Calculus for Functions of Noncommuting Operators and General Phase-Space Methods in Quantum Mechanics. II. Quantum Mechanics in Phase Space*

G. S. Agarwal and E. Wolf
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

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In Paper I of this investigation a new calculus for functions of noncommuting operators was developed, based on the notion of mapping of operators onto c-number functions. With the help of this calculus, a general theory is formulated, in the present paper, of phase-space representation of quantum-mechanical systems. It is shown that there is a whole class of such representations, one associated with each type of mapping, the simplest one being the well-known representation due to Weyl. For each representation, the quantum-mechanical expectation value of an operator is found to be expressible in the form of a phase-space average of classical statistical mechanics. The phase-space distribution functions are, however, not true probabilities, in general. The phase-space forms of the main quantum-mechanical equations of motion are obtained and are found to have the form of a generalized Liouville equation. The phase-space form of the Bloch equation for the density operator of a quantum system in thermal equilibrium is also derived. The generalized characteristic functions of boson systems are defined and their main properties are studied. The equations of motion for the characteristic functions are also derived. As an illustration of the theory, a generalized stochastic description of a quantized electromagnetic field is obtained.

I. INTRODUCTION

In Paper I of this investigation[1] (hereafter referred to as I) we developed a new calculus for functions of noncommuting operators. This calculus is based on the notion of mapping functions of operators onto functions of c-numbers and vice versa. We studied in detail a class of mappings, each member of which is characterized by an entire analytic function of two complex variables. We have shown that the most commonly encountered rules of association between operators and c-numbers (the Weyl, the normal, the antinormal, the standard, and the antistandard rules) belong to this class and are, in fact, the simplest ones in a clearly defined sense. We have also shown that the problem of expressing an operator in an ordered form according to some prescribed ordering rule is equivalent to an appropriate mapping of the operator onto a c-number space.

In the present paper we obtain, on the basis of this calculus, a general theory of phase-space representations of boson systems. There is a whole class of such representations, one associated with each type of mapping. In Sec. II we show that the quantum-mechanical expectation values may be expressed in the same mathematical form as the averages of classical statistical mechanics. The distribution functions, however, are not true probabilities in general, but can, nevertheless, be used with great advantage as an aid in calculations. In Sec. III we discuss the mapping of the product of two operators. In Sec. IV we derive the phase-space form of the main quantum-mechanical equations of motion. All these equations are found to have the form of a generalized Liouville equation. Some special forms of these equations are discussed in Sec. V. In Sec. VI we derive the phase-space form of the Bloch equation for the density operator of a system in thermal equilibrium. In Sec. VII we define the generalized characteristic functions of a boson system and study their main properties. The equations of motion for the generalized characteristic functions are also obtained. In Sec. VIII we outline the generalization of the theory to systems with more than one degree of freedom. As an example of the theory we discuss in Sec. IX the stochastic description of a quantized electromagnetic field.

Our generalized phase-space description provides a new representation of boson systems, which closely resembles classical statistical mechanics and the theory of stochastic processes. Numerous results previously obtained by specialized techniques follow logically as special cases from our general formulation.

II. QUANTUM-MECHANICAL EXPECTATION VALUES AS GENERALIZED PHASE-SPACE AVERAGES

We will now make use of the calculus developed in the first paper of this series to show that it is possible to express quantum-mechanical expectation values in the same mathematical form as phase-space averages of classical statistical mechanics. We will see that there is an infinite number of ways of doing this, one for each rule of association Ω.

To begin with, we will express the trace of the product of two operators $G_1(\hat{a},\hat{a}^\dagger)$ and $G_2(\hat{a},\hat{a}^\dagger)$ in terms of the c-number equivalents of the two operators. Let Ω be any linear analytic mapping, whose filter function $\Omega(\alpha,\alpha^*)$ *has no zeros*,[2] and let $\tilde{\Omega}$ be the mapping antireciprocal to Ω, i.e., the mapping characterized by the filter function $\tilde{\Omega}(\alpha,\alpha^*)=[\Omega(-\alpha,-\alpha^*)]^{-1}$. Let $F_1^{(\Omega)}(z,z^*)$

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[1] G. S. Agarwal and E. Wolf, preceding paper, Phys. Rev. D 2, 2161 (1970).

[2] This assumption will be retained throughout this paper.

be the Ω equivalent of $\hat{G}_1$, and $F_2^{(\tilde{\Omega})}(z,z^*)$ the $\tilde{\Omega}$ equivalent of $\hat{G}_2$. Then,[3] according to Theorem III [Eq. (I. 3.25)],

$$F_1^{(\Omega)}(z_1,z_1^*)=\pi\ \mathrm{Tr}[\hat{G}_1\Delta^{(\tilde{\Omega})}(z_1-\hat{a},\ z_1^*-\hat{a}^\dagger)],\quad (2.1)$$

$$F_2^{(\tilde{\Omega})}(z_2,z_2^*)=\pi\ \mathrm{Tr}[\hat{G}_2\Delta^{(\Omega)}(z_2-\hat{a},\ z_2^*-\hat{a}^\dagger)],\quad (2.2)$$

where $\hat{\Delta}^{(\Omega)}$ and $\hat{\Delta}^{(\tilde{\Omega})}$ are the corresponding mapping Δ operators defined by Eqs. (I.3.14) and (I.3.26). The inverse relations are given by Theorem II [Eq. (I.3.13)]:

$$G_1(\hat{a},\hat{a}^\dagger)=\int F_1^{(\Omega)}(z_1,z_1^*)\Delta^{(\Omega)}(z_1-\hat{a},\ z_1^*-\hat{a}^\dagger)d^2z_1,\quad (2.3)$$

$$G_2(\hat{a},\hat{a}^\dagger)=\int F_2^{(\tilde{\Omega})}(z_2,z_2^*)\Delta^{(\tilde{\Omega})}(z_2-\hat{a},\ z_2^*-\hat{a}^\dagger)d^2z_2.\quad (2.4)$$

Let us now take the trace of the product of $\hat{G}_1$ and $\hat{G}_2$, expressed by Eqs. (2.3) and (2.4). If we interchange the order of the trace operation and the integrations and make use of the relation (I.4.8), viz.,

$$\mathrm{Tr}[\Delta^{(\Omega)}(z_1-\hat{a},\ z_1^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega})}(z_2-\hat{a},\ z_2^*-\hat{a}^\dagger)]=(1/\pi)\delta^{(2)}(z_1-z_2),\quad (2.5)$$

we obtain the following theorem.

Theorem IV. The trace of the product of two operators $G_1(\hat{a},\hat{a}^\dagger)$ and $G_2(\hat{a},\hat{a}^\dagger)$ is expressible in the form

$$\mathrm{Tr}(\hat{G}_1\hat{G}_2)=\frac{1}{\pi}\int F_1^{(\Omega)}(z,z^*)F_2^{(\tilde{\Omega})}(z,z^*)d^2z,\quad (2.6)$$

where the c-number equivalents $F_1^{(\Omega)}(z,z^*)$ and $F_2^{(\tilde{\Omega})}(z,z^*)$ are given by Eqs. (2.1) and (2.2), respectively.

Consider now a quantum-mechanical system in a pure or mixed state, characterized by a density operator $\hat{\rho}$, and let $G(\hat{a},\hat{a}^\dagger)$ be some dynamical variable of the system. If we set $\hat{G}_1=\hat{G}$ and $\hat{G}_2=\hat{\rho}$ in (2.6), we obtain the following expression for the expectation value of $\hat{G}$:

$$\mathrm{Tr}(\hat{\rho}\hat{G})=\frac{1}{\pi}\int F_\rho^{(\tilde{\Omega})}(z,z^*)F_G^{(\Omega)}(z,z^*)d^2z,\quad (2.7)$$

where, of course, $F_\rho^{(\tilde{\Omega})}$ is the $\tilde{\Omega}$ equivalent of $\hat{\rho}$ and $F_G^{(\Omega)}$ is the Ω equivalent of $\hat{G}$. We see that the choice of Ω in (2.7) is quite *arbitrary*. It will be convenient to set

$$\Phi^{(\tilde{\Omega})}(z,z^*)=(1/\pi)F_\rho^{(\tilde{\Omega})}(z,z^*).\quad (2.8)$$

Then (2.7) becomes

$$\mathrm{Tr}(\hat{\rho}\hat{G})=\int \Phi^{(\tilde{\Omega})}(z,z^*)F_G^{(\Omega)}(z,z^*)d^2z.\quad (2.9)$$

The integral on the right-hand side of (2.9) is of the same form as the phase-space average of classical statistical mechanics for the average, denoted by $\langle\ \rangle_{\mathrm{p.s.}}$, of $F_G^{(\Omega)}$ with respect to the phase-space distribution function $\Phi^{(\tilde{\Omega})}$. If we denote quantum-mechanical expectation values b angular brackets without a suffix, we may express (2.9) in the compact symbolic form

$$\langle\hat{G}\rangle=\langle F_G^{(\Omega)}\rangle_{\mathrm{p.s.}}.\quad (2.10)$$

In spite of the formal similarity just noted, the right-hand side of (2.9) cannot, in general, be identified with a true phase-space average. For the function $\Phi^{(\tilde{\Omega})}$ may not possess all the properties of a probability density; it is not necessarily non-negative,[4,5] and it may become singular. It cannot therefore, in general, represent a true statistical distribution function.

If we combine Eq. (2.8) and Eq. (I.3.25), we obtain a more explicit expression for $\Phi^{(\tilde{\Omega})}$:

$$\Phi^{(\tilde{\Omega})}(z_0,z_0^*)=\mathrm{Tr}[\hat{\rho}\Delta^{(\Omega)}(z_0-\hat{a},\ z_0^*-\hat{a}^\dagger)].\quad (2.8')$$

Further, if we make use of the relation (I.3.21), which expresses the mapping Δ operator in terms of the Dirac δ function, we obtain the interesting formula

$$\Phi^{(\tilde{\Omega})}(z_0,z_0^*)=\langle\Omega\{\delta^{(2)}(z_0-z)\}\rangle.\quad (2.8'')$$

We see that the distribution function for Ω mapping is the expectation value of the operator onto which the Dirac δ function is mapped by Ω mapping.[6]

This formula corresponds, in a sense, to the following expression of classical probability theory:

$$p(x)=\int p(x_0)\delta(x-x_0)dx_0=\langle\delta(x-x_0)\rangle.$$

Finally, we note that $\Phi^{(\tilde{\Omega})}$ is correctly normalized; for if in (2.9) we take for $\hat{G}$ the identity operator 1, then since $F_G^{(\Omega)}=1$ and $\mathrm{Tr}(\hat{\rho})=1$, we obtain

$$\int \Phi^{(\tilde{\Omega})}(z,z^*)d^2z=1.\quad (2.11)$$

A c-number function such as $\Phi^{(\tilde{\Omega})}$ which has some, but not all, of the attributes of a probability density and which may be used for the computation of expectation values by means of integrals of the form (2.9) may be

[3] Equations prefixed by I will refer to equations of Ref. 1.

[4] Cf. M. S. Bartlett and J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 545 (1949).

[5] An example of a phase-space distribution function, which is non-negative for all values of its argument is provided by the c-number equivalent of the density operator for the normal rule of association, i.e., the phase-space distribution function for antinormal mapping. This distribution function corresponds to the choice $\Omega(\alpha,\alpha^*)=\exp(\frac{1}{2}\alpha\alpha^*)$. In Appendix A, we show that there is a whole class of Ω equivalents, namely, those corresponding to the class of filter functions $\Omega(\alpha,\alpha^*)=\exp(\lambda\alpha\alpha^*)$, $\lambda\geqslant\frac{1}{2}$, for which the phase-space distribution functions are non-negative.

[6] In a recent interesting paper M. Lax [Phys. Rev. **172**, 350 (1968)] also introduced a class of generalized phase-space distribution functions. He defined them as the expectation values of the Dirac δ function (with operator arguments) when this function was expressed in a "chosen order."

said to be a *quasiprobability* or a *generalized distribution function*. In the past such functions have been frequently used in special cases as aids in calculations, the oldest one being the Wigner distribution function[7]; it is nothing else than our function $\Phi^{(\tilde{\Omega})}$ for the special case of the Weyl rule of association.

Since according to (2.9) the function $\Phi^{(\tilde{\Omega})}(z,z^*)$ is the weighting function in integrals which contain the Ω equivalents of the operators $\hat{G}$, we will refer to $\Phi^{(\tilde{\Omega})}$ as the (generalized) distribution function for Ω mapping (not for $\tilde{\Omega}$ mapping). Of course, since the choice of Ω in (2.9) is quite arbitrary, we may, in particular, write in place of (2.9)

$$\mathrm{Tr}(\hat{\rho}\hat{G})=\int\Phi^{(\Omega)}(z,z^*)F_G{}^{(\tilde{\Omega})}(z,z^*)d^2z. \tag{2.9$'$}$$

The expectation value of an operator $\hat{G}$ is now expressed in terms of the (generalized) distribution function for $\tilde{\Omega}$ mapping.

We have now generated a whole class of generalized distribution functions associated with a given state of a quantum-mechanical system, each such function being associated with a particular choice of mapping. It seems worthwhile to stress once again that in evaluating the expectation value of an operator $\hat{G}$ by means of the "phase-space integral" (2.9), the c-number equivalent F_G of $\hat{G}$ and the generalized distribution function Φ are obtained from $\hat{G}$ and $\hat{\rho}$ via *mappings that are mutually antireciprocal*. Only in the special case of Weyl's mapping (which is self-reciprocal) will the two associated mappings be of the same kind.

Often one wishes to evaluate the average of an operator which is ordered in some particular way, e.g., normally ordered field correlations in the quantum theory of photoelectric detection.[8] In other words, $\hat{G}(\hat{a},\hat{a}^\dagger)$ is given in the form $\mathcal{G}^{(\Omega^{(1)})}(\hat{a},\hat{a}^\dagger)$ where $\mathcal{G}^{(\Omega^{(1)})}$ is in an *ordered form* [see Eqs. (I.2.14) and (I.2.15)] for some particular rule $\Omega^{(1)}$. In such a case it is convenient to map $\hat{G}$ onto the phase space by means of the mapping $\Omega^{(1)}$. We then have (with $\Theta^{(1)}$ being the mapping inverse to $\Omega^{(1)}$)

$$\begin{aligned}F_G{}^{(\Omega^{(1)})}(z,z^*)&=\Theta^{(1)}\{G(\hat{a},\hat{a}^\dagger)\}\\&=\Theta^{(1)}\{\mathcal{G}^{(\Omega^{(1)})}(\hat{a},\hat{a}^\dagger)\}.\end{aligned} \tag{2.12}$$

In particular, we see from Eqs. (I.2.13a), (I.2.13b), and (I.2.13c), and from the linearity of the mapping operator, that if $\Omega^{(1)}$ represents the normal, the antinormal, or the Weyl rule of association, then

$$\Theta^{(1)}\{\mathcal{G}^{(\Omega^{(1)})}(\hat{a},\hat{a}^\dagger)\}=\mathcal{G}^{(\Omega^{(1)})}(z,z^*), \tag{2.13}$$

where

$$\mathcal{G}^{(\Omega^{(1)})}(z,z^*)=\mathcal{G}^{(\Omega^{(1)})}(\hat{a},\hat{a}^\dagger)\,|_{\hat{a}\to z,\hat{a}^\dagger\to z^*}. \tag{2.14}$$

It follows from (2.12) and (2.13) that

$$F_G{}^{(\Omega^{(1)})}(z,z^*)=\mathcal{G}^{(\Omega^{(1)})}(z,z^*), \tag{2.15}$$

i.e., the c-number equivalent of the operator function $\hat{G}$ may be written in the same functional form as $\hat{G}$ itself. Using (2.15), one obtains from (2.9) the interesting result that

$$\mathrm{Tr}[\hat{\rho}\mathcal{G}^{(\Omega^{(1)})}(\hat{a},\hat{a}^\dagger)]=\int\Phi^{(\tilde{\Omega}^{(1)})}(z,z^*)\mathcal{G}^{(\Omega^{(1)})}(z,z^*)d^2z. \tag{2.16}$$

This formula brings into evidence even more clearly than before the close formal analogy between the present representation and classical statistical mechanics. Specialized to the case when $\Omega^{(1)}$ represents the normal rule of association, this result is the essence of Sudarshan's theorem on the equivalence between the semiclassical and the quantum theory of optical coherence.[9,10]

We will illustrate these remarks by a simple example. Let $\mathcal{G}^{(N)}(\hat{a},\hat{a}^\dagger)$ be the normally ordered monomial

$$\mathcal{G}^{(N)}(\hat{a},\hat{a}^\dagger)\equiv\hat{a}^{\dagger m}\hat{a}^n, \tag{2.17}$$

where m and n are non-negative integers. Then according to (2.17) and (2.15)

$$F_G{}^{(N)}(z,z^*)=z^{*m}z^n. \tag{2.18}$$

On substitution from (2.17) and (2.18) into (2.16), with $\Omega^{(1)}$ representing the normal rule (N) and $\tilde{\Omega}^{(1)}$ the antinormal rule (A) of association, we have

$$\mathrm{Tr}(\hat{\rho}\hat{a}^{\dagger m}\hat{a}^n)=\int\Phi^{(A)}(z,z^*)z^{*m}z^n d^2z, \tag{2.19}$$

or, more compactly,

$$\langle\hat{a}^{\dagger m}\hat{a}^n\rangle=\langle z^{*m}z^n\rangle_{\text{p.s.}}, \tag{2.20}$$

where $\langle\ \rangle_{\text{p.s.}}$ represents the phase-space average with respect to the generalized distribution function $\Phi^{(A)}(z,z^*)$; the function $\Phi^{(A)}(z,z^*)$ is of course, $1/\pi$ times the c-number equivalent of the density operator for the antinormal rule of association.

