\}$.
Expectation value of $\hat{G}$: $\langle\hat{G}\rangle=\mathrm{Tr}(\hat{\rho}\hat{G})$.	Expectation value of $\hat{G}$ expressed as phase-space average: $\langle\hat{G}\rangle=\int\Phi^{(\bar{\Omega})}(z,z^*)F_G^{(\Omega)}(z,z^*)d^2z$.
Schrödinger's equation of motion for the unitary time-evolution operator $\hat{U}(t,t_0)$: $i\hbar\partial\hat{U}/\partial t=\hat{H}\hat{U}$.	Phase-space equation of motion for the Ω equivalent $F_U^{(\Omega)}$ of $\hat{U}(t,t_0)$: $i\hbar\partial F_U^{(\Omega)}/\partial t=\mathfrak{L}_+F_U^{(\Omega)}$.
Schrödinger's equation of motion for density operator $\hat{\rho}$: $i\hbar\partial\hat{\rho}/\partial t=[\hat{H},\hat{\rho}]$.	Phase-space equation of motion for the distribution function $\Phi^{(\Omega)}$: $i\hbar\partial\Phi^{(\Omega)}/\partial t=(\mathfrak{L}_+-\mathfrak{L}_-)\Phi^{(\Omega)}$.
Equation of motion for Heisenberg operator $\hat{G}$: $i\hbar d\hat{G}/dt=-[\hat{H},\hat{G}]+i\hbar\partial\hat{G}/\partial t$.	Phase-space equation of motion for Ω equivalent $F_G^{(\Omega)}$ of $\hat{G}$: $i\hbar dF_G^{(\Omega)}/dt=-(\mathfrak{L}_+-\mathfrak{L}_-)F_G^{(\Omega)}+i\hbar\partial F_G^{(\Omega)}/\partial t$.
Bloch's equation for unnormalized density operator $\hat{\rho}$ of system in thermal equilibrium: $\partial\hat{\rho}/\partial\beta=-\hat{H}\hat{\rho}$.	Equation for Ω equivalent $F_\rho^{(\Omega)}$ of $\hat{\rho}$: $\partial F_\rho^{(\Omega)}/\partial\beta=-\mathfrak{L}_+F_\rho^{(\Omega)}$.
Time evolution of density operator $\hat{\rho}$: $\hat{\rho}(t)=\hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)$.	Time evolution of phase-space distribution function $\Phi^{(\Omega)}$: $\Phi^{(\Omega)}(z,z^*,t)=\int K^{(\Omega)}(z,z^*,t\,\vert\,z_0,z_0^*,t_0)\Phi^{(\Omega)}(z_0,z_0^*,t_0)d^2z_0$.
Time evolution of a Heisenberg operator $\hat{G}$: $\hat{G}(t)=\hat{U}^\dagger(t)\hat{G}(0)\hat{U}(t)$.	Time evolution of Ω equivalent $F_G^{(\Omega)}$ of $\hat{G}$: $F_G^{(\Omega)}(z,z^*;t)=\int K^{(\bar{\Omega})}(z,z^*,t\,\vert\,z_0,z_0^*,t_0)F_G^{(\Omega)}(z_0,z_0^*;t_0)d^2z_0$.
Basic (group) property of time evolution operator $\hat{U}$: $\hat{U}(t,t_0)=\hat{U}(t,t_1)\hat{U}(t_1,t_0)\quad(t\geqslant t_1\geqslant t_0)$.	Basic property of time-evolution kernel $K^{(\Omega)}$: $K^{(\Omega)}(z,z^*,t\,\vert\,z_0,z_0^*,t_0)=\int K^{(\Omega)}(z,z^*,t\,\vert\,z_1,z_1^*,t_1)K^{(\Omega)}(z_1,z_1^*,t_1\,\vert\,z_0,z_0^*,t_0)d^2z_1\quad(t\geqslant t_1\geqslant t_0)$.
Product of M operators: $\hat{G}_1\hat{G}_2\cdots\hat{G}_M$.	Phase-space equivalent of product of M operators: $F_{12\cdots M}^{(\Omega)}=\Theta\{\hat{G}_1\hat{G}_2\cdots\hat{G}_M\}=\exp\{\sum_j\sum_{i<j}\Lambda_{ij}\}\mathfrak{U}_{12\cdots M}^{(\Omega)}\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*)\vert_{z_m=z;\,z_m^*=z^*}$.
Time-ordered product of M operators: $T\{\hat{G}_1(t_1)\hat{G}_2(t_2)\cdots\hat{G}_M(t_M)\}$.	Phase-space equivalent of time-ordered product of M operators: $F_{12\cdots M}^{(\Omega)}=\Theta\{T\{\hat{G}_1(t_1)\cdots\hat{G}_M(t_M)\}\}=\exp\{\sum_j\sum_{i<j}\Lambda_{ij}\epsilon(t_i-t_j)\}\mathfrak{U}_{12\cdots M}^{(\Omega)}\times\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*;t_m)\vert_{z_m=z;\,z_m^*=z^*}$.
Wick's theorem for ordinary product of two operators[27] that are linear in $\hat{a}$ and $\hat{a}^\dagger$: $\hat{G}_1\hat{G}_2=:\hat{G}_1\hat{G}_2:+\hat{G}_1^{\cdot}\hat{G}_2^{\cdot}$.	New identity for the ordinary product of two operators[27] that are linear in $\hat{a}$ and $\hat{a}^\dagger$: $\hat{G}_1\hat{G}_2=S^{(\Omega)}\{F_1^{(\Omega)}F_2^{(\Omega)}\}+[\hat{G}_1^{\cdot}\hat{G}_2^{\cdot}]_P^{(\Omega)}$.

TABLE I. (*Continued*)

Conventional operator theory	Present phase-space theory
Wick's theorem for time-ordered product of two operators[27] that are linear in $\hat{a}$ and $\hat{a}^\dagger$: $$T\{\hat{G}_1(t_1)\hat{G}_2(t_2)\} = :\hat{G}_1(t_1)\hat{G}_2(t_2): + \hat{G}_1^\cdot(t_1)\hat{G}_2^\cdot(t_2).$$	New identity for time-ordered product of two operators[27] that are linear in $\hat{a}$ and $\hat{a}^\dagger$: $$T\{\hat{G}_1(t_1)\hat{G}_2(t_2)\} = S^{(\Omega)}\{F_1^{(\Omega)}(t_1)F_2^{(\Omega)}(t_2)\} + [\hat{G}_1^\cdot(t_1)\hat{G}_2^\cdot(t_2)]^{(\Omega)}.$$ In the special case when Ω represents the normal rule of mapping this identity is equivalent to Wick's theorem. A more general identity for time ordering of the product of n operators that are not necessarily linear in $\hat{a}$ and $\hat{a}^\dagger$ is given by Eq. (2.10).
Normally ordered time-ordered correlation function: $$\Gamma_T^{(N)} = \mathrm{Tr}\{\hat{\rho}[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}[\hat{a}(t_n)]^{j_n}\cdots[\hat{a}(t_1)]^{j_1}\}.$$	Normally ordered time-ordered correlation function expressed as phase-space average: $$\Gamma_T^{(N)} = \int\cdots\int \Phi^{(A)}(z_0,z_0^*,t_0) \times\prod_{\lambda=1}^{n}\{(z_\lambda^*)^{i_\lambda}(z_\lambda)^{j_\lambda}K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda\|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})\}d^2\{z_\lambda\}d^2z_0.$$

