Calculus for Functions of Noncommuting Operators and General Phase-Space Methods in Quantum Mechanics. I. Mapping Theorems and Ordering of Functions of Noncommuting Operators*

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A new calculus for functions of noncommuting operators is developed, based on the notion of mapping of functions of operators onto *c*-number functions. The class of linear mappings, each member of which is characterized by an entire analytic function of two complex variables, is studied in detail. Closed-form solutions for such mappings and for the inverse mappings are obtained and various properties of these mappings are studied. It is shown that the most commonly occurring rules of association between operators and *c*-numbers (the Weyl, the normal, the antinormal, the standard, and the antistandard rules) belong to this class and are, in fact, the simplest ones in a clearly defined sense. It is shown further that the problem of expressing an operator in an ordered form according to some prescribed rule is equivalent to an appropriate mapping of the operator on a *c*-number space. The theory provides a systematic technique for the solution of numerous quantum-mechanical problems that were treated in the past by *ad hoc* methods, and it furnishes a new approach to many others. This is illustrated by a number of examples relating to mappings and ordering of operators.

I. INTRODUCTION

In recent years increasing use has been made of generalized phase-space techniques in the treatment of various quantum-mechanical problems. Similar techniques have previously been used chiefly in quantum statistical mechanics.[1] They have their origin in a well-known paper by Wigner,[2] dealing with quantum corrections to thermodynamics. Wigner associated with the state of a quantum-mechanical system a certain *c*-number function—now generally known as the Wigner distribution function—and he showed that by means of it quantum-mechanical expectation values can be expressed in the same mathematical form as the averages of classical statistical mechanics. Wigner also pointed out that his distribution function is not the only one that makes this possible. Later Groenewold[3] and Moyal[4] developed Wigner's technique further. It became apparent from their work that the association of the Wigner distribution function with the state function of the system involves implicitly a certain rule of correspondence between functions of noncommuting operators and *c*-number functions investigated earlier by Weyl[5] in his group-theoretical studies on the correspondence between classical and quantum mechanics. Weyl's correspondence is intimately related to a certain rule of *ordering* of functions of noncommuting operators (the so-called Weyl symmetrization rule—see Appendix A and Ref. 6).

Since these early investigations, other rules of associations between operators and *c*-numbers have been studied[7–9] and some of them formed the basis for introducing other distribution functions, by means of which the solution to quantum-mechanical problems could be expressed in a quasiclassical form. During the course of the last few years such methods have become of central importance in quantum optics, especially in studies of coherence properties of light,[10–12] in the theory of the laser,[13–15] and in investigations of non linear processes such as parametric oscillation.[16] One of the main underlying reasons why these methods have proved so useful arises from the fact that the quantities of interest are often the expectation values of various operators arranged in some particular ordered form. For

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[1] See, e.g., (a) R. Kubo, J. Phys. Soc. Japan **19**, 2127 (1964); (b) R. Kubo, in *Lectures in Theoretical Physics, 1965*, edited by W. E. Brittin (University of Colorado Press, Boulder, Colo., 1966), Vol. VIII A, p. 239; (c) K. Imre, E. Ozizmir, M. Rosenbaum, and P. F. Zweifel, J. Math. Phys. **8**, 1097 (1967); (d) H. Mori, I. Oppenheim, and J. Ross, in *Studies in Statistical Mechanics*, edited by J. de Boer and G. E. Uhlenbeck (North-Holland, Amsterdam, 1962), Vol. I, p. 217.

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[6] The symmetrization rule and some properties of the Wigner distribution function have been discussed by E. C. G. Sudarshan, in *Lectures in Theoretical Physics* (Benjamin, New York, 1961), Vol. II, p. 143; see also T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. **33**, 515 (1961).

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[16] R. Graham, Z. Physik **210**, 319 (1968).

example, in the study of the statistical properties of light by means of photoelectric correlation and coincidence experiments, the outcome of the experiments is most naturally expressed in terms of the normally ordered products of the photon annihilation and creation operators.[12,17,18] Another example is light scattering from fluctuations in liquids and gases. In this case the results of observation are expressed in terms of time-ordered expressions that involve the fluctuating density of the particles in the medium.[19]

In spite of the considerable use of the phase-space techniques in several areas of quantum physics, most of the theoretical results have up to now been derived by various *ad hoc* methods[20] (see, for example, Refs. 21–26). In the present series of papers we develop a general phase-space calculus from which most of the earlier results and many new ones may be derived in a systematic manner. First we show, in Sec. II, that the problem of expressing a function $G(\hat{a},\hat{a}^\dagger)$ of two noncommuting operators that obey the commutation relation $[\hat{a},\hat{a}^\dagger]=1$ in some *ordered form* is equivalent to the problem of mapping it onto a c-number function $F(z,z^*)$, where z is a complex variable and z^* is its complex conjugate. The class of linear analytic mappings is then defined, and it is shown that this class includes all the mappings that are required for ordering $G(\hat{a},\hat{a}^\dagger)$ according to the usual ordering rules, such as the Weyl, the normal, the antinormal, the standard, and the antistandard rules. With each mapping we associate a mapping operator Ω and its inverse Θ, such that

$$\Omega\{F(z,z^*)\}=G(\hat{a},\hat{a}^\dagger), \tag{1.1}$$

$$\Theta\{G(\hat{a},\hat{a}^\dagger)\}=F(z,z^*). \tag{1.2}$$

In Sec. III we present closed-form solutions for such mappings and illustrate the results by considering in detail the mappings that involve the normal, the antinormal, and the Weyl rules of associations. In particular it is found that for the antinormal rule of association the mapping $F(z,z^*)\rightleftarrows G(\hat{a},\hat{a}^\dagger)$ leads at once to the Sudarshan-Glauber diagonal coherent-state representation[10,11] of the operator $\hat{G}$. This result throws some new light on the true significance of the weighting function that appears in their representation.

Closed-form solutions for the mappings (1.1) and (1.2) are expressed as transforms involving certain new operators that we call the mapping delta operators. They are basic for the present theory. The properties of these operators are studied in Sec. IV. In Sec. V relations connecting the c-number functions $F(z,z^*)$, derived from one and same operator $G(\hat{a},\hat{a}^\dagger)$ by different mappings, are established. In Sec. VI our main results are illustrated by examples, relating both to mapping and ordering. In Sec. VII the main theorems are expressed in a form appropriate to the mapping of functions of the canonical operators $\hat{q}$, $\hat{p}$ ($[\hat{q},\hat{p}]=i\hbar$) rather than $\hat{a}$ and $\hat{a}^\dagger$.

For the sake of simplicity we consider in this paper systems with one degree of freedom only; generalizations to systems with a finite or denumerably infinite number of degrees of freedom are straightforward and will be briefly discussed in the second paper of this series. Our analysis may be also readily extended to problems that involve the mapping of functions of operators $\hat{b}$ and $\hat{b}^\dagger$, which obey the commutation relation $[\hat{b},\hat{b}^\dagger]=c$, where c is any real constant. Thus a system of pseudo-oscillators (corresponding to the choice $c=-1$) may also be treated by the present techniques.[27] Moreover, this technique may also be applied to problems that involve the spin angular momentum, if use is made of Schwinger's coupled boson representation.[28–30]

In Paper II of this investigation,[31] we define a wide class of generalized phase-space distribution functions of a quantum-mechanical system in terms of the functions obtained by mappings of the density operator onto a c-number space (the phase space) and we derive the phase-space form of the basic equations of quantum dynamics and quantum statistics. In Paper III our phase-space calculus is applied to various time-ordering problems.[32] In particular we show that it provides a new derivation and a deeper understanding of Wick's theorem and that it leads to some new generalizations of it. A connection with some recent work of Lax[33,34] on multitime correspondence between quantum and stochastic systems is also established in that paper.

It will become apparent that in view of the very wide scope of this theory, a rigorous formulation of it can

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[21] W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw Hill, New York, 1964), Chap. III.

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[28] J. Schwinger, in *Quantum Theory of Angular Momentum*, edited by J. Schwinger, L. C. Biedenharn, and H. VanDam (Academic, New York, 1965), p. 229.

[29] See, e.g., T. L. Wang and H. B. Callen, Phys. Rev. **148**, 433 (1966).

[30] This technique has already been applied by one of the authors [G. S. Agarwal, Phys. Rev. **178**, 2025 (1969)] in the treatment of a two-level system interacting with a reservoir.

[31] G. S. Agarwal and E. Wolf, first following paper, Phys. Rev. D **2**, 2187 (1970).

[32] G. S. Agarwal and E. Wolf, second following paper, Phys. Rev. D **2**, 2206 (1970).

[33] M. Lax, Phys. Rev. **157**, 213 (1967).

[34] M. Lax, Phys. Rev. **172**, 350 (1968).

only be carried out within the framework of the theory of generalized functions. We do not attempt to provide a mathematically rigorous basis for our theory; rather we show how a great wealth of old and new results on ordering of operators and on phase-space descriptions of quantum-mechanical systems may be derived in a systematic manner from a few new principles. These principles are brought out by the concept of a linear analytic mapping of functions of noncommuting operators on c-number functions.

II. MAPPING OPERATORS AND ORDERING OF FUNCTIONS OF NONCOMMUTING OPERATORS

In the first part of this investigation we will be concerned with mapping of functions of c-numbers onto functions of noncommuting operators and with the inverse mapping. It will be helpful to begin by summarizing first some of the better known rules of association[7–9] between elementary c-number functions of two real variables p, q, and functions of two Hermitian operators[35] $\hat{p}$ and $\hat{q}$, satisfying the commutation relation (see Table I)

$$[\hat{q},\hat{p}]=i\hbar\,. \tag{2.1}$$

In Table I m and n are non-negative integers. The symbol $(\hat{p}^m\hat{q}^n)_W$ denotes the Weyl-symmetrized form of the product $\hat{p}^m\hat{q}^n$, i.e., the linear combination of all possible products, involving m $\hat{p}$'s and n $\hat{q}$'s, divided by the total number of such products. For example,

$$(\hat{p}^2\hat{q})_W=\tfrac{1}{3}(\hat{p}^2\hat{q}+\hat{p}\hat{q}\hat{p}+\hat{q}\hat{p}^2)\,. \tag{2.2}$$

We may also consider rules of association between functions of complex c-numbers z and z^* (where z^* denotes the complex conjugate of z) and functions of the boson annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$, obeying the commutation relation

$$[\hat{a},\hat{a}^\dagger]=1\,. \tag{2.3}$$

The better-known rules of association in this case are listed in Table II.

In Table II the symbol $(\hat{a}^{\dagger m}\hat{a}^n)_W$ is defined in a similar way as $(\hat{p}^m\hat{q}^n)_W$, i.e., it represents the sum of all possible products involving m creation operators $\hat{a}^\dagger$ and n annihilation operators $\hat{a}$, divided by the total number of such products.

With each rule of association indicated in Table II, and, more generally, with each rule of a wider class to be defined in Sec. III, we associate a *linear mapping operator* Ω, which transforms an arbitrary function of the c-numbers z, z^* onto a function of the operators $\hat{a}$, $\hat{a}^\dagger$ that obey the commutation relation (2.3).[36] If we denote by $G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ the function onto which the monomial $z^{*m}z^n$ is mapped, the operator Ω is defined by the equation

(i) $\Omega\{z^{*m}z^n\}=G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$

and by the linearity conditions

(ii) $\Omega\{c_1F_1(z,z^*)+c_2F_2(z,z^*)\}=c_1\Omega\{F_1(z,z^*)\}+c_2\Omega\{F_2(z,z^*)\}$,

(iii) $\Omega\{c\}=c1$.

Here F, F_1, and F_2 are arbitrary functions, c is a c-number, and 1 is the identity operator.

When we wish to stress that a mapping is associated with a particular rule, we will attach an appropriate superscript to Ω. Thus for normal,[37] antinormal, and Weyl rules of association (which we denote by suffices N, A, and W, respectively), condition (i) becomes

$$\Omega^{(N)}\{z^{*m}z^n\}=\hat{a}^{\dagger m}\hat{a}^n\,, \tag{2.4a}$$

$$\Omega^{(A)}\{z^{*m}z^n\}=\hat{a}^n\hat{a}^{\dagger m}\,, \tag{2.4b}$$

$$\Omega^{(W)}\{z^{*m}z^n\}=(\hat{a}^{\dagger m}\hat{a}^n)_W\,. \tag{2.4c}$$

From the definition of the operator Ω, it is evident that Ω will map a c-number function $F(z,z^*)$, which has the power-series expansion

$$F(z,z^*)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}z^{*m}z^n\,, \tag{2.5}$$

TABLE I. Some of the commonly employed rules of association between functions of the two real c-number variables p and q and functions of two canonical operators $\hat{p}$ and $\hat{q}$.

Rule of association	
Standard	$p^mq^n\to\hat{q}^n\hat{p}^m$
Antistandard	$p^mq^n\to\hat{p}^m\hat{q}^n$
Weyl	$p^mq^n\to(\hat{p}^m\hat{q}^n)_W$

TABLE II. Some of the commonly employed rules of association between functions of c-number variables z and z^* and functions of the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$.

Rule of association	
Normal	$z^{*m}z^n\to\hat{a}^{\dagger m}\hat{a}^n$
Antinormal	$z^{*m}z^n\to\hat{a}^n\hat{a}^{\dagger m}$
Weyl	$z^{*m}z^n\to(\hat{a}^{\dagger m}\hat{a}^n)_W$

[35] Throughout this paper the caret denotes an operator. Functions of operators, e.g., $G(\hat{q},\hat{p})$, will sometimes be denoted by $\hat{G}$.

[36] Results relating to mapping from the c-number space (p,q) onto q-number space $(\hat{p},\hat{q})$ are strictly similar and may be formally obtained via the following transformations:

$$\hat{a}=(2\hbar)^{-1/2}(\hat{q}+i\hat{p}),\quad \hat{a}^\dagger=(2\hbar)^{-1/2}(\hat{q}-i\hat{p}),$$
$$z=(2\hbar)^{-1/2}(q+ip),\quad z^*=(2\hbar)^{-1/2}(q-ip).$$

We will briefly discuss mapping from the (p,q) space onto the space $(\hat{p},\hat{q})$ in Sec. VII.

[37] The operator $\Omega^{(N)}$ is essentially the same as the normal-ordering operator discussed by Louisell in Ref. 21, Sec. 3.3.

onto the operator function

$$G^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\Omega\{F(z,z^*)\}$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}\Omega\{z^{*m}z^n\}$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger). \qquad (2.6)$$

By using the commutation relation (2.3), $G^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ may, of course, be expressed in many *different functional forms*, but all the different functional forms represent one and the same operator $G^{(\Omega)}(\hat{a},\hat{a}^\dagger)$.

The Ω operator expresses formally the mapping of a c-number function $F(z,z^*)$ onto a q-number function $G(\hat{a},\hat{a}^\dagger)$, according to a particular rule of association. We will now introduce for each rule of association an *inverse operator* Θ, which maps a q-number function[38] $G(\hat{a},\hat{a}^\dagger)$ onto a c-number function $F^{(\Omega)}(z,z^*)$:

$$F^{(\Omega)}(z,z^*)=\Theta\{G(\hat{a},\hat{a}^\dagger)\}. \qquad (2.7)$$

We assume that the inverse exists and is unique. Since the Θ mapping is inverse to the Ω mapping, we have, of course,

$$\Omega\{F^{(\Omega)}(z,z^*)\}=G(\hat{a},\hat{a}^\dagger). \qquad (2.8)$$

We will refer to the c-number function $F^{(\Omega)}(z,z^*)$ as the *Ω equivalent of $G(\hat{a},\hat{a}^\dagger)$ or the phase-space representation of $\hat{G}$ for Ω mapping.*

From (2.7) and (2.8) we have the symbolic relations

$$\Theta\Omega=1, \qquad (2.9)$$

$$\Omega\Theta=1. \qquad (2.10)$$

The Θ operator is obviously also linear, having the following properties:

(i) $\Theta\{G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)\}=z^{*m}z^n$,

(ii) $\Theta\{c_1G_1(\hat{a},\hat{a}^\dagger)+c_2G_2(\hat{a},\hat{a}^\dagger)\}=c_1\Theta\{G_1(\hat{a},\hat{a}^\dagger)\}+c_2\Theta\{G_2(\hat{a},\hat{a}^\dagger)\}$,

(iii) $\Theta\{c\}=c$,

for arbitrary q-number functions $\hat{G}$, $\hat{G}_1$, and $\hat{G}_2$. The operator function $G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ in (i) is, of course, the operator function onto which $z^{*m}z^n$ is mapped by Ω.