III. MAPPING OF PRODUCT OF TWO OPERATORS

In order to determine the phase-space form of the basic quantum-mechanical equations of motion, we need to know how the product of two operators is mapped onto a c-number space. The result, derived in Appendix B, is expressed by the following theorem.

Theorem V (Product Theorem). The Ω equivalent of the product $G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)$ of two operators $\hat{G}_1$ and $\hat{G}_2$, i.e., the c-number function $F_{12}{}^{(\Omega)}(z,z^*)$ such that

$$G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)=\Omega\{F_{12}{}^{(\Omega)}(z,z^*)\}, \tag{3.1}$$

$$F_{12}{}^{(\Omega)}(z,z^*)=\Theta\{G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)\}, \tag{3.2}$$

[7] E. Wigner, Phys. Rev. **40**, 749 (1932).

[8] See, e.g., L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965); see also R. J. Glauber, Phys. Rev. **130**, 2529 (1963).

[9] E. C. G. Sudarshan, (a) Phys. Rev. Letters **10**, 277 (1963); (b) in *Proceedings of the Symposium on Optical Masers* (Wiley, New York, 1963), p. 45.

[10] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968).

is given by

$$F_{12}^{(\Omega)}(z,z^*)=\exp(\Lambda_{12})\,\mathfrak{U}_{12}^{(\Omega)}F_1^{(\Omega)}(z_1,z_1^*)\times F_2^{(\Omega)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}, \quad (3.3)$$

where $F_1^{(\Omega)}(z,z^*)$ *and* $F_2^{(\Omega)}(z,z^*)$ *are the* Ω *equivalents of the two operators, and* Λ_{12} *and* $\mathfrak{U}_{12}^{(\Omega)}$ *are the differential operators defined by*

$$\Lambda_{12}=\frac{1}{2}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2^*}-\frac{\partial}{\partial z_1^*}\frac{\partial}{\partial z_2}\right), \quad (3.4)$$

$$\mathfrak{U}_{12}^{(\Omega)}=\Omega\left(\frac{\partial}{\partial z_1^*},-\frac{\partial}{\partial z_1}\right)\Omega\left(\frac{\partial}{\partial z_2^*},-\frac{\partial}{\partial z_2}\right)\times\bar{\Omega}\left(\frac{\partial}{\partial z_1^*}+\frac{\partial}{\partial z_2^*},-\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right). \quad (3.5)$$

In (3.5) $\bar{\Omega}(\alpha,\beta)$ denotes, again, the filter function for mapping reciprocal to $\Omega(\alpha,\beta)$, i.e., $\bar{\Omega}(\alpha,\beta)=[\Omega(\alpha,\beta)]^{-1}$; $\bar{\Omega}(\alpha,\beta)$ will be *nonsingular*, since we assumed that $\Omega(\alpha,\beta)$ *has no zeros.*

The operator Λ_{12} has a simple meaning. If we use the relations $z_j=(q_j+ip_j)/(2\hbar)^{1/2}$, $z_j^*=(q_j-ip_j)/(2\hbar)^{1/2}$, then (3.4) becomes

$$\Lambda_{12}=\left(\frac{i\hbar}{2}\right)\left(\frac{\partial}{\partial q_1}\frac{\partial}{\partial p_2}-\frac{\partial}{\partial q_2}\frac{\partial}{\partial p_1}\right), \quad (3.4')$$

i.e., $\Lambda_{12}/(i\hbar/2)$ is just the Poisson-bracket operator.[11]

We note that Λ_{12} is antisymmetric and $\mathfrak{U}_{12}^{(\Omega)}$ is symmetric with respect to the two indices 1 and 2:

$$\Lambda_{21}=-\Lambda_{12}, \quad (3.6)$$

$$\mathfrak{U}_{12}^{(\Omega)}=\mathfrak{U}_{21}^{(\Omega)}. \quad (3.7)$$

It immediately follows from Theorem V and the relations (3.6) and (3.7) that the Ω equivalent of the product $G_2(\hat{a},\hat{a}^\dagger)G_1(\hat{a},\hat{a}^\dagger)$ is the c-number function

$$F_{21}^{(\Omega)}(z,z^*)=\exp(-\Lambda_{12})\mathfrak{U}_{12}^{(\Omega)}F_1^{(\Omega)}(z_1,z_1^*)\times F_2^{(\Omega)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}. \quad (3.8)$$

For the important class of mappings characterized by filter functions of the form given by Eq. (I.3.38), viz.,

$$\Omega(\alpha,\beta)=\exp(\mu\alpha^2+\nu\beta^2+\lambda\alpha\beta), \quad (3.9)$$

the differential operator $\mathfrak{U}_{12}^{(\Omega)}$ is readily found to have the following form:

$$\mathfrak{U}_{12}^{(\Omega)}=\exp\left[-2\nu\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2}-2\mu\frac{\partial}{\partial z_1^*}\frac{\partial}{\partial z_2^*}+\lambda\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2^*}+\frac{\partial}{\partial z_1^*}\frac{\partial}{\partial z_2}\right)\right]. \quad (3.10)$$

[11] F. Strocchi, Rev. Mod. Phys. **38**, 36 (1966).

With the help of Theorem V, we may immediately write down a necessary and sufficient condition for the distribution function $\Phi^{(\Omega)}(z,z^*)$ to represent a pure state. The density operator $\hat{\rho}$ of a pure state satisfies the condition (which is both necessary and sufficient)

$$\hat{\rho}^2=\hat{\rho}. \quad (3.11)$$

On taking the Ω equivalent of this equation, we obtain the relation

$$F_{\rho^2}^{(\Omega)}=F_\rho^{(\Omega)}. \quad (3.12)$$

Now according to Theorem V,

$$F_{\rho^2}^{(\Omega)}(z,z^*)=\exp(\Lambda_{12})\,\mathfrak{U}_{12}^{(\Omega)}F_\rho^{(\Omega)}(z_1,z_1^*)\times F_\rho^{(\Omega)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}. \quad (3.13)$$

From (3.12) and (3.13) we obtain, if we make use of relation (2.8), the required phase-space form of the condition for a pure state:

$$\pi\exp(\Lambda_{12})\mathfrak{U}_{12}^{(\Omega)}\Phi^{(\Omega)}(z_1,z_1^*)\Phi^{(\Omega)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}=\Phi^{(\Omega)}(z,z^*). \quad (3.14)$$

As an example, one may show on using (3.14) that the function $\Phi^{(N)}(z,z^*)=(1/\pi)\exp(-|z|^2)$ represents a distribution function of a system in a pure state. The corresponding density operator is

$$\hat{\rho}=\Omega^{(N)}\{\exp(-|z|^2)\}=|0\rangle\langle 0|.$$

In making use of the product theorem (Theorem V) to derive the phase-space form of the basic quantum-mechanical equations of motion, one of the operators will be the Hamiltonian operator $\hat{H}$ of the system. We will find it convenient to express the Ω equivalent of the products $\hat{H}\hat{G}$ and $\hat{G}\hat{H}$ in more compact form. We will then write

$$\Theta\{\hat{H}\hat{G}\}=\mathfrak{L}_+F_G^{(\Omega)}, \quad (3.15a)$$

$$\Theta\{\hat{G}\hat{H}\}=\mathfrak{L}_-F_G^{(\Omega)}, \quad (3.15b)$$

where, in accordance with Theorem V,

$$\mathfrak{L}_+F_G^{(\Omega)}=\exp(\Lambda_{12})\,\mathfrak{U}_{12}^{(\Omega)}F_H^{(\Omega)}(z_1,z_1^*)\times F_G^{(\Omega)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}, \quad (3.16a)$$

$$\mathfrak{L}_-F_G^{(\Omega)}=\exp(-\Lambda_{12})\,\mathfrak{U}_{12}^{(\Omega)}F_H^{(\Omega)}(z_1,z_1^*)\times F_G^{(\Omega)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}, \quad (3.16b)$$

$F_H^{(\Omega)}(z,z^*)$ and $F_G^{(\Omega)}(z,z^*)$ being the Ω equivalents of the operators $\hat{H}$ and $\hat{G}$, respectively.

In this notation the Ω equivalent of the commutator $[\hat{H},\hat{G}]\equiv\hat{H}\hat{G}-\hat{G}\hat{H}$ evidently is[12]

$$\Theta\{[\hat{H},\hat{G}]\}=(\mathcal{L}_+-\mathcal{L}_-)F_G{}^{(\Omega)}. \tag{3.17}$$

For mappings given by the filter function of the form (3.9), i.e., $\Omega(\alpha,\alpha^*)=\exp(\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\alpha^*)$, Eqs. (3.16a) and (3.16b) become

$$\mathcal{L}_+F_G{}^{(\Omega)}=F_H{}^{(\Omega)}\left[z_1-2\nu\frac{\partial}{\partial z_2}+(\lambda+\tfrac{1}{2})\frac{\partial}{\partial z_2{}^*},\; z_1{}^*-2\mu\frac{\partial}{\partial z_2{}^*}+(\lambda-\tfrac{1}{2})\frac{\partial}{\partial z_2}\right]\times F_G{}^{(\Omega)}(z_2,z_2{}^*)\big|_{z_1=z_2=z;\; z_1{}^*=z_2{}^*=z^*}, \tag{3.18a}$$

$$\mathcal{L}_-F_G{}^{(\Omega)}=F_H{}^{(\Omega)}\left[z_1-2\nu\frac{\partial}{\partial z_2}+(\lambda-\tfrac{1}{2})\frac{\partial}{\partial z_2{}^*},\; z_1{}^*-2\mu\frac{\partial}{\partial z_2{}^*}+(\lambda+\tfrac{1}{2})\frac{\partial}{\partial z_2}\right]\times F_G{}^{(\Omega)}(z_2,z_2{}^*)\big|_{z_1=z_2=z;\; z_1{}^*=z_2{}^*;=z^*}. \tag{3.18b}$$

IV. PHASE-SPACE FORM OF QUANTUM-MECHANICAL EQUATIONS OF MOTION

We will now derive the equations of motion for the c-number equivalents (phase-space representations) of the time-evolution operator, the density operator, and of a Heisenberg operator.

A. Schrödinger Equation of Motion for Time-Evolution Operator

The unitary time-evolution operator[13] $\hat{U}(t,t_0)$ of a quantum-mechanical system satisfies the Schrödinger equation

$$i\hbar\partial\hat{U}(t,t_0)/\partial t=\hat{H}\hat{U}(t,t_0), \tag{4.1}$$

where $\hat{H}$ is the Hamiltonian of the system. This equation must be solved subject to the initial condition

$$\hat{U}(t_0,t_0)=1. \tag{4.2}$$

If we apply the inverse mapping operator Θ to both sides of (4.1), we obtain the equation

$$i\hbar\frac{\partial}{\partial t}\Theta\{\hat{U}\}=\Theta\{\hat{H}\hat{U}\}. \tag{4.3}$$

If $F_U{}^{(\Omega)}=\Theta\{\hat{U}\}$ and $F_H{}^{(\Omega)}=\Theta\{\hat{H}\}$ are the Ω equivalents of $\hat{U}$ and $\hat{H}$, respectively, we have at once from (4.3) and (3.15a)

$$i\hbar\partial F_U{}^{(\Omega)}/\partial t=\mathcal{L}_+F_U{}^{(\Omega)}. \tag{4.4}$$

This then is the phase-space form of the Schrödinger equation (4.1) for the time-evolution operator $\hat{U}$. It is to be solved subject to the initial condition

$$F_U{}^{(\Omega)}(z,z^*;\,t,t_0)=1 \quad \text{when } t=t_0 \text{ for all } z,z^*, \tag{4.5}$$

as is evident from Eq. (4.2) on applying the inverse mapping operator Θ to both sides of it.

B. Schrödinger Equation for Density Operator

If we apply the inverse mapping operator Θ to the Schrödinger equation for the density operator $\hat{\rho}$, i.e., to the equation

$$i\hbar\partial\hat{\rho}/\partial t=[\hat{H},\hat{\rho}], \tag{4.6}$$

and we use formula (3.17) for the Ω equivalent of a commutator, we obtain

$$i\hbar\partial F_\rho{}^{(\Omega)}/\partial t=(\mathcal{L}_+-\mathcal{L}_-)F_\rho{}^{(\Omega)}. \tag{4.7}$$

Now according to (2.8), $F_\rho{}^{(\Omega)}=\pi\Phi^{(\Omega)}$, so that Eq. (4.7) may be expressed as the equation of motion for the phase-space distribution function:

$$i\hbar\partial\Phi^{(\Omega)}/\partial t=(\mathcal{L}_+-\mathcal{L}_-)\Phi^{(\Omega)}. \tag{4.8}$$

One can also easily derive equations of motion for the macroscopic averages (expectation values) of observables by combining (4.8) with Eq. (2.9).

C. Heisenberg Equation of Motion

In a strictly similar way, we obtain from the equation of motion for a Heisenberg operator $G(\hat{a}(t),\hat{a}^\dagger(t))$, viz.,

$$i\hbar d\hat{G}/dt=-[\hat{H},\hat{G}]+i\hbar\partial\hat{G}/\partial t, \tag{4.9}$$

the following phase-space equation of motion for the

---

[12] More generally, let us associate with any two (sufficiently well behaved) c-number functions $F_1(z,z^*)$ and $F_2(z,z^*)$ the symbol

$$(F_1,F_2|\Omega)\equiv[\exp(\Lambda_{12})-\exp(-\Lambda_{12})]\times\mathcal{U}_{12}{}^{(\Omega)}F_1(z_1,z_1{}^*)F_2(z_2,z_2{}^*)\big|_{z_1=z_2=z;\,z_1{}^*=z_2{}^*=z^*}.$$

Evidently, $(F_1,F_2|\Omega)$ is the Ω equivalent of the commutator $[\hat{G}_1,\hat{G}_2]$, where $\hat{G}_1=\Omega\{F_1\}$, $\hat{G}_2=\Omega\{F_2\}$. It may be shown that $(F_1,F_2|\Omega)$ is, for each Ω, a Lie bracket, i.e., it satisfies the following conditions.

(1) *Antisymmetry:*

$$(F_1,F_2|\Omega)=-(F_2,F_1|\Omega).$$

(2) *Linearity:*

$$(F_1,\alpha_2F_2+\alpha_3F_3|\Omega)=\alpha_2(F_1,F_2|\Omega)+\alpha_3(F_1,F_3|\Omega).$$

(3) *Jacobi identity:*

$$(F_1,(F_2,F_3|\Omega)|\Omega)+(F_2,(F_3,F_1|\Omega)|\Omega)+(F_3,(F_1,F_2|\Omega)|\Omega)=0.$$

The ansitymmetry and linearity are obvious from the defining equation of this bracket. The validity of the Jacobi identity may be established by a straightforward, but long, calculation involving the Fourier transforms of F_1, F_2, and F_3.