expectation values by c-number techniques. We found in Sec. II of II that the expectation values may be expressed in the same mathematical form as the expectation values in classical statistical mechanics, i.e., as weighted averages of c-number functions in a (generalized) phase space. The phase-space distribution function associated with a given (generally mixed) state of a quantum-mechanical system, one for each choice of mapping Ω, is proportional to the c-number function onto which the density operator is mapped. The phase-space equation of motion of the distribution function and of the c-number equivalent of the time-evolution operator and of a Heisenberg operator were derived in Sec. IV of II and the phase-space form of the Bloch equation for the unnormalized density operator $\hat{\rho}$ of a system in thermodynamic equilibrium was given in Sec. VI of II. All these phase-space equations were found to be of the form of generalized Liouville equations.

In the present paper, we applied this new technique to various time-ordering problems. In Sec. II of III we found an interesting generalization of Wick's theorem for boson systems, and we showed that the phase-space representation provides a clear insight into the meaning of this theorem. In Sec. V of III we discussed the mapping of functions of boson operators, taken at different times, onto c-number functions, and we provided methods for calculating the normally and the antinormally ordered time-ordered correlation functions.

The present series of papers dealt with closed systems only. The extension of our techniques to open systems (e.g., a system interacting with a reservoir) was given in another paper.[26]

Finally we display in Table I the phase-space form and the conventional form of the main quantum-mechanical formulas.

APPENDIX A: Ω EQUIVALENT OF PRODUCT OF M BOSON OPERATORS AND GENERALIZED WICK THEOREM FOR ORDINARY PRODUCTS

In this appendix we derive formula (2.5), which expresses the Ω equivalent of the product of M boson operators in terms of the Ω equivalents of each of the operators. This result, which is a generalization of Theorem V (Sec. III of II), will be shown to lead to an interesting generalization of the Wick theorem for ordinary products of boson operators.

Let $F_m^{(\Omega)}(z,z^*)$ be the Ω equivalents of the operators $G_m(\hat{a},\hat{a}^\dagger)$ $(m=1,2,\ldots,M)$. An expression for each of the Ω equivalents $F_m^{(\Omega)}$ in terms of the operator $G_m^{(\Omega)}$ is given by Theorem III [Eqs. (I.3.25) and (I.3.26)]. This expression may be written as

$$F_m^{(\Omega)}(z,z^*) = \int \bar{\Omega}(\alpha,\alpha^*)g_m(\alpha,\alpha^*)\exp(\alpha z^* - \alpha^* z)d^2\alpha, \quad \text{(A1)}$$

where

$$g_m(\alpha,\alpha^*) = (1/\pi)\,\mathrm{Tr}\{G_m(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha)\}, \quad \text{(A2)}$$

and $\hat{D}^\dagger(\alpha)$ is the Hermitian adjoint of the displacement operator (cf. Appendix B of I).

Next we make use of the operator analog of the Fourier theorem [Eq. (I.C1)] to express the product of the M operators $\hat{G}_m$ in the form

$$G_1(\hat{a},\hat{a}^\dagger)\cdots G_M(\hat{a},\hat{a}^\dagger) = \int\cdots\int \prod_{m=1}^{M} g_m(\alpha_m,\alpha_m^*)\hat{D}(\alpha_m)d^2\alpha_m. \quad \text{(A3)}$$

[26] G. S. Agarwal, Phys. Rev. **178**, 2025 (1969).

[27] For the sake of simplicity, we display the Wick identities and our new identities in forms that involve the product of two operators only. The general forms of our new identities involving the product of any number of operators are given by Eqs. (2.21) and (2.26).

We recall that the product of two displacement operators may be expressed in the form [Eq. (I.B10)]

$$\hat{D}(\alpha_m)\hat{D}(\alpha_n)=\hat{D}(\alpha_m+\alpha_n)\exp(\psi_{mn}), \tag{A4}$$

where

$$\psi_{mn}=\tfrac{1}{2}(\alpha_m\alpha_n^*-\alpha_m^*\alpha_n). \tag{A5}$$

Identity (A4) may be generalized to the product of an arbitrary number of displacement operators. The result, which may readily be proved by induction, is

$$\prod_{m=1}^{M}\hat{D}(\alpha_m)=\hat{D}(\sum_{m=1}^{M}\alpha_m)\exp[\sum_{m<n}\sum_{n}\psi_{m,n}]. \tag{A6}$$

Next we express the first term on the right-hand side of (A6) in the form

$$\hat{D}(\sum_{m=1}^{M}\alpha_m)=\bar{\Omega}(\sum_{m=1}^{M}\alpha_m,\sum_{m=1}^{M}\alpha_m^*)\times\Omega(\sum_{m=1}^{M}\alpha_m,\sum_{m=1}^{M}\alpha_m^*)\hat{D}(\sum_{m=1}^{M}\alpha_m), \tag{A7}$$

which obviously holds because the filter function $\bar{\Omega}(\alpha,\alpha^*)$ was defined as the reciprocal of $\Omega(\alpha,\alpha^*)$, and it was assumed that $\Omega(\alpha,\alpha^*)$ has no zeros [cf. (I.3.23)]. Now in view of relation (I.3.17), which may be written as

$$\Omega\{\exp(\alpha z^*-\alpha^*z)\}=\Omega(\alpha,\alpha^*)\hat{D}(\alpha),$$

it is evident that (A7) may be rewritten in the form

$$\hat{D}(\sum_{m=1}^{M}\alpha_m)=\bar{\Omega}(\sum_{m=1}^{M}\alpha_m,\sum_{m=1}^{M}\alpha_m^*)\times\Omega\{\exp[\sum_{m=1}^{M}(\alpha_m z^*-\alpha_m^*z)]\}. \tag{A8}$$