From the definition of Θ it is obvious that an arbitrary operator function $G(\hat{a},\hat{a}^\dagger)$ which is of the form

$$G(\hat{a},\hat{a}^\dagger)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger), \qquad (2.11)$$

where the d_{mn}'s are c-numbers, will have the Ω equivalent

$$F^{(\Omega)}(z,z^*)=\Theta\{G(\hat{a},\hat{a}^\dagger)\}$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}\Theta\{G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger)\}$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}z^{*m}z^n. \qquad (2.12)$$

We see, on comparing Eqs. (2.11) and (2.12), that if $G(\hat{a},\hat{a}^\dagger)$ is expressed as a series in the $G_{mn}^{(\Omega)}$'s, the Ω equivalent of $G(\hat{a},\hat{a}^\dagger)$ is obtained at once by simply replacing each $G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ by $z^{*m}z^n$. This immediately raises the question of how to determine the coefficients in the expansion (2.11). We will now show that this problem is intimately related to one that frequently arises in the solution of various quantum-mechanical problems, namely, how to express a given operator in an *ordered form,* associated with a prescribed rule of ordering.

To answer this question, we introduce for each rule of association the set of ordered forms of the $G_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ which we denote by $\mathcal{G}_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$. For each pair of non-negative integers m and n, $\mathcal{G}_{mn}^{(\Omega)}$ is equal to $G_{mn}^{(\Omega)}$, but has a *particular functional form.* Thus, for example, for the three rules of association listed in Table II, we would naturally choose

$$\mathcal{G}_{mn}^{(N)}(\hat{a},\hat{a}^\dagger)\equiv\hat{a}^{\dagger m}\hat{a}^n, \qquad (2.13a)$$

$$\mathcal{G}_{mn}^{(A)}(\hat{a},\hat{a}^\dagger)\equiv\hat{a}^n\hat{a}^{\dagger m}, \qquad (2.13b)$$

$$\mathcal{G}_{mn}^{(W)}(\hat{a},\hat{a}^\dagger)\equiv(\hat{a}^{\dagger m}\hat{a}^n)_W, \qquad (2.13c)$$

where the identity sign indicates that the particular forms displayed on the right-hand sides are implied. More generally, the identity sign between any two expressions involving operators is to be understood to mean that the two expressions may be shown to be equal to each other *without the use of the commutation relation (2.3).* The following simple example illustrates these definitions: If we take $m=n=1$ and consider the normal rule of association, then $\mathcal{G}_{11}^{(N)}(\hat{a},\hat{a}^\dagger)\equiv\hat{a}^\dagger\hat{a}$. On the other hand, $G_{11}^{(N)}(\hat{a},\hat{a}^\dagger)$ is the same operator as $\mathcal{G}_{11}^{(N)}(\hat{a},\hat{a}^\dagger)$ expressed in *any* form, e.g., as $\hat{a}\hat{a}^\dagger-1$.

We now define the *Ω-ordered form $\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ of an arbitrary operator $G(\hat{a},\hat{a}^\dagger)$* by the *following two properties:*

(i) $\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=G(\hat{a},\hat{a}^\dagger)$, (2.14)

(ii) $\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ is identically equal to a linear combination of the $\mathcal{G}_{mn}^{(\Omega)}$'s, i.e.,

$$\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^\dagger)\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{mn}'\mathcal{G}_{mn}^{(\Omega)}(\hat{a},\hat{a}^\dagger). \qquad (2.15)$$

The identity sign rather than the equality sign is essential in (2.15).

Finally we introduce, for each mapping Ω, an operator $S^{(\Omega)}$ which we will call the *substitution operator for Ω*

[38] $\hat{G}$ need not be an explicit function of $\hat{a}$ and $\hat{a}^\dagger$.

mapping. It is defined by the property

$$S^{(\Omega)}\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}z^{*m}z^{n}\}\equiv\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}\mathcal{G}_{mn}{}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})\,. \quad (2.16)$$

Thus $S^{(\Omega)}$ operating on a power series in z and z^* turns it into an Ω-ordered form, the form being obtained by replacing each $z^{*m}z^n$ by $\mathcal{G}_{mn}{}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})$. If we set

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}z^{*m}z^{n}=F(z,z^*)\,, \quad (2.17)$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_{mn}\mathcal{G}_{mn}{}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})\equiv\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})\,, \quad (2.18)$$

then (2.16) implies that

$$S^{(\Omega)}\{F(z,z^*)\}\equiv\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})\,. \quad (2.19)$$

It is also obvious from the fact that

$$S^{(\Omega)}\{z^{*m}z^{n}\}\equiv\mathcal{G}_{mn}{}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})$$

and from the linearity of the substitution operator $S^{(\Omega)}$ that $\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})$ is the operator $G(\hat{a},\hat{a}^{\dagger})$, expressed in the particular functional form, onto which $F(z,z^*)$ is mapped by the mapping operator Ω:

$$\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})=G(\hat{a},\hat{a}^{\dagger})=\Omega\{F(z,z^*)\}\,, \quad (2.20)$$

or, taking the inverse of (2.20),

$$\Theta\{\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})\}=\Theta\,\{G(\hat{a},\hat{a}^{\dagger})\}=F(z,z^*)\,. \quad (2.21)$$

Hence from (2.19) and (2.21) we obtain the following theorem.

Theorem I. The Ω-ordered form $\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})$ of an arbitrary function $G(\hat{a},\hat{a}^{\dagger})$ of the boson operators $\hat{a}$ and $\hat{a}^{\dagger}$ is given by

$$\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^{\dagger})\equiv S^{(\Omega)}\{F(z,z^*)\}\,, \quad (2.22)$$

where

$$F(z,z^*)=\Theta\{G(\hat{a},\hat{a}^{\dagger})\}\,. \quad (2.23)$$

The theorem that we have just established reduces the problem of determining the Ω-ordered form of a given operator function to the problem of mapping the operator function onto a c-number function via the appropriate mapping. The theorem implies that to find the Ω-ordered form[39] of $G(\hat{a},\hat{a}^{\dagger})$ we only need to find its Ω equivalent $F(z,z^*)$, expand $F(z,z^*)$ into a power series in z and z^*, and then make the trivial substitutions indicated by Eq. (2.16). In Sec. III we derive a closed expression for $F(z,z^*)$ in terms of $G(\hat{a},\hat{a}^{\dagger})$ and also a closed expression for $G(\hat{a},\hat{a}^{\dagger})$ in terms of $F(z,z^*)$. Examples of application of Theorem I to ordering problems will be given in Sec. VI B. We show, in a later paper in this series,[32] that our technique may also be applied to problems involving time ordering.

[39] For the special cases of normal and antinormal ordering, our theorem is equivalent to the result recently noted by Lax and Louisell in Ref. 13.

III. CLOSED-FORM SOLUTIONS FOR LINEAR ANALYTIC MAPPINGS

Let $F(z,z^*)$ be an arbitrary c-number function. We assume that $F(z,z^*)$ may be represented as a Fourier integral,

$$F(z,z^*)=\int f(\alpha,\alpha^*)\exp(\alpha z^*-\alpha^*z)d^2\alpha\,, \quad (3.1)$$

where

$$f(\alpha,\alpha^*)=\frac{1}{\pi^2}\int F(z,z^*)\exp[-(\alpha z^*-\alpha^*z)]d^2z\,. \quad (3.2)$$

Suppose now that the operator Ω will map the c-number function $F(z,z^*)$ onto a q-number function $G(\hat{a},\hat{a}^{\dagger})$,

$$G(\hat{a},\hat{a}^{\dagger})=\Omega\{F(z,z^*)\}\,, \quad (3.3)$$

and let Θ again denote the operator for the inverse mapping, i.e.,

$$F(z,z^*)=\Theta\{G(\hat{a},\hat{a}^{\dagger})\}\,. \quad (3.4)$$

Now under fairly general conditions an arbitrary q-number function may be expressed in the form of an "operator Fourier theorem," discussed in Appendix C. Let us represent $G(\hat{a},\hat{a}^{\dagger})$ in this way [Eqs. (C1) and (C2)]:

$$G(\hat{a},\hat{a}^{\dagger})=\int g(\alpha,\alpha^*)\exp(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})d^2\alpha\,, \quad (3.5)$$

where

$$g(\alpha,\alpha^*)=\frac{1}{\pi}\operatorname{Tr}\{G(\hat{a},\hat{a}^{\dagger})\exp[-(\alpha\hat{a}^{\dagger}-\alpha^*\hat{a})]\}\,. \quad (3.6)$$

The function $f(\alpha,\alpha^*)$ in (3.1) is, of course, the *Fourier spectrum of the c-number function*. In view of the formal similarity between (3.1) and (3.5), we will refer to the (c-number) function $g(\alpha,\alpha^*)$ in (3.5) as the *Fourier spectrum of the q-number function* $G(\hat{a},\hat{a}^{\dagger})$. It is evident that the relation (3.3) [or, equivalently, (3.4)] connects the two Fourier spectra $f(\alpha,\alpha^*)$ and $g(\alpha,\alpha^*)$. We will first investigate this relationship for the rules of association defined in Tables I and II.

We have from Eqs. (3.3) and (3.1), on using the linearity of the Ω operator,

$$G(\hat{a},\hat{a}^{\dagger})=\int f(\alpha,\alpha^*)\Omega\,\{\exp(\alpha z^*-\alpha^*z)\}d^2\alpha\,. \quad (3.7)$$

Consider now the normal rule of association (superscript N). We then have

$$\begin{aligned}\Omega^{(N)}\,\{\exp(\alpha z^*-\alpha^*z)\}&=\Omega^{(N)}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha z^*)^m(-\alpha^*z)^n}{m!n!}\right]\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^m(-\alpha^*)^n}{m!n!}\hat{a}^{\dagger m}\hat{a}^n\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^{\dagger})^m}{m!}\frac{(-\alpha^*\hat{a})^n}{n!}\\&=\exp(\alpha\hat{a}^{\dagger})\exp(-\alpha^*\hat{a})\,. \quad (3.8)\end{aligned}$$

We rewrite the last expression with the help of the Baker-Hausdorff identity[40] and obtain the formula

$$\Omega^{(N)}\{\exp(\alpha z^*-\alpha^* z)\} =\exp(+\tfrac{1}{2}|\alpha|^2)\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a}). \quad (3.9)$$

From (3.7) and (3.9) it follows that, for the normal rule of association,

$$G^{(N)}(\hat{a},\hat{a}^\dagger)=\int \exp(+\tfrac{1}{2}|\alpha|^2)f(\alpha,\alpha^*) \times\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})d^2\alpha. \quad (3.10)$$

From (3.10) we see that the Fourier spectrum $g^{(N)}(\alpha,\alpha^*)$ of $G^{(N)}(\hat{a},\hat{a}^\dagger)$ is given by

$$g^{(N)}(\alpha,\alpha^*)=\exp(+\tfrac{1}{2}|\alpha|^2)f(\alpha,\alpha^*). \quad (3.11)$$

We may carry out similar calculations for the other rules of association defined in Sec. II and we find that for each of them the relation between the spectrum of $G(\hat{a},\hat{a}^\dagger)$ and the spectrum of $F(z,z^*)$ is of the form

$$g(\alpha,\alpha^*)=\Omega(\alpha,\alpha^*)f(\alpha,\alpha^*), \quad (3.12)$$

where the function $\Omega(\alpha,\alpha^*)$ is characteristic of the particular rule of association Ω. The explicit forms of the functions $\Omega(\alpha,\alpha^*)$ are shown in Table III. It is seen that, in each case, $\Omega(\alpha,\alpha^*)$ is obtained from a function $\Omega(\alpha,\beta)$, which satisfies the following conditions:

(i) $\Omega(\alpha,\beta)$ is an entire analytic function of two complex variables α, β.
(ii) $\Omega(\alpha,\beta)$ has no zeros.

In addition, $\Omega(\alpha,\beta)$ also has the following properties:

(iii) $\Omega(0,0)=1$.
(iv) $\Omega(-\alpha,-\beta)=\Omega(\alpha,\beta)$.

We now define a broader class of mappings $F(z,z^*)\to G(\hat{a},\hat{a}^\dagger)$, which we call the class of *linear* mappings, by the property that for each mapping of this class, the Fourier spectra $f(\alpha,\alpha^*)$ and $g(\alpha,\alpha^*)$ of $F(z,z^*)$ and $G(\hat{a},\hat{a}^\dagger)$ are related by Eq. (3.12). A linear mapping for which the condition (i) is satisfied will be said to be analytic. Throughout this investigation we will mainly consider the *subclass of linear analytic mappings for which the functions $\Omega(\alpha,\beta)$ satisfy condition*[41] *(ii) and the trivial normalization condition (iii).*

TABLE III. The filter function $\Omega(\alpha,\alpha^*)$ for the commonly employed rules of association.

Rule of association	$\Omega(\alpha,\alpha^*)$
Normal	$\exp(+\frac{1}{2}\alpha^*\alpha)$
Antinormal	$\exp(-\frac{1}{2}\alpha^*\alpha)$
Weyl	1
Standard	$\exp(\frac{1}{4}\alpha^2-\frac{1}{4}\alpha^{*2})$
Antistandard	$\exp(-\frac{1}{4}\alpha^2+\frac{1}{4}\alpha^{*2})$

It is clear that a linear mapping $F(z,z^*)\to G(\hat{a},\hat{a}^\dagger)$ is, in a sense, analogous to a time-invariant linear filter, with the function $\Omega(\alpha,\beta)$ playing the role of the *filter function.* A linear mapping for which condition (ii) is satisfied is analogous to a *minimal filter*[42]; that for which condition (iv) is satisfied is analogous to a *symmetric filter.*

A closed expression for a linear mapping is now readily obtained by substituting from (3.12) into (3.5) and then expressing $f(\alpha,\alpha^*)$ in terms of $F(z,z^*)$ from the relation (3.2). This result is expressed by the following theorem.[43]

Theorem II. The q-number function $G(\hat{a},\hat{a}^\dagger)$ obtained from a c-number function $F(z,z^*)$ by the linear mapping Ω ($\hat{G}=\Omega\{F\}$) is given by

$$G(\hat{a},\hat{a}^\dagger)=\int F(z,z^*)\Delta^{(\Omega)}(z-\hat{a},\, z^*-\hat{a}^\dagger)d^2z, \quad (3.13)$$

where

$$\Delta^{(\Omega)}(z-\hat{a},\, z^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int\Omega(\alpha,\alpha^*) \times\exp\{-[\alpha(z^*-\hat{a}^\dagger)-\alpha^*(z-\hat{a})]\}d^2\alpha. \quad (3.14)$$

The operator $\Delta^{(\Omega)}(z-\hat{a},\, z^*-\hat{a}^\dagger)$, defined by (3.14), plays a central role in the present theory. We will refer to it as the mapping Δ *operator for the* Ω mapping and we will study its main properties in Sec. IV. Here we only note that this operator has a clear symbolic meaning. To see this we note, first of all, that if we choose

$$f(\alpha,\alpha^*)=\delta^{(2)}(\alpha-\alpha_0), \quad (3.15)$$

where $\delta^{(2)}(\alpha-\alpha_0)$ is the two-dimensional Dirac δ function [i.e., $\delta^{(2)}(\alpha-\alpha_0)=\delta(\alpha^{(r)}-\alpha_0^{(r)})\delta(\alpha^{(i)}-\alpha_0^{(i)})$, where $\alpha=\alpha^{(r)}+i\alpha^{(i)}$, $\alpha_0=\alpha_0^{(r)}+i\alpha_0^{(i)}$, $\alpha^{(r)}$, $\alpha^{(i)}$, $\alpha_0^{(r)}$, and $\alpha^{(i)}$ are real, and δ is the ordinary Dirac δ function], then according to (3.12),

$$g(\alpha,\alpha^*)=\Omega(\alpha,\alpha^*)\delta^{(2)}(\alpha-\alpha_0), \quad (3.16)$$

and (3.1), (3.3), (3.5), (3.15), and (3.16) then give the relation

$$\Omega\{\exp(\alpha_0 z^*-\alpha_0^* z)\}=\Omega(\alpha_0,\alpha_0^*)\exp(\alpha_0\hat{a}^\dagger-\alpha_0^*\hat{a}). \quad (3.17)$$

[40] See Ref. 21, p. 102, or A. Messiah, *Quantum Mechanics* (Wiley, New York, 1961), Vol. I, p. 442.

[41] It will be seen shortly [Theorem III: Eqs. (3.25)–(3.27)] that when condition (ii) is not satisfied, the kernel of the inverse transform is expressed in terms of a function that has singularities. An example of such a mapping is provided by Rivier's rule of association [D. C. Rivier, Phys. Rev. **83**, 862(L) (1951)]. In this case,

$$\Omega\{\exp(\alpha z^*-\alpha^* z)\}=\tfrac{1}{2}[\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})+\exp(-\alpha^*\hat{a})\exp(\alpha\hat{a}^\dagger)].$$

With the help of this result, one can show that for Rivier's rule, $\Omega(\alpha,\beta)=\cosh(\frac{1}{2}\alpha\beta)$. Since this function has zeros, the corresponding function $\tilde{\Omega}(\alpha,\beta)=\Omega(-\alpha,-\beta)$ that enters the expression for the inverse transform has singularities.