In the special case when Ω represents the Weyl rule of association, $(1/i\hbar)(F_1,\,F_2|\Omega)$ is the so-called Moyal bracket [J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949), Eq. (7.10); see also H. J. Groenwold, Physica **12**, 405 (1946), Eq. (4.38)] when expressed in terms of z and z^* rather than q and p [see Eq. (3.4')]. That the Moyal bracket is a Lie bracket was first noted by T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. **33**, 515 (1961); see also C. L. Mehta, J. Math. Phys. **5**, 677 (1964). The importance of the Lie bracket in the structure of dynamical theories has been discussed by E. C. G. Sudarshan, in *Lectures in Theoretical Physics* (Benjamin, New York, 1961), Vol. II, p. 143.

[13] We do not display explicitly the dependence of $\hat{U}$ on $\hat{a}$ and $\hat{a}^\dagger$. Similar abbreviated notation will be used in connection with other operators considered in this section.

Ω equivalent $F_G^{(\Omega)}(z(t),z^*(t))$ of $\hat{G}$:

$$i\hbar dF_G^{(\Omega)}/dt=-(\mathcal{L}_+-\mathcal{L}_-)F_G^{(\Omega)}+i\hbar\partial F_G^{(\Omega)}/\partial t. \quad (4.10)$$

We note that the two phase-space equations (4.4) and (4.8) are of first order in time. Since they are equations for c-number functions they are, in general, easier to solve than the original equations for the operators. Each of them is of the form of a Liouville equation

$$dX/dt=-i\mathcal{L}X, \quad (4.11)$$

where $\mathcal{L}$ is a differential operator that does not involve differentiation with respect to time. Approximate techniques for solving such equations are well known.[14] If all the operators are assumed to be in the interaction picture, the perturbation series expansion of the solution of (4.11) is

$$X(t)=\sum_{n=0}^{\infty} X^{(n)}(t), \quad (4.12)$$

where

$$X^{(n)}(t)=(-i)^n\int_0^t dt_1\int_0^{t_1} dt_2\cdots\int_0^{t_{n-1}} dt_n \times\mathcal{L}(t_1)\mathcal{L}(t_2)\cdots\mathcal{L}(t_n)X(0). \quad (4.13)$$

In the special case when the "Liouville operator" $\mathcal{L}$ is independent of time, one can immediately write down the following formal solution of (4.11):

$$X(t)=\exp(-i\mathcal{L}t)\,X(0). \quad (4.14)$$

However, in practical applications this exact formal solution is of little use and one has to resort to approximations. For example, if

$$\mathcal{L}=\mathcal{L}_0+\epsilon\mathcal{L}_1, \quad (4.15)$$

where ϵ is a small perturbation parameter, then, by using standard resolvent techniques,[14] one can show that

$$X(t)=-\frac{1}{2\pi i}\oint_C dz\, e^{-izt}\sum_{n=0}^{\infty}(\mathcal{L}_0-z)^{-1} \times[-\epsilon\mathcal{L}_1(\mathcal{L}_0-z)^{-1}]^n X(0), \quad (4.16)$$

where C is any contour that encloses the real axis.

V. EQUATION OF MOTION OF DISTRIBUTION FUNCTION FOR SYSTEMS WITH QUADRATIC HAMILTONIAN

We return to the equation of motion (4.8) for the distribution function $\Phi^{(\Omega)}(z,z^*;t)$, viz.,

$$i\hbar\partial\Phi^{(\Omega)}/\partial t=(\mathcal{L}_+^{(\Omega)}-\mathcal{L}_-^{(\Omega)})\Phi^{(\Omega)}, \quad (5.1)$$

and consider the form of it in the special but important case when the Hamiltonian is a quadratic function of $\hat{a}$ and $\hat{a}^\dagger$, i.e., when

$$\hat{H}=\omega\hat{a}^\dagger\hat{a}+\delta\hat{a}^2+\delta^*\hat{a}^{\dagger 2}+\gamma\hat{a}+\gamma^*\hat{a}^\dagger. \quad (5.2)$$

Here ω, δ, and γ are parameters which may depend on time and ω is real.

We will restrict ourselves to the class of mappings for which the filter function is of the form

$$\Omega(\alpha,\alpha^*)=\exp(\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\alpha^*). \quad (5.3)$$

As we saw at the end of Sec. III of I, the filter functions for the usual rules of associations are of this form.

We will first determine the Ω equivalent $F_H^{(\Omega)}$ of the Hamiltonian. Since the Hamiltonian is assumed to be a quadratic function of $\hat{a}$ and $\hat{a}^\dagger$, we know from the general result derived in Appendix E of I *that* $F_H^{(\Omega)}$ *will be quadratic in* z *and* z^*. Moreover, it is obvious from (5.2) that for the special case of the *normal* rule of association,

$$F_H^{(N)}=\omega z^*z+\delta z^2+\delta^*z^{*2}+\gamma z+\gamma^*z^*. \quad (5.4)$$

To obtain $F_H^{(\Omega)}$ for other rules of association, we apply to (5.4) the connecting relation (I.5.25), which relates the Ω equivalents for two different rules of association, viz.,

$$F^{(\Omega^{(2)})}(z,z^*)=L_{21}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)F^{(\Omega^{(1)})}(z,z^*), \quad (5.5)$$

where

$$L_{21}(\alpha,\alpha^*)=\bar{\Omega}^{(2)}(-\alpha,\,-\alpha^*)\Omega^{(1)}(-\alpha,\,-\alpha^*), \quad (5.6)$$

and $\bar{\Omega}^{(2)}(-\alpha,\,-\alpha^*)=[\Omega^{(2)}(-\alpha,\,-\alpha^*)]^{-1}$. If $\Omega^{(1)}$ is the filter function for the normal rule of association, i.e., the function $\exp(\frac{1}{2}\alpha^*\alpha)$ (see Table III of I) and if $\Omega^{(2)}(\alpha,\alpha^*)$ is the filter function (5.3), then

$$L_{21}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)=\exp\left[-\mu\frac{\partial^2}{\partial z^{*2}}-\nu\frac{\partial^2}{\partial z^2}+(\lambda-\tfrac{1}{2})\frac{\partial^2}{\partial z^*\partial z}\right], \quad (5.7)$$

and (5.5) becomes

$$F_H^{(\Omega)}(z,z^*)=\exp\left[-\mu\frac{\partial^2}{\partial z^{*2}}-\nu\frac{\partial^2}{\partial z^2}+(\lambda-\tfrac{1}{2})\frac{\partial^2}{\partial z\partial z^*}\right]\times F_H^{(N)}(z,z^*). \quad (5.8)$$

On substituting from (5.4) into (5.8), we obtain the following expression for $F_H^{(\Omega)}(z,z^*)$:

$$F_H^{(\Omega)}(z,z^*)=[\omega z^*z+\delta z^2+\delta^*z^{*2}+\gamma z+\gamma^*z^*] +[-2\mu\delta^*-2\nu\delta+(\lambda-\tfrac{1}{2})\omega], \quad (5.9)$$

and this expression is indeed quadratic in z and z^* as it ought to be.

On substituting (5.9) and (3.18) into (5.1), we obtain the required equation of motion:

[14] See e.g., P. Résibois, in *Cargèse Lectures in Theoretical Physics*, edited by B. Jancovici (Gordon and Breach, New York, 1966), p. 139; see also R. Balescu, *Statistical Mechanics of Charged Particles* (Interscience, New York, 1963), Chaps. I and XIV.

$$i\hbar\frac{\partial\Phi^{(\Omega)}}{\partial t}=A\frac{\partial^2\Phi^{(\Omega)}}{\partial z^2}+B\frac{\partial^2\Phi^{(\Omega)}}{\partial z^{*2}}+C\frac{\partial^2\Phi^{(\Omega)}}{\partial z^*\partial z}+D\frac{\partial\Phi^{(\Omega)}}{\partial z}-D^*\frac{\partial\Phi^{(\Omega)}}{\partial z^*},\quad(5.10)$$

where

$$A=2\nu\omega-2\lambda\delta^*,\qquad B=-2\mu\omega+2\lambda\delta,$$
$$C=-4\nu\delta+4\mu\delta^*,\quad D=-\omega z-\gamma^*-2\delta^*z^*.\quad(5.11)$$

Equation (5.10) has the form of the Fokker-Planck equation.[15] However, since the quadratic form involving the dffusion terms on the right-hand side of (5.10) is not in general positive definite, the solution of this equation is not necessarily non-negative and may be singular. This observation illustrates our earlier remark that in general the phase-space distribution function is not a true probability.

For the special case of mapping via the Weyl rule of association ($\mu=\nu=\lambda=0$; see Table IV of I), $\Phi^{(\Omega)}$ becomes the Wigner distribution function $\Phi^{(W)}$ and (5.10) reduces to

$$i\hbar\frac{\partial\Phi^{(W)}}{\partial t}=D\frac{\partial\Phi^{(W)}}{\partial z}-D^*\frac{\partial\Phi^{(W)}}{\partial z^*}.\quad(5.12)$$

Also for the Weyl rule of association

$$F_H{}^{(W)}=\omega z^*z+\delta z^2+\delta^*z^{*2}+\gamma z+\gamma^*z^*-\tfrac{1}{2}\omega,\quad(5.13)$$

so that

$$\partial F_H{}^{(W)}/\partial z=\omega z^*+2\delta z+\gamma,$$
$$\partial F^{(W)}/\partial z^*=\omega z+2\delta^*z^*+\gamma^*.\quad(5.14)$$

On comparing (5.14) with the expression for the coefficient D in (5.12), we see that

$$\partial F_H{}^{(W)}/\partial z=-D^*,\quad \partial F_H{}^{(W)}/\partial z^*=-D.\quad(5.15)$$

Hence, the equation of motion (5.12) for the Wigner distribution function $\Phi^{(W)}$ now reduces to

$$i\hbar\frac{\partial\Phi^{(W)}}{\partial t}=-\frac{\partial F_H{}^{(W)}}{\partial z^*}\frac{\partial\Phi^{(W)}}{\partial z}+\frac{\partial F_H{}^{(W)}}{\partial z}\frac{\partial\Phi^{(W)}}{\partial z^*}.\quad(5.16)$$

Now according to (3.4) and (3.4′), the differential operator

$$\frac{1}{i\hbar}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2{}^*}-\frac{\partial}{\partial z_1{}^*}\frac{\partial}{\partial z_2}\right)$$

is just the Poisson-bracket operator. Hence (5.16) may be written in the form

$$\partial\Phi^{(W)}/\partial t=-[\Phi^{(W)},F_H{}^{(W)}]_P,\quad(5.16')$$

where $[\cdots]_P$ denotes the Poisson bracket. This result, which in a somewhat less general form was first obtained by Moyal,[16] shows that the Wigner distribution function for a system with a quadratic Hamiltonian obeys the classical equation of motion.

VI. PHASE-SPACE FORM OF BLOCH EQUATION

For a system in thermodynamic equilibrium, the *unnormalized* density operator $\hat{\rho}$ is given by

$$\hat{\rho}=\exp(-\beta\hat{H}),\quad(6.1)$$

where $\hat{H}$ is the Hamiltonian of the system and $\beta=1/kT$, k being the Boltzmann constant and T the absolute temperature. The operator (6.1) evidently satisfies the differential equation

$$\partial\hat{\rho}/\partial\beta=-\hat{H}\hat{\rho},\quad(6.2)$$

known as the Bloch equation.[17] It is to be solved subject to the condition

$$\hat{\rho}=1\quad\text{for}\quad\beta=0.\quad(6.3)$$

In a way strictly similar to that used in connection with the Schrödinger equation (4.1) for the time-evolution operator, we obtain from the Bloch equation (6.2) the following equation for the phase-space equivalent $F_\rho{}^{(\Omega)}$ of the unnormalized density operator $\hat{\rho}$:

$$\partial F_\rho{}^{(\Omega)}/\partial\beta=-\mathcal{L}_+F_\rho{}^{(\Omega)},\quad(6.4)$$

where the operator $\mathcal{L}_+$ is again defined by Eq. (3.16a). On taking the Ω equivalent of (6.3), we see that (6.4) must be solved subject to the condition

$$F_\rho{}^{(\Omega)}(z,z^*;\beta)=1\quad\text{when}\quad\beta=0,\quad\text{for all } z \text{ and } z^*.\quad(6.5)$$

The phase-space form (6.4) of the Bloch equation may be used to determine the partition function of the system and provides a new way for determining the density operator of a system in thermal equilibirum. Since (6.4) has the form of Liouville's equation, with time t replaced by the variable $-i\beta$, similar remarks apply here as were made at the end of Sec. IV.

Equation (6.4) provides also a new way for determining *ordered forms* of exponential operators $\exp(-\beta\hat{H})$. We will illustrate this by determining the antinormally ordered form of the operator

$$\hat{\rho}=\exp(-\beta\omega\hat{a}^\dagger\hat{a}),\quad(6.6)$$

where ω is a constant. This operator is of the form (6.1), with the Hamiltonian

$$\hat{H}=\omega\hat{a}^\dagger\hat{a}.\quad(6.7)$$

Now the antinormally ordered form of (6.7) evidently is $\omega(\hat{a}\hat{a}^\dagger-1)$, so that the c-number equivalent $F_H{}^{(A)}$ of $\hat{H}$, for the antinormal rule of association, is

$$F_H{}^{(A)}(z,z^*)=\omega(z^*z-1).\quad(6.8)$$

Now the filter function for the antinormal rule of association is given by (3.9), with $\mu=\nu=0$, $\lambda=-\frac{1}{2}$ (see

[15] For discussions of the Fokker-Planck equation, see, e.g., M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945); or M. Lax, *ibid.* **38**, 359 (1966).

[16] J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

[17] For a discussion of the Bloch equation see, e.g., T. Matsubara, Progr. Theoret. Phys. (Kyoto) **14**, 351 (1955).

Table IV of I), so that according to (3.18a) and (6.8),

$$\begin{aligned}\mathcal{L}_+^{(A)}F_\rho^{(A)} &= F_H^{(A)}(z_1, z_1^*-\partial/\partial z_2)F_\rho^{(A)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*} \\ &=\omega[z_1(z_1^*-\partial/\partial z_2)-1]\times F_\rho^{(A)}(z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*} \\ &=\omega(z^*z-1-z\partial/\partial z)F_\rho^{(A)}. \end{aligned} \quad (6.9)$$

Hence Eq. (6.4) becomes, in this case,

$$\partial F_\rho^{(A)}/\partial\beta = -\omega(z^*z-1-z\partial/\partial z)F_\rho^{(A)}, \quad (6.10)$$

and is to be solved[18] subject to the condition (6.5) (with $F_\rho^{(\Omega)}$ replaced by $F_\rho^{(A)}$). The solution is

$$F_\rho^{(A)}(z,z^*;\beta)=\exp[\beta\omega+(1-e^{\beta\omega})z^*z]. \quad (6.11)$$

Hence, by Theorem I [Eq. (I.2.20)] the antinormally ordered form of the operator $\hat{\rho}=\exp(-\beta\omega\hat{a}^\dagger\hat{a})$ is obtained by applying to (6.11) the substitution operator $S^{(A)}$ for antinormal ordering:

$$\begin{aligned}\exp(-\beta\omega\hat{a}^\dagger\hat{a}) &= S^{(A)}\{\exp[\beta\omega+(1-e^{\beta\omega})z^*z]\} \\ &= S^{(A)}\left\{e^{\beta\omega}\sum_{n=0}^{\infty}\frac{(1-e^{\beta\omega})^n}{n!}z^{*n}z^n\right\} \\ &\equiv e^{\beta\omega}\sum_{n=0}^{\infty}\frac{(1-e^{\beta\omega})^n}{n!}\hat{a}^n\hat{a}^{\dagger n}. \end{aligned} \quad (6.12)$$

VII. GENERALIZED CHARACTERISTIC FUNCTIONS OF QUANTUM-MECHANICAL SYSTEM

In the theory of probability, the characteristic function,[19] i.e., the Fourier transform of the probability distribution, plays an important role. In particular, the moments of the distribution may, in general, be easily derived from it simply by differentiation. In the present theory we have associated a class of quasi-probabilities with a quantum-mechanical system, namely, the phase-space distribution functions $\Phi^{(\Omega)}(z,z^*)$. By analogy with classical probability theory, we will now introduce also the corresponding "characteristic functions" $C^{(\Omega)}(\alpha,\alpha^*)$. However, since $\Phi^{(\Omega)}$ is not necessarily non-negative, $C^{(\Omega)}$ will, in general, not satisfy the criterion for characteristic functions, expressed by Bochner's theorem.[20] Nevertheless, functions of this kind, which we will call *generalized characteristic functions,* are of considerable value in applications of phase-space formalism, as is clearly evident from treatments of special problems.[21,22] A generalized characteristic function appears to have been first employed by Moyal,[16] for the case of the Weyl correspondence, and played a central role in his important investigation on the statistical foundations of quantum theory.