From (A6) and (A8) it follows that the product $\prod_{m=1}^{M}\hat{D}(\alpha_m)$ of the M displacement operators may be expressed as

$$\prod_{m=1}^{M}\hat{D}(\alpha_m)=\bar{\Omega}(\sum_{m=1}^{M}\alpha_m,\sum_{m=1}^{M}\alpha_m^*)\exp[\sum_{m<n}\sum_{n}\psi_{mn}]\times\Omega\{\exp[\sum_{m=1}^{M}(\alpha_m z^*-\alpha_m^*z)]\}, \tag{A9}$$

and hence (A3) may be written as

$$\prod_{m=1}^{M}G_m(\hat{a},\hat{a}^\dagger)=\Omega\{F_{12\cdots M}^{(\Omega)}(z,z^*)\}, \tag{A10}$$

where

$$F_{12\cdots M}^{(\Omega)}(z,z^*)=\int\cdots\int\bar{\Omega}(\sum_{i=1}^{M}\alpha_i,\sum_{i=1}^{M}\alpha_i^*)\times\prod_{i=1}^{M}g_i(\alpha_i,\alpha_i^*)\exp\{\sum_{i<j}\sum_{j}\psi_{ij}\}\times\exp[\sum_{i=1}^{M}(\alpha_i z^*-\alpha_i^*z)]d^2\alpha_1\cdots d^2\alpha_M. \tag{A11}$$

Equation (A10) shows that the c-number function $F_{12\cdots M}^{(\Omega)}(z,z^*)$ is the Ω equivalent of the product $G_1(\hat{a},\hat{a}^\dagger)\cdots G_M(\hat{a},\hat{a}^\dagger)$ of the M operators $\hat{G}_m$. We will next express $F_{12\cdots M}^{(\Omega)}$ in terms of the Ω equivalents of each of these operators. For this purpose we rewrite $F_{12\cdots M}^{(\Omega)}$ in the form

$$F_{12\cdots M}^{(\Omega)}(z,z^*)=f(z_1,z_1^*;\ldots;z_M,z_M^*)|_{z_m=z;\,z_m^*=z^*}, \tag{A12}$$

where

$$f(z_1,z_1^*;\ldots;z_M,z_M^*)=\int\cdots\int\prod_{m=1}^{M}[g_m(\alpha_m,\alpha_m^*)\times\bar{\Omega}(\alpha_m,\alpha_m^*)\exp(\alpha_m z_m^*-\alpha_m^*z_m)]\times\prod_{m=1}^{M}\Omega(\alpha_m,\alpha_m^*)\bar{\Omega}(\sum_{m=1}^{M}\alpha_m,\sum_{m=1}^{M}\alpha_m^*)\times\exp[\sum_{n}\sum_{m<n}\psi_{mn}]d^2\alpha_1\cdots d^2\alpha_M. \tag{A13}$$

Now by a strictly similar procedure as was used in deriving (II.B15), one may express (A13) in the form

$$f(z_1,z_1^*;\ldots;z_M,z_M^*)=\prod_{m=1}^{M}\Omega\left(\frac{\partial}{\partial z_m^*},-\frac{\partial}{\partial z_m}\right)\bar{\Omega}\left(\sum_{m=1}^{M}\frac{\partial}{\partial z_m^*},-\sum_{m=1}^{M}\frac{\partial}{\partial z_m}\right)\times\exp\left[\tfrac{1}{2}\sum_{n}\sum_{m<n}\left(\frac{\partial}{\partial z_m}\frac{\partial}{\partial z_n^*}-\frac{\partial}{\partial z_m^*}\frac{\partial}{\partial z_n}\right)\right]\times\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*), \tag{A14}$$

where relation (A1) was used. From (A12) and (A14) it follows that the Ω equivalent $F_{12\cdots M}^{(\Omega)}(z,z^*)$ of the product of the M operators $G_1(\hat{a},\hat{a}^\dagger)\cdots G_M(\hat{a},\hat{a}^\dagger)$ may be written as follows:

$$F_{12\cdots M}^{(\Omega)}(z,z^*)=\exp\{\sum_{j}\sum_{i<j}\Lambda_{ij}\}\times\mathfrak{U}_{12\cdots M}^{(\Omega)}\prod_{m=1}^{M}F_m^{(\Omega)}(z_m,z_m^*)\Big|_{z_m=z;\,z_m^*=z^*}. \tag{A15}$$

Here

$$\Lambda_{ij}=\frac{1}{2}\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j^*}-\frac{\partial}{\partial z_i^*}\frac{\partial}{\partial z_j}\right), \tag{A16}$$

$$\mathfrak{U}_{12\cdots M}^{(\Omega)}=\prod_{m=1}^{M}\Omega\left(\frac{\partial}{\partial z_m^*},-\frac{\partial}{\partial z_m}\right)\times\bar{\Omega}\left(\sum_{m=1}^{M}\frac{\partial}{\partial z_m^*},-\sum_{m=1}^{M}\frac{\partial}{\partial z_m}\right). \tag{A17}$$

Formula (A10) together with (A15) expresses the *product theorem* for an arbitrary number of boson operators.

The product $G_1(\hat{a},\hat{a}^\dagger)\cdots G_M(\hat{a},\hat{a}^\dagger)$ may, of course, be expressed in many different forms. In particular, it may be expressed as an Ω-ordered form, for a chosen rule of ordering. According to (A10) and Theorem I [Eqs. (I.2.22) and (I.2.23)] the Ω-ordered form of this product is given by

$$\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=[\Omega\text{-ordered form of }\prod_{m=1}^{M}G_m(\hat{a},\hat{a}^\dagger)]$$
$$\equiv S^{(\Omega)}\{F_{12\cdots M}{}^{(\Omega)}(z,z^*)\}\,, \quad \text{(A18)}$$

where $S^{(\Omega)}$ is the substitution operator for Ω mapping [Eq. (I.2.16)]. Formula (A18), together with expression (A15), for $F_{12\cdots M}{}^{(\Omega)}(z,z^*)$ may be regarded as a *generalization of the Wick theorem for ordinary products of boson operators*.[3,4] To see the connection between (A18) and the Wick theorem for ordinary products, consider the special case when each of the operators is a linear combination of the creation and annihilation operators, i.e., of the form

$$G_m(\hat{a},\hat{a}^\dagger)=A_m\hat{a}+B_m\hat{a}^\dagger\,, \quad \text{(A19)}$$

where A_m and B_m are c-numbers. We again consider the class of mappings whose filter functions are given by (2.12), viz.,

$$\Omega(\alpha,\beta)=\exp(\mu\alpha^2+\nu\beta^2+\lambda\alpha\beta)\,. \quad \text{(A20)}$$