[42] See, e.g., V. V. Solodovnikov, *Introduction to the Statistical Dynamics of Automatic Control Systems*, translation edited by J. B. Thomas and L. A. Zadeh (Dover, New York, 1960), p. 45.

[43] Theorem II is valid regardless of whether or not the filter function $\Omega(\alpha,\alpha^*)$ has any zeros.

Next let us choose

$$F(z,z^*)=\delta^{(2)}(z-z_0). \tag{3.18}$$

If we represent $\delta^{(2)}$ in the form of a Fourier integral, i.e.,

$$\delta^{(2)}(z-z_0) =\frac{1}{\pi^2}\int \exp\{-[\alpha(z_0^*-z^*)-\alpha^*(z_0-z)]\}d^2\alpha, \tag{3.19}$$

and then apply the operator Ω to both sides of (3.19) and use the linearity of Ω, we obtain the relation

$$\Omega\delta^{(2)}(z-z_0)=\frac{1}{\pi^2}\int \exp[-(\alpha z_0^*-\alpha^* z_0)] \times\Omega\{\exp(\alpha z^*-\alpha^* z)\}d^2\alpha. \tag{3.20}$$

If on the right-hand side of (3.20) we now make use of relation (3.17) and compare the resulting expression with (3.14), we obtain the formula

$$\Omega\{\delta^{(2)}(z-z_0)\}=\Delta^{(\Omega)}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger). \tag{3.21}$$

Equation (3.21) shows that *the operator* $\Delta^{(\Omega)}(z_0-\hat{a}, z_0^*-\hat{a}^\dagger)$ *is simply the operator onto which the two-dimensional Dirac* δ *function is mapped via the* Ω *mapping.*

Next we will derive a closed expression for the inverse mapping (3.4). We have, from (3.12),

$$f(\alpha,\alpha^*)=\bar{\Omega}(\alpha,\alpha^*)g(\alpha,\alpha^*), \tag{3.22}$$

where

$$\bar{\Omega}(\alpha,\alpha^*)=[\Omega(\alpha,\alpha^*)]^{-1}. \tag{3.23}$$

Since we assumed that $\Omega(\alpha,\alpha^*)$ had no zeros, $\bar{\Omega}(\alpha,\alpha^*)$ is defined for all values of α, real or complex. If now we substitute from (3.22) into (3.1) and then use (3.6) we find that

$$F(z,z^*)=\frac{1}{\pi}\int \bar{\Omega}(\alpha,\alpha^*)\,\mathrm{Tr}\{G(\hat{a},\hat{a}^\dagger)\exp[-(\alpha\hat{a}^\dagger-\alpha^*\hat{a})]\}\times\exp[(\alpha z^*-\alpha z)]d^2\alpha$$

$$=\pi\,\mathrm{Tr}\left\{G(\hat{a},\hat{a}^\dagger)\frac{1}{\pi^2}\int\bar{\Omega}(\alpha,\alpha^*)\times\exp[\alpha(z^*-\hat{a}^\dagger)-\alpha^*(z-\hat{a})]d^2\alpha\right\}. \tag{3.24}$$

Finally, on changing the variable of integration from α to $-\alpha$, we obtain the following theorem.

Theorem III. The c-number function $F(z,z^*)$, obtained from a q-number function $G(\hat{a},\hat{a}t)$ by the mapping Θ inverse to Ω ($F=\Theta$, $\{\hat{G}\}$), is given by

$$F(z,z^*)=\pi\,\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\Delta^{(\tilde{\Omega})}(z-\hat{a},\, z^*-\hat{a}^\dagger)], \tag{3.25}$$

where

$$\Delta^{(\tilde{\Omega})}(z-\hat{a},\, z^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int\tilde{\Omega}(\alpha,\alpha^*)\times\exp\{-[\alpha(z^*-\hat{a}^\dagger)-\alpha^*(z-\hat{a})]\}d^2\alpha \tag{3.26}$$

and

$$\tilde{\Omega}(\alpha,\alpha^*)=[\Omega(-\alpha,\,-\alpha^*)]^{-1}. \tag{3.27}$$

In formulating Theorem III we were led to the introduction of the two functions $\bar{\Omega}(\alpha,\alpha^*)$ and $\tilde{\Omega}(\alpha,\alpha^*)$, defined by Eqs. (3.23) and (3.27), respectively. It seems appropriate, theorefore, to associate with each mapping Ω two other mappings, which we will call the mapping *reciprocal* to Ω and the mapping *antireciprocal to* Ω, and will denote the corresponding mapping operators by $\bar{\Omega}$ and by $\tilde{\Omega}$, respectively. These are the mappings (3.13), with the function $\Omega(\alpha,\alpha^*)$ replaced by the functions $\bar{\Omega}(\alpha,\alpha^*)$ and $\tilde{\Omega}(\alpha,\alpha^*)$ defined by Eqs. (3.23) and (3.27), respectively. The mapping operators for the corresponding inverse mappings will be denoted by $\bar{\Theta}$ and $\tilde{\Theta}$, respectively.

In Appendix D, we derive the following two *reciprocity theorems*, valid for any two arbitrary mappings $\Omega^{(1)}$ and $\Omega^{(2)}$:

$$\Theta^{(2)}\bar{\Omega}^{(1)}=\Theta^{(1)}\bar{\Omega}^{(2)}, \tag{3.28}$$

$$\bar{\Omega}^{(2)}\Theta^{(1)}=\bar{\Omega}^{(1)}\Theta^{(2)}. \tag{3.29}$$

Here $\Theta^{(1)}$ and $\Theta^{(2)}$ represent, of course, the mappings inverse to $\Omega^{(1)}$ and $\Omega^{(2)}$, respectively. We will make use of these relations later.

In the special case when $\Omega(\alpha,\alpha^*)$ is symmetric [i.e., when $\Omega(-\alpha,\,-\alpha^*)=\Omega(\alpha,\alpha^*)$], the distinction between the reciprocal and the antireciprocal mapping disappears. We note that this is so for each of the five mappings listed in Table III and that, moreover, the Weyl mapping is also self-reciprocal:

$$\bar{\Omega}^{(W)}(\alpha,\alpha^*)=\Omega^{(W)}(\alpha,\alpha^*).$$

In Appendix E, we prove the following interesting result: Each mapping Ω maps a polynomial of degree M in z^* and degree N in z onto a polynomial of degree M in $\hat{a}^\dagger$ and degree N in $\hat{a}$ and vice versa.

Finally, we note that in view of the assumptions (i) and (ii) that we have made about the filter functions $\Omega(\alpha,\beta)$, namely, that it is an entire analytic function of the two complex variables α and β, which has no zeros, $\Omega(\alpha,\beta)$ must necessarily be of the form[44]

$$\Omega(\alpha,\beta)=e^{\chi(\alpha,\beta)}, \tag{3.30}$$

where $\chi(\alpha,\beta)$ is itself an entire analytic function of α, β. Let us expand $\chi(\alpha,\beta)$ in a power series around the origin:

$$\chi(\alpha,\beta)=C+A\alpha+B\beta+\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\beta+\cdots, \tag{3.31}$$

where C, A, B, μ, ν, and λ are parameters. In view of our

[44] The corresponding result for a function of one complex variable is derived, e.g., in E. T. Copson, *An Introduction to the Theory of Functions of a Complex Variable* (Clarendon, Oxford, 1960), p. 159.

assumption (iii), namely, that $\Omega(0,0)=1$, we must have

$$C=0. \tag{3.32}$$

Now one may readily verify, by the use of Theorem II [Eq. (3.13)] and Eqs. (3.30) and (3.31), that for a mapping Ω for which the filter function is given by (3.30) and (3.31),

$$\Omega\{z\}=\hat{a}-B, \tag{3.33a}$$

$$\Omega\{z^*\}=\hat{a}^\dagger+A. \tag{3.33b}$$

Further, one can show with the help of Theorem III [Eq. (3.25)] that the inverse mapping Θ is such that

$$\Theta\{\hat{a}\}=z+B, \tag{3.34a}$$

$$\Theta\{\hat{a}^\dagger\}=z^*-A. \tag{3.34b}$$

Hence we see that the presence of linear terms in the expansion (3.31) implies translations in both the c-number and the q-number spaces and that in order to have the correspondence $z\rightleftarrows\hat{a}$, $z^*\rightleftarrows\hat{a}^\dagger$, both the linear terms in (3.31) must be absent, i.e.,

$$A=B=0. \tag{3.34c}$$

This condition is, of course, equivalent to the condition

$$\frac{\partial\Omega}{\partial\alpha}=\frac{\partial\Omega}{\partial\beta}=0 \quad \text{when} \quad \alpha=\beta=0, \tag{3.35}$$

which is satisfied by all analytic symmetric filter functions $\Omega(\alpha,\beta)$.

It is now evident that all the filter functions $\Omega(\alpha,\beta)$ of the class that we are considering and which, moreover, ensure the correspondence

$$z\rightleftarrows\hat{a}, \quad z^*\rightleftarrows\hat{a}^\dagger \tag{3.36}$$

may be expressed in the form (3.30), with

$$\chi(\alpha,\beta)=\mu\alpha^2+\nu\beta^2+\lambda\alpha\beta+\text{higher-order terms in } \alpha \text{ and } \beta. \tag{3.37}$$

If in (3.37) the higher-order terms are absent, then

$$\Omega(\alpha,\beta)=\exp(\mu\alpha^2+\nu\beta^2+\lambda\alpha\beta). \tag{3.38}$$

It is seen from Table III that all the five rules of associations listed there belong to the subclass of mappings characterized by filter functions of the form (3.38). We note that the Weyl rule of association is the simplest, with $\Omega(\alpha,\beta)=1$ [$\chi(\alpha,\beta)=0$]. The values of the parameters μ, ν, and λ for these five rules of association are listed in Table IV.

TABLE IV. The values of the coefficients μ, ν, and λ [Eq. (3.38)] for some of the five commonly employed rules of association.

Rule of association	μ	ν	λ
Weyl	0	0	0
Normal	0	0	$\frac{1}{2}$
Antinormal	0	0	$-\frac{1}{2}$
Standard	$\frac{1}{4}$	$-\frac{1}{4}$	0
Antistandard	$-\frac{1}{4}$	$\frac{1}{4}$	0

We study in Sec. VI and in other papers of this series some of the special properties of the subclass of linear analytic mappings whose filter functions are of the form (3.38).

We will now illustrate our results by a few examples. Let us consider the representation (3.13) of an arbitrary operator function $G(\hat{a},\hat{a}^\dagger)$ via the antinormal rule of association (superscript A). From (3.20) and (3.21) we readily find that

$$\Delta^{(A)}\{z_0-\hat{a},\, z_0^*-\hat{a}^\dagger\}-\frac{1}{\pi^2}\int \exp(\alpha^*\hat{a})\times\exp(-\alpha\hat{a}^\dagger)\exp(\alpha z_0^*-\alpha^* z_0)d^2\alpha. \tag{3.39}$$

We can simplify this expression by inserting, between the first two exponential terms on the right-hand side of (3.39), the identity operator, expressed in terms of the projection operators of the coherent states [the resolution of the identity, Eq. (B3)]. Then using also the fact that the coherent states are the eigenstates of the annihilation operator [Eq. (B1)], (3.39) reduces to

$$\begin{aligned}\Delta^{(A)}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger)&=\frac{1}{\pi^3}\int \exp(\alpha^* z')\,|z'\rangle\langle z'|\,\exp(-\alpha z'^*)\times\exp(\alpha z_0^*-\alpha^* z_0)d^2\alpha d^2z'\\&=\frac{1}{\pi}\int \delta^{(2)}(z_0-z')\,|z'\rangle\langle z'|\,d^2z'\\&=\frac{1}{\pi}|z_0\rangle\langle z_0|\,.\end{aligned} \tag{3.40}$$

Hence we see that the mapping Δ operator for the antinormal rule of association is proportional to the projection operator of the coherent state $|z_0\rangle$.

From Theorem II [Eq. (3.13)] and (3.40), it follows that

$$G(\hat{a},\hat{a}^\dagger)=\frac{1}{\pi}\int F^{(A)}(z,z^*)\,|z\rangle\langle z|\,d^2z. \tag{3.41}$$

The possibility of expressing an arbitrary density operator of a quantum-mechanical system in the form (3.41), known as the *diagonal coherent-state representation*, was first noted by Sudarshan.[10] A representation of this type for a more restricted class of operators was also discussed by Glauber, under the name of P representation. Since the original derivation was somewhat heuristic, the generality of this representation has been widely argued about. It is now known[25,45] that the

[45] J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 733 (1965), and references therein; J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966); J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Benjamin, New York, 1968), Chap. VIII, and references therein; M. M. Miller and E. A. Mishkin, Phys. Rev. **164**, 1610 (1967).

TABLE V. The c-number equivalents of some operators for the normal, antinormal, and Weyl rules of association. In this table, $\partial^{m+n}/\partial z^{*m}\partial z^{n}$ has been abbreviated by $\partial_{m,n}$ and $\mathcal{H}_{m,n}$ is defined by Eq. (E14).

$G(\hat{a},\hat{a}^\dagger)$	$F^{(W)}(z,z^*)$	$F^{(N)}(z,z^*)$	$F^{(A)}(z,z^*)$
$\hat{a}^{\dagger m}\hat{a}^{n}$	$\exp(2\lvert z\rvert^2)(-\frac{1}{2})^{m+n}\partial_{m,n}\exp(-2\lvert z\rvert^2)$	$z^{*m}z^{n}$	$\exp(\lvert z\rvert^2)(-1)^{m+n}\partial_{m,n}\exp(-\lvert z\rvert^2)$
$\hat{a}^{n}\hat{a}^{\dagger m}$	$(-1)^m\mathcal{H}_{m,n}(z,\ -z^*)$	$(-1)^m(2)^{(m+n)/2}\mathcal{H}_{m,n}(z/\sqrt{2},\ -z^*/\sqrt{2})$	$z^{n}z^{*m}$
$\lvert z_0\rangle\langle z_0\rvert$	$2\exp(-2\lvert z-z_0\rvert^2)$	$\exp(-\lvert z-z_0\rvert^2)$	$\delta^{(2)}(z-z_0)$
$\exp(-\lambda\hat{a}^\dagger\hat{a})$	$[2/(1+e^{-\lambda})]\exp[-2\lvert z\rvert^2(1-e^{-\lambda})/(1+e^{-\lambda})]$	$\exp[-\lvert z\rvert^2(1-e^{-\lambda})]$	$\exp[\lambda-(e^{\lambda}-1)\lvert z\rvert^2]$
$f(\hat{a}^\dagger\hat{a})$	$2\exp(-2\lvert z\rvert^2)\sum_{n=0}^{\infty}\frac{f(n)}{n!}(-1)^nL_n(4\lvert z\rvert^2)$	$\exp(-\lvert z\rvert^2)\sum_{n=0}^{\infty}\frac{\lvert z\rvert^{2n}}{n!}f(n)$	$\pi\exp(\lvert z\rvert^2)\sum_{0}^{\infty}\frac{f(n)}{n!}\{\partial_{n,n}\delta^{(2)}(z)\}$

diagonal coherent-state representation holds under very general conditions provided that it is interpreted in the sense of generalized function theory. We have just seen that this representation follows at once from our general Theorem II, which also shows the significance of the weighting function which appears in this representation. An expression for this weighting function $F^{(A)}(z,z^*)$ may readily be derived with the help of Theorem III (see Appendix F, where a similar calculation is carried out for the c-number equivalent for the Weyl rule of association). The result is (with α labeling coherent states)

$$F^{(A)}(z,z^*)=\frac{1}{\pi}\exp(\lvert z\rvert^2)\int\langle-\alpha\rvert\hat{G}\lvert\alpha\rangle\exp(\lvert\alpha\rvert^2)\times\exp[-(\alpha z^*-\alpha^*z)]d^2\alpha\,,\quad(3.42)$$

provided that $\langle-\alpha\rvert\hat{G}\lvert\alpha\rangle\exp(\lvert\alpha\rvert^2)$ is square integrable. If this expression is not square integrable, (3.42) must be interpreted in the sense of generalized function theory. Formula (3.42) was first derived by Metha[46] by inverting Eq. (3.41).

From Theorem III [Eq. (3.25)] and from (3.40) it immediately follows that the c-number equivalent of $G(\hat{a},\hat{a}^\dagger)$ for the normal rule of association is given by

$$F^{(N)}(z,z^*)=\langle z\rvert G(\hat{a},\hat{a}^\dagger)\lvert z\rangle\,.\quad(3.43)$$

The properties[47,48] of $F^{(N)}$ when $\hat{G}$ is the density operator of a quantum-mechanical system were studied by Kano,[26] Mehta and Sudarshan,[25] Glauber,[24] and Mandel.[49]

For the sake of completeness, we also give a closed expression for the c-number equivalent $F^{(W)}$ for the Weyl rule of association:

$$F^{(W)}(z,z^*)=\frac{2}{\pi}\exp(2\lvert z\rvert^2)\int\langle-\alpha\rvert\hat{G}\lvert\alpha\rangle\times\exp[-2(\alpha z^*-\alpha^*z)]d^2\alpha\,.\quad(3.44)$$

The derivation of (3.44) is given in Appendix F.