By analogy with the classical theory, we define the generalized characteristic function $C^{(\Omega)}(\alpha,\alpha^*)$ of a quantum system for $\tilde{\Omega}$ mapping as the two-dimensional Fourier transform of the phase-space distribution function $\Phi^{(\Omega)}(z,z^*)$:

$$C^{(\Omega)}(\alpha,\alpha^*)=\int\Phi^{(\Omega)}(z,z^*)\exp[-(\alpha z^*-\alpha^*z)]d^2z. \quad (7.1)$$

The integral in (7.1) may be expressed as a trace of two operators by the use of the relation (2.9′), and one then obtains the following expression for $C^{(\Omega)}$:

$$C^{(\Omega)}(\alpha,\alpha^*)=\mathrm{Tr}(\hat{\rho}\tilde{\Omega}\{\exp[-(\alpha z^*-\alpha^*z)]\}). \quad (7.2)$$

Thus the generalized characteristic function $C^{(\Omega)}$ is the expectation value of the operator that is obtained by mapping the c-number function $\exp(-\alpha z^*+\alpha^*z)$ via the mapping that is antireciprocal to Ω. This result corresponds to the fact that, in classical theory, the characteristic function is the average of the exponential function.

Since according to Eq. (I.3.17)

$$\tilde{\Omega}\{\exp(-\alpha z^*+\alpha^*z)\}=\bar{\Omega}(\alpha,\alpha^*)\exp[-(\alpha\hat{a}^\dagger-\alpha^*\hat{a})]. \quad (7.3)$$

Equation (7.2) may also be expressed in the form

$$C^{(\Omega)}(\alpha,\alpha^*)=\bar{\Omega}(\alpha,\alpha^*)\,\mathrm{Tr}\{\hat{\rho}\exp[-(\alpha\hat{a}^\dagger-\alpha^*\hat{a})]\}, \quad (7.4)$$

where in accordance with Eq. (I.3.23),

$$\bar{\Omega}(\alpha,\alpha^*)=[\Omega(\alpha,\alpha^*)]^{-1}.$$

For certain states of the system, and for certain rules of association, the distribution function $\Phi^{(\Omega)}(z,z^*)$ may not exist as an ordinary function, and hence the definition of the generalized characteristic function $C^{(\Omega)}$ by means of Eq. (7.1) has to be interpreted with some care. However, *if $C^{(\Omega)}$ is defined by the formula (7.4) it will exist for every linear analytic mapping Ω, whose filter function $\Omega(\alpha,\beta)$ has no zeros*; this follows from the fact that the operator $\exp(-\alpha\hat{a}^\dagger+\alpha^*\hat{a})$ is unitary and that the expectation value of a unitary operator is bounded. In fact, this expectation value is bounded by unity and hence (7.4) implies that

$$|C^{(\Omega)}(\alpha,\alpha^*)|\leqslant|\bar{\Omega}(\alpha,\alpha^*)|. \quad (7.5)$$

We also note that since by our earlier assumption $\Omega(0,0)=1$ and since $\mathrm{Tr}\hat{\rho}=1$, Eq. (7.4) also implies that

$$C^{(\Omega)}(0,0)=1. \quad (7.6)$$

[18] The technique for solving differential equations of the type (6.10) is similar to the one described in J. H. Marburger, J. Math. Phys. **7**, 829 (1966).

[19] For a discussion of the characteristic function, in the classical theory of probability, see, e.g., E. Lukacs, *Characteristic Functions* (C. Griffin, London, 1960).

[20] For a discussion of Bochner's theorem see Ref. 19 or R. R. Goldberg, *Fourier Transforms* (Cambridge U. P., New York, 1961), Chap. V.

[21] J. P. Gordon, W. H. Louisell, and L. R. Walker, Phys. Rev. **129**, 481 (1963); J. P. Gordon, L. R. Walker, and W. H. Louisell, *ibid.* **130**, 806 (1963); B. R. Mollow and R. J. Glauber, *ibid.* **160**, 1097 (1967); J. H. Marburger, thesis, Microwave Laboratory, Stanford University [M. L. Report No. 1490 (unpublished)].

[22] See also, A. Yariv, IEEE J. Quant. Electron. **QE-1**, 28 (1965); W. G. Wagner and R. W. Hellwarth, Phys. Rev. **133**, A915 (1964); A. E. Glassgold and D. Holliday, *ibid.* **139**, A1717 (1965).

From (7.4) we also obtain at once the following relation between the generalized characteristic functions $C^{(\Omega^{(2)})}(\alpha,\alpha^*)$ and $C^{(\Omega^{(1)})}(\alpha,\alpha^*)$ of the same system, obtained via two different mappings $\Omega^{(1)}$ and $\Omega^{(2)}$:

$$C^{(\Omega^{(2)})}(\alpha,\alpha^*)=\left[\frac{\tilde{\Omega}^{(2)}(\alpha,\alpha^*)}{\tilde{\Omega}^{(1)}(\alpha,\alpha^*)}\right]C^{(\Omega^{(1)})}(\alpha,\alpha^*). \qquad (7.7)$$

In Table I we list the generalized characteristic functions for the normal rule of mapping for some typical density operators.

The moments of the phase-space distribution functions may be defined by the expression

$$M_{mn}{}^{(\Omega)}\equiv\int\Phi^{(\Omega)}(z,z^*)z^{*m}z^n d^2z. \qquad (7.8)$$

If we apply to the right-hand side of (7.8) the identity (2.9′), we see that

$$M_{mn}{}^{(\Omega)}=\mathrm{Tr}[\hat{\rho}\tilde{\Omega}\{z^{*m}z^n\}]. \qquad (7.9)$$

This formula shows that $M_{mn}{}^{(\Omega)}$ is the expectation value of the operator obtained by mapping the c-number function $z^{*m}z^n$ via the mapping that is antireciprocal to Ω.

It follows from (7.8) and (7.1) that the usual expression for the moments of a distribution in terms of the characteristic function has a strict analog in the present theory, i.e.,

$$M_{mn}{}^{(\Omega)}=\left.\frac{\partial^{m+n}C^{(\Omega)}(\alpha,\alpha^*)}{\partial(-\alpha)^m\partial(\alpha^*)^n}\right|_{\alpha=\alpha^*=0}. \qquad (7.10)$$

One may also derive an equation of motion for the generalized characteristic function. The derivation is given in Appendix C, and the result is

$$i\hbar\partial C^{(\Omega)}/\partial t=(\mathfrak{N}_+{}^{(\Omega)}-\mathfrak{N}_-{}^{(\Omega)})C^{(\Omega)}, \qquad (7.11)$$

where the operators $\mathfrak{N}_+{}^{(\Omega)}$ and $\mathfrak{N}_-{}^{(\Omega)}$ are defined by the formulas

$$\mathfrak{N}_+{}^{(\Omega)}C^{(\Omega)}=\exp(\Lambda_{12}')\,\mathcal{V}^{(\Omega)}F_H{}^{(\tilde{\Omega})}(\alpha_1,\alpha_1^*,t)\times C^{(\Omega)}(\alpha_2,\alpha_2^*,t)|_{\alpha_1=\alpha/2;\,\alpha_2=\alpha;\,\alpha_1^*=\alpha^*/2;\,\alpha_2^*=\alpha^*}, \qquad (7.12a)$$

$$\mathfrak{N}_-{}^{(\Omega)}C^{(\Omega)}=\exp(\Lambda_{12}')\,\mathcal{V}^{(\Omega)}F_H{}^{(\tilde{\Omega})}(\alpha_1,\alpha_1^*,t)\times C^{(\Omega)}(\alpha_2,\alpha_2^*,t)|_{\alpha_1=-\alpha/2;\,\alpha_2=\alpha;\,\alpha_1^*=-\alpha^*/2;\,\alpha_2^*=\alpha^*}. \qquad (7.12b)$$

Here Λ_{12}' and $\mathcal{V}^{(\Omega)}$ are the differential operators defined as follows:

$$\Lambda_{12}'=\left(\frac{\partial}{\partial\alpha_1}\frac{\partial}{\partial\alpha_2^*}-\frac{\partial}{\partial\alpha_1^*}\frac{\partial}{\partial\alpha_2}\right), \qquad (7.13)$$

$$\mathcal{V}^{(\Omega)}=\tilde{\Omega}(\alpha,\alpha^*)\tilde{\Omega}\left(\frac{\partial}{\partial\alpha_1^*},-\frac{\partial}{\partial\alpha_1}\right)\times\Omega\left(\alpha-\frac{\partial}{\partial\alpha_1^*},\alpha^*+\frac{\partial}{\partial\alpha_1}\right). \qquad (7.14)$$

Table I. The form of the generalized characteristic function for the normal rule of association for some density operators. Here L_n is the Laguerre polynomial of degree n, J_0 is the Bessel function of the first kind and zero order, and $\langle n\rangle=(e^\beta-1)^{-1}$.

ρ	$C^{(A)}(\alpha,\alpha^*)$
$\vert z_0\rangle\langle z_0\vert$	$\exp[-(\alpha z_0^*-\alpha^* z_0)]$
$\vert n\rangle\langle n\vert$	$L_n(\vert\alpha\vert^2)$
$\dfrac{\exp(-\beta\hat{a}^\dagger\hat{a})}{\mathrm{Tr}\exp(-\beta\hat{a}^\dagger\hat{a})}$	$\exp(-\langle n\rangle\vert\alpha\vert^2)$
$\dfrac{1}{2\pi}\int_0^{2\pi}d\theta\vert r\exp(i\theta)\rangle\langle r\exp(i\theta)\vert$	$J_0(2r\vert\alpha\vert)$

In Eqs. (7.12a) and (7.12b), $F_H{}^{(\tilde{\Omega})}$ is, of course, the $\tilde{\Omega}$ equivalent of the Hamiltonian operator. The differential operator Λ_{12}' is proportional to the Poisson-bracket operator [see the remark in the second paragraph that follows Eq. (3.5)].

For the important class of mappings for which the filter function is of the form (3.9), viz.,

$$\Omega(\alpha,\alpha^*)=\exp(\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\alpha^*), \qquad (7.15)$$

the differential operator $\mathcal{V}^{(\Omega)}$ defined by (7.14) is readily seen to be given by

$$\mathcal{V}^{(\Omega)}=\exp\left(-2\mu\alpha\frac{\partial}{\partial\alpha_1^*}+2\nu\alpha^*\frac{\partial}{\partial\alpha_1}-\lambda\alpha^*\frac{\partial}{\partial\alpha_1^*}+\lambda\alpha\frac{\partial}{\partial\alpha_1}\right), \qquad (7.16)$$

and expressions (7.12a) and (7.12b) then become

$$\mathfrak{N}_+{}^{(\Omega)}C^{(\Omega)}=F_H{}^{(\tilde{\Omega})}\left[\frac{\alpha}{2}+2\nu\alpha^*+\lambda\alpha+\frac{\partial}{\partial\alpha_2^*},\ \frac{\alpha^*}{2}-2\mu\alpha-\lambda\alpha^*-\frac{\partial}{\partial\alpha_2}\right]\times C^{(\Omega)}(\alpha_2,\alpha_2^*,t)|_{\alpha_2=\alpha;\ \alpha_2^*=\alpha^*}, \qquad (7.17)$$

$$\mathfrak{N}_-{}^{(\Omega)}C^{(\Omega)}=F_H{}^{(\tilde{\Omega})}\left[-\frac{\alpha}{2}+2\nu\alpha^*+\lambda\alpha+\frac{\partial}{\partial\alpha_2^*},\ -\frac{\alpha^*}{2}-2\mu\alpha-\lambda\alpha^*-\frac{\partial}{\partial\alpha_2}\right]\times C^{(\Omega)}(\alpha_2,\alpha_2^*,t)|_{\alpha_2=\alpha;\ \alpha_2^*=\alpha^*}. \qquad (7.18)$$

Let us consider the special form of the equation of motion (7.11) for the generalized characteristic function, when the filter function $\Omega(\alpha,\alpha^*)$ is given by Eq. (7.15) and when the Hamiltonian of the system is a quadratic function of $\hat{a}$ and $\hat{a}^\dagger$, given by Eq. (5.2). On using Eqs. (7.11), (7.17), (7.18), and (5.9), it may be shown by a straightforward but rather long calculation that the equation of motion for $C^{(\Omega)}$ is, in this case,

$$i\hbar\partial C^{(\Omega)}/\partial t=(a\alpha^{*2}+b\alpha^2+c\alpha\alpha^*+\gamma\alpha+\gamma^*\alpha^*)C^{(\Omega)}+d\partial C^{(\Omega)}/\partial\alpha-d^*\partial C^{(\Omega)}/\partial\alpha^*,\quad(7.19)$$

where

$$a=2\nu\omega-2\lambda\delta^*,\quad b=-2\mu\omega+2\lambda\delta,\quad c=4\nu\delta-4\mu\delta^*,\quad d=-(\omega\alpha+2\delta^*\alpha^*).\quad(7.20)$$

We note that Eq. (7.19) for the characteristic function is of the first order in the variables t, α, and α^*.

We will illustrate the use of Eq. (7.19) by considering a simple example, namely, an ensemble of driven harmonic oscillators. We will derive the generalized characteristic function for this ensemble for Weyl correspondence.

The interaction Hamiltonian in the interaction picture of a driven harmonic oscillator is given by (with H.c. denoting the Hermitian conjugate)

$$\hat{H}_I(t)=\hbar\{f(t)\exp[-ig(t)]\hat{a}+\text{H.c.}\}.\quad(7.21)$$

Here $f(t)$ is the external time-dependent force and

$$g(t)=\int\omega(t')dt',\quad(7.22)$$

$\omega(t)$ being the (time-varying) frequency of the oscillator. Since the Hamiltonian (7.21) is linear in $\hat{a}$ and $\hat{a}^\dagger$ {corresponding to (5.2) with $\omega=\delta=0$, $\gamma=\hbar f(t)\times\exp[-ig(t)]$}, and the mapping is of the form (5.3) (with $\mu=\nu=\lambda=0$), (7.19) applies in this case and one obtains

$$i\frac{\partial C^{(W)}(\alpha,\alpha^*,t)}{\partial t}=\{f(t)\exp[-ig(t)]\alpha+\text{c.c.}\}\times C^{(W)}(\alpha,\alpha^*,t).\quad(7.23)$$

The solution of (7.23) is readily seen to be

$$C^{(W)}(\alpha,\alpha^*,t)=C^{(W)}(\alpha,\alpha^*,0)\times\exp\{-i[\alpha\varphi(t)+\alpha^*\varphi^*(t)]\},\quad(7.24)$$

where

$$\varphi(t)=\int_0^t f(t')\exp[-ig(t')]dt'.\quad(7.25)$$

By applying the general formula (7.10), one may obtain from (7.24) expressions for the (time-dependent) moments $M_{mn}^{(W)}$ of this system.

In recent publications[21,22] already referred to, which deal with problems of quantum fluctuations and noise in parametric devices, extensive use has been made of generalized characteristic functions. In these investigations the time dependence of the characteristic function and of various moments was obtained by first solving the Heisenberg equation of motion for annihilation and creation operators. It would seem simpler and more appropriate to base such calculations directly on our Eq. (7.11) for the characteristic function rather than on the Heisenberg equation of motion. The example that we just considered illustrates this point.

Finally, we stress that since $C^{(\Omega)}$ and $\Phi^{(\Omega)}$ are Fourier transforms of each other [with the pairs (α,α^*) and (z,z^*) being the conjugate Fourier variables], the time dependence of the characteristic function leads to the time dependence of the Ω equivalent of the density operator and vice versa.

VIII. GENERALIZATIONS TO SYSTEMS WITH MORE THAN ONE DEGREE OF FREEDOM

For the sake of simplicity, we have up to now restricted ourselves to systems with only one degree of freedom. However, the theory may readily be extended to systems with any number of degrees of freedom.[23] We will now briefly present the appropriate generalizations of some of our main results.