The Ω equivalent of the operator $\hat{G}_m$ is then, in view of Eqs. (I.3.34) and (I.3.36), given by

$$F_m{}^{(\Omega)}(z,z^*)=A_mz+B_mz^*\,, \quad \text{(A21)}$$

and is independent of the particular choice of Ω. We substitute from (A21) in (A15) and expand the expressions on the right-hand side of (A15), which involve the differential operators Λ_{ij} and $\mathcal{U}_{12\cdots M}{}^{(\Omega)}$, in power series. We then obtain in a way strictly similar to the derivation of (2.21) the following expression for the product $\hat{G}_1\cdots\hat{G}_M$:

$$G_1(\hat{a},\hat{a}^\dagger)\cdots G_M(\hat{a},\hat{a}^\dagger)=\mathcal{H}_0{}^{(\Omega)}+\mathcal{H}_1{}^{(\Omega)}+\cdots\,, \quad \text{(A22)}$$

where

$$\mathcal{H}_0{}^{(\Omega)}=S^{(\Omega)}\{\prod_{m=1}^{M}F_m{}^{(\Omega)}(z,z^*)\}\,,$$

$$\mathcal{H}_1{}^{(\Omega)}=\sum_j\sum_{i<j}S^{(\Omega)}\{\prod_{m=1;\,m\neq i,j}^{M}F_m{}^{(\Omega)}(z,z^*)\times[\hat{G}_i\cdot\hat{G}_j\cdot]_P{}^{(\Omega)}\}\,, \quad \text{(A23)}$$

$$\mathcal{H}_2{}^{(\Omega)}=\frac{1}{2!}\sum_j\sum_{i<j}\sum_l\sum_{k<l}S^{(\Omega)}\{\prod_{m=1;\,m\neq i,j,k,l}^{M}F_m{}^{(\Omega)}(z,z^*)\times[\hat{G}_i\cdot\hat{G}_j\cdot]_P{}^{(\Omega)}[\hat{G}_k\cdot\hat{G}_l\cdot]_P{}^{(\Omega)}\}\,,$$

etc. Here $[\hat{G}_i\cdot\hat{G}_j\cdot]_P{}^{(\Omega)}$, which we call the *pairing of the operators* $\hat{G}_i$ *and* $\hat{G}_j$ *for* Ω *mapping*, is given by

$$[\hat{G}_1\cdot\hat{G}_2\cdot]_P{}^{(\Omega)}=-2\mu B_1B_2-2\nu A_1A_2+A_1B_2(\lambda+\tfrac{1}{2})+A_2B_1(\lambda-\tfrac{1}{2})\,. \quad \text{(A24)}$$

Equation (A22) expresses the ordinary product of a set of operators, which are linear in the creation and the annihilation operators, as the sum of all Ω-ordered products of the $\hat{G}$'s, with all possible pairings for Ω mapping, including the term with no pairing.

In the special case when Ω represents the normal rule of association ($\mu=\nu=0$, $\lambda=\frac{1}{2}$), Eq. (A24) reduces to

$$[\hat{G}_1\cdot\hat{G}_2\cdot]_P{}^{(N)}=A_1B_2\,. \quad \text{(A25)}$$

This pairing has a simple physical meaning as is seen at once by taking the vacuum expectation value of (A22), for the case $M=2$, with Ω representing the normal rule of association. One then obtains

$$[\hat{G}_1\cdot\hat{G}_2\cdot]_P{}^{(N)}=\langle 0|\hat{G}_1\hat{G}_2|0\rangle\,, \quad \text{(A26)}$$

i.e., $[\hat{G}_1\cdot\hat{G}_2\cdot]_P{}^{(N)}$ is the vacuum expectation value of the product $\hat{G}_1\hat{G}_2$. Equation (A22) with the choice $\Omega=\Omega^{(N)}$ together with (A26) is the usual Wick theorem for an ordinary product[3,4] of boson operators.

APPENDIX B: PROOF OF IDENTITY (4.27)

We will now derive identity (4.27), viz.,

$$\mathcal{L}_+(t_1)\cdots\mathcal{L}_+(t_{n-1})F_H{}^{(\Omega)}(z,z^*;t_n)=\exp\{\sum_j\sum_{i<j}\Lambda_{ij}\}\times\mathcal{U}_{12\cdots n}{}^{(\Omega)}\prod_{m=1}^{n}F_H{}^{(\Omega)}(z_m,z_m{}^*;t_m)\Big|_{z_m=z;\,z_m{}^*=z^*}\,. \quad \text{(B1)}$$

Consider first the expression $\mathcal{L}_+(t_i)F_H{}^{(\Omega)}(t_j)$ $(t_i<t_j)$. From (4.6) and (A1) it follows that

$$\mathcal{L}_+(t_i)F_H{}^{(\Omega)}(t_j)=\int\int d^2\alpha_id^2\alpha_j\bar{\Omega}(\alpha_i,\alpha_i{}^*)\bar{\Omega}(\alpha_j,\alpha_j{}^*)\times g_H(\alpha_i,\alpha_i{}^*;t_i)g_H(\alpha_j,\alpha_j{}^*;t_j)\times\exp(\Lambda_{ij})\mathcal{U}_{ij}{}^{(\Omega)}\exp(\alpha_iz_i{}^*-\alpha_i{}^*z_i)\times\exp(\alpha_jz_j{}^*-\alpha_j{}^*z_j)\,|_{z_i=z_j=z;\,z_i{}^*=z_j{}^*=z^*}\,, \quad \text{(B2)}$$

where

$$g_H(\alpha_i,\alpha_i{}^*;t_i)=(1/\pi)\,\mathrm{Tr}\{H(\hat{a},\hat{a}^\dagger;t_i)\hat{D}^\dagger(\alpha_i)\}\,. \quad \text{(B3)}$$

Since the differential operators Λ_{ij} and $\mathcal{U}_{ij}{}^{(\Omega)}$ in (B2) [defined by Eqs. (II.3.4) and (II.3.5)] act on the exponential functions, we may replace their arguments $\partial/\partial z_i$ by $-\alpha_i{}^*$, $\partial/\partial z_i{}^*$ by α_i, etc., and we obtain the formula

$$\exp(\Lambda_{ij})\mathcal{U}_{ij}{}^{(\Omega)}\exp(\alpha_iz_i{}^*-\alpha_i{}^*z_i)\exp(\alpha_jz_j{}^*-\alpha_j{}^*z_j)=\exp[\tfrac{1}{2}(-\alpha_i{}^*\alpha_j+\alpha_i\alpha_j{}^*)]\,\Omega(\alpha_i,\alpha_i{}^*)\Omega(\alpha_j,\alpha_j{}^*)\times\bar{\Omega}(\alpha_i+\alpha_j,\,\alpha_i{}^*+\alpha_j{}^*)\exp(\alpha_iz_i{}^*-\alpha_i{}^*z_i)\times\exp(\alpha_jz_j{}^*-\alpha_j{}^*z_j)\,. \quad \text{(B4)}$$

Using (B4), (B2) becomes

$$\mathcal{L}_+(t_i)F_H^{(\Omega)}(t_j)=\int\int d^2\alpha_i d^2\alpha_j \bar{\Omega}(\alpha_i+\alpha_j,\alpha_i^*+\alpha_j^*)$$
$$\times g_H(\alpha_i,\alpha_i^*;t_i)g_H(\alpha_j,\alpha_j^*;t_j)$$
$$\times\exp[(\alpha_i+\alpha_j)z^*-(\alpha_i^*+\alpha_j^*)z]\exp(\psi_{ij}),\quad \text{(B5)}$$

where ψ_{ij} is the quantity defined by (A5), viz.,

$$\psi_{ij}=\tfrac{1}{2}(\alpha_i\alpha_j^*-\alpha_i^*\alpha_j).\quad \text{(B6)}$$