In Sec. VI, the general formulas derived in this section and certain relations derived in Sec. V will be used to determine the c-number equivalents for various rules of association and the corresponding ordered forms of various commonly occurring operators. (See also Table V.)

IV. PROPERTIES OF MAPPING Δ OPERATORS AND SOME RELATED RESULTS

According to Eq. (3.14), each of the mapping Δ operators $\hat{\Delta}^{(\Omega)}$ may be expressed in the form

$$\Delta^{(\Omega)}(z_0-\hat{a},\,z_0^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int\hat{D}(\alpha)\Omega(\alpha,\alpha^*)\times\exp[-(\alpha z_0^*-\alpha^*z_0)]d^2\alpha\,,\quad(4.1)$$

where

$$\hat{D}(\alpha)=\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})\,.\quad(4.2)$$

The operators $\hat{D}(\alpha)$ are the well-known displacement operators for coherent states (the eigenstates of the annihilation operator $\hat{a}$) and their properties have been extensively studied. We summarize them in Appendix B. Using the properties of $\hat{D}(\alpha)$ and the representation (4.1) of $\hat{\Delta}^{(\Omega)}$, one may readily determine the basic properties of the mapping Δ operators.

Let us first integrate both sides of (4.1) over the whole complex z_0 plane and use the fact that $\Omega(0,0)=1$. We then obtain the following result:

$$\int\Delta^{(\Omega)}(z_0-\hat{a},\,z_0^*-\hat{a}^\dagger)d^2z_0=1\,.\quad(4.3)$$

Next let us consider the trace of $\hat{\Delta}^{(\Omega)}$. We have, from Eqs. (4.1) and (B11), again using the fact that $\Omega(0,0)=1$,

$$\mathrm{Tr}[\Delta^{(\Omega)}(z_0-\hat{a},\,z_0^*-\hat{a}^\dagger)]=1/\pi\,.\quad(4.4)$$

From the property (B9) of the displacement operator and from (4.1), we find that

$$\hat{D}(z_2-z_1)\Delta^{(\Omega)}(z_1-\hat{a},\,z_1^*-\hat{a}^\dagger)\hat{D}^\dagger(z_2-z_1)=\Delta^{(\Omega)}(z_2-\hat{a},\,z_2^*-\hat{a}^\dagger)\,.\quad(4.5)$$

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[46] C. L. Mehta, Phys. Rev. Letters **18**, 752 (1967).

[47] This function in a slightly different context and in a different form was first introduced by K. Husimi [Proc. Phys. Math. Soc. Japan **22**, 264 (1940)].

[48] In some recent publications, the role of the indices A and N is interchanged. Our notation appears to be preferable in view of the Theorems I and III.

[49] L. Mandel, Phys. Rev. **152**, 438 (1966).

Next let us consider the trace of the product of two mapping Δ operators

$$\Delta^{(\Omega_j)}(z_j-\hat{a},\, z_j^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int \Omega_j(\alpha,\alpha^*)\hat{D}(\alpha)\times\exp[-(\alpha z_j^*-\alpha^* z_j)]d^2\alpha \quad (j=1,2), \quad (4.6)$$

corresponding to two different rules of association. It readily follows, with the help of (B10) and (B12), that

$$\mathrm{Tr}[\Delta^{(\Omega_1)}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger)\Delta^{(\Omega_2)}(z_2-\hat{a},\, z_2^*-\hat{a}^\dagger)]=\frac{1}{\pi^3}\int \Omega_1(\alpha,\alpha^*)\Omega_2(-\alpha,\,-\alpha^*)\times\exp\{-[\alpha(z_1^*-z_2^*)-\alpha^*(z_1-z_2)]\}d^2\alpha. \quad (4.7)$$

In the special but important case when the mappings Ω_1 and Ω_2 are mutually antireciprocal, i.e., when $\Omega_2=\tilde{\Omega}_1$, the corresponding filter functions being related by the formula $\Omega_2(\alpha,\alpha^*)=[\Omega_1(-\alpha,\,-\alpha^*)]^{-1}$, Eq. (4.7) reduces to

$$\mathrm{Tr}[\Delta^{(\Omega)}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega})}(z_2-\hat{a},\, z_2^*-\hat{a}^\dagger)]=\frac{1}{\pi}\delta^{(2)}(z_1-z_2). \quad (4.8)$$

This relation expresses the orthogonality of the mapping Δ operators for any two mappings that are mutually antireciprocal.

We also have, on taking the adjoint of (4.1),

$$\Delta^{(\Omega)\dagger}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int \Omega^*(\alpha,\alpha^*)\hat{D}^\dagger(\alpha)\times\exp[(\alpha z_0^*-\alpha^* z_0)]d^2\alpha. \quad (4.9)$$

From Eqs. (4.1), (4.2), and (B12), we find that

$$\mathrm{Tr}[\Delta^{(\Omega)}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger)\Delta^{(\Omega)\dagger}(z_2-\hat{a},\, z_2^*-\hat{a}^\dagger)]=\frac{1}{\pi^3}\int |\Omega(\alpha,\alpha^*)|^2\times\exp[\alpha(z_2^*-z_1^*)-\alpha^*(z_2-z_1)]d^2\alpha. \quad (4.10)$$

We have up to now made no special assumptions about the function $\Omega(\alpha,\alpha^*)$, beyond those made in Sec. III, where these functions were introduced. We shall now consider separately the implications of some additional assumptions.

A. $\Omega(-\alpha,\,-\alpha^*)=\Omega^*(\alpha,\alpha^*)$

We have from (4.9), if we change the variable of integration from α to $-\alpha$,

$$\Delta^{(\Omega)\dagger}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int \hat{D}(\alpha)\Omega^*(-\alpha,\,-\alpha^*)\times\exp[-(\alpha z_0^*-\alpha^* z_0)]d^2\alpha. \quad (4.11)$$

Comparison of (4.11) with (4.1) shows that *if* $\Omega(-\alpha,\,-\alpha^*)=\Omega^*(\alpha,\alpha^*)$, then

$$\Delta^{(\Omega)\dagger}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger)=\Delta^{(\Omega)}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger), \quad (4.12)$$

i.e., the mapping operator is then *Hermitian.*

Let us now consider the Ω equivalent $F^{(\Omega)}(z,z^*)$ of a Hermitian operator function $G(\hat{a},\hat{a}^\dagger)$ when the filter function that characterizes this mapping satisfies the relation $\Omega(-\alpha,\,-\alpha^*)=\Omega^*(\alpha,\alpha^*)$, i.e., when the two-dimensional Fourier transform of $\Omega(\alpha,\alpha^*)$ is real. According to Theorem III [Eq. (3.25)],

$$F^{(\Omega)}(z,z^*)=\pi\,\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\Delta^{(\tilde{\Omega})}(z-\hat{a},\, z^*-\hat{a}^\dagger)]. \quad (4.13)$$

If we take the complex conjugate of $F^{(\Omega)}$, use elementary properties of the adjoint and of the trace of the product of two operators, we readily find that

$$[F^{(\Omega)}(z,z^*)]^*=\pi\,\mathrm{Tr}[G^\dagger(\hat{a},\hat{a}^\dagger)\Delta^{(\tilde{\Omega})\dagger}(z-\hat{a},\, z^*-\hat{a}^\dagger)]. \quad (4.14)$$

Now if $\Omega(\alpha,\alpha^*)$ satisfies the condition

$$\Omega(\alpha,\alpha^*)=\Omega^*(-\alpha,\,-\alpha^*),$$

so does $\tilde{\Omega}(\alpha,\alpha^*)$ and hence, according to (4.12), $\hat{\Delta}^{(\tilde{\Omega})}$ is Hermitian. If $G(\hat{a},\hat{a}^\dagger)$ is also Hermitian, the right-hand side of (4.14) is obviously equal to the right-hand side of (4.13), so that

$$[F^{(\Omega)}(z,z^*)]^*=F^{(\Omega)}(z,z^*). \quad (4.15)$$

This result shows that *when* $\Omega^*(-\alpha,\,-\alpha^*)=\Omega(\alpha,\alpha^*)$, *the* Ω *equivalent of a Hermitian operator is a real* (*c-number*) *function.*

It is seen from Table III that for normal, antinormal, and Weyl rules of association, the filter functions $\Omega(\alpha,\alpha^*)$ are real. Hence, according to the results that we have just established, the mapping operators $\hat{\Delta}^{(\tilde{\Omega})}$ for each of these rules of association are Hermitian and each of them maps Hermitian operators onto real *c*-number functions. We will consider these three rules of association more fully in Sec. VI.

B. $\Omega(\alpha,\alpha^*)$ Unimodular

The class of mappings for which $\Omega(\alpha,\alpha^*)$ is unimodular is also of interest and we will now briefly consider it.

It has been shown in Sec. III that any arbitrary operator function $G(\hat{a},\hat{a}^\dagger)$ can be written, in terms of its Ω equivalent $F^{(\Omega)}(z,z^*)$, in the following form:

$$G(\hat{a},\hat{a}^\dagger)=\int F^{(\Omega)}(z,z^*)\Delta^{(\Omega)}(z-\hat{a},\, z^*-\hat{a}^\dagger)d^2z. \quad (4.16)$$

On taking the adjoint of (4.16), we obtain the formula

$$G^\dagger(\hat{a},\hat{a}^\dagger)=\int [F^{(\Omega)}(z,z^*)]^*\Delta^{(\Omega)\dagger}(z-\hat{a},\, z^*-\hat{a}^\dagger)d^2z. \quad (4.17)$$

It follows from (4.16) and (4.17) that

$$\mathrm{Tr}(\hat{G}\hat{G}^\dagger)=\int\int F^{(\Omega)}(z_1,z_1^*)[F^{(\Omega)}(z_2,z_2^*)]^* \times\mathrm{Tr}[\Delta^{(\Omega)}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger)\Delta^{(\Omega)\dagger}(z_2-\hat{a},\, z_2^*-\hat{a}^\dagger)] \times d^2z_1d^2z_2. \quad (4.18)$$

Now the trace in Eq. (4.18) is given by formula (4.10). If we assume now that $\Omega(\alpha,\alpha^*)$ is unimodular, i.e., of the form

$$\Omega(\alpha,\alpha^*)=\exp[i\phi(\alpha,\alpha^*)], \quad (4.19)$$

where $\phi(\alpha,\alpha^*)$ is real, then Eq. (4.10) gives

$$\mathrm{Tr}[\Delta^{(\Omega)}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger)\Delta^{(\Omega)\dagger}(z_2-\hat{a},\, z_2^*-\hat{a}^\dagger)] =(1/\pi)\delta^{(2)}(z_1-z_2), \quad (4.20)$$

and (4.18) reduces to

$$\mathrm{Tr}[G^\dagger(\hat{a},\hat{a}^\dagger)G(\hat{a},\hat{a}^\dagger)]=\frac{1}{\pi}\int |F^{(\Omega)}(z,z^*)|^2d^2z. \quad (4.21)$$

This relation shows that *when* $\Omega(\alpha,\alpha^*)$ *is unimodular, operators of the Hilbert-Schmidt class are mapped onto square-integrable functions and vice versa.*

It is seen from Table III that the result we have just established applies to the Weyl, the standard, and the antistandard rules of association.

C. $\Omega(\alpha,\alpha^*)$ Square Integrable

We have, from (4.10),

$$\mathrm{Tr}[\Delta^{(\Omega)}(z-\hat{a},\, z^*-\hat{a}^\dagger)\Delta^{(\Omega)\dagger}(z-\hat{a},\, z^*-\hat{a}^\dagger)] =\frac{1}{\pi^3}\int |\Omega(\alpha,\alpha^*)|^2d^2\alpha. \quad (4.22)$$

Hence in order that the mapping Δ operator $\hat{\Delta}^{(\Omega)}$ belong to the Hilbert-Schmidt class, $\Omega(\alpha,\alpha^*)$ must be square integrable. It is seen from Table III that this is so for the antinormal rule of association.

D. $\Omega(\alpha,\alpha^*)\exp(-\frac{1}{2}|\alpha|^2)$ Absolutely Integrable

We will now derive sufficiency conditions for $\hat{\Delta}^{(\Omega)}$ to be a finite operator, i.e., that the diagonal matrix elements of $\hat{\Delta}^{(\Omega)\dagger}\hat{\Delta}^{(\Omega)}$ with respect to a complete set of states exist.

Let us evaluate the matrix elements with respect to the coherent states, i.e., the matrix elements

$$\langle z|\Delta^{(\Omega)}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger)\Delta^{(\Omega)\dagger}(z_0-\hat{a},\, z_0^*-\hat{a}^\dagger)|z\rangle.$$

In view of (4.5) we may, without loss of generality, take $z_0=0$. We then obtain, on using (4.1), (4.9), (B8), (B10), and (B13),

$$\langle z|\Delta^{(\Omega)}(-\hat{a},\, -\hat{a}^\dagger)\Delta^{(\Omega)\dagger}(-\hat{a},\, -\hat{a}^\dagger)|z\rangle=\frac{1}{\pi^4}\int\int \Omega(\alpha,\alpha^*) \times\exp(-\tfrac{1}{2}|\alpha|^2+z^*\alpha-z\alpha^*)\exp(-\tfrac{1}{2}|\beta|^2-z^*\beta+z\beta^*) \times\exp(\alpha^*\beta)\Omega^*(\beta,\beta^*)d^2\alpha d^2\beta. \quad (4.23)$$

If $\mathfrak{F}(z,z^*)$ denotes the Fourier transform of $\Omega(\alpha,\alpha^*)\times\exp(-\frac{1}{2}|\alpha|^2)$, i.e.,

$$\mathfrak{F}(z,z^*)=\frac{1}{\pi^2}\int \Omega(\alpha,\alpha^*) \times\exp(-\tfrac{1}{2}|\alpha|^2+\alpha z^*-\alpha^*z)d^2\alpha, \quad (4.24)$$

then (4.23) may be expressed in the form

$$\langle z|\Delta^{(\Omega)}(-\hat{a},\, -\hat{a}^\dagger)\Delta^{(\Omega)\dagger}(-\hat{a},\, -\hat{a}^\dagger)|z\rangle=\exp\left(+\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_2^*}\right) \times\mathfrak{F}(z_1,z_1^*)\mathfrak{F}^*(z_2,z_2^*)|_{z_1=z_2=z;\, z_1^*=z_2^*=z^*}. \quad (4.25)$$

We see that in order that $\hat{\Delta}^{(\Omega)}$ be a finite operator, the Fourier transform $\mathfrak{F}(z,z^*)$, defined by (4.24), must exist and be infinitely differentiable with respect to both arguments. A sufficiency condition for the existence of $\mathfrak{F}$ is the absolute integrability of the integrand of $\mathfrak{F}$, i.e.,

$$\int|\Omega(\alpha,\alpha^*)|\,\exp(-\tfrac{1}{2}|\alpha|^2)d^2\alpha<\infty. \quad (4.26)$$

This condition will certainly be satisfied if

$$|\Omega(\alpha,\alpha^*)|=\exp(+\mu|\alpha|^2), \quad (4.27)$$

where μ is a real constant less than $\frac{1}{2}$. We see from Table III that this is so for the Weyl and the antinormal rules of association.

V. CONNECTING RELATIONS FOR DIFFERENT MAPPINGS

Many formulas may be found in the literature which in our terminology express relations between two *c*-number equivalents of the same operator for different rules of association. Such relations may be derived in a systematic way from the present theory by the application of certain general formulas that we will now derive.