Let $\{z_k\}\equiv(z_1,z_2,\ldots,z_N)$ be a set of N complex c-numbers and $\{z_k^*\}\equiv(z_1^*,z_2^*,\ldots,z_N^*)$ be the set of its complex conjugates. Further, let $\{\hat{a}_k\}\equiv(\hat{a}_1,\hat{a}_2,,\ldots\hat{a}_N)$ be a set of N annihilation operators and $\{\hat{a}_k^\dagger\}\equiv(\hat{a}_1^\dagger,\hat{a}_2^\dagger,\ldots,\hat{a}_N^\dagger)$ be the set of its adjoints, which obey the commutation relations

$$[a_k,a_{k'}^\dagger]=\delta_{kk'},\quad(8.1a)$$

$$[a_k,a_{k'}]=[a_k^\dagger,a_{k'}^\dagger]=0.\quad(8.1b)$$

We are concerned with the mapping of functions $F(\{z_k\},\{z_k^*\})$ of the c-numbers onto functions $G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})$ of the operators and vice versa, expressed symbolically by the formulas

$$\Omega\{F(\{z_k\},\{z_k^*\})\}=G(\{\hat{a}\},\{\hat{a}_k^\dagger\})\quad(8.2)$$

and

$$\Theta\{G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})\}=F^{(\Omega)}(\{z_k\},\{z_k^*\}).\quad(8.3)$$

The class of mappings that we consider will be defined by a straightforward generalization of the class that we introduced in Sec. III of I. Suppose that F is represented as a $2N$-dimensional Fourier integral

$$F(\{z_k\},\{z_k^*\})=\int f(\{\alpha_k\},\{\alpha_k^*\})\times\exp[\sum_k(\alpha_k z_k^*-\alpha_k^* z_k)]d^2\{\alpha_k\},\quad(8.4a)$$

where

$$f(\{\alpha_k\},\{\alpha_k^*\})=\frac{1}{\pi^{2N}}\int F(\{z_k\},\{z_k^*\})\times\exp[-\sum_k(\alpha_k z_k^*-\alpha_k^* z_k)]d^2\{z_k\},\quad(8.4b)$$

and $\{\alpha_k\}\equiv(\alpha_1,\alpha_2,\ldots,\alpha_N)$ denotes, of course, a set of N complex c-numbers and $\{\alpha_k^*\}\equiv(\alpha_1^*,\alpha_2^*,\ldots,\alpha_N^*)$ the set of its complex conjugates. In (8.4a) the integration extends over the N complex α_k planes, and in (8.4b) it extends over the N complex z_k planes $(k=1,2,\ldots,N)$.

Next suppose that $G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})$ is represented, as a $2N$-dimensional "operator" Fourier integral [see

[23] Our results are true both for finite and countably infinite number of degrees of freedom; see also Ref. 10, Chap. VIII.

Eq. (I.C1)],

$$G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})=\int g(\{\alpha_k\},\{\alpha_k^*\})\times\exp[\sum_k(\alpha_k\hat{a}_k^\dagger-\alpha_k^*\hat{a}_k)]d^2\{\alpha_k\}\,,\quad(8.5a)$$

where

$$g(\{\alpha_k\},\{\alpha_k^*\})=(1/\pi^N)\,\mathrm{Tr}\{G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})\times\exp[-\sum_k(\alpha_k\hat{a}_k^\dagger-\alpha_k^*\hat{a}_k)]\}\,.\quad(8.5b)$$

The class of mappings under consideration is defined by the property that for each mapping the multidimensional "Fourier spectra" $f(\{\alpha_k\},\{\alpha_k^*\})$ and $g(\{\alpha_k\},\{\alpha_k^*\})$ are related by an expression of the form

$$g(\{\alpha_k\},\{\alpha_k^*\})=\Omega(\{\alpha_k\},\{\alpha_k^*\})f(\{\alpha_k\},\{\alpha_k^*\})\,,\quad(8.6)$$

where the function $\Omega(\{\alpha_k\},\{\alpha_k^*\})$ that characterizes a particular mapping is assumed to have the following properties:

(1) It is an entire analytic function of the $2N$ complex variables $\{\alpha_k\}\equiv(\alpha_1,\alpha_2,\ldots,\alpha_N)$, $\{\beta_k\}\equiv(\beta_1,\beta_2,\ldots,\beta_N)$.
(2) $\Omega(\{\alpha_k\},\{\beta_k\})$ has no zeros.
(3) $\Omega(\{0\},\{0\})=1$, where $\{0\}\equiv(0,0,\ldots,0)$.

The mapping expressed symbolically by Eqs. (8.2) and (8.3) may be written down in a closed form with the help of an appropriate mapping Δ operator, which is defined as a straightforward generalization of Eq. (I.3.14) for the one-dimensional case:

$$\Delta^{(\Omega)}(\{z_k'-\hat{a}_k\},\{z_k'^*-\hat{a}_k^\dagger\})=\frac{1}{\pi^{2N}}\int\Omega(\{\alpha_k\},\{\alpha_k^*\})\exp[-\{\sum_k\alpha_k(z_k'^*-\hat{a}_k^\dagger)-\alpha_k^*(z_k'-\hat{a}_k)\}]d^2(\{\alpha_k\})\,.\quad(8.7)$$

The required expressions for the mappings $F\to\hat{G}$ and $\hat{G}\to F$, which are generalizations of the results expressed by Theorems II and III of I, are

$$G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})=\Omega\{F(\{z_k\},\{z_k^*\})\}=\int F(\{z_k\},\{z_k^*\})\Delta^{(\Omega)}(\{z_k-\hat{a}_k\},\{z_k^*-\hat{a}_k^\dagger\})\times d^2(\{z_k\})\,,\quad(8.8)$$

$$F^{(\Omega)}(\{z_k\},\{z_k^*\})=\Theta\{G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})\}=\pi^N\,\mathrm{Tr}[G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})\times\Delta^{(\tilde{\Omega})}(\{z_k-\hat{a}_k\},\{z_k^*-\hat{a}_k^\dagger\})]\,.\quad(8.9)$$

In (8.9), $\Delta^{(\tilde{\Omega})}$ denotes the Δ operator for the mapping that is antireciprocal to Ω, i.e., the mapping for which the filter function $\tilde{\Omega}(\{\alpha_k\},\{\alpha_k^*\})=[\Omega(\{-\alpha_k\},\{-\alpha_k^*\})]^{-1}$, where $\{-\alpha_k\}=(-\alpha_1,-\alpha_2,\ldots,-\alpha_N)$, $\{-\alpha_k^*\}=(-\alpha_1^*,\ldots,-\alpha_N^*)$.

In a strictly similar way, as in connection with Eq. (I.3.21), the mapping Δ operator may be expressed in the following symbolic form:

$$\Delta^{(\Omega)}(\{z_k'-\hat{a}_k\},\{z_k'^*-\hat{a}_k^\dagger\})=\Omega\{\prod_k\delta^{(2)}(z_k'-z_k)\}\,.\quad(8.10)$$

The generalization of Theorem IV [Eq. (2.6)] for the trace of the product of two operators is readily seen to be

$$\mathrm{Tr}(\hat{G}_1\hat{G}_2)=\frac{1}{\pi^N}\int F_1^{(\Omega)}(\{z_k\},\{z_k^*\})\times F_2^{(\tilde{\Omega})}(\{z_k\},\{z_k^*\})d^2(\{z_k\})\,.\quad(8.11)$$

By analogy with Eq. (2.8), we may define the generalized phase-space distribution function for Ω mapping of a quantum-mechanical system with any number of degrees of freedom by the relation

$$\Phi^{(\tilde{\Omega})}(\{z_k\},\{z_k^*\})=(1/\pi^N)F_\rho^{(\tilde{\Omega})}(\{z_k\},\{z_k^*\})\,,\quad(8.12)$$

where $F_\rho^{(\tilde{\Omega})}$ is the c-number equivalent for $\tilde{\Omega}$ mapping of the density operator $\rho(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})$ of the system. From (8.11) and (8.12) it then follows that the expectation value of a dynamical variable $G(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})$ may be expressed in the form of a phase-space average:

$$\mathrm{Tr}(\hat{\rho}\hat{G})=\int\Phi^{(\tilde{\Omega})}(\{z_k\},\{z_k^*\})\times F_G^{(\Omega)}(\{z_k\},\{z_k^*\})d^2(\{z_k\})\,,\quad(8.13)$$

where, of course, $F_G^{(\Omega)}$ is the Ω equivalent of $\hat{G}$.

Formula (3.3) of Theorem V may readily be generalized to systems of many degrees of freedom. If we assume, for the sake of simplicity, that the filter function $\Omega(\{\alpha_k\},\{\alpha_k^*\})$ is of the form

$$\Omega(\{\alpha_k\},\{\alpha_k^*\})=\prod_k\Omega_k(\alpha_k,\alpha_k^*)\,,$$

then one readily finds that the Ω equivalent

$$F_{12}^{(\Omega)}(\{z_k\},\{z_k^*\})=\Theta\{G_1(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})G_2(\{\hat{a}_k\},\{\hat{a}_k^\dagger\})\}\quad(8.14)$$

of the product of two operators $\hat{G}_1$ and $\hat{G}_2$ is

$$F_{12}^{(\Omega)}(\{z_k\},\{z_k^*\})=\exp(\sum_k\Lambda_{12k})\times\prod_k\mathfrak{U}_{12k}^{(\Omega)}F_1^{(\Omega)}(\{z_{k1}\},\{z_{k1}^*\})\times F_2^{(\Omega)}(\{z_{k2}\},\{z_{k2}^*\})\big|_{\{z_{k1}\}=\{z_{k2}\}=\{z_k\};\ \{z_{k1}^*\}=\{z_{k2}^*\}=\{z_k^*\}}\,,\quad(8.15)$$

where $F_1^{(\Omega)}$ and $F_2^{(\Omega)}$ are the Ω equivalents of $\hat{G}_1$ and $\hat{G}_2$, respectively, and Λ_{12k} and $\mathfrak{U}_{12k}^{(\Omega)}$ are the differential operators defined by the formulas

$$\Lambda_{12k}=\frac{1}{2}\left(\frac{\partial}{\partial z_{k1}}\frac{\partial}{\partial z_{k2}^*}-\frac{\partial}{\partial z_{k1}^*}\frac{\partial}{\partial z_{k2}}\right) \tag{8.16}$$

and

$$\mathfrak{U}_{12k}{}^{(\Omega)}=\Omega_k\left(\frac{\partial}{\partial z_{k1}^*},-\frac{\partial}{\partial z_{k1}}\right)\Omega_k\left(\frac{\partial}{\partial z_{k2}^*},-\frac{\partial}{\partial z_{k2}}\right)$$
$$\times\bar{\Omega}_k\left(\frac{\partial}{\partial z_{k1}^*}+\frac{\partial}{\partial z_{k2}^*},-\frac{\partial}{\partial z_{k1}}-\frac{\partial}{\partial z_{k2}}\right). \tag{8.17}$$

In a manner strictly similar to the one-dimensional case (Secs. IV–VI), the relation (8.15) may be used to derive the phase-space form of various quantum-mechanical equations for systems with any number of degrees of freedom.

IX. EXAMPLE: STOCHASTIC DESCRIPTION OF QUANTIZED ELECTROMAGNETIC FIELD

In the last few years many investigations have been carried out concerning the statistical properties of light,[8] partly in order to elucidate the basic differences between laser light and light generated by conventional sources. In some of these investigations phase-space techniques have proved very useful. In this section we show how with the help of our theory one may introduce in a systematic way various quasiprobabilities that characterize the statistical properties of the quantized electromagnetic field and how the coherence functions of the field may be expressed in terms of them. We will restrict our discussion to a free field only.

Let

$$\hat{\mathbf{A}}(\mathbf{r},t)=\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)+\hat{\mathbf{A}}^{(-)}(\mathbf{r},t) \tag{9.1}$$

be the operator that represents the vector potential of the field at the space-time point $(\mathbf{r},t)$, with $\hat{\mathbf{A}}^{(+)}$ and $\hat{\mathbf{A}}^{(-)}$ denoting its positive- and negative-frequency parts, respectively. We expand $\hat{\mathbf{A}}^{(+)}$ and $\hat{\mathbf{A}}^{(-)}$ in the usual way:

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)=\left(\frac{\hbar c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}\hat{a}_{ks}\boldsymbol{\varepsilon}_{ks}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)], \tag{9.2a}$$

$$\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)=\left(\frac{\hbar c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}\hat{a}_{ks}^\dagger\boldsymbol{\varepsilon}_{ks}^*$$
$$\times\exp[-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)]. \tag{9.2b}$$

Here L^3 denotes the volume to which the field is confined, $\hat{a}_{ks}$ is the annihilation operator for a photon of momentum $\mathbf{p}=\hbar\mathbf{k}$ and spin s, and the $\boldsymbol{\varepsilon}_{ks}$ are unit polarization vectors.

Let us now map the operators $\hat{\mathbf{A}}^{(+)}$ and $\hat{\mathbf{A}}^{(-)}$ onto c-number functions $\mathbf{V}(\mathbf{r},t)$ and $\mathbf{V}^*(\mathbf{r},t)$, respectively, via the Ω mapping:

$$\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)=\Omega\{\mathbf{V}(\mathbf{r},t)\}, \tag{9.3a}$$

$$\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)=\Omega\{\mathbf{V}^*(\mathbf{r},t)\}. \tag{9.3b}$$

We have the following relations [see Eq. (I.3.36)] for any mapping Ω of the class that we are considering:

$$\hat{a}_k=\Omega\{z_k\},\quad \hat{a}_k^\dagger=\Omega\{z_k^*\}. \tag{9.4}$$

If we make use of (9.4) and the linearity of the mapping operator, it is evident from (9.2) and (9.3) that[24]

$$\mathbf{V}(\mathbf{r},t)=\left(\frac{\hbar c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}z_{ks}\boldsymbol{\varepsilon}_{ks}\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)], \tag{9.5a}$$

$$\mathbf{V}^*(\mathbf{r},t)=\left(\frac{\hbar c}{L^3}\right)^{1/2}\sum_{ks}\frac{1}{\sqrt{k}}z_{ks}^*\boldsymbol{\varepsilon}_{ks}^*$$
$$\times\exp[-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)]. \tag{9.5b}$$

The statistical properties of the quantized field may be characterized in different ways. Of particular interest is its description in terms of the normally ordered correlation functions (the normally ordered coherence functions)

$$\Gamma_{j_1,j_2,\ldots,j_n;j_{n+1},\ldots,j_{n+m}}{}^{(n,m)}(x_1,x_2,\ldots,x_n;\,x_{n+1},\ldots,x_{n+m})$$
$$=\langle\hat{A}_{j_1}{}^{(-)}(x_1)\hat{A}_{j_2}{}^{(-)}(x_2)\cdots\hat{A}_{j_n}{}^{(-)}(x_n)\hat{A}_{j_{n+1}}{}^{(+)}(x_{n+1})\cdots$$
$$\hat{A}_{j_{n+m}}{}^{(+)}(x_{n+m})\rangle. \tag{9.6}$$

Here the arguments $x_\alpha\equiv(r_\alpha,t_\alpha)$ label various space-time points and the subscripts $j_i, j_{,2}, \ldots, j_{n+m}$, each of which can take on the value 1, 2, or 3, label Cartesian components. Some of the correlations functions of this type occur naturally in the analysis of results of photoelectric correlation and coincidence experiments on the electromagnetic field.[8,10]

It is evident at once from the structure of formulas (9.4) and (9.5) that the correlation function (9.6) may be expressed in the form

$$\Gamma^{(n,m)}=\langle\Omega^{(N)}\{\prod_{\alpha=1}^{n}V_{j_\alpha}{}^*(x_\alpha)\prod_{\beta=n+1}^{m+n}V_{j_\beta}(x_\beta)\}\rangle$$
$$=\mathrm{Tr}[\hat{\rho}\Omega^{(N)}\{\prod_{\alpha=1}^{n}V_{j_\alpha}{}^*(x_\alpha)\prod_{\beta=n+1}^{n+m}V_{j_\beta}(x_\beta)\}], \tag{9.7}$$

where $\hat{\rho}$ is the density operator of the field and $\Omega^{(N)}$ is the mapping operator for the normal rule of association. For the sake of simplicity, we have suppressed the numerous subscripts and arguments on the left-hand side of Eq. (9.7). The trace in (9.7) may be expressed as a phase-space integral by the use of Eq. (8.13), so that

$$\Gamma^{(n,m)}=\int\Phi^{(A)}(\{z_{ks}\},\{z_{ks}^*\})$$
$$\times\prod_{\alpha=1}^{n}V_{j_\alpha}{}^*(x_\alpha)\prod_{\beta=n+1}^{n+m}V_{j_\beta}(x_\beta)d^2(\{z_{ks}\}). \tag{9.8}$$

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[24] Cf. L. Mandel, Phys. Letters 7, 117 (1963).