We rewrite (B5) in the form

$$\mathcal{L}_+(t_i)F_H^{(\Omega)}(t_j)=\int d^2\alpha\, \bar{\Omega}(\alpha,\alpha^*)g_{ij}(\alpha,\alpha^*)$$
$$\times\exp(\alpha z^*-\alpha^*z),\quad \text{(B7)}$$

where

$$g_{ij}(\alpha,\alpha^*)=\int\int d^2\alpha_i d^2\alpha_j g_H(\alpha_i,\alpha_i^*;t_i)g_H(\alpha_j,\alpha_j^*;t_j)$$
$$\times\exp(\psi_{ij})\delta^{(2)}(\alpha-\alpha_i-\alpha_j).\quad \text{(B8)}$$

Next we apply to (B7) the operator $\mathcal{L}_+(t_k)$. An expression for the resulting quantity may be obtained at once with the help of (B5) by noting that the right-hand side of (B7) is of the same mathematical form as formula (A1), with the function g_{ij} of (B7) corresponding to g_m of (A1). Hence we see at once that

$$\mathcal{L}_+(t_k)\mathcal{L}_+(t_i)F_H^{(\Omega)}(t_j)$$
$$=\int\int d^2\alpha_k d^2\alpha_0 \bar{\Omega}(\alpha_k+\alpha_0,\alpha_k^*+\alpha_0^*)g_H(\alpha_k,\alpha_k^*;t_k)$$
$$\times g_{ij}(\alpha_0,\alpha_0^*)\exp[(\alpha_k+\alpha_0)z^*-(\alpha_k^*+\alpha_0^*)z]$$
$$\times\exp(\psi_{k0})\quad (t_k\leqslant t_i\leqslant t_j).\quad \text{(B9)}$$

On substituting from (B8) into (B9) we are led to the expression

$$\mathcal{L}_+(t_k)\mathcal{L}_+(t_i)\mathcal{L}_+(t_j)$$
$$=\int\int\int d^2\alpha_i d^2\alpha_j d^2\alpha_k \bar{\Omega}(\alpha_i+\alpha_j+\alpha_k,\ \alpha_i^*+\alpha_j^*+\alpha_k^*)$$
$$\times g_H(\alpha_k,\alpha_k^*;t_k)g_H(\alpha_i,\alpha_i^*;t_i)g_H(\alpha_j,\alpha_j^*;t_j)$$
$$\times\exp[(\alpha_i+\alpha_j+\alpha_k)z^*-(\alpha_i^*+\alpha_j^*+\alpha_k^*)z]$$
$$\times\exp(\psi_{ki}+\psi_{ij}+\psi_{kj}).\quad \text{(B10)}$$

By repeating the same procedure again and again, we finally arrive at the formula

$$\mathcal{L}_+(t_1)\cdots\mathcal{L}_+(t_{n-1})F_H^{(\Omega)}(t_n)$$
$$=\int\cdots\int\bar{\Omega}\Big(\sum_{i=1}^n\alpha_i,\sum_{i=1}^n\alpha_i^*\Big)$$
$$\times\prod_{i=1}^n g_H(\alpha_i,\alpha_i^*;t_i)\exp\Big[\sum_{i<j}\sum_j\psi_{ij}\Big]$$
$$\times\exp\Big[\sum_{i=1}^n(\alpha_i z^*-\alpha_i^* z)\Big]d^2\alpha_1\cdots d^2\alpha_n.\quad \text{(B11)}$$

The expression on the right-hand side of (B11) is of the same form as the expression on the right-hand side of (A11) and hence the same procedure can be applied to (B11) as was applied to (A11). The resulting formula, which corresponds to (A15), is the required identity (B1).

APPENDIX C: DERIVATION OF CHAPMAN-KOLMOGOROV RELATION FOR GREEN'S FUNCTION $K^{(\Omega)}$

In this appendix we will establish relation (5.26), viz.,

$$K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)=\int K^{(\Omega)}(z,z^*,t|z_1,z_1^*,t_1)$$
$$\times K^{(\Omega)}(z_1,z_1^*,t_1|z_0,z_0^*,t_0)d^2z_1,\quad \text{(C1)}$$

which holds for all values of t such that $t\geqslant t_1\geqslant t_0$.

The Green's function $K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)$ is given by Eq. (5.21), viz.,

$$K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)$$
$$=\pi\,\mathrm{Tr}\{\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger)\hat{U}^\dagger(t,t_0)$$
$$\times\Delta^{(\tilde{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger)\hat{U}(t,t_0)\}.\quad \text{(C2)}$$

Here $\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger)$ is the mapping Δ operator in the Schrödinger picture and is, therefore, independent of time. We now make use of relations (5.22) and (5.28) in (C2) and obtain the identity

$$K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)=\pi\int K^{(\Omega)}(z,z^*,t|z_1,z_1^*,t_1)$$
$$\times\mathrm{Tr}\{\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger)$$
$$\times\Delta^{(\tilde{\Omega})}(z_1-\hat{a},z_1^*-\hat{a}^\dagger;t_1)\}d^2z_1,\quad \text{(C3)}$$

where it is assumed that $t\geqslant t_1\geqslant t_0$. Let us use relation (5.28) once again to express $\Delta^{(\tilde{\Omega})}(z_1-\hat{a},z_1^*-\hat{a}^\dagger;t_1)$ in (C3) in terms of $\Delta^{(\tilde{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger;t_0)$. We then find that

$$K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)$$
$$=\pi\int\int K^{(\Omega)}(z,z^*,t|z_1,z_1^*,t_1)K^{(\Omega)}(z_1,z_1^*,t_1|z_2,z_2^*,t_0)$$
$$\times\mathrm{Tr}\{\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger)$$
$$\times\Delta^{(\Omega)}(z_2-\hat{a},z_2^*-\hat{a}^\dagger;t_0)\}d^2z_1d^2z_2.\quad \text{(C4)}$$

Now $\Delta^{(\Omega)}(z-\hat{a}, z^*-\hat{a}^\dagger; t_0)$ is the Δ operator for the $\tilde{\Omega}$ mapping in the Schrödinger picture. Since the Schrödinger and the Heisenberg picture were assumed to coincide at the time $t=t_0$, we have

$$\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger; t_0) = \Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger). \quad \text{(C5)}$$

Using (C5), it follows at once from the orthogonality relation (I.4.8) for the mapping Δ operators that

$$\mathrm{Tr}\{\Delta^{(\Omega)}(z_0-\hat{a}, z_0^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega})}(z_2-\hat{a}, z_2^*-\hat{a}^\dagger; t_0)\} = (1/\pi)\delta^{(2)}(z_2-z_0). \quad \text{(C6)}$$