A. Integral Relations

Consider two mappings $\Omega^{(1)}$ and $\Omega^{(2)}$ and let $\hat{\Delta}^{(\Omega^{(1)})}$ and $\hat{\Delta}^{(\Omega^{(2)})}$ be the corresponding $\hat{\Delta}$ operators:

$$\Delta^{(\Omega^{(1)})}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int\Omega^{(1)}(\alpha,\alpha^*)\hat{D}(\alpha) \times\exp[-(\alpha z_1^*-\alpha^*z_1)]d^2\alpha, \quad (5.1)$$

$$\Delta^{(\Omega^{(2)})}(z_2-\hat{a},\, z_2^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int\Omega^{(2)}(\alpha,\alpha^*)\hat{D}(\alpha) \times\exp[-(\alpha z_2^*-\alpha^*z_2)]d^2\alpha. \quad (5.2)$$

From (5.1) it follows, on taking the Fourier inverse, that

$$\Omega^{(1)}(\alpha,\alpha^*)\hat{D}(\alpha)=\int\Delta^{(\Omega^{(1)})}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger) \times\exp[(\alpha z_1^*-\alpha^*z_1)]d^2z_1. \quad (5.3)$$

We now substitute for $\hat{D}(\alpha)$ from (5.3) into (5.2) and obtain the following relation between the mapping $\hat{\Delta}$ operators for two different rules of association:

$$\Delta^{(\Omega^{(2)})}(z_2-\hat{a},\, z_2^*-\hat{a}^\dagger)=\int \Delta^{(\Omega^{(1)})}(z_1-\hat{a},\, z_1^*-\hat{a}^\dagger) \times \mathcal{K}_{21}(z_2-z_1,\, z_2^*-z_1^*)d^2z, \quad (5.4)$$

where

$$\mathcal{K}_{21}(\xi,\xi^*)=\frac{1}{\pi^2}\int \bar{\Omega}^{(1)}(\alpha,\alpha^*)\Omega^{(2)}(\alpha,\alpha^*) \times\exp[-(\alpha\xi^*-\alpha^*\xi)]d^2\alpha. \quad (5.5)$$

With the help of (5.4), we may easily derive relations between the Ω equivalents $F^{(\Omega^{(1)})}(z,z^*)$ and $F^{(\Omega^{(2)})}(z,z^*)$ of an operator $G(\hat{a},\hat{a}^\dagger)$ for two different rules of association. We have by Theorem III [Eq. (3.25)]

$$F^{(\tilde{\Omega}^{(1)})}(z,z^*)=\pi\,\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\Delta^{(\Omega^{(1)})}(z-\hat{a},\, z^*-\hat{a}^\dagger)]. \quad (5.6)$$

From (5.4) and (5.6), we immediately obtain the relation

$$F^{(\tilde{\Omega}^{(2)})}(z_2,z_2^*)=\int F^{(\tilde{\Omega}^{(1)})}(z_1,z_1^*) \times\mathcal{K}_{21}(z_2-z_1,\, z_2^*-z_1^*)d^2z_1. \quad (5.7)$$

If we let $\tilde{\Omega}^{(2)}\to\Omega^{(2)}$, $\tilde{\Omega}^{(1)}\to\Omega^{(1)}$, and use the fact, which follows from (3.23) and (3.27), that $\tilde{\Omega}(\alpha,\alpha^*)=\bar{\Omega}(-\alpha,\,-\alpha^*)$, we finally obtain the required relation between $F^{(\Omega^{(2)})}$ and $F^{(\Omega^{(1)})}$:

$$F^{(\Omega^{(2)})}(z_2,z_2^*)=\int F^{(\Omega^{(1)})}(z_1,z_1^*) \times K_{21}(z_2-z_1,\, z_2^*-z_1^*)d^2z_1, \quad (5.8)$$

where

$$K_{21}(\xi,\xi^*)=\mathcal{K}_{12}(-\xi,\,-\xi^*) =\frac{1}{\pi^2}\int \Omega^{(1)}(\alpha,\alpha^*)\bar{\Omega}^{(2)}(\alpha,\alpha^*) \times\exp(\alpha\xi^*-\alpha^*\xi)d^2\alpha. \quad (5.9)$$

A necessary condition for relations of the type (5.4) and (5.8) to exist is, of course, the convergence of the integrals (5.5) and (5.9).

We will now illustrate the relation (5.8) by a few examples. If suffixes N, A, and W denote the normal, the antinormal, and the Weyl rules of association, we have from Table III

$$\Omega^{(A)}\bar{\Omega}^{(W)}=\Omega^{(W)}\bar{\Omega}^{(N)}=\exp(-\tfrac{1}{2}\alpha^*\alpha), \quad (5.10)$$

$$\Omega^{(A)}\bar{\Omega}^{(N)}=\exp(-\alpha^*\alpha). \quad (5.11)$$

On substituting from (5.10) and (5.11) into (5.9) in turn and evaluating the integrals, we obtain the following expressions for the three kernels:

$$K_{WA}(\xi,\xi^*)=K_{NW}(\xi,\xi^*)=(2/\pi)\exp(-2\xi^*\xi), \quad (5.12)$$

$$K_{NA}(\xi,\xi^*)=(1/\pi)\exp(-\xi^*\xi). \quad (5.13)$$

From (5.8), (5.12), and (5.13), we obtain the formulas

$$F^{(W)}(z,z^*)=\frac{2}{\pi}\int F^{(A)}(z_0,z_0^*) \times\exp(-2|z-z_0|^2)d^2z_0, \quad (5.14)$$

$$F^{(N)}(z,z^*)=\frac{2}{\pi}\int F^{(W)}(z_0,z_0^*) \times\exp(-2|z-z_0|^2)d^2z_0, \quad (5.15)$$

$$F^{(N)}(z,z^*)=\frac{1}{\pi}\int F^{(A)}(z_0,z_0^*) \times\exp(-|z-z_0|^2)d^2z_0. \quad (5.16)$$

The relations (5.14) and (5.16) appear to have been first derived by Glauber [Ref. 24, Eqs. (13.36) and (13.37)]. The relation (5.16) was also derived by Mehta and Sudarshan [Ref. 25, Eq. (4.4)], who used it in their study of the diagonal coherent representation of operators.

From (5.16) one may also easily derive two useful identities due to Schwinger. If we take

$$F^{(A)}(z,z^*)=\exp(\beta z^*z)\, f(z^*), \quad (5.17)$$

where $f(z^*)$ is an arbitrary function of z^* and β is a constant, (5.16) gives, after a straightforward calculation,

$$F^{(N)}(z,z^*)=\frac{1}{1-\beta}f\left(\frac{z^*}{1-\beta}\right)\exp\left[\left(\frac{\beta}{1-\beta}\right)z^*z\right]. \quad (5.18)$$

Let $G(\hat{a},\hat{a}^\dagger)$ be the operator onto which $F^{(A)}(z,z^*)$ is mapped via the antinormal rule of association. Then $G(\hat{a},\hat{a}^\dagger)$ is also the operator onto which $F^{(N)}(z,z^*)$ is mapped via the normal rule of association, i.e.,

$$\Omega^{(A)}\{F^{(A)}(z,z^*)\}=\Omega^{(N)}\{F^{(N)}(z,z^*)\}, \quad (5.19)$$

or, on substituting from (5.17) and (5.18) into (5.19), we obtain the identity

$$\Omega^{(A)}\{e^{\beta z^*z}\}f(\hat{a}^\dagger) =\frac{1}{1-\beta}f\left(\frac{\hat{a}^\dagger}{1-\beta}\right)\Omega^{(N)}\left\{\exp\left[\left(\frac{\beta}{1-\beta}\right)z^*z\right]\right\}. \quad (5.20)$$

In a strictly similar way it can be shown that

$$f(\hat{a})\Omega^{(A)}\{e^{\beta z^*z}\} =\frac{1}{1-\beta}\Omega^{(N)}\left\{\exp\left[\left(\frac{\beta}{1-\beta}\right)z^*z\right]\right\}f\left(\frac{\hat{a}}{1-\beta}\right). \quad (5.21)$$

Relations (5.20) and (5.21) are identical (except for notation) with the identities (B7) derived by Schwinger[50] in connection with the quantum theory of angular momentum.

B. Differential Relations

The integral representation (3.14) may be expressed in the form (taking $\Omega=\Omega^{(2)}$)

[50] Reference 28, pp. 274–276.

$$\Delta^{(\Omega^{(2)})}(z-\hat{a},\, z^*-\hat{a}^\dagger)$$
$$=\frac{1}{\pi^2}\int[\Omega^{(2)}(\alpha,\alpha^*)\bar{\Omega}^{(1)}(\alpha,\alpha^*)]\Omega^{(1)}(\alpha,\alpha^*)\hat{D}(\alpha)\times\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha$$
$$=\Omega^{(2)}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)\bar{\Omega}^{(1)}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)\frac{1}{\pi^2}\int\Omega^{(1)}(\alpha,\alpha^*)\times\hat{D}(\alpha)\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha,$$

i.e.,

$$\Delta^{(\Omega^{(2)})}(z-\hat{a},\, z^*-\hat{a}^\dagger)=\mathcal{L}_{21}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)\Delta^{(\Omega^{(1)})}(z-\hat{a},\, z^*-\hat{a}^\dagger),\quad(5.22)$$

where

$$\mathcal{L}_{21}(\xi,\xi^*)=\Omega^{(2)}(\xi,\xi^*)\bar{\Omega}^{(1)}(\xi,\xi^*).\quad(5.23)$$

From (5.23) and (5.6) it follows that

$$F^{(\tilde{\Omega}^{(2)})}(z,z^*)=\mathcal{L}_{21}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)F^{(\tilde{\Omega}^{(1)})}(z,z^*).\quad(5.24)$$

Again letting $\tilde{\Omega}^{(2)}\to\Omega^{(2)}$, $\tilde{\Omega}^{(1)}\to\Omega^{(1)}$, and using the fact that $\tilde{\Omega}(\alpha,\alpha^*)=\bar{\Omega}(-\alpha,\,-\alpha^*)$, we obtain the following differential relation[51] between $F^{(\Omega^{(2)})}$ and $F^{(\Omega^{(1)})}$:

$$F^{(\Omega^{(2)})}(z,z^*)=L_{21}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)F^{(\Omega^{(1)})}(z,z^*),\quad(5.25)$$

where

$$L_{21}(\xi,\xi^*)=\mathcal{L}_{12}(-\xi,\,-\xi^*)=\bar{\Omega}^{(2)}(-\xi,\,-\xi^*)\Omega^{(1)}(-\xi,\,-\xi^*).\quad(5.26)$$

We will illustrate the relation (5.25) by a few examples. From (5.26) and Table III, we have

$$L_{AW}(\xi,\xi^*)=L_{WN}(\xi,\xi^*)=\exp(\tfrac{1}{2}\xi^*\xi),\quad(5.27)$$
$$L_{AN}(\xi,\xi^*)=\exp(\xi^*\xi).\quad(5.28)$$

From (5.25), (5.27), and (5.28), we obtain the relations

$$F^{(A)}(z,z^*)=\exp\left(-\frac{1}{2}\frac{\partial^2}{\partial z\partial z^*}\right)F^{(W)}(z,z^*),\quad(5.29)$$
$$F^{(W)}(z,z^*)=\exp\left(-\frac{1}{2}\frac{\partial^2}{\partial z\partial z^*}\right)F^{(N)}(z,z^*),\quad(5.30)$$
$$F^{(A)}(z,z^*)=\exp\left(-\frac{\partial^2}{\partial z\partial z^*}\right)F^{(N)}(z,z^*).\quad(5.31)$$

These differential relations, which we have just derived, may also be obtained from formulas (5.14)–(5.16) by noting that each of them is a Weierstrass transform. Such a transform may be inverted by standard mathematical techniques.[52] The inversion gives again formulas (5.29)–(5.31).

[51] A special case of formula (5.25) was obtained by N. H. McCoy [Proc. Natl. Acad. Sci. (U. S.) 18, 674 (1932)]. Several other special cases were found by C. L. Mehta in Ref. 8.

VI. EXAMPLES

We will now illustrate some of the general results derived in the previous sections by a few examples relating to the mapping of operators onto c-number functions and expressing operators in ordered forms.

A. Mapping of Operators onto c-Number Functions

1. $\hat{G}=f(\hat{a}^\dagger\hat{a})$; *Normal Rule of Association*

Let us first determine the c-number equivalent, for the normal rule of association, of an arbitrary function $f(\hat{N})$ of the number operator $\hat{N}=\hat{a}^\dagger\hat{a}$. According to (3.43), we now have

$$F^{(N)}(z,z^*)=\Theta^{(N)}\{f(\hat{a}^\dagger\hat{a})\}=\langle z|f(\hat{a}^\dagger\hat{a})|z\rangle,\quad(6.1)$$

where $|z\rangle$ is the coherent state, labeled by the eigenvalue z. Using the well-known expansion of the coherent states in terms of the Fock states $|n\rangle$ [see Ref. 11, Eq. (3.7)], viz.,

$$|z\rangle=\exp(-\tfrac{1}{2}|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle,\quad(6.2)$$

Eq. (6.1) may be expressed in the form

$$F^{(N)}(z,z^*)=\exp(-|z|^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{z^{*m}z^n}{(m!n!)^{1/2}}\times\langle m|f(\hat{a}^\dagger\hat{a})|n\rangle.\quad(6.3)$$

Since $|n\rangle$ is the eigenstate of the number operator, $f(\hat{a}^\dagger\hat{a})|n\rangle=f(n)|n\rangle$. If we make use of this result and of the orthogonality of the Fock states, (6.3) reduces to

$$F^{(N)}(z,z^*)=\Theta^{(N)}\{f(\hat{a}^\dagger\hat{a})\}=\exp(-|z|^2)\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}f(n).\quad(6.4)$$

It is of interest to note that (6.4) implies that the c-number equivalent of $f(\hat{a}^\dagger\hat{a})$ for the normal rule of association is the weighted sum of Poisson distributions, each with the same parameter $|z|^2$, the weighting factors being $f(n)$.

2. $\hat{G}=\mathfrak{N}\exp(-\beta\hat{a}^\dagger\hat{a})$; *Rule of Association Characterized by* $\Omega(\alpha,\alpha^*)=\exp(\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\alpha^*)$.

As another example, we will derive the c-number equivalent, for rules of associations characterized by the filter function

$$\Omega(\alpha,\alpha^*)=\exp(\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\alpha^*),\quad(6.5)$$

of the operator

$$\hat{G}=\mathfrak{N}\exp(-\beta\hat{a}^\dagger\hat{a}),\quad(6.6)$$

[52] I. I. Hirschman and D. V. Widder, in *The Convolution Transform* (Princeton U. P., Princeton, N. J., 1955), Chap. VIII.

where $\mathfrak{N}$ and β are c-numbers. With the choice

$$\beta=\hbar\omega/kT,\quad \mathfrak{N}=[1-\exp(-\hbar\omega/kT)],\qquad (6.7)$$

where k is the Boltzmann constant, (6.6) represents the density operator of a mode labeled by energy $\hbar\omega$, of an electromagnetic field in equilibrium at temperature T.

As pointed out in Sec. III (see also Table IV), the function $\Omega(\alpha,\alpha^*)$ is, for each of the five usual rules of association, of the form (6.5). Consider first the case of the *normal* rule of association. Since the operator (6.6) is a function of the number operator, we may readily obtain its c-number equivalent for the normal rule by applying the formula (6.4), with the choice $f(n)=\mathfrak{N}\exp(-\beta n)$. We then obtain the following expression for $F^{(N)}$:

$$F^{(N)}(z,z^*)=\mathfrak{N}\exp(-|z|^2)\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}\exp(-\beta n).\qquad (6.8)$$

The series on the right-hand side of (6.8) will be recognized as the power-series expansion of

$$\exp\{-|z|^2[1-\exp(-\beta)]\},$$

so that

$$F^{(N)}(z,z^*)=\Theta^{(N)}\{\mathfrak{N}\exp(-\beta\hat{a}^\dagger\hat{a})\}=\mathfrak{N}\exp(-|z|^2/\gamma),\qquad (6.9)$$

where

$$\gamma=[1-\exp(-\beta)]^{-1}.\qquad (6.10)$$

We see from (6.9) that the c-number equivalent of the operator $\hat{G}=\mathfrak{N}\exp(-\beta\hat{a}^\dagger\hat{a})$ for the normal rule of association is proportional to the Gaussian distribution in $|z|$, with variance $\sqrt{\gamma}$.