If now we introduce the function[25]

$$p^{(N)}[\mathbf{V}(1),\mathbf{V}(2),\ldots,\mathbf{V}(n+m);\,x_1,x_2,\ldots,x_{n+m}]$$
$$=\langle\Omega^{(N)}\{\prod_{i=1}^{n+m}\delta[\mathbf{V}(i)-\mathbf{V}(x_i)]\}\rangle$$
$$=\mathrm{Tr}[\hat{\rho}\Omega^{(N)}\{\prod_{i=1}^{n+m}\delta[\mathbf{V}(i)-\mathbf{V}(x_i)]\}]$$
$$=\int\Phi^{(A)}(\{z_{ks}\},\{z_{ks}^*\})\prod_{i=1}^{n+m}\delta[\mathbf{V}(i)-\mathbf{V}(x_i)]\times d^2(\{z_{ks}\}),\quad(9.9)$$

Eq. (9.8) may then be written in the form

$$\Gamma^{(n,m)}=\int p^{(N)}[\mathbf{V}(1),\mathbf{V}(2),\ldots,\mathbf{V}(n+m);x_1,x_2,\ldots,x_{n+m}]\times\prod_{\alpha=1}^{n}V_{j_\alpha}^*(\alpha)\prod_{\beta=n+1}^{n+m}V_{j_\beta}(\beta)d^2\mathbf{V}(1)d^2\mathbf{V}(2)\cdots d^2\mathbf{V}(n+m).\quad(9.10)$$

Equation (9.10) expresses the normally ordered correlation function of the quantized field in a form that is mathematically identical with that occurring in the classical stochastic description of the field.[26] In general, $p^{(N)}$ is, of course, not a true probability. We will call $p^{(N)}$ a (space-time) quasiprobability distribution of the quantized field. It is clear that the statistical behavior of the field is characterized not by a single such quasiprobability distribution but rather by an infinite sequence of them, each successive member of the sequence having more arguments:

$$p^{(N)}[\mathbf{V}(1);\,x_1],\quad p^{(N)}[\mathbf{V}(1),\mathbf{V}(2);\,x_1,x_2],\quad p^{(N)}[\mathbf{V}(1),\mathbf{V}(2),\mathbf{V}(3);\,x_1,x_2,x_3],\ldots.\quad(9.11)$$

In principle all these quasiprobabilities may, of course, be derived from an appropriate characteristic functional.[27,28]

As an illustration of these results, let us determine the space-time quasiprobabilities, for the normal rule of association, of a free electromagnetic field in thermal equilibrium. The density operator of such a field is given by

$$\hat{\rho}=\frac{\exp(-\theta\hat{H})}{\mathrm{Tr}[\exp(-\theta\hat{H})]},\quad(9.12)$$

where the Hamiltonian $\hat{H}$ is

$$\hat{H}=\sum_{ks}\hbar\omega_k\hat{a}_{ks}^{\dagger}\hat{a}_{ks},\quad(9.13)$$

and $\theta=1/kT$, k being the Boltzmann constant and T the absolute temperature. Now by a multidimensional generalization of the formula (I.6.17), specialized to the antinormal rule of association ($\mu=\nu=0$, $\lambda=-\frac{1}{2}$) for each mode, the phase-space distribution function $\Phi^{(A)}$, associated with the density operator (9.12), is the multivariate Gaussian distribution

$$\Phi^{(A)}(\{z_{ks}\},\{z_{ks}^*\})=\prod_{ks}\frac{1}{\pi\tau_{ks}}\exp\left(-\frac{|z_{ks}|^2}{\tau_{ks}}\right),\quad(9.14)$$

where

$$\tau_{ks}=[1-\exp(-\theta\omega_k)]^{-1}-1.\quad(9.15)$$

The quasiprobability distribution $p^{(N)}$ for the system under consideration is obtained on substituting from (9.14) into the integral (9.9). The integral is evaluated in Appendix D, and the result is

$$p^{(N)}[\mathbf{V}(1),\mathbf{V}(2),\ldots,\mathbf{V}(n+m);\,x_1,x_2,\ldots,x_{n+m}]=\frac{1}{\pi^{3(n+m)}|\det R^{(N)}|}\exp[-\mathcal{V}^{\dagger}(R^{(N)})^{-1}\mathcal{V}].\quad(9.16)$$

Here $R^{(N)}$ is the covariance matrix

$$R^{(N)}=\int\Phi^{(A)}(\{z_{ks}\},\{z_{ks}^*\})\mathcal{V}'\mathcal{V}'^{\dagger}d^2(\{z_{ks}\}),\quad(9.17)$$

$\det R^{(N)}$ denotes the determinant of $R^{(N)}$, and $\mathcal{V}$ and $\mathcal{V}'$ are the column matrices given by

$$\mathcal{V}=\begin{bmatrix}V_1(1)\\V_2(1)\\V_3(1)\\\vdots\\V_1(n+m)\\V_2(n+m)\\V_3(n+m)\end{bmatrix},\quad\mathcal{V}'=\begin{bmatrix}V_1(x_1)\\V_2(x_1)\\V_3(x_1)\\\vdots\\V_1(x_{n+m})\\V_2(x_{n+m})\\V_3(x_{n+m})\end{bmatrix}.\quad(9.18)$$

Equation (9.16) shows that all the (space-time) quasiprobabilities of a thermal field are multivariate Gaussian distributions, so that the quantized field is described as a Gaussian random process. If we make use of the moment theorem[29] for such a process, it

[25] In (9.9), $\delta[\mathbf{V}(l)-\mathbf{V}(x_l)]$ stands for the expression

$$\prod_{j=1}^{3}\delta[V_j^{(r)}(l)-V_j^{(r)}(x_l)]\delta[V_j^{(i)}(l)-V_j^{(i)}(x_l)],$$

where $V_j^{(r)}$ and $V_j^{(i)}$ are the real and the imaginary parts of the Cartesian component V_j ($j=1$, 2, 3) of $\mathbf{V}$ and δ denotes the Dirac δ function.

[26] E. Wolf, in *Proceedings of the Symposium on Optical Masers* (Wiley, New York, 1963), p. 29.

[27] For a brief discussion of the characteristic functional, see Appendix D or Ref. 10, Chap. IV, and references therein.

[28] The method of the characteristic functional to calculate the correlation functions of the form $\langle\mathbf{A}^{\alpha_1}(x_1)\cdots\mathbf{A}^{\alpha_n}(x_n)\rangle$ and the associated quasiprobability distribution functions for the case of a thermal field has also been employed by E. F. Keller, Phys. Rev. **139**, B202 (1965).

[29] I. S. Reed, IRE Trans. Inform. Theory **IT-8**, 194 (1962); see also C. L. Mehta, in *Lectures in Theoretical Physics*, edited by W. E. Brittin (University of Colorado Press, Boulder, Colo., 1965), Vol. VII C, p. 398.

follows that

$$\Gamma_{j_1,j_2,\ldots,j_n;j_{n+1},\ldots,j_{2n}}{}^{(n,n)}(x_1,x_2,\ldots,x_n;\,x_{n+1},\ldots,x_{2n})$$
$$=\sum_{\Pi}\Gamma_{j_1,j_{n+1}}{}^{(1,1)}(x_1,x_{n+1})\cdots\Gamma_{j_n,j_{2n}}{}^{(1,1)}(x_n,x_{2n}),\quad(9.19)$$

$$\Gamma^{(n,m)}=0\quad\text{if}\quad n\neq m,\quad(9.20)$$

where $\sum_{\Pi}$ stands for the sum over all $n!$ possible permutations of the indices 1 to n.

In this section we have restricted ourselves entirely to normally ordered correlation functions and the various associated quasiprobabilities. It is clear, of course, that strictly similar results will apply to correlation functions ordered in different ways (e.g., the antinormally ordered correlations occurring in Mandel's theory of quantum counters[30]) and that one may introduce the associated space-time quasiprobabilities by similar formulas. In particular, if the quasiprobabilities are introduced by formulas analogous to (9.9), for a mapping whose filter function is of the form

$$\Omega(\{\alpha_{ks}\},\{\alpha_{ks}^*\})=\exp(\lambda\sum_{ks}|\alpha_{ks}|^2),\quad(9.21)$$

with $\lambda\leqslant\frac{1}{2}$ (see Appendix A), then one finds that for a field in thermal equilibrium Eqs. (9.16) and (9.17) remain valid, with trivial modifications. In place of (9.16) one now has

$$p^{(\Omega)}[\mathbf{V}(1),\mathbf{V}(2),\ldots,\mathbf{V}(n+m);\,x_1,x_2,\ldots,x_{n+m}]$$
$$=\frac{1}{\pi^{3(n+m)}|\det R^{(\Omega)}|}\exp[-\mathcal{V}^\dagger(R^{(\Omega)})^{-1}\mathcal{V}],\quad(9.22)$$

where $R^{(\Omega)}$ is the covariance matrix,

$$R^{(\Omega)}=\int\Phi^{(\tilde{\Omega})}(\{z_{ks}\},\{z_{ks}^*\})\mathcal{V}'\mathcal{V}'^\dagger d^2(\{z_{ks}\}),\quad(9.23)$$

and $\mathcal{V}$ and $\mathcal{V}'$ are again the column vectors (9.18). The phase-space distribution function $\Phi^{(\tilde{\Omega})}$ that occurs in (9.23) is now given by the following generalization of formula (9.14):

$$\Phi^{(\tilde{\Omega})}(\{z_{ks}\},\{z_{ks}^*\})=\prod_{ks}\frac{1}{\pi\tau_{ks}}\exp\left(-\frac{|z_{ks}|^2}{\tau_{ks}}\right),\quad(9.24)$$

where

$$\tau_{ks}=[1-\exp(-\theta\omega_k)]^{-1}-\lambda-\tfrac{1}{2}.\quad(9.25)$$

One may readily show that the covariance matrix (9.23) is *positive definite*.

It is seen that both the space-time quasiprobabilities as well as the phase-space distribution functions, given by (9.21) and (9.24), respectively, are multivariate Gaussian distributions with positive-definite covariance matrices. Hence these quantities are *true probabilities*. It seems remarkable that the c-number representation, via any rule of association characterized by the mapping function (9.20) with $\lambda\leqslant\frac{1}{2}$ (which includes the normal, antinormal, and Weyl rules), of a quantized field in thermal equilibrium leads to a statistical description of the field as a true classical stochastic process. Thus the usual arguments[31] (based on the noncommutability of conjugate operators) as to why the various c-number distribution functions of a quantum system cannot be true probabilities seem to oversimplify the problem.

[30] L. Mandel, Phys. Rev. 152, 438 (1966).

APPENDIX A: PROPERTIES OF Ω EQUIVALENT OF DENSITY OPERATOR WHEN $\Omega(\alpha,\alpha^*)=\exp(\lambda\alpha\alpha^*)$ $(\lambda\geqslant\frac{1}{2})$

In this appendix, we study the properties of the Ω equivalent of the density operator $\hat{\rho}$ when the rule of association is characterized by the filter function

$$\Omega(\alpha,\alpha^*)=\exp(\lambda\alpha\alpha^*)\quad(\lambda\geqslant\tfrac{1}{2}),\quad(\text{A}1)$$

where λ is real.

According to Theorem III [Eq. (I.3.25)]

$$F_\rho{}^{(\Omega)}(z,z^*)=\pi\,\mathrm{Tr}[\hat{\rho}\Delta^{(\tilde{\Omega})}(z-\hat{a},\,z^*-\hat{a}^\dagger)],\quad(\text{A}2)$$

where $\Delta^{(\tilde{\Omega})}(z-\hat{a},\,z^*-\hat{a}^\dagger)$ is given by Eq. (I.3.26) and Eq. (A1), i.e.,

$$\Delta^{(\tilde{\Omega})}(z-\hat{a},\,z^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int\exp(-\lambda\alpha\alpha^*)\hat{D}(\alpha)\times\exp[-(\alpha z^*-\alpha^*z)]d^2\alpha.\quad(\text{A}3)$$

Here $\hat{D}(\alpha)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})$ is the displacement operator for the coherent states [Eq. (I.B4)]. Since $\Omega(\alpha,\alpha^*)$, given by (A1), satisfies the condition $\Omega^*(-\alpha,\,-\alpha^*)=\Omega(\alpha,\alpha^*)$, it follows from (I.4.12) that $\hat{\Delta}^{(\tilde{\Omega})}$ is a Hermitian operator. Since $\hat{\rho}$ is a density operator, it is necessarily a Hermitian, positive-definite, bounded operator. In fact every $\hat{\rho}$ belongs to operators of the trace class. Because $\hat{\rho}$ and $\hat{\Delta}^{(\tilde{\Omega})}$ are Hermitian, it follows that $F_\rho{}^{(\Omega)}$ is real [see (I.4.15)], i.e.,

$$[F_\rho{}^{(\Omega)}(z,z^*)]^*=F_\rho{}^{(\Omega)}(z,z^*).\quad(\text{A}4)$$

The mapping Δ operator (A3) has a number of interesting properties. If we make use of the Baker-Hausdorff identity, we immediately see that

$$\Delta^{(\tilde{\Omega})}(z_0-\hat{a},\,z_0^*-\hat{a}^\dagger)$$
$$=\Omega^{(N)}\left\{\frac{1}{\pi^2}\int\exp[-(\lambda+\tfrac{1}{2})|\alpha|^2]\times\exp[\alpha(z^*-z_0^*)-\alpha^*(z-z_0)]\right\}$$
$$=\Omega^{(N)}\left\{\frac{1}{\pi(\lambda+\frac{1}{2})}\exp\left[-\frac{|z-z_0|^2}{(\lambda+\frac{1}{2})}\right]\right\}.\quad(\text{A}5)$$

[31] See, e.g., E. C. G. Sudarshan, in *Lectures in Theoretical Physics* (Benjamin, New York, 1961), Vol. II, p. 143.

We also have the identity[32]

$$\exp[-\beta(\hat{a}^\dagger - z_0^*)(\hat{a}-z_0)] = \Omega^{(N)}\{\exp[-|z-z_0|^2(1-e^{-\beta})]\}. \quad (A6)$$

From (A5) and (A6) it follows that, for $\lambda > \frac{1}{2}$,

$$\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger) = \frac{1}{\pi(\lambda+\frac{1}{2})}\exp\left[(\hat{a}^\dagger - z)(\hat{a}-z)\ln\frac{\lambda-\frac{1}{2}}{\lambda+\frac{1}{2}}\right]. \quad (A7)$$

For $\lambda = \frac{1}{2}$, the filter function (A1) is that for the normal rule of association and (A5) reduces to

$$\Delta^{(\tilde{\Omega})}(z_0-\hat{a}, z_0^*-\hat{a}^\dagger) = \Omega^{(N)}\{(1/\pi)\exp(-|z-z_0|^2)\} = (1/\pi)|z_0\rangle\langle z_0|, \quad (A8)$$

where $|z_0\rangle$ is a coherent state. Equation (A8) is in agreement with the formula (I.3.40), obtained by a more direct argument.

It should be noted that if we make use of the property (I.B9) of the displacement operators, (A7) may be expressed in the form

$$\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger) = [1/\pi(\lambda+\tfrac{1}{2})]\hat{D}(z)(\sigma)^{\hat{a}^\dagger\hat{a}}\hat{D}^\dagger(z), \quad (A9)$$

where

$$\sigma = (\lambda-\tfrac{1}{2})/(\lambda+\tfrac{1}{2}). \quad (A10)$$

It is evident that $\hat{D}(z)|n\rangle$ is the eigenfunction of $\hat{\Delta}^{(\tilde{\Omega})}$ with the eigenvalue $[1/\pi(\lambda+\frac{1}{2})]\sigma^n$, i.e.,

$$\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)\hat{D}(z)|n\rangle = E_n\hat{D}(z)|n\rangle, \quad (A11)$$

where

$$E_n = [1/\pi(\lambda+\tfrac{1}{2})]\sigma^n. \quad (A12)$$

For $\lambda=\frac{1}{2}$, the corresponding eigenvalue problem is

$$\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)\hat{D}(z)|0\rangle = (1/\pi)\hat{D}(z)|0\rangle. \quad (A13)$$

It is thus seen that when $\lambda \geqslant \frac{1}{2}$, all the eigenvalues of $\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)$ are non-negative. Hence we conclude that *when the filter function* $\Omega(\alpha,\alpha^*)$ *is of the form* $\exp(\lambda\alpha\alpha^*)$ *and if* $\lambda\geqslant\frac{1}{2}$, *then the mapping* Δ *operator* $\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)$ (*for mapping* $\tilde{\Omega}$ *antireciprocal to* Ω) *is a non-negative definite Hermitian operator.* In our subsequent discussion the limiting case $\lambda=\frac{1}{2}$ will be included, since the appropriate formula for this case may be obtained by the formal substitution $\lambda=\frac{1}{2}$, $n=0$.