On substituting from (C6) into (C4) and carrying out the trivial integration over z_2, we obtain the required identity (C1), i.e., a relation of the Chapman-Kolmogorov type[28] for the Green's function $K^{(\Omega)}$. This relation is essentially a reflection of the well-known group property of the unitary evolution operator $\hat{U}$, viz.,

$$\hat{U}(t,t_0) = \hat{U}(t,t_1)\hat{U}(t_1,t_0) \quad (t \geq t_1 \geq t_0). \quad \text{(C7)}$$

APPENDIX D: EXPRESSION FOR NORMALLY ORDERED TIME-ORDERED CORRELATION FUNCTION $\Gamma_T^{(N)}$ IN TERMS OF GREEN'S FUNCTION $K^{(\Omega)}$ AND PHASE-SPACE DISTRIBUTION FUNCTION $\Phi^{(\Omega)}$

In Sec. V we defined the normally ordered time-ordered correlation function $\Gamma_T^{(N)}$ [Eq. (5.7)], and we derived an expression for it in terms of the Green's function $K^{(A)}$ and the phase-space distribution function $\Phi^{(A)}$ [Eq. (5.39)]. In this appendix we will derive an expression[25] for $\Gamma_T^{(N)}$ in terms of the Green's function $K^{(\Omega)}$ and the phase-space distribution function $\Phi^{(\Omega)}$, where Ω represents any particular mapping characterized by a filter function of the form (2.12), viz.,

$$\Omega(\alpha,\beta) = \exp(\mu\alpha^2+\nu\beta^2+\lambda\alpha\beta). \quad \text{(D1)}$$

It will be useful to introduce the *generating function* $N(\xi_1,\xi_1^*,t_1; \ldots; \xi_n,\xi_n^*,t_n)$ defined as follows:

$$\begin{aligned} N(\xi_1,\xi_1^*,t_1; \ldots; \xi_n,\xi_n^*,t_n) &\equiv \langle \tilde{T}\{\exp[\sum_i \xi_i\hat{a}^\dagger(t_i)]\} T\{\exp[-\sum_i \xi_i^*\hat{a}(t_i)]\}\rangle \\ &= \langle \exp[\xi_1\hat{a}^\dagger(t_1)]\cdots\exp[\xi_n\hat{a}^\dagger(t_n)] \\ &\quad \times\exp[-\xi_n^*\hat{a}(t_n)]\cdots\exp[-\xi_1^*\hat{a}(t_1)]\rangle. \quad \text{(D2)} \end{aligned}$$

In terms of this generating function the normally ordered time-ordered correlation function $\Gamma_T^{(N)}$ defined by Eq. (5.7) may be expressed in the form

$$\Gamma_T^{(N)} = \frac{\partial^{i_1+i_2+\cdots+i_n+j_1+\cdots+j_n}}{(\partial\xi_1)^{i_1}\cdots(\partial\xi_n)^{i_n}(-\partial\xi_n^*)^{j_n}\cdots(-\partial\xi_1^*)^{j_1}} \times N(\xi_1,\xi_1^*,t_1; \ldots; \xi_n,\xi_n^*,t_n)\bigg|_{\{\xi_i\}=0,\{\xi_i^*\}=0}. \quad \text{(D3)}$$

We will now find an expression for the generating function N.

According to Eq. (II.2.8), the generating function may be expressed in the form

$$N = \int \Phi^{(\Omega)}(z_0,z_0^*; t_0) \times F_G^{(\tilde{\Omega})}(z_0,z_0^*; t_0,\{\xi_i\},\{\xi_i^*\},\{t_i\})d^2z_0, \quad \text{(D4)}$$

where $F_G^{(\tilde{\Omega})} = \tilde{\Theta}\{\hat{G}\}$ is the $\tilde{\Omega}$ equivalent of the operator

$$\hat{G} = \exp[\xi_1\hat{a}^\dagger(t_1)]\cdots\exp[\xi_n\hat{a}^\dagger(t_n)] \times\exp[-\xi_n^*\hat{a}(t_n)]\cdots\exp[-\xi_1^*\hat{a}(t_1)]. \quad \text{(D5)}$$

To find an expression for the $\tilde{\Omega}$ equivalent of $\hat{G}$, we consider first the operator

$$\hat{g}_n = \exp(\xi_n\hat{a}^\dagger)\exp(-\xi_n^*\hat{a}). \quad \text{(D6)}$$

According to Theorem III [Eq. (I.3.25)], the $\tilde{\Omega}$ equivalent $F_n^{(\tilde{\Omega})}$ of $\hat{g}_n$ is given by

$$\begin{aligned} F_n^{(\tilde{\Omega})} &= \pi\,\mathrm{Tr}\{\exp[\xi_n\hat{a}^\dagger]\exp[-\xi_n^*\hat{a}]\Delta^{(\Omega)}(z-\hat{a}, z^*-\hat{a}^\dagger)\} \\ &= \frac{1}{\pi}\int \Omega(\alpha,\alpha^*)\,\mathrm{Tr}[\hat{D}(\xi_n)\hat{D}^\dagger(\alpha)] \times\exp(\tfrac{1}{2}|\xi_n|^2)\exp(\alpha z^*-\alpha^* z)d^2\alpha, \quad \text{(D7)} \end{aligned}$$

where Eqs. (I.3.14) and (I.3.9) were used. $\hat{D}(\alpha)$ is, as before, the displacement operator for the coherent state $|\alpha\rangle$. On using the orthogonality property of the displacement operators expressed by Eq. (I.B12), Eq. (D7) reduces to

$$F_n^{(\tilde{\Omega})} = \Omega(\xi_n,\xi_n^*)\exp(\xi_n z^*-\xi_n^* z)\exp(\tfrac{1}{2}|\xi_n|^2). \quad \text{(D8)}$$

We now assume that the filter function $\Omega(\alpha,\beta)$ is of the form (D1). If we use Eqs. (5.29) and (D8) we find that

$$\begin{aligned} \hat{g}_n &\equiv \exp[\xi_n\hat{a}^\dagger(t_n)]\exp[-\xi_n^*\hat{a}(t_n)] \\ &= \int\int d^2z_n d^2z'\,\Omega(\xi_n,\xi_n^*)\exp(\xi_n z_n^*-\xi_n^* z_n) \\ &\quad \times\exp(\tfrac{1}{2}|\xi_n|^2)K^{(\Omega)}(z_n,z_n^*,t_n|z',z'^*,t_{n-1}) \\ &\quad \times\Delta^{(\tilde{\Omega})}(z'-\hat{a}, z'^*-\hat{a}^\dagger; t_{n-1}). \quad \text{(D9)} \end{aligned}$$

[28] The Chapman-Kolmogorov relation for the Green's function $K^{(\Omega)}$ in the special case when Ω represents the Weyl rule was first obtained by J. E. Moyal, Proc. Cambridge Phil. Soc. 45, 99 (1949).