To determine the Ω equivalent of the operator $\hat{G}=\mathfrak{N}\exp(-\beta\hat{a}^\dagger\hat{a})$ for any other rule of association characterized by (6.5), we make use of the connecting relation (5.25). We then obtain from (5.25), (5.26), (6.5), and (6.9), if we recall that for the normal rule of association $\mu=\nu=0$, $\lambda=\frac{1}{2}$,

$$F^{(\Omega)}(z,z^*)=L_{21}\left(-\frac{\partial}{\partial z^*},\frac{\partial}{\partial z}\right)\mathfrak{N}\exp(-|z|^2|\gamma),\qquad (6.11)$$

where

$$L_{21}(\xi,\xi^*)=\exp[-\mu\xi^2-\nu\xi^{*2}-(\lambda-\tfrac{1}{2})\xi\xi^*].\qquad (6.12)$$

Hence

$$F^{(\Omega)}(z,z^*)=\mathfrak{N}\exp\left[-\mu\frac{\partial^2}{\partial z^{*2}}-\nu\frac{\partial^2}{\partial z^2}+(\lambda-\tfrac{1}{2})\frac{\partial^2}{\partial z\partial z^*}\right]\times\exp(-|z|^2/\gamma).\qquad (6.13)$$

Now we have the identity

$$\exp(-|z|^2/\gamma)=\frac{\gamma}{\pi}\int\exp(-\gamma|\alpha|^2)\times\exp(\alpha^*z-\alpha z^*)d^2\alpha,\qquad (6.14)$$

and if we use it in (6.13), we obtain

$$F^{(\Omega)}(z,z^*)=\frac{\mathfrak{N}\gamma}{\pi\tau}\int\exp(-|\alpha|^2)\times\exp\left(-\frac{\mu}{\tau}\alpha^2-\frac{\nu}{\tau}\alpha^{*2}-\frac{z^*}{\sqrt{\tau}}\alpha+\frac{z}{\sqrt{\tau}}\alpha^*\right)d^2\alpha,\qquad (6.15)$$

where

$$\tau=(\gamma+\lambda-\tfrac{1}{2}).\qquad (6.16)$$

The integral appearing in (6.15) may be evaluated with the help of a formula derived in Ref. 53, Eq. (1.18), and one obtains the following expression for $F^{(\Omega)}(z,z^*)$:

$$F^{(\Omega)}(z,z^*)=\Theta\{\mathfrak{N}\exp(-\beta\hat{a}^\dagger\hat{a})\}=\frac{\mathfrak{N}\gamma}{(\tau^2-4\mu\nu)^{1/2}}\exp\left(-\frac{\mu z^2+\nu z^{*2}+\tau zz^*}{\tau^2-4\mu\nu}\right),\qquad (6.17)$$

provided that

$$\gamma+\lambda-\mu-\nu-\tfrac{1}{2}>0.\qquad (6.18)$$

Condition (6.18) ensures the absolute convergence of the integral (6.15).

With the choice of the normalization constant

$$\mathfrak{N}=1/\gamma\pi,\qquad (6.19)$$

Eq. (6.17) represents a joint Gaussian probability distribution in the two complex variables z and z^*, whose first and second moments are given by

$$\langle z\rangle=\langle z^*\rangle=0,\quad \langle z^2\rangle=-2\nu,\quad \langle z^{*2}\rangle=-2\mu,\quad \langle zz^*\rangle=\tau.\qquad (6.20)$$

It is of interest to note that the two moments $\langle z^2\rangle$ and $\langle z^{*2}\rangle$ depend only on the particular choice of mapping.

If $\hat{G}$ is the density operator of a single mode of the electromagnetic field in thermal equilibrium at temperature T, then we have from (6.10) and (6.7)

$$\gamma=1/\mathfrak{N}=[1-\exp(-\hbar\omega/kT)]^{-1}=(\bar{n}+1),\qquad (6.21)$$

where

$$\bar{n}=[\exp(\hbar\omega/kT)-1]^{-1}\qquad (6.22)$$

is the average occupation number of the mode. Equation (6.17) then gives

$$F^{(\Omega)}(z,z^*)=\frac{1}{\sigma}\exp\left(-\frac{\mu z^2+\nu z^{*2}+\tau' zz^*}{\sigma^2}\right).\qquad (6.23)$$

where

$$\tau'=(\bar{n}+\lambda+\tfrac{1}{2}),\qquad (6.24a)$$

$$\sigma=(\tau'^2-4\mu\nu)^{1/2}.\qquad (6.24b)$$

Condition (6.18) now becomes

$$\bar{n}+\tfrac{1}{2}+\lambda-\mu-\nu>0.\qquad (6.25)$$

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[53] V. Bargmann, Commun. Pure Appl. Math. 14, 187 (1961).

We see from Table IV that for all the five rules of association listed there, $\lambda-\mu-\nu\geq-\frac{1}{2}$. Since $\bar{n}$ is necessarily positive, condition (6.25) is satisfied in all the five cases.

3. $\hat{G}=\hat{a}^{\dagger m}\hat{a}^m$; *Antinormal Rule of Association*

Next we determine the c-number equivalent, for the antinormal rule of association, of the operator

$$\hat{G}=\hat{a}^{\dagger m}\hat{a}^m \quad (m \text{ a non-negative integer}). \tag{6.26}$$

We have, according to Eq. (3.42),

$$F^{(A)}(z,z^*)=\Theta^{(A)}\{\hat{a}^{\dagger m}\hat{a}^m\} = \frac{1}{\pi}\exp(|z|^2)\int\langle-\alpha|\hat{a}^{\dagger m}\hat{a}^m|\alpha\rangle e^{|\alpha|^2} \times\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha. \tag{6.27}$$

Using the fact that the coherent states $|\alpha\rangle$ are eigenstates of the annihilation operator $\hat{a}$, Eq. (6.27) reduces to

$$F^{(A)}(z,z^*)=\frac{1}{\pi}\exp(|z|^2)\int(-\alpha^*)^m(\alpha)^m\exp(-|\alpha|^2) \times\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha$$
$$=\frac{1}{\pi}\exp(|z|^2)\frac{\partial^{2m}}{\partial z^m\partial z^{*m}}\int\exp(-|\alpha|^2) \times\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha$$
$$=\exp(|z|^2)\frac{\partial^{2m}}{\partial z^m\partial z^{*m}}\exp(-|z|^2). \tag{6.28}$$

We now make use of the following identity that may be verified by induction:

$$\frac{\partial^{2m}}{\partial z^{*m}\partial z^m}\exp(-\beta z^* z) =(-\beta)^m\exp(-\beta|z|^2)L_m(\beta|z|^2), \tag{6.29}$$

where $L_m(x)$ is the Laguerre polynomial of order m. From (6.28) and (6.29) with $\beta=1$, we obtain

$$F^{(A)}(z,z^*)=\Theta^{(A)}\{\hat{a}^{\dagger m}\hat{a}^m\}=(-1)^m L_m(z^* z). \tag{6.30}$$

Next we give two examples of mapping of operators that are not explicitly given as functions of $\hat{a}$ and $\hat{a}^\dagger$.

4. $\hat{G}=|n\rangle\langle n|$; *Weyl Rule of Association*

We will now determine the c-number equivalent for the Weyl rule of association of the projection operator

$$\hat{G}=|n\rangle\langle n|, \tag{6.31}$$

for the Fock state $|n\rangle$. According to (3.44), it is given by

$$F^{(W)}(z,z^*)=\Theta^{(W)}\{|n\rangle\langle n|\} =\frac{2}{\pi}\exp(2|z|^2)\int\langle-\alpha|n\rangle\langle n|\alpha\rangle \times\exp[-2(\alpha z^*-\alpha^* z)]d^2\alpha. \tag{6.32}$$

Now the scalar product of a coherent state $|\alpha\rangle$ and a Fock state $|n\rangle$ is[11]

$$\langle n|\alpha\rangle=\exp(-\tfrac{1}{2}|\alpha|^2)\alpha^n/\sqrt{(n!)}, \tag{6.33}$$

so that (6.32) becomes

$$F^{(W)}(z,z^*)=\left(\frac{2}{\pi n!}\right)\exp(2|z|^2)\int(-\alpha^*)^n(\alpha)^n\exp(-|\alpha|^2) \times\exp(2\alpha^* z-2\alpha z^*)d^2\alpha$$
$$=\frac{2}{n!(4)^n}\exp(2|z|^2)\left(\frac{\partial^2}{\partial z\partial z^*}\right)^n \times\exp(-4|z|^2). \tag{6.34}$$

Making again use of the identity (6.29) we find that

$$F^{(W)}(z,z^*)=\Theta^{(W)}\{|n\rangle\langle n|\} =\frac{2(-1)^n}{n!}\exp(-2|z|^2)L_n(4|z|^2), \tag{6.35}$$

where $L_n(x)$ again denotes the Laguerre polynomial of order n.

5. $\hat{G}=|z_0\rangle\langle z_0|$; *Normal Rule of Association*

As the last example of mapping of operators onto c-number functions, let us determine the c-number equivalent, for the normal rule of association, of the operator

$$\hat{G}=|z_0\rangle\langle z_0|, \tag{6.36}$$

where $|z_0\rangle$ is a coherent state. The c-number equivalent of this operator for the normal rule of association is, according to (3.43) and (6.36), given by

$$F^{(N)}(z,z^*)=\Theta^{(N)}\{|z_0\rangle\langle z_0|\}=\langle z|z_0\rangle\langle z_0|z\rangle. \tag{6.37}$$

The scalar product of two coherent states is given by formula (B2), and on using it we obtain

$$F^{(N)}(z,z^*)=\Theta^{(N)}\{|z_0\rangle\langle z_0|\} =\exp(-|z_0|^2)\exp(z_0 z^*+z_0^* z-z^* z). \tag{6.38}$$

In Table V we list, for convenience, the c-number equivalents for the normal, the antinormal, and the Weyl rules of association of the various operators[54] considered in this and in the next section.

B. Ordering of Operators

In Sec. II we have reduced the problem of expressing an arbitrary operator $\hat{G}$ in an ordered form, appropriate

[54] It should be noted that the c-number equivalents correspodning to the projection operator $|m\rangle\langle m|$ on Fock state $|m\rangle$ may be obtained from those corresponding to the operator $f(\hat{a}^\dagger\hat{a})$ by the replacement $f(n)\rightarrow\delta_{n,m}$. This result follows from the fact that if the operator f is expressed in terms of Fock states, i.e.,

$$f(\hat{a}^\dagger\hat{a})=\sum_{n=0}^{\infty}f(n)|n\rangle\langle n|,$$

then the operator $|m\rangle\langle m|$ corresponds to the choice $f(n)=\delta_{n,m}$.

to some prescribed type of ordering, to the problem of determining the corresponding c-number equivalent $F(z,z^*)$ of $G(\hat{a},\hat{a}^\dagger)$. The result is expressed by our Theorem I [Eqs. (2.22) and (2.23)], according to which the Ω-ordered form $\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$ of $G(\hat{a},\hat{a}^\dagger)$ is given by

$$\mathcal{G}^{(\Omega)}(\hat{a},\hat{a}^\dagger)\equiv S^{(\Omega)}\{F^{(\Omega)}(z,z^*)\}, \tag{6.39}$$

where

$$F^{(\Omega)}(z,z^*)=\Theta\{G(\hat{a},\hat{a}^\dagger)\}. \tag{6.40}$$

Here $S^{(\Omega)}$ denotes the substitution operator for Ω ordering and Θ represents the mapping inverse to Ω. We will now illustrate this theorem by a few examples.

1. *Normally Ordered Form of Operator* $\hat{G}=f(\hat{a}^\dagger\hat{a})$

As a first example we will determine the normally ordered form of an arbitrary function $f(\hat{N})$ of the number operator $\hat{N}=\hat{a}^\dagger\hat{a}$, i.e., of the operator

$$G(\hat{a},\hat{a}^\dagger)=f(\hat{a}^\dagger\hat{a}). \tag{6.41}$$

According to (6.39) and (6.40), the normally ordered form of this operator is obtained by applying the substitution operator $S^{(N)}$ for normal ordering to the c-number equivalent $F^{(N)}(z,z^*)$ of $\hat{G}$. We have already determined this equivalent; it is given by Eq. (6.4), viz.,

$$F^{(N)}(z,z^*)=\Theta^{(N)}\{f(\hat{a}^\dagger\hat{a})\}$$
$$=\exp(-|z|^2)\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!}f(n). \tag{6.42}$$

We expand $F^{(N)}(z,z^*)$ in a power series in z and z^* and obtain

$$F^{(N)}(z,z^*)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m f(n)}{m!n!}z^{*m+n}z^{m+n}$$
$$=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^s f(r-s)}{s!(r-s)!}z^{*r}z^r, \tag{6.43}$$

where, on going from the first to the second line, we have made the substitution $m+n=r$, $m=s$. Applying now the substitution operator $S^{(N)}$ to both sides of (6.43) [see Eq. (6.39)], we obtain the required normally ordered form of $f(\hat{a}^\dagger\hat{a})$:

$$f(\hat{a}^\dagger\hat{a})=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^s f(r-s)}{s!(r-s)!}\hat{a}^{\dagger r}\hat{a}^r. \tag{6.44}$$

Formula (6.44) was derived previously by Louisell (Ref. 21, p. 114) in a less direct way.

From (6.43) several results of interest may be derived as special cases. For example, if $\hat{G}$ is the Kth power of the number operator, i.e., if

$$\hat{G}=(\hat{a}^\dagger\hat{a})^K, \tag{6.45}$$

where K is a non-negative integer, then $f(n)=n^K$ and (6.44) gives

$$(\hat{a}^\dagger\hat{a})^K=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{(-1)^s(r-s)^K}{s!(r-s)!}\hat{a}^{\dagger r}\hat{a}^r. \tag{6.46}$$

Now the infinite series over r may be reduced to a finite series by the use of the identity[55]

$$\sum_{s=0}^{r}\frac{(-1)^s(r-s)^K}{s!(r-s)!}=0 \quad\text{for}\quad r>K\geq 0, \tag{6.47}$$

and (6.46) reduces to

$$(\hat{a}^\dagger\hat{a})^K=\sum_{r=0}^{K}\sum_{s=0}^{r}\frac{(-1)^s(r-s)^K}{s!(r-s)!}\hat{a}^{\dagger r}\hat{a}^r. \tag{6.48}$$

This formula for the normally ordered form of the Kth moment of the number operator may be used, for example, to compute the counting moments in terms of the counting correlations in photoelectric measurements of light fluctuations.[56]

2. *Antinormally Ordered Form of Operator* $\hat{a}^{\dagger m}\hat{a}^m$

Next let us express the operator

$$\hat{G}=\hat{a}^{\dagger m}\hat{a}^m \tag{6.49}$$

in antinormally ordered form. We have, according to Eq. (6.30),

$$F^{(A)}(z,z^*)=\Theta^{(A)}\{\hat{a}^{\dagger m}\hat{a}^m\}=(-1)^m L_m(zz^*), \tag{6.50}$$

where L_m is the Laguerre polynomial of order m:

$$L_m(zz^*)=\sum_{r=0}^{m}\frac{(-1)^r}{r!}\binom{m}{r}(zz^*)^r. \tag{6.51}$$

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[55] The identity (6.47) is a special case of the identity

$$S\equiv\sum_{K=0}^{M}\binom{M}{K}(-1)^K(\alpha+K)^N=0 \quad\text{for}\quad M>N\geqslant 0,$$

which may be established as follows:

$$S=\sum_{K=0}^{M}\binom{M}{K}(-1)^K\sum_{L=0}^{N}\binom{N}{L}\alpha^{N-L}K^L$$
$$=\sum_{K=0}^{M}\binom{M}{K}(-1)^K\sum_{L=0}^{N}\binom{N}{L}\alpha^{N-L}\left(x\frac{\partial}{\partial x}\right)^L x^K\bigg|_{x=1}$$
$$=\left(\alpha+x\frac{\partial}{\partial x}\right)^N(1-x)^M\bigg|_{x=1}.$$

From the last expression it is evident that $S=0$ if $M>N\geqslant 0$. We are indebted to Dr. J. H. Eberly for bringing this identity and its proof to our attention.

[56] L. Mandel, Phys. Rev. **136**, B1221 (1964). The corresponding expression for the operator $(\hat{a}^\dagger\hat{a})^k$ in the antinormal form may be obtained from the identity

$$\exp(-\beta\hat{a}^\dagger\hat{a})=\Omega^{(A)}\{\exp[(1-e^{-\beta})|z|^2+\beta]\},$$

which is a special case (for $\mu=\nu=0$, $\lambda=-\frac{1}{2}$) of result (6.17). Expanding both sides of this identity in powers of β and equating the coefficients of equal powers of β, we obtain the required form

$$(\hat{a}^\dagger\hat{a})^K=\sum_{r=0}^{K}\sum_{s=0}^{r}\frac{(-1)^{K+s}(s+1)^K}{s!(r-s)!}\hat{a}^r\hat{a}^{\dagger r}.$$

This formula for the antinormally ordered form of the Kth moment of the number operator may be used to compute the counting moments in terms of the counting correlations in the measurement of light fluctuations by a quantum counter; see also Ref. 49.

Hence we have from (6.50) and (6.51)

$$F^{(A)}(z,z^*)=\sum_{r=0}^{m}\frac{(-1)^{m+r}}{r!}\binom{m}{r}(zz^*)^r. \qquad (6.52)$$

On applying to both sides of (6.52) the substitution operator $S^{(A)}$ for antinormal ordering, we obtain the required representation of the operator (6.49):

$$\hat{a}^{\dagger m}\hat{a}^m=\sum_{r=0}^{m}\frac{(-1)^{m+r}}{r!}\binom{m}{r}\hat{a}^r\hat{a}^{\dagger r}. \qquad (6.53)$$

3. *Weyl-Ordered Form of Operator* $G=\exp(-\beta\hat{a}^\dagger\hat{a})$

As another example let us determine the Weyl-ordered form of the operator

$$\hat{G}=\exp(-\beta\hat{a}^\dagger\hat{a}), \qquad (6.54)$$

where β is a c-number.