Next we will show that $\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)$ is a bounded operator. From Eqs. (A11) and (A12), we obtain the following expression for the norm of $\hat{\Delta}^{(\tilde{\Omega})}$:

$$\|\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)\hat{D}(z)|n\rangle\| = [1/\pi(\lambda+\tfrac{1}{2})]\sigma^n\|\hat{D}(z)|n\rangle\|. \quad (A14)$$

It is obvious that

$$\|\hat{D}(z)|n\rangle\|^2 = \langle n|\hat{D}^\dagger(z)\hat{D}(z)|n\rangle = 1,$$

[32] This identity follows from the result [Eq. (I.6.42) with $f(n)=e^{-\beta n}$]

$$\exp(-\beta\hat{a}^\dagger\hat{a}) = \Omega^{(N)}\{\exp[-|z|^2(1-e^{-\beta})]\},$$

and the property (I.B9) of the displacement operator for the coherent state and the linearity of the mapping operator Θ.

and hence (A14) reduces to

$$\|\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)\hat{D}(z)|n\rangle\| = [1/\pi(\lambda+\tfrac{1}{2})]\sigma^n.$$

Since according to (A10), $\sigma<1$, it follows that

$$\|\Delta^{(\tilde{\Omega})}(z-\hat{a}, z^*-\hat{a}^\dagger)\hat{D}(z)|n\rangle\| < 1/\pi(\lambda+\tfrac{1}{2}). \quad (A15)$$

This inequality shows that $\hat{\Delta}^{(\tilde{\Omega})}$ is a bounded operator.

Next we show that $F_\rho^{(\Omega)}(z,z^*)$, as given by (A2) and (A3), satisfies the inequality

$$0 \leqslant F_\rho^{(\Omega)}(z,z^*) \leqslant 1 \quad \text{for all } z,z^*. \quad (A16)$$

The non-negativeness of $F_\rho^{(\Omega)}(z,z^*)$ for all z and z^* follows immediately from (A2) and the fact that both $\hat{\rho}$ and $\hat{\Delta}^{(\tilde{\Omega})}$ are *positive*-definite operators. To prove that $F_\rho^{(\Omega)}$ does not exceed unity, we combine Eqs. (A9) and (A2) and obtain the following expansion for $F_\rho^{(\Omega)}$:

$$F_\rho^{(\Omega)}(z,z^*) = \frac{1}{(\lambda+\frac{1}{2})}\sum_0^\infty \sigma^n\langle n|\hat{D}^\dagger(z)\hat{\rho}\hat{D}(z)|n\rangle. \quad (A17)$$

Let us express $\hat{\rho}$ in the form

$$\hat{\rho} = \sum_\lambda \rho_\lambda|\psi_\lambda\rangle\langle\psi_\lambda|, \quad (A18)$$

where ρ_λ are the eigenvalues and $|\psi_\lambda\rangle$ are the corresponding eigenstates of $\hat{\rho}$. Since $0\leqslant\rho_\lambda\leqslant 1$, we find that

$$F_\rho^{(\Omega)}(z,z^*) = \frac{1}{(\lambda+\frac{1}{2})}\sum_{n=0}^\infty\sum_\lambda \sigma^n\rho_\lambda\langle n|\hat{D}^\dagger(z)|\psi_\lambda\rangle\langle\psi_\lambda|\hat{D}(z)|n\rangle$$

$$\leqslant \frac{1}{(\lambda+\frac{1}{2})}\sum_0^\infty\sigma^n\sum_\lambda|\langle\psi_\lambda|\hat{D}(z)|n\rangle|^2$$

$$= \frac{1}{(\lambda+\frac{1}{2})}\sum_0^\infty\sigma^n$$

$$=1,$$

and hence

$$F_\rho^{(\Omega)}(z,z^*) \leqslant 1 \quad \text{for all } z,z^*.$$

If we employ the method of Ref. 33 (where the result is established for the special case $\lambda=\frac{1}{2}$), one can derive the following important result.

The function $F_\rho^{(\Omega)}(z,z^*)$, *regarded as a function of two real variables* x *and* y ($z=x+iy$), *is the boundary value of an entire analytic function of two complex variables* α *and* β ($x\to\alpha$, $y\to\beta$).

APPENDIX B: PROOF OF THEOREM V (PRODUCT THEOREM), EQ. (3.3)

Let $F_1^{(\Omega)}(z,z^*)$ and $F_2^{(\Omega)}(z,z^*)$ be the Ω equivalents of two operators $G_1(\hat{a},\hat{a}^\dagger)$ and $G_2(\hat{a},\hat{a}^\dagger)$, respectively. According to Theorem III [Eqs. (I.3.25) and (I.3.26)]

[33] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).

and Eq. (I.3.6), we have

$$F_j^{(\Omega)}(z,z^*)=\int \bar{\Omega}(\alpha,\alpha^*)g_j(\alpha,\alpha^*)\exp(\alpha z^*-\alpha^* z)d^2\alpha\,, \quad \text{(B1)}$$

where

$$g_j(\alpha,\alpha^*)=(1/\pi)\,\mathrm{Tr}[\hat{G}_j\hat{D}^\dagger(\alpha)] \quad (j=1,\,2)\,, \quad \text{(B2)}$$

$\hat{D}(\alpha)$ being the displacement operator for the coherent state $|\alpha\rangle$ [Eq. (I.B4)]. On the other hand, we have from the "operator Fourier integral theorem" [Eq. (I.C1)] the following representation for $\hat{G}_j$:

$$G_j(\hat{a},\hat{a}^\dagger)=\int g_j(\alpha,\alpha^*)\hat{D}(\alpha)d^2\alpha \quad (j=1,\,2)\,, \quad \text{(B3)}$$

where $g_j(\alpha,\alpha^*)$ is defined by (B2). It follows that

$$\hat{G}_1\hat{G}_2=\int\int g_1(\alpha,\alpha^*)g_2(\beta,\beta^*)\hat{D}(\alpha)\hat{D}(\beta)d^2\alpha d^2\beta\,. \quad \text{(B4)}$$

Now according to (I.B10),

$$\hat{D}(\alpha)\hat{D}(\beta)=\hat{D}(\alpha+\beta)\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]\,,$$

so that (B4) reduces to

$$G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)=\int\int g_1(\alpha,\alpha^*)g_2(\beta,\beta^*)\hat{D}(\alpha+\beta)\times\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]d^2\alpha d^2\beta\,. \quad \text{(B5)}$$

We now use the fact that

$$\Omega(\alpha+\beta,\,\alpha^*+\beta^*)\bar{\Omega}(\alpha+\beta,\,\alpha^*+\beta^*)=1$$

to rewrite (B5) in the form

$$\hat{G}_1\hat{G}_2=\int\int g_1(\alpha,\alpha^*)g_2(\beta,\beta^*)\bar{\Omega}(\alpha+\beta,\,\alpha^*+\beta^*)\times\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)][\Omega(\alpha+\beta,\,\alpha^*+\beta^*)\hat{D}(\alpha+\beta)]\times d^2\alpha d^2\beta\,. \quad \text{(B6)}$$

Next we make use of Eq. (I.3.17) to rewrite (B6) as

$$\hat{G}_1\hat{G}_2=\int\int g_1(\alpha_1\alpha^*)g_2(\beta,\beta^*)\times\bar{\Omega}(\alpha+\beta,\,\alpha^*+\beta^*)\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]\times[\Omega\{\exp[(\alpha+\beta)z^*-(\alpha^*+\beta^*)z]\}]d^2\alpha d^2\beta\,. \quad \text{(B7)}$$

In view of the linearity of the mapping operator Ω, Eq. (B7) may also be expressed in the form

$$\hat{G}_1\hat{G}_2=\Omega\left\{\int\int g_1(\alpha_1\alpha^*)g_2(\beta,\beta^*)\times\bar{\Omega}(\alpha+\beta,\,\alpha^*+\beta^*)\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]\times\exp[(\alpha+\beta)z^*-(\alpha^*+\beta^*)z]d^2\alpha d^2\beta\right\}. \quad \text{(B8)}$$

Let $F_{12}^{(\Omega)}(z,z^*)$ be the Ω equivalent of $G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)$, i.e.,

$$G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)=\Omega\{F_{12}^{(\Omega)}(z,z^*)\}\,. \quad \text{(B9)}$$

On comparing (B8) with (B9), we see that

$$F_{12}^{(\Omega)}(z,z^*)=\int\int g_1(\alpha,\alpha^*)g_2(\beta,\beta^*)\times\bar{\Omega}(\alpha+\beta,\,\alpha^*+\beta^*)\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]\times\exp[(\alpha+\beta)z^*-(\alpha^*+\beta^*)z]d^2\alpha d^2\beta\,. \quad \text{(B10)}$$

Now we express $F_{12}^{(\Omega)}(z,z^*)$, as given by (B10), in terms of $F_1^{(\Omega)}(z,z^*)$, $F_2^{(\Omega)}(z,z^*)$, and their derivatives. For this purpose we note that (B10) may be expressed in the following form:

$$F_{12}^{(\Omega)}(z,z^*)=f(z_1,z_1^*;\,z_2,z_2^*)\,|_{z_1=z_2=z;\,z_1^*=z_2^*=z^*}\,, \quad \text{(B11)}$$

where

$$f(z_1,z_1^*;\,z_2,z_2^*)=\int\int g_1(\alpha,\alpha^*)g_2(\beta,\beta^*)\bar{\Omega}(\alpha+\beta,\,\alpha^*+\beta^*)\times\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]\exp(\alpha z_1^*-\alpha^* z_1)\exp(\beta z_2^*-\beta^* z_2)d^2\alpha d^2\beta \quad \text{(B12)}$$

$$=\int\int[g_1(\alpha,\alpha^*)\bar{\Omega}(\alpha,\alpha^*)\exp(\alpha z_1^*-\alpha^* z_1)][g_2(\beta,\beta^*)\bar{\Omega}(\beta,\beta^*)\exp(\beta z_2^*-\beta^* z_2)]\times\Omega(\alpha,\alpha^*)\Omega(\beta,\beta^*)\bar{\Omega}(\alpha+\beta,\,\alpha^*+\beta^*)\exp[\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]d^2\alpha d^2\beta\,. \quad \text{(B13)}$$

In passing from (B12) to (B13), we have in appropriate places multiplied and divided by the product $\Omega(\alpha,\alpha^*)\times\Omega(\beta,\beta^*)$. Is is easily seen that (B13) may be written as

$$f(z_1,z_1^*;\,z_2,z_2^*)=\Omega\left(\frac{\partial}{\partial z_1^*},\,-\frac{\partial}{\partial z_1}\right)\Omega\left(\frac{\partial}{\partial z_2^*},\,-\frac{\partial}{\partial z_2}\right)\bar{\Omega}\left(\frac{\partial}{\partial z_1^*}+\frac{\partial}{\partial z_2^*},\,-\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right)\times\exp\left[\frac{1}{2}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2^*}-\frac{\partial}{\partial z_1^*}\frac{\partial}{\partial z_2}\right)\right]\int\int[g_1(\alpha,\alpha^*)\bar{\Omega}(\alpha,\alpha^*)\exp(\alpha z_1^*-\alpha^* z_1)]\times[g_2(\beta,\beta^*)\bar{\Omega}(\beta,\beta^*)\exp(\beta z_2^*-\beta^* z_2)]d^2\alpha d^2\beta\,. \quad \text{(B14)}$$

Here we have made use of the identity

$$\exp(\alpha)\ \exp(\alpha z^*-\alpha^* z)=\exp(\partial/\partial z^*)\ \exp(\alpha z^*-\alpha^* z).$$

On making use of (B1), Eq. (B14) simplifies to

$$f(z_1,z_1^*;\, z_2,z_2^*)=\exp(\Lambda_{12})\mathcal{U}_{12}{}^{(\Omega)}F_1{}^{(\Omega)}(z_1,z_1^*)\times F_2{}^{(\Omega)}(z_2,z_2^*)\,, \quad \text{(B15)}$$

where the operators Λ_{12} and $\mathcal{U}_{12}{}^{(\Omega)}$ are defined by

$$\Lambda_{12}=\frac{1}{2}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2^*}-\frac{\partial}{\partial z_1^*}\frac{\partial}{\partial z_2}\right) \quad \text{(B16)}$$

and

$$\mathcal{U}_{12}{}^{(\Omega)}=\Omega\left(\frac{\partial}{\partial z_1^*},-\frac{\partial}{\partial z_1}\right)\Omega\left(\frac{\partial}{\partial z_2^*},-\frac{\partial}{\partial z_2}\right)\times\tilde{\Omega}\left(\frac{\partial}{\partial z_1^*}+\frac{\partial}{\partial z_2^*},-\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right). \quad \text{(B17)}$$

Finally, on combining (B11) and (B15), we obtain the desired product theorem [Eq. (3.3)].

APPENDIX C: DERIVATION OF EQUATION OF MOTION (7.11) FOR GENERALIZED CHARACTERISTIC FUNCTION

It is evident from the definition (7.2) of the generalized characteristic function $C^{(\Omega)}$ for $\tilde{\Omega}$ mapping and from the Schrödinger equation of motion (4.6) for the density operator that

$$i\hbar\frac{\partial C^{(\Omega)}}{\partial t}=\mathrm{Tr}\left[i\hbar\frac{\partial\hat{\rho}}{\partial t}\tilde{\Omega}\{\exp[-(\alpha z^*-\alpha^* z)]\}\right]$$
$$=\mathrm{Tr}[\hat{H}\hat{\rho}\tilde{\Omega}\{\exp[-(\alpha z^*-\alpha^* z)]\}]-\mathrm{Tr}[\hat{H}\hat{\rho}\tilde{\Omega}\{\exp[-(\alpha z^*-\alpha^* z)]\}] \quad \text{(C1)}$$

or

$$i\hbar\partial C^{(\Omega)}/\partial t=\bar{\Omega}(\alpha,\alpha^*)\ \mathrm{Tr}[\hat{H}\hat{\rho}\hat{D}^\dagger(\alpha)]-\bar{\Omega}(\alpha,\alpha^*)\ \mathrm{Tr}[\hat{\rho}\hat{H}\hat{D}^\dagger(\alpha)], \quad \text{(C2)}$$

where $\hat{D}(\alpha)$ is the displacement operator (I.B4) for the coherent states. To simplify the right-hand side of (C2), we make use of the operator convolution theorem discussed in Appendix C [Eq. (I.C4)] of I. It follows from this theorem that

$$\mathrm{Tr}[\hat{H}\hat{\rho}\hat{D}^\dagger(\alpha)]=\int g_H(\beta,\beta^*)C^{(W)}(\alpha-\beta,\,\alpha^*-\beta^*)\times\exp[-\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]d^2\beta, \quad \text{(C3)}$$

where

$$g_H(\beta,\beta^*)=\frac{1}{\pi}\mathrm{Tr}[\hat{H}\hat{D}^\dagger(\alpha)]. \quad \text{(C4)}$$

Next we use the identity

$$\Omega(-\beta,\,-\beta^*)\tilde{\Omega}(\beta,\beta^*)=1$$

and the relation

$$C^{(W)}(\alpha,\alpha^*)=\Omega(\alpha,\alpha^*)C^{(\Omega)}(\alpha,\alpha^*),$$

which follows from Eq. (7.7), to rewrite Eq. (C4) in the form

$$\mathrm{Tr}[\hat{H}\hat{\rho}\hat{D}^\dagger(\alpha)]=\int[\Omega(-\beta,\,-\beta^*)g_H(\beta,\beta^*)]\times C^{(\Omega)}(\alpha-\beta,\,\alpha^*-\beta^*)\tilde{\Omega}(\beta,\beta^*)\Omega(\alpha-\beta,\,\alpha^*-\beta^*)\times\exp[-\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]d^2\beta. \quad \text{(C5)}$$