Using (D9), it follows that the operator $\hat{G}$, defined by (D5), may be expressed in the form

$$\hat{G}=\int\int d^2z_n d^2z' \Omega(\xi_n,\xi_n^*)\exp(\xi_n z_n^*-\xi_n^* z_n)$$
$$\times\exp(\tfrac{1}{2}|\xi_n|^2)K^{(\Omega)}(z_n,z_n^*,t_n|z',z'^*,t_{n-1})$$
$$\times\exp[\xi_1\hat{a}^\dagger(t_1)]\cdots\exp[\xi_{n-1}\hat{a}^\dagger(t_{n-1})]$$
$$\times\Delta^{(\tilde{\Omega})}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-1})$$
$$\times\exp[-\xi_{n-1}^*\hat{a}(t_{n-1})]\cdots\exp[-\xi_1^*\hat{a}(t_1)]. \quad \text{(D10)}$$

To simplify (D10), we make use of an identity that involves the $\tilde{\Omega}$ equivalent of the operator

$$\hat{g}_{n-1}=\exp[\xi_{n-1}\hat{a}^\dagger(t_{n-1})]\Delta^{(\tilde{\Omega})}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-1})\times\exp[-\xi_{n-1}^*\hat{a}(t_{n-1})]. \quad \text{(D11)}$$

Denoting the $\tilde{\Omega}$ equivalent of this operator by $F_{n-1}^{(\tilde{\Omega})}$, we have according to Theorem III [Eq. (I.3.25)]

$$F_{n-1}^{(\tilde{\Omega})}=\pi\,\mathrm{Tr}\{\hat{g}_{n-1}\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger)\}. \quad \text{(D12)}$$

Now by substituting from (D11) into (D12), expressing each of the two mapping Δ operators in the integral form (I.3.14), making use of the orthogonality relation (I.B12), and recalling that $\Omega(\alpha,\beta)$ is given by (D1), we readily obtain the following expression for $F_{n-1}^{(\tilde{\Omega})}$:

$$F_{n-1}^{(\tilde{\Omega})}=\frac{1}{\pi^2}\int\exp[\mu\xi_{n-1}^2+\nu\xi_{n-1}^{*2}+(\lambda+\tfrac{1}{2})|\xi_{n-1}|^2+\xi_{n-1}z^*-\xi_{n-1}^*z+2\mu\alpha\xi_{n-1}$$
$$+2\nu\alpha^*\xi_{n-1}^*+(\lambda+\tfrac{1}{2})(\alpha\xi_{n-1}^*+\alpha^*\xi_{n-1})]\exp[-\alpha(z'^*-z^*)+\alpha^*(z'-z)]d^2\alpha$$
$$=\exp[\mu\xi_{n-1}^2+\nu\xi_{n-1}^{*2}+(\lambda+\tfrac{1}{2})|\xi_{n-1}|^2+\xi_{n-1}z^*-\xi_{n-1}^*z]$$
$$\times\exp\left[-2\mu\xi_{n-1}\frac{\partial}{\partial z'^*}+2\nu\xi_{n-1}^*\frac{\partial}{\partial z'}-(\lambda+\tfrac{1}{2})\xi_{n-1}^*\frac{\partial}{\partial z'^*}+(\lambda+\tfrac{1}{2})\xi_{n-1}\frac{\partial}{\partial z'}\right]\delta^{(2)}(z'-z). \quad \text{(D13)}$$

On representing the operator $\hat{g}_{n-1}$ defined by (D11) in terms of its $\tilde{\Omega}$ equivalent $F_{n-1}^{(\tilde{\Omega})}$ [Eq. (D13)] via the identity (5.29) and on making use of that representation in (D10), it follows that $\hat{G}$ may be expressed in the form

$$\hat{G}=\int\int\int d^2z_n d^2z_{n-1}d^2z'\prod_{\lambda=n-1}^{n}\exp[\mu\xi_\lambda^2+\nu\xi_\lambda^{*2}+(\lambda+\tfrac{1}{2})|\xi_\lambda|^2+\xi_\lambda z_\lambda^*-\xi_\lambda^*z_\lambda]$$
$$\times K^{(\Omega)}(z_n,z_n^*,t_n|z_{n-1}-2\nu\xi_{n-1}^*-(\lambda+\tfrac{1}{2})\xi_{n-1},\,z_{n-1}^*+2\mu\xi_{n-1}+(\lambda+\tfrac{1}{2})\xi_{n-1}^*,t_{n-1})$$
$$\times K^{(\Omega)}(z_{n-1},z_{n-1}^*,t_{n-1}|z',z'^*,t_{n-2})\exp[\xi_1\hat{a}^\dagger(t_1)]\cdots\exp[\xi_{n-2}\hat{a}^\dagger(t_{n-2})]$$
$$\times\Delta^{(\tilde{\Omega})}(z'-\hat{a},z'^*-\hat{a}^\dagger;t_{n-2})\exp[-\xi_{n-2}^*\hat{a}(t_{n-2})]\cdots\exp[-\xi_1^*\hat{a}(t_1)]. \quad \text{(D14)}$$

On repeating the kind of procedure which led from (D10) to (D14), we eventually arrive at the following expression for the operator $\hat{G}$:

$$\hat{G}=\int\cdots\int d^2\{z_\lambda\}d^2z_0\prod_{\lambda=1}^{n}\exp[\mu\xi_\lambda^2+\nu\xi_\lambda^{*2}+(\lambda+\tfrac{1}{2})|\xi_\lambda|^2+\xi_\lambda z_\lambda^*-\xi_\lambda^*z_\lambda]$$
$$\times\prod_{\lambda=2}^{n}K^{(\Omega)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1}-2\nu\xi_{\lambda-1}^*-(\lambda+\tfrac{1}{2})\xi_{\lambda-1},\,z_{\lambda-1}^*+2\mu\xi_{\lambda-1}+(\lambda+\tfrac{1}{2})\xi_{\lambda-1}^*,t_{\lambda-1})$$
$$\times K^{(\Omega)}(z_1,z_1^*,t_1|z_0,z_0^*,t_0)\Delta^{(\tilde{\Omega})}(z_0-\hat{a},z_0^*-\hat{a}^\dagger;t_0). \quad \text{(D15)}$$

From (D15) and Theorem II [Eq. (I.3.13)] it follows at once that the $\tilde{\Omega}$ equivalent of $\hat{G}$ is given by

$$F_G^{(\tilde{\Omega})}=\int\cdots\int d^2\{z_\lambda\}\prod_{\lambda=1}^{n}\exp[\mu\xi_\lambda^2+\nu\xi_\lambda^{*2}+(\lambda+\tfrac{1}{2})|\xi_\lambda|^2+\xi_\lambda z_\lambda^*-\xi_\lambda^*z_\lambda]$$
$$\times\prod_{\lambda=2}^{n}K^{(\tilde{\Omega})}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1}-2\nu\xi_{\lambda-1}^*-(\lambda+\tfrac{1}{2})\xi_{\lambda-1},\,z_{\lambda-1}^*+2\mu\xi_{\lambda-1}+(\lambda+\tfrac{1}{2})\xi_{\lambda-1}^*,t_{\lambda-1})K^{(\Omega)}(z_1,z_1^*,t_1|z_0,z_0^*,t_0). \quad \text{(D16)}$$

Finally, making use of (D16) we obtain the following expression[29] for the generating function $N(\xi_1,\xi_1^*,t_1;\ldots;$

[29] Since the analysis of this appendix was completed, the following papers appeared, containing some special cases of our formula (D17): F. Graham, F. Haake, H. Haken, and F. Weidlich, Z. Physik **213**, 21 (1968); R. Graham and F. Haake, Phys. Letters **26A**, 385 (1968).