We have already determined the c-number equivalents of this operator, for the class of rules of associations characterized by the filter function $\Omega(\alpha,\alpha^*)=\exp(\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\alpha^*)$. (See Sec. VI A 2). For the Weyl rule of association, one has (see Table IV) $\mu=\nu=\lambda=0$ and hence we obtain from (6.17), on making this substitution and taking $\mathfrak{N}=1$,

$$F^{(W)}(z,z^*)=\Theta^{(W)}\{\exp(-\beta\hat{a}^\dagger\hat{a})\}$$
$$=(\gamma/\tau)\exp(-zz^*/\tau), \qquad (6.55)$$

where, according to (6.10) and (6.16), with $\lambda=0$,

$$\gamma=(1-\exp(-\beta))^{-1}, \qquad (6.56)$$

$$\tau=\gamma-\tfrac{1}{2}=\frac{1}{2}\left[\frac{1+\exp(-\beta)}{1-\exp(-\beta)}\right]. \qquad (6.57)$$

Hence (6.55) may be written in the form

$$F^{(W)}(z,z^*)=2(1+\exp(-\beta))^{-1}\exp(-zz^*/\tau)$$
$$=2(1+\exp(-\beta))^{-1}\sum_{n=0}^{\infty}\left(-\frac{1}{\tau}\right)^n z^n z^{*n}. \qquad (6.58)$$

On applying the substitution operator for Weyl ordering to (6.58), we obtain the required Weyl-ordered form of the operator (6.54):

$$\exp(-\beta\hat{a}^\dagger\hat{a})=2[1+\exp(-\beta)]^{-1}$$
$$\times\sum_{n=0}^{\infty}\left(-\frac{1}{\tau}\right)^n(\hat{a}^{\dagger n}\hat{a}^n)_W. \qquad (6.59)$$

In this formula $(\hat{a}^{\dagger n}\hat{a}^n)_W$ denotes, as before, the Weyl-symmetrized form of the operator $\hat{a}^{\dagger n}\hat{a}^n$, i.e., the sum of all possible products involving n creation operators and n annihilation operators, divided by the total number of such products.

4. *Normally Ordered Form of Operator* $\hat{G}=|z_0\rangle\langle z_0|$

As the last example let us consider the normally ordered form of the operator

$$\hat{G}=|z_0\rangle\langle z_0|, \qquad (6.60)$$

where $|z_0\rangle$ is a coherent state.

According to (6.38), the c-number equivalent of this operator for the normal rule of association is

$$F^{(N)}(z,z^*)=\Theta^{(N)}\{|z_0\rangle\langle z_0|\}$$
$$=\exp(-|z_0|^2)\exp(z_0z^*+z_0^*z-z^*z). \qquad (6.61)$$

This function may be expanded into a power series in z and z^* by the use of the following identity [Ref. 57, Eq. (A1)]:

$$\exp(\lambda w+\mu z+\nu wz)$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\nu^n\mu^{m-n}}{m!}L_n^{m-n}\left(\frac{-\mu\lambda}{\nu}\right)w^nz^m, \qquad (6.62)$$

where $L_n^m(x)$ are the associated Laguerre polynomials. From this identity, with $\lambda=z_0$, $w=z^*$, $\mu=z_0^*$, $\nu=-1$, and from (6.61) we obtain

$$F^{(N)}(z,z^*)=\exp(-|z_0|^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(z_0^*)^{m-n}}{m!}$$
$$\times L_n^{m-n}(|z_0|^2)z^{*n}z^m. \qquad (6.63)$$

Applying now the substitution operator $S^{(N)}$ to both sides of (6.63), we obtain the required normally ordered form of the operator $|z_0\rangle\langle z_0|$:

$$|z_0\rangle\langle z_0|=\exp(-|z_0|^2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(z_0^*)^{m-n}}{m!}$$
$$\times L_n^{m-n}(|z_0|^2)\hat{a}^{\dagger n}\hat{a}^m. \qquad (6.64)$$

In particular, when $z_0=0$, we have, on using the fact that $L_n^0(0)=1$,

$$|0\rangle\langle 0|=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\hat{a}^{\dagger n}\hat{a}^n. \qquad (6.65)$$

Formula (6.65) for the normally ordered form of the projection operator of the vacuum state has been derived previously by other methods.[58]

VII. MAPPING IN COORDINATE-MOMENTUM REPRESENTATION

For the sake of completeness, we will now briefly consider the form of our main theorems when the basic operators are the coordinate and the momentum operators $\hat{q}$ and $\hat{p}$, obeying the usual commutation relation (2.1), viz., $[\hat{q},\hat{p}]=i\hbar$. As in Sec. II, we denote the corresponding c-number variables q and p, respectively.

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[57] B. R. Mollow and R. J. Glauber, Phys. Rev. 160, 1076 (1967).

[58] See, e.g., J. Schwinger, J. Math. Phys. 2, 407 (1961).

Let $F(q,p)$ be an arbitrary function of the c-numbers q, p and let $G(\hat{q},\hat{p})$ be the corresponding function of the operators $\hat{q}$, $\hat{p}$ onto which $F(q,p)$ is mapped by means of some rule of association, characterized by a mapping operator Ω. Let Θ again characterize the inverse mapping,

$$G(\hat{q},\hat{p})=\Omega\{F(q,p)\}\,,\qquad F(q,p)=\Theta\{G(\hat{q},\hat{p})\}\,. \qquad (7.1)$$

Each mapping Ω is characterized by a c-number function $\Omega(u,v)$. The associated mapping $\hat{\Delta}$ operator[59] is defined by the formulas [see (3.14) and (3.21)]

$$\Delta^{(\Omega)}(q_0-\hat{q},\,p_0-\hat{p})=\frac{1}{(2\pi)^2}\int\Omega(u,v)\,\exp\{-i[u(q_0-\hat{q})+v(p_0-\hat{p})]\}\,du\,dv \qquad (7.2\text{a})$$

$$=\Omega\{\delta(q_0-q)\delta(p_0-p)\}\,. \qquad (7.2\text{b})$$

By strictly similar arguments to those given in Sec. III, one obtains the following closed expressions for the two mappings (7.1):

$$G(\hat{q},\hat{p})=\int F(q,p)\Delta^{(\Omega)}(q-\hat{q},\,p-\hat{p})\,dq\,dp\,, \qquad (7.3)$$

$$F^{(\Omega)}(q,p)=2\pi\hbar\,\mathrm{Tr}[G(\hat{q},\hat{p})\Delta^{(\tilde{\Omega})}(q-\hat{q},\,p-\hat{p})]\,. \qquad (7.4)$$

In (7.4), $\Delta^{(\tilde{\Omega})}$ is the mapping Δ operator (7.2a) for mapping antireciprocal to Ω, i.e., for mapping characterized by the filter function

$$\tilde{\Omega}(u,v)=[\Omega(-u,\,-v)]^{-1}\,. \qquad (7.5)$$

We may readily define expressions for the Ω equivalents $F(q,p)$ of a given operator function $G(\hat{q},\hat{p})$, for each of the three rules of associations defined in Table I. For the standard rule of association (superscript S), for example, we have, by a strictly similar argument to that given in connection with Eq. (3.8),

$$\Omega^{(S)}(u,v)=\exp(-\tfrac{1}{2}i\hbar uv)\,, \qquad (7.6)$$

so that, according to (7.6), (7.5), and (7.2b),

$$\Delta^{(\tilde{S})}(q_0-\hat{q},\,p_0-\hat{p})=\frac{1}{(2\pi)^2}\int\exp(\tfrac{1}{2}iuv\hbar)\times\exp\{-i[u(q_0-\hat{q})+v(p_0-\hat{p})]\}\,du\,dv$$

$$=\frac{1}{(2\pi)^2}\int\exp(iv\hat{p})\,\exp(iu\hat{q})\times\exp\{-i(uq_0+vp_0)\}\,du\,dv\,, \qquad (7.7)$$

where on going from the first to the second line of (7.7) we used the Baker-Hausdorff identity.[40] Hence according to (7.4) and (7.7),

$$F^{(S)}(q,p)=\frac{\hbar}{2\pi}\int\mathrm{Tr}[G(\hat{q},\hat{p})\,\exp(iv\hat{p})\,\exp(iu\hat{q})]\times\exp[-i(uq+vp)]\,du\,dv\,. \qquad (7.8)$$

One may readily evaluate the trace in (7.8) either in the coordinate or in the momentum representation. In the coordinate representation we have, if we use the completeness of the q-states and the relation

$$\exp(-i\xi\hat{p}/\hbar)\,|q'\rangle=|q'+\xi\rangle\,,$$

$$\mathrm{Tr}[G(\hat{q},\hat{p})\,\exp(iv\hat{p})\,\exp(iu\hat{q})]$$

$$=\mathrm{Tr}\left[\int dq'\,|q'\rangle\langle q'|\,G(\hat{q},\hat{p})\,\exp(iv\hat{p})\,\exp(iu\hat{q})\right]$$

$$=\int\langle q'|\,G(\hat{q},\hat{p})\,\exp(iv\hat{p})\,|q'\rangle\,\exp(iuq')\,dq'$$

$$=\int\langle q'|\,G(\hat{q},\hat{p})\,|q'-\hbar v\rangle\,\exp(iuq')\,dq'\,. \qquad (7.9)$$

From (7.8) and (7.9) it follows that

$$F^{(S)}(q,p)=\hbar\int\langle q|\,G(\hat{q},\hat{p})\,|q-\hbar v\rangle\,\exp(-ivp)\,dv\,. \qquad (7.10)$$

In a similar way we may derive an expression for $F^{(S)}(q,p)$ in terms of the matrix elements of $G(\hat{q},\hat{p})$ in the momentum representation, and we find that

$$F^{(S)}(q,p)=\hbar\int\langle p-\hbar u|\,G(\hat{q},\hat{p})\,|p\rangle\,\exp(-iuq)\,du\,. \qquad (7.10')$$

In a similar manner we may obtain expressions for c-number equivalents of $G(\hat{q},\hat{p})$ for the antistandard rule of association (suffix AS) and the Weyl rule of association (suffix W). The results are

$$F^{(AS)}(q,p)=\hbar\int\langle q-\hbar v|\,G(\hat{q},\hat{p})\,|q\rangle\,\exp(ivp)\,dv \qquad (7.11\text{a})$$

$$=\hbar\int\langle p|\,G(\hat{q},\hat{p})\,|p+\hbar u\rangle\,\exp(-iuq)\,du \qquad (7.11\text{b})$$

and

$$F^{(W)}(q,p)=\hbar\int\langle q-\tfrac{1}{2}\hbar v|\,G(\hat{q},\hat{p})\,|q+\tfrac{1}{2}\hbar v\rangle\times\exp(ivp)\,dv \qquad (7.12\text{a})$$

$$=\hbar\int\langle p-\tfrac{1}{2}\hbar u|\,G(\hat{q},\hat{p})\,|p+\tfrac{1}{2}\hbar u\rangle\times\exp(-iuq)\,du\,. \qquad (7.12\text{b})$$

Expressions (7.12a) and (7.12b) are, of course, well

[59] In the special case, when Ω refers to Weyl's rule of correspondence, $\hat{\Delta}^{(\Omega)}$ is essentially the operator introduced by Kubo in Ref. 1(a).

known.[1a,2–4,59a] A special case of (7.10′) was also derived by Mehta.[8]

We see from (7.10) and (7.11) that in the special case when $\hat{G}$ is a Hermitian operator,

$$F^{(S)*}(q,p)=F^{(AS)}(q,p)\,, \tag{7.13}$$

and from (7.12) we see that

$$F^{(W)*}(q,p)=F^{(W)}(q,p)\,. \tag{7.14}$$

Thus the c-number equivalent of a Hermitian operator is real for the Weyl rule of correspondence and is complex for the standard and antistandard rules. In the latter two cases, the equivalents are complex conjugates of each other.

As an example of these formulas, we may consider the c-number equivalents for the antistandard and the Weyl rules of association of the operator

$$\hat{G}=\hat{q}^n\hat{p}^m \quad (n,\ m \text{ are non-negative integers})\,. \tag{7.15}$$

It is shown in Appendix G that the c-number equivalent of the operator (7.15) for the antistandard rule of association is

$$F^{(AS)}(q,p)=\sum_{r=0}^{\infty}\frac{(i\hbar)^r}{r!}\frac{n!m!}{(n-r)!(m-r)!}q^{n-r}p^{m-r}\,. \tag{7.16}$$

In a similar way we may show that for the operator (7.15), the Weyl equivalent $F^{(W)}(q,p)$ is given by

$$F^{(W)}(q,p)=\sum_{r=0}^{\infty}\left(\frac{i\hbar}{2}\right)^r\frac{n!m!}{r!(n-r)!(m-r)!}q^{n-r}p^{m-r}\,. \tag{7.17}$$

Since the operator (7.15) is already in standard-ordered form, obviously we have

$$F^{(S)}(q,p)=q^np^m\,. \tag{7.18}$$

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APPENDIX A: WEYL RULE OF ASSOCIATION AND WEYL SYMMETRIZATION OF OPERATORS

The correspondence rule of Weyl[5] between a c-number function $F(p,q)$ and an operator function $G(\hat{p},\hat{q})$ is as follows: One represents $F(p,q)$ as a Fourier integral,

$$F(p,q)=\int\!\!\int f(u,v)\exp[i(up+vq)]dudv\,. \tag{A1}$$

Then the corresponding operator function is defined as

$$G(\hat{p}\,\hat{q})=\int\!\!\int f(u,v)\exp[i(u\hat{p}+v\hat{q})]dudv\,. \tag{A2}$$

[59a] A beautiful demonstration of (7.12a) from general physical considerations was recently given by E. Wigner, in *Prospective in Quantum Theory* (MIT, Cambridge, Mass., 1971). We are indebted to Professor Wigner for pointing out this paper.

We will now show that the Weyl rule associates with each monomial

$$F_{mn}(p,q)=p^mq^n \tag{A3}$$

(m,n are non-negative integers) an operator that is readily expressible in a certain symmetrized form.

Let $f_{mn}(u,v)$ be the inverse Fourier transform of $F_{mn}(p,q)$:

$$\begin{aligned}f_{mn}(u,v)&=\frac{1}{(2\pi)^2}\int\!\!\int p^mq^n\exp[-i(up+vq)]dpdq\\&=(i)^{m+n}\frac{\partial^{n+m}}{\partial u^m\partial v^n}\frac{1}{(2\pi)^2}\\&\qquad\times\int\!\!\int\exp[-i(up+vq)]dpdq\\&=(i)^{m+n}\delta^{(m)}(u)\delta^{(n)}(v)\,,\end{aligned} \tag{A4}$$

where $\delta^{(m)}(u)$ is the mth derivative of the Dirac δ function. It follows from (A4) and (A2) that the operator $G_{mn}(\hat{p},\hat{q})$ that corresponds to the monomial $F_{mn}(p,q)=p^mq^n$ in Weyl's correspondence is

$$\begin{aligned}G_{mn}(\hat{p},\hat{q})&=(i)^{m+n}\int\!\!\int\delta^{(m)}(u)\delta^{(n)}(v)\\&\qquad\times\exp[i(u\hat{p}+v\hat{q})]dudv\\&=\sum_{r=0}^{\infty}\frac{(i)^{m+n}(i)^r}{r!}\int\!\!\int\delta^{(m)}(u)\delta^{(n)}(v)\\&\qquad\times(u\hat{p}+v\hat{q})^rdudv\,.\end{aligned} \tag{A5}$$

On using the result that

$$\int_{-\infty}^{+\infty}f(u)\delta^{(m)}(u)du=(-1)^mf^{(m)}(u)\,, \tag{A6}$$

where $f^{(m)}(u)$ is the mth derivative of $f(u)$, we readily see that the only terms that survive under the summation sign in (A5) are some of the terms arising from the expansion of $(u\hat{p}+v\hat{q})^r$ for $r=m+n$. More precisely, one deduces from (A5) that

$$G_{mn}(\hat{p},\hat{q})=(1/N)\times[\text{coefficient of }\xi^m\eta^n\text{ in the expansion of }(\xi\hat{p}+\eta\hat{q})^{m+n}]\,, \tag{A7}$$

where

$$N=(m+n)/m!n!\,. \tag{A8}$$

Equations (A7) and (A8) imply that $G_{mn}(\hat{p},\hat{q})$ is the sum of all possible products involving m $\hat{p}$'s and n $\hat{q}$'s, divided by the total number of such products. [Equation (2.2) of the main text provides an explicit example.] We will denote this expression by $(\hat{p}^m\hat{q}^n)_W$,

$$G_{m,n}(\hat{p},\hat{q})=(\hat{p}^m\hat{q}^n)_W\,, \tag{A9}$$

and refer to $(\hat{p}^m\hat{q}^n)_W$ as the Weyl-symmetrized form of $\hat{p}^m\hat{q}^n$. Of course, $(\hat{p}^m\hat{q}^n)_W\neq\hat{p}^m\hat{q}^n$, except when either m or n is equal to zero.