On substituting for $\Omega(-\beta,\,-\beta^*)g_H(\beta,\beta^*)$ in terms of $F_H{}^{(\tilde{\Omega})}$ [Eqs. (I.3.25) and (I.3.26)], we obtain

$$\mathrm{Tr}\{\hat{H}\hat{\rho}\hat{D}^\dagger(\alpha)\}=\frac{1}{\pi^2}\int\int F_H{}^{(\tilde{\Omega})}(z,z^*)C^{(\Omega)}(\alpha-\beta,\,\alpha^*-\beta^*)\times\exp[-(\beta z^*-\beta^* z)]\tilde{\Omega}(\beta,\beta^*)\Omega(\alpha-\beta,\,\alpha^*-\beta^*)\times\exp[-\tfrac{1}{2}(\alpha\beta^*-\alpha^*\beta)]d^2\beta d^2z. \quad \text{(C6)}$$

We also have the obvious identity that follows from Taylor's expansion of $C^{(\Omega)}(\alpha-\beta,\,\alpha^*-\beta^*)$ around α, α^*:

$$C^{(\Omega)}(\alpha-\beta,\,\alpha^*-\beta^*)=\exp\left(-\beta\frac{\partial}{\partial\alpha}-\beta^*\frac{\partial}{\partial\alpha^*}\right)\times C^{(\Omega)}(\alpha,\alpha^*). \quad \text{(C7)}$$

From (C6) and (C7) it follows that

$$\mathrm{Tr}[\hat{H}\hat{\rho}\hat{D}^\dagger(\alpha)]=\frac{1}{\pi^2}\int\int F_H{}^{(\tilde{\Omega})}(z,z^*)\times\exp[\beta(\tfrac{1}{2}\alpha^*-z^*)-\beta^*(\tfrac{1}{2}\alpha-z)]\times\tilde{\Omega}(\beta,\beta^*)\Omega(\alpha-\beta,\,\alpha^*-\beta^*)\times\exp(-\beta\partial/\partial\alpha-\beta^*\partial/\partial\alpha^*)C^{(\Omega)}(\alpha,\alpha^*)d^2\beta d^2z. \quad \text{(C8)}$$

Now it can be shown by straightforward but long calculations that (C8) may be rewritten in the form

$$\mathrm{Tr}[\hat{H}\hat{\rho}\hat{D}^\dagger(\alpha)]=\tilde{\Omega}\left(\frac{\partial}{\partial\alpha_1^*},-\frac{\partial}{\partial\alpha_1}\right)\Omega\left(\alpha-\frac{\partial}{\partial\alpha_1^*},\alpha^*+\frac{\partial}{\partial\alpha_1}\right)\times\exp\left(\frac{\partial}{\partial\alpha_1}\frac{\partial}{\partial\alpha_2^*}-\frac{\partial}{\partial\alpha_1^*}\frac{\partial}{\partial\alpha_2}\right)F_H{}^{(\tilde{\Omega})}(\alpha_1,\alpha_1^*)\times C^{(\Omega)}(\alpha_2,\alpha_2^*)\big|_{\alpha_1=\alpha/2;\,\alpha_2=\alpha;\,\alpha_1^*=\alpha^*/2;\,\alpha_2^*=\alpha^*}. \quad \text{(C9)}$$

Further, proceeding in a manner strictly similar to the one which led to (C9), we find that

$$\mathrm{Tr}[\hat{\rho}\hat{H}\hat{D}^\dagger(\alpha)]=\tilde{\Omega}\left(\frac{\partial}{\partial\alpha_1^*},-\frac{\partial}{\partial\alpha_1}\right)\Omega\left(\alpha-\frac{\partial}{\partial\alpha_1^*},\alpha^*+\frac{\partial}{\partial\alpha_1}\right)\times\exp\left(\frac{\partial}{\partial\alpha_1}\frac{\partial}{\partial\alpha_2^*}-\frac{\partial}{\partial\alpha_1^*}\frac{\partial}{\partial\alpha_2}\right)F_H{}^{(\tilde{\Omega})}(\alpha_1,\alpha_1^*)\times C^{(\Omega)}(\alpha_2,\alpha_2^*)\big|_{\alpha_1=-\alpha/2;\,\alpha_2=\alpha;\,\alpha_1^*=-\alpha^*/2;\,\alpha_2^*=\alpha^*}. \quad \text{(C10)}$$

On combining (C2), (C9), and (C10), we obtain the desired equation of motion for the generalized characteristic function $C^{(\Omega)}$:

$$i\hbar\partial C^{(\Omega)}/\partial t=(\mathfrak{N}_+^{(\Omega)}-\mathfrak{N}_-^{(\Omega)})C^{(\Omega)}, \qquad \text{(C11)}$$

where

$$\mathfrak{N}_+C^{(\Omega)}=\exp(\Lambda_{12}')\,\mathcal{V}^{(\Omega)}F_H^{(\tilde\Omega)}(\alpha_1,\alpha_1^*)\times C^{(\Omega)}(\alpha_2,\alpha_2^*;t)\,|_{\alpha_1=\alpha/2,\alpha_2=\alpha;\alpha_1^*=\alpha^*/2,\alpha_2^*=\alpha^*}, \qquad \text{(C12a)}$$

and

$$\mathfrak{N}_-C^{(\Omega)}=\exp(\Lambda_{12}')\,\mathcal{V}^{(\Omega)}F_H^{(\tilde\Omega)}(\alpha_1,\alpha_1^*)\times C^{(\Omega)}(\alpha_2,\alpha_2^*;t)\,|_{\alpha_1=-\alpha/2;\alpha_2=\alpha;\alpha_1^*=-\alpha^*/2;\alpha_2^*=\alpha^*}. \qquad \text{(C12b)}$$

The operators Λ_{12}' and $\mathcal{V}^{(\Omega)}$ are defined by the formulas

$$\Lambda_{12}'=\left(\frac{\partial}{\partial\alpha_1}\frac{\partial}{\partial\alpha_2^*}-\frac{\partial}{\partial\alpha_1^*}\frac{\partial}{\partial\alpha_2}\right) \qquad \text{(C13)}$$

and

$$\mathcal{V}^{(\Omega)}=\bar\Omega(\alpha,\alpha^*)\times\tilde\Omega\left(\frac{\partial}{\partial\alpha_1^*},-\frac{\partial}{\partial\alpha_1}\right)\Omega\left(\alpha-\frac{\partial}{\partial\alpha_1^*},\alpha^*+\frac{\partial}{\partial\alpha_1}\right). \qquad \text{(C14)}$$

APPENDIX D: CHARACTERISTIC FUNCTIONAL OF QUANTIZED FIELD AND PROOF OF (9.16)

The space-time quasiprobability of a quantized field, for the normal rule of association, was defined by the first expression of the right-hand side of Eq. (9.9), viz.,

$$p^{[N]}[\mathbf{V}(1),\mathbf{V}(2),\ldots,\mathbf{V}(M);x_1,\ldots,x_M]=\langle\Omega^{(N)}\{\prod_{i=1}^{M}\delta[\mathbf{V}(i)-\mathbf{V}(x_i)]\}\rangle. \qquad \text{(D1)}$$

If in (D1) we express the Dirac δ function in the form of a Fourier integral, we obtain the following expression for $p^{(N)}$:

$$p^{(N)}[\mathbf{V}(1),\mathbf{V}(2),\ldots,\mathbf{V}(M);x_1,\ldots,x_M]=\left(\frac{1}{\pi^2}\right)^{3M}\int\left\langle\Omega^{(N)}\left\{\prod_{i=1}^{M}\exp\{i[\mathbf{U}(i)\cdot\mathbf{V}^*(x_i)+\mathbf{U}^*(i)\cdot\mathbf{V}(x_i)]\}\right\}\right\rangle\times\prod_{i=1}^{M}\exp\{-i[\mathbf{U}(i)\cdot\mathbf{V}^*(i)+\mathbf{U}^*(i)\cdot\mathbf{V}(i)]\}d^2U(1)\cdots d^2U(M). \qquad \text{(D2)}$$

The expectation value on the right-hand side of (D2) may conveniently be expressed in terms of the characteristic functional

$$\mathcal{C}^{(N)}[\mathbf{W}(\cdot)]=\left\langle\Omega^{(N)}\left\{\exp\left[i\int\mathbf{W}(x)\cdot\mathbf{V}^*(x)dx+i\int\mathbf{W}^*(x)\cdot\mathbf{V}(x)dx\right]\right\}\right\rangle, \qquad \text{(D3)}$$

where $\mathbf{W}(x)$ is an arbitrary vector function of the space-time variable $x\equiv(\mathbf{r},t)$. Let us substitute for $\mathbf{V}(x)$ and $\mathbf{V}^*(x)$ the series expansions (9.5a) and (9.5b). We then obtain the following expression for the characteristic functional $\mathcal{C}^{(N)}$:

$$\mathcal{C}^{(N)}[\mathbf{W}(\cdot)]=\langle\Omega^{(N)}\{\exp[i\sum_{ks}(W_{ks}z_{ks}^*+W_{ks}^*z_{ks})]\}\rangle, \qquad \text{(D4)}$$

where

$$W_{ks}=\frac{1}{L^{3/2}}\left(\frac{\hbar c}{k}\right)^{1/2}\int\mathbf{W}(\mathbf{r},t)\cdot\epsilon_{ks}^*\times\exp[-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)]d^3r\,dt. \qquad \text{(D5)}$$

We may express the characteristic functional $\mathcal{C}^{(N)}$ as a phase-space integral by applying to (D4) the identity (8.13). The result is

$$\mathcal{C}^{(N)}[\mathbf{W}(\cdot)]=\int\Phi^{(A)}(\{z_{ks}\},\{z_{ks}^*\})\times\exp[i\sum_{ks}(W_{ks}z_{ks}^*+W_{ks}^*z_{ks})]d^2(\{z_{ks}\}). \qquad \text{(D6)}$$

From the characteristic functional, all the statistical properties of the quantized field may be derived. For example, the normally ordered correlation functions $\Gamma^{(n,m)}$ of the quantized field, defined by Eq. (9.6), may be obtained from the formula

$$\Gamma_{j_1,j_2,\ldots,j_{n+m}}^{(n,m)}(x_1,\ldots,x_n;x_{n+1},\ldots,x_{n+m})=\left.\frac{(-i)^{n+m}\delta^{n+m}\mathcal{C}^{(N)}[\mathbf{W}(\cdot)]}{\delta W_{j_1}(x_1)\cdots\delta W_{j_n}(x_n)\delta W_{j_{n+1}}^*(x_{n+1})\cdots\delta W_{j_{n+m}}^*(x_{n+m})}\right|_{W=0}, \qquad \text{(D7)}$$

where $\delta/\delta W(x)$ denotes the functional derivative.[34] Further, we see from (D2) and (D3) that the space-time quasiprobabilities may be expressed in the form

$$p^{(N)}[\mathbf{V}(1),\ldots,\mathbf{V}(M);x_1,\ldots,x_M]=\frac{1}{\pi^{6M}}\int \mathcal{C}^{(N)}[\mathbf{U}(\cdot)]\times\exp\{-i\sum_{i=1}^{M}[\mathbf{U}^*(i)\cdot\mathbf{V}(i)+\mathbf{U}(i)\cdot\mathbf{V}^*(i)]\}\times d^2U(1)\cdots d^2U(M),\quad \text{(D8)}$$

where

$$\mathbf{U}(x)=\sum_{i=1}^{M}\mathbf{U}(i)\delta^{(4)}(x-x_i).\quad \text{(D9)}$$

We will now derive with the help of the characteristic functional an explicit expression for the space-time quasiprobabilities for an electromagnetic field that is in thermal equilibrium at temperature T. We have from (D6) and (9.14)

$$\mathcal{C}^{(N)}[\mathbf{W}(\cdot)]=\int\prod_{ks}\frac{1}{\pi\tau_{ks}}\exp\left(-\frac{|z_{ks}|^2}{\tau_{ks}}\right)\times\exp[i\sum_{ks}(W_{ks}z_{ks}^*+W_{ks}^*z_{ks})]d^2(\{z_{ks}\})=\exp[-\sum_{ks}\tau_{ks}|W_{ks}|^2],\quad \text{(D10)}$$

where, in accordance with (9.15),

$$\tau_{ks}=[1-\exp(-\theta\omega_k)]^{-1}-1,\quad \text{(D11)}$$

and $\theta=1/kT$. It is clear from (9.14) that τ_{ks} is the variance of the distribution $\Phi^{(A)}$,

$$\tau_{ks}=\int\Phi^{(A)}(\{z_{ks}\},\{z_{ks}^*\})z_{ks}^*z_{ks}d^2(\{z_{ks}\}).\quad \text{(D12)}$$

From now on *we will denote by* $\langle\ \rangle_{\text{p.s.}}$ *the phase-space average with respect to the distribution function* $\Phi^{(A)}$, so that (D12) may be written as

$$\tau_{ks}=\langle z_{ks}^*z_{ks}\rangle_{\text{p.s.}}.\quad \text{(D12}')$$

[34] For the definition of the functional derivative, see, e.g., R. J. Glauber, in *Quantum Optics and Electronics*, edited by C. deWitt, A. Blanden, and C. Cohen-Tannoudji (Gordon and Breach, New York, 1965), p. 65.

It follows from (D10), (D5) and (D12′) that

$$\mathcal{C}^{(N)}[\mathbf{W}(\cdot)]=\exp\left\{-\int\int\langle[\mathbf{W}(x_1)\cdot\mathbf{V}^*(x_1)]\times[\mathbf{W}^*(x_2)\cdot\mathbf{V}(x_2)]\rangle_{\text{p.s.}}dx_1dx_2\right\}.\quad \text{(D13)}$$

If we choose for $\mathbf{W}(x)$ the function $\mathbf{U}(x)$ defined by (D9), we obtain the formula

$$\mathcal{C}^{(N)}[\mathbf{U}(\cdot)]=\exp\{-\sum_i\sum_j\langle[\mathbf{U}(i)\cdot\mathbf{V}^*(x_i)]\times[\mathbf{U}^*(j)\cdot\mathbf{V}(x_j)]\rangle_{\text{p.s.}}\}.\quad \text{(D14)}$$

It will be convenient to introduce the column matrices

$$\mathcal{V}=\begin{bmatrix}V_1(1)\\V_2(1)\\V_3(1)\\\vdots\\V_1(M)\\V_2(M)\\V_3(M)\end{bmatrix},\quad \mathcal{V}'=\begin{bmatrix}V_1(x_1)\\V_2(x_1)\\V_3(x_1)\\\vdots\\V_1(x_M)\\V_2(x_M)\\V_3(x_M)\end{bmatrix},\quad \text{(D15)}$$

and the column matrices $\mathcal{U}$ defined in a similar way as the column vector $\mathcal{V}$, with $V(i)$'s replaced by $U(i)$'s. The expression (D14) for $\mathcal{C}^{(N)}[\mathbf{U}(\cdot)]$ may then be written in the compact form

$$\mathcal{C}^{(N)}[\mathbf{U}(\cdot)]=\exp(-\mathcal{U}^\dagger R^{(N)}\mathcal{U}),\quad \text{(D16)}$$

where

$$R^{(N)}=\langle\mathcal{V}'\mathcal{V}'^\dagger\rangle_{\text{p.s.}}.\quad \text{(D17)}$$

We now substitute from (D16) into (D2) and obtain the following expression for the space-time quasiprobability $p^{(N)}$:

$$p^{(N)}[\mathbf{V}(1),\ldots,\mathbf{V}(M);x_1,\ldots,x_M]=\frac{1}{\pi^{6m}}\int\exp(-\mathcal{U}^\dagger R^{(N)}\mathcal{U})\exp[-i(\mathcal{U}^\dagger\mathcal{V}+\mathcal{V}^\dagger\mathcal{U})]\times d^2U(1)\cdots d^2U(m).\quad \text{(D18)}$$

The integral on the right-hand side of (D18) is well known[35] and leads to the following expression for $p^{(N)}$:

$$p^{(N)}=\frac{1}{\pi^{3M}|\det R^{(N)}|}\exp[-\mathcal{V}^\dagger(R^{(N)})^{-1}\mathcal{V}].\quad \text{(D19)}$$

This is formula (9.16).

[35] For such identities involving the quadratic form, see, e.g., K. S. Miller, *Multidimensional Gaussian Distributions* (Wiley, New York, 1964), p. 15.