$\xi_n,\xi_n^*,t_n)$ in terms of the Green's function $K^{(\Omega)}$ and the phase-space distribution function $\Phi^{(\Omega)}$:

$$N=\int\cdots\int d^2\{z_\lambda\}d^2z_0\prod_{\lambda=1}^{n}\exp[\mu\xi_\lambda^2+\nu\xi_\lambda^{*2}+(\lambda+\tfrac{1}{2})|\xi_\lambda|^2+\xi_\lambda z_\lambda^*-\xi_\lambda^* z_\lambda]$$
$$\times\prod_{\lambda=2}^{n}K^{(\Omega)}(z_\lambda,z_\lambda^*,t_{\lambda-1}|z_{\lambda-1}-2\nu\xi_{\lambda-1}^*-(\lambda+\tfrac{1}{2})\xi_{\lambda-1},\,z_{\lambda-1}^*+2\mu\xi_{\lambda-1}+(\lambda+\tfrac{1}{2})\xi_{\lambda-1}^*,t_{\lambda-1})$$
$$\times K^{(\Omega)}(z_1,z_1^*,t_1|z_0,z_0^*,t_0)\Phi^{(\Omega)}(z_0,z_0^*,t_0)\,. \quad \text{(D17)}$$

The normally ordered time-ordered correlation function $\Gamma_T^{(N)}$ may be obtained from (D3) and (D17). We stress that in (D17) Ω is *any* mapping characterized by a filter function of the form given by (D1). With the special choice corresponding to mapping according to the antinormal rule one has $\mu=\nu=0$, $\lambda=-\frac{1}{2}$ (cf. Table IV of I) and Eqs. (D3) and (D17) may then be readily shown to give formula (5.38) derived in the text.

PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Derivation of Equal-Time Commutators Involving the Symmetric Energy-Momentum Tensor and Applications*

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The use of covariance and the Jacobi identity in the study of equal-time commutators is investigated. Denoting by $T_{\mu\nu}$ the conserved and symmetric tensor density of Poincaré tranformations and by X any of the operators ϕ, $\partial_0\phi$, J_0, J_l, or J_{0l}, we use the most general form of the equal-time commutators $[iT_{0\mu}(x), X(y)]$ and $[iT_{00}(x), iT_{00}(y)]$ compatible with covariance, together with the Jacobi identities for $[[iT_{00}(x), iT_{00}(y)], X(z)]$, to derive relations between the equal-time commutators $[iT_{0m}(x), X(y)]$ and $[iT_{00}(x), Y(y)]$, where Y is any of the operators denoted by X or $\Box\phi$, $\partial^\mu\bar{\psi}\gamma_\mu$, $\partial^\mu J_\mu$, and $\partial^0 J_{0m}$. This information is first used in deriving equal-time commutators in canonical models. We then show that the assumption of $SU(2)\otimes SU(2)$ charge-current commutators together with $[A_0^\alpha(x),\bar{\psi}(y)]_{x_0=y_0}\propto\bar{\psi}(x)\tau^\alpha\gamma_5\delta(\mathbf{x}-\mathbf{y})$ (where A_μ^α denotes the axial-vector current and ψ denotes a spinor field) implies (as obtained earlier by the authors under different assumptions) $[A_k^\alpha(x),\bar{\psi}(y)_0]_{x_0=y_0}=\frac{1}{2}\bar{\psi}(x)\gamma_5\gamma_k\tau^\alpha\delta(\mathbf{x}-\mathbf{y})+i(y-x)_k[A_0^\alpha(x),f_m^\dagger(y)\gamma_0]_{x_0=y_0}$ [where f denotes $(i\gamma^\mu\partial_\mu-m)\psi$]. For the conserved vector current an analogous relation holds. The incompatibility of field-algebra current commutators with $\int d^3x[A_k^\alpha(x),\bar{\psi}(y)\gamma_0]_{x_0=y_0}\propto\bar{\psi}(y)\gamma_5\gamma_k$ is noted. Taking ψ to be the nucleon field, it is shown that a certain form of the nucleon current leads to the above unless the right-hand side vanishes. Imposing this requirement, one then obtains $g_{A_1}=g_\rho$, where $g_{A_1}a_\mu^\alpha(x)\gamma_5\gamma^\mu(\tau^\alpha/2)\psi(x)$ $[g_\rho v_\mu^\alpha(x)\gamma^\mu\times(\tau^\alpha/2)\psi(x)]$ denotes the contribution of A_1 (ρ) to f_m in terms of the renormalized field a_μ^α (v_μ^α). From this and the usual saturation of the Weinberg spectral-function sum rules by single-particle intermediate states, we obtain the universality relations $g_\rho=m_\rho^2/f_\rho$ and $g_{A_1}=(m_\rho/m_{A_1})^2m_{A_1}^2/f_{A_1}$, where f_{A_1} (f_ρ) is defined by $\rho_{A_1}(m^2)=f_{A_1}^2\delta(m^2-m_{A_1}^2)$ $[\rho_\rho(m^2)=f_\rho^2\delta(m^2-m_\rho^2)]$. For currents obeying the algebra-of-fields commutators, we obtain restrictions on Schwinger terms contained in equal-time commutators involving time derivatives of the currents. These relations show, for example, that in canonical realizations of current-field identities one needs derivative couplings of the spin-1 field.

I. INTRODUCTION

It is generally assumed[1–7] that in relativistic local field theories a conserved and symmetric local tensor operator $T_{\mu\nu}(x)$ exists with the property that the generators of Poincaré transformations may be written as

$$P_\mu=\int d^3x\, T_{0\mu}(x) \quad (1.1)$$

and

$$M_{\mu\nu}=\int d^3x[x_\mu T_{0\nu}(x)-x_\nu T_{0\mu}(x)]. \quad (1.2)$$

Denoting by ϕ, $\bar{\psi}$, J_μ, and $J_{\mu\nu}$ (defined as $J_{\mu\nu}\equiv\partial_\mu J_\nu-\partial_\nu J_\mu$) local operators with spins 0, $\frac{1}{2}$, 1, and 2, respec-

* Supported in part by the DAAD through a NATO grant and in part by the U. S. Atomic Energy Commission.
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