The fact that $p^m q^n$ and $(\hat{p}^m \hat{q}^n)_W$ correspond to each other in the Weyl association may be used, with the help of Theorem I, to express an arbitrary operator function in the "Weyl-symmetrized form," i.e., in the form

$$G(\hat{p},\hat{q})=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{mn}(\hat{p}^m\hat{q}^n)_W. \tag{A10}$$

APPENDIX B: PROPERTIES OF COHERENT STATES AND OF ASSOCIATED DISPLACEMENT OPERATORS

The coherent states[11] $|z\rangle$ are defined as the eigenstates of the annihilation operator $\hat{a}$:

$$\hat{a}|z\rangle = z|z\rangle. \tag{B1}$$

Since $\hat{a}$ is not Hermitian, the eigenvalues z are, in general, complex. In fact every value of z, real or complex, may be shown to be an eigenvalue of $\hat{a}$. The states $|z\rangle$ are not orthogonal; the scalar product $\langle z'|z\rangle$ has the value

$$\langle z'|z\rangle=\exp\{zz'^*-\tfrac{1}{2}|z|^2-\tfrac{1}{2}|z'|^2\}. \tag{B2}$$

The coherent states form an overcomplete set,[60] yielding the resolution of identity:

$$\frac{1}{\pi}\int |z\rangle\langle z|\, d^2z=1. \tag{B3}$$

The integration in (B3) extends over the whole complex z plane.

The coherent states may be generated from the vacuum state $|0\rangle$ by the application of the displacement operator[11]

$$\hat{D}(z)=\exp(z\hat{a}^\dagger - z^*\hat{a}). \tag{B4}$$

In fact,

$$\hat{D}(z)|0\rangle=|z\rangle, \tag{B5}$$

and more generally,

$$\hat{D}(z')|z\rangle=|z+z'\rangle \exp[\tfrac{1}{2}(z'z^*-z'^*z)]. \tag{B6}$$

The following properties of the displacement operators are among the most important ones[11,61].

Unitarity:

$$\hat{D}^{-1}(z)=\hat{D}^\dagger(z). \tag{B7}$$

Hermitian adjoint:

$$\hat{D}^\dagger(z)=\hat{D}(-z). \tag{B8}$$

Displacement: For any arbitrary operator function $G(\hat{a},\hat{a}^\dagger)$,

$$\hat{D}^\dagger(z)G(\hat{a},\hat{a}^\dagger)\hat{D}(z)=G(\hat{a}+z,\, \hat{a}^\dagger+z^*). \tag{B9}$$

Product:

$$\hat{D}(z')\hat{D}(z)=\hat{D}(z+z')\exp[\tfrac{1}{2}(z'z^*-z'^*z)]. \tag{B10}$$

[60] J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).
[61] A. E. Glassgold and D. Holliday, Phys. Rev. **139**, A1717 (1965).

Trace:

$$\mathrm{Tr}[\hat{D}(z)]=\pi\delta^{(2)}(z). \tag{B11}$$

Orthogonality:

$$\mathrm{Tr}[\hat{D}(z)\hat{D}^\dagger(z')]=\pi\delta^{(2)}(z-z'). \tag{B12}$$

Matrix elements with respect to coherent states:

$$\langle z'|\hat{D}(\alpha)|z\rangle=\langle z'|z\rangle \exp(\alpha z'^*-\alpha^*z-\tfrac{1}{2}|\alpha|^2), \tag{B13}$$

where $\langle z'|z\rangle$ is given by (B2).

The displacement operators form a complete set in the sense that any operator function $G(\hat{a},\hat{a}^\dagger)$ of the Hilbert-Schmidt class may be expressed as a linear combination of the displacement operators. The proof of this result is given in Appendix C.

APPENDIX C: COMPLETENESS OF DISPLACEMENT OPERATORS. OPERATOR ANALOG OF FOURIER THEOREM AND SOME RELATED THEOREMS

Since the completeness of the displacement operators is basic for the present theory, we will now establish this result.[62] It may be expressed in the form of the following *theorem* that may be regarded as an operator analog of the Fourier integral theorem and of Plancheral's theorem for c-number functions[63]:

Every operator function $G(\hat{a},\hat{a}^\dagger)$ of the Hilbert-Schmidt class, i.e., such that $\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)G^\dagger(\hat{a},\hat{a}^\dagger)]<\infty$, *may be expressed uniquely in the form*

$$G(\hat{a},\hat{a}^\dagger)=\int g(\alpha,\alpha^*)\hat{D}(\alpha)d^2\alpha, \tag{C1}$$

where

$$g(\alpha,\alpha^*)=\frac{1}{\pi}\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha)]. \tag{C2}$$

Moreover,

$$\frac{1}{\pi}\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)G^\dagger(\hat{a},\hat{a}^\dagger)]=\int |g(\alpha,\alpha^*)|^2 d^2\alpha. \tag{C3}$$

We will also establish the following generalization of (C3). *If $G_1(\hat{a},\hat{a}^\dagger)$ and $G_2(\hat{a},\hat{a}^\dagger)$ are two operators of the Hilbert-Schmidt class and $g_1(\alpha,\alpha^*)$ and $g_2(\alpha,\alpha^*)$ are their "Fourier transforms," given by equations of the form (C2), then*

$$\frac{1}{\pi}\mathrm{Tr}[G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha)]=\int g_1(\beta,\beta^*) \times g_2(\alpha-\beta,\, \alpha^*-\beta^*)\exp[\tfrac{1}{2}(\beta\alpha^*-\beta^*\alpha)]d^2\beta. \tag{C4}$$

The identity (C4) is in some respects analogous to the convolution theorem[64] on Fourier transforms.

[62] Our attention has been drawn to a paper by J. C. T. Pool [J. Math. Phys. **7**, 66 (1966)], where another proof of this result is given.
[63] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Clarendon, Oxford, 1948).
[64] R. R. Goldberg, *Fourier Transforms* (Cambridge U. P., New York, 1961), pp. 18–20.

In order to establish (C1) we will consider the coherent-state matrix elements of the operator function

$$H(\hat{a},\hat{a}^\dagger)=\frac{1}{\pi}\int \operatorname{Tr}[G(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha)]\hat{D}(\alpha)d^2\alpha. \quad \text{(C5)}$$

Let us first evaluate the coherent-state matrix elements of the displacement operator. We have from (B4), (B1), and (B2),

$$\langle z|\hat{D}(\alpha)|z'\rangle=\exp(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|z|^2-\tfrac{1}{2}|z'|^2 +z^*z'+\alpha z^*-\alpha^*z'). \quad \text{(C6)}$$

Next let us evaluate the trace appearing on the right-hand side of (C5). For this purpose we represent $\hat{G}$ in terms of its coherent-state matrix elements:

$$G(\hat{a},\hat{a}^\dagger)=\frac{1}{\pi^2}\int\int\langle\beta|\hat{G}|\gamma\rangle|\beta\rangle\langle\gamma|d^2\beta d^2\gamma. \quad \text{(C7)}$$

One then readily finds, with the help of (B8) and (C6), that

$$\operatorname{Tr}[G(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha)] =\frac{1}{\pi^2}\int\int\langle\beta|\hat{G}|\gamma\rangle\exp(-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2-\tfrac{1}{2}|\gamma|^2 -\alpha\gamma^*+\alpha^*\beta+\gamma^*\beta)d^2\beta d^2\gamma. \quad \text{(C8)}$$

If we use (C6) and (C8), we readily find that the coherent-state matrix elements of the operator H, defined by (C4), are

$$\langle z|H(\hat{a},\hat{a}^\dagger)|z'\rangle =\frac{1}{\pi^3}\int\int I(\beta,\gamma^*)\langle\beta|\hat{G}|\gamma\rangle\exp(-\tfrac{1}{2}|\beta|^2-\tfrac{1}{2}|\gamma|^2+\gamma^*\beta -\tfrac{1}{2}|z|^2-\tfrac{1}{2}|z'|^2+z^*z')d^2\beta d^2\gamma, \quad \text{(C9)}$$

where

$$I(\beta,\gamma^*)=\int\exp(-|\alpha|^2) \times\exp[\alpha(z^*-\gamma^*)-\alpha^*(z'-\beta)]d^2\alpha. \quad \text{(C10)}$$

The integral $I(\beta,\gamma^*)$ may readily be evaluated, and one finds that

$$I(\beta,\gamma^*)=\pi\exp(-z^*z'-\beta\gamma^*+\gamma^*z'+\beta z^*). \quad \text{(C11)}$$

On substituting (C11) into (C9), one readily finds, with the help of (B2), that

$$\langle z|H(\hat{a},\hat{a}^\dagger)|z'\rangle =\frac{1}{\pi^2}\int\int\langle z|\beta\rangle\langle\beta|\hat{G}|\gamma\rangle\langle\gamma|z'\rangle d^2\beta d^2\gamma. \quad \text{(C12)}$$

On using the resolution of the identity [Eq. (B3)], (C12) reduces to

$$\langle z|H(\hat{a},\hat{a}^\dagger)|z'\rangle=\langle z|G(\hat{a},\hat{a}^\dagger)|z'\rangle. \quad \text{(C13)}$$

Thus we have shown that all the coherent-state matrix elements of $\hat{H}$ are equal to the coherent-state matrix elements of $\hat{G}$ and we may therefore conclude that $H(\hat{a},\hat{a}^\dagger)=G(\hat{a},\hat{a}^\dagger)$. Hence (C5) gives the identity

$$G(\hat{a},\hat{a}^\dagger)=\frac{1}{\pi}\int\operatorname{Tr}[G(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha)]\hat{D}(\alpha)d^2\alpha, \quad \text{(C14)}$$

which, of course, is equivalent to the two relations (C1) and (C2). That the representation (C1) is unique follows immediately on multiplying both sides of (C1) by $\hat{D}^\dagger(\beta)$, taking the trace of the product, and making use of (B12).

From (C1) we also have

$$\operatorname{Tr}[\hat{G}^\dagger(\hat{a},\hat{a}^\dagger)G(\hat{a},\hat{a}^\dagger)]=\int\int g(\alpha,\alpha^*)g^*(\beta,\beta^*) \times\operatorname{Tr}[\hat{D}(\alpha)\hat{D}^\dagger(\beta)]d^2\alpha d^2\beta, \quad \text{(C15)}$$

which, on using (B12), reduces to (C3).

Consider now two operator functions $G_1(\hat{a},\hat{a}^\dagger)$ and $G_2(\hat{a},\hat{a}^\dagger)$ of the Hilbert-Schmidt class. According to (C1) and (C2), each may be represented in the form

$$G_j(\hat{a},\hat{a}^\dagger)=\int g_j(\beta,\beta^*)\hat{D}(\beta)d^2\beta, \quad \text{(C16)}$$

where

$$g_j(\beta,\beta^*)=\frac{1}{\pi}\operatorname{Tr}[G_j(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\beta)] \quad (j=1,2). \quad \text{(C17)}$$

It follows from (C16) and (C17) that

$$\operatorname{Tr}[G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)\hat{D}^\dagger(\alpha)]=\int\int g_1(\beta,\beta^*)g_2(\gamma,\gamma^*) \times\operatorname{Tr}[\hat{D}(\beta)\hat{D}(\gamma)\hat{D}^\dagger(\alpha)]d^2\beta d^2\gamma. \quad \text{(C18)}$$

Now according to (B12) and (B10),

$$\operatorname{Tr}[\hat{D}(\beta)\hat{D}(\gamma)\hat{D}^\dagger(\alpha)]=\pi\delta(\beta+\gamma-\alpha) \times\exp[\tfrac{1}{2}(\beta\gamma^*-\beta^*\gamma)], \quad \text{(C19)}$$

and, on substituting from (C19) into (C18), the identity (C4) follows.

APPENDIX D: PROOF OF RECIPROCITY RELATIONS (3.28) AND (3.29)

We will now establish the two reciprocity relations given by Eqs. (3.28) and (3.29), viz.,

$$\Theta^{(2)}\bar{\Omega}^{(1)}=\Theta^{(1)}\bar{\Omega}^{(2)}, \quad \text{(D1)}$$

$$\bar{\Omega}^{(2)}\Theta^{(1)}=\bar{\Omega}^{(1)}\Theta^{(2)}. \quad \text{(D2)}$$

The superscripts (1) and (2) specify any two mappings and $\bar{\Omega}$ denotes the mappings reciprocal to Ω, i.e., the mapping whose filter function is $[\Omega(\alpha,\alpha^*)]^{-1}$.

Let us consider the effect of operating with $\Theta^{(2)}\bar{\Omega}^{(1)}$ on an arbitrary c-number function $F(z,z^*)$, according to Theorems II and III [Eqs. (3.13) and (3.25)]:

$$\Theta^{(2)}\bar{\Omega}^{(1)}\{F(z,z^*)\} =\pi\int d^2z' F(z',z'^*)\,\mathrm{Tr}[\Delta^{(\tilde{\bar{\Omega}}^{(1)})}(z'-\hat{a},\,z'^*-\hat{a}^\dagger)\times\Delta^{(\tilde{\Omega}^{(2)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)]. \quad \text{(D3)}$$

According to Eq. (4.7), with $\Omega^{(1)}\rightarrow\bar{\Omega}^{(1)}$, $\Omega^{(2)}\rightarrow\tilde{\Omega}^{(2)}$, the trace under the integral sign may be expressed in the form

$$\mathrm{Tr}[\Delta^{(\tilde{\bar{\Omega}}^{(1)})}(z'-\hat{a},\,z'^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega}^{(2)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)] =\frac{1}{\pi^3}\int\bar{\Omega}^{(1)}(\alpha,\alpha^*)\tilde{\Omega}^{(2)}(-\alpha,\,-\alpha^*) \times\exp\{-[\alpha(z'^*-z^*)-\alpha^*(z'-z)]\}d^2\alpha. \quad \text{(D4)}$$

Similarly,

$$\Theta^{(1)}\bar{\Omega}^{(2)}\{F(z,z^*)\}=\pi\int d^2z\,F(z',z'^*) \times\mathrm{Tr}[\Delta^{(\tilde{\bar{\Omega}}^{(2)})}(z'-\hat{a},\,z'^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega}^{(1)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)] \quad \text{(D5)}$$

and

$$\mathrm{Tr}[\Delta^{(\tilde{\bar{\Omega}}^{(2)})}(z'-\hat{a},\,z'^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega}^{(1)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)] =\frac{1}{\pi^3}\int\bar{\Omega}^{(2)}(\alpha,\alpha^*)\tilde{\Omega}^{(1)}(-\alpha,\,-\alpha^*) \times\exp\{-[\alpha(z'^*-z^*)-\alpha^*(z'-z)]\}d^2\alpha. \quad \text{(D6)}$$

Now from the definitions (3.23) and (3.27) of $\bar{\Omega}$ and $\tilde{\Omega}$, it immediately follows that $\bar{\Omega}^{(2)}(\alpha,\alpha^*)\tilde{\Omega}^{(1)}(-\alpha,\,-\alpha^*)=\tilde{\Omega}^{(2)}(-\alpha,\,-\alpha^*)\bar{\Omega}^{(1)}(\alpha,\alpha^*)$, so that (D6) may also be expressed in the form

$$\mathrm{Tr}[\Delta^{(\tilde{\bar{\Omega}}^{(2)})}(z'-\hat{a},\,z'^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega}^{(1)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)] =\frac{1}{\pi^3}\int\tilde{\Omega}^{(2)}(-\alpha,\,-\alpha^*)\bar{\Omega}^{(1)}(\alpha,\alpha^*) \times\exp\{-[\alpha(z'^*-z^*)-\alpha^*(z'-z)]\}d^2\alpha. \quad \text{(D7)}$$

Comparison of (D4) with (D7) shows that

$$\mathrm{Tr}[\Delta^{(\tilde{\bar{\Omega}}^{(1)})}(z'-\hat{a},\,z'^*-\hat{a}^\dagger)\Delta^{(\tilde{\Omega}^{(2)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)] =\mathrm{Tr}[\Delta^{(\tilde{\bar{\Omega}}^{(2)})}(z'-\hat{a},\,z'^*-\hat{a}^\dagger)\times\Delta^{(\tilde{\Omega}^{(1)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)]. \quad \text{(D8)}$$

On comparing (D3) and (D5) and on using (D8), we obtain the first reciprocity relation (D1).

To establish the second reciprocity relation, let us consider the effect of operating with $\bar{\Omega}^{(2)}\Theta^{(1)}$ on an arbitrary operator function $G(\hat{a},\hat{a}^\dagger)$. We have from Theorems II and III [Eqs. (3.13) and (3.25)]

$$\bar{\Omega}^{(2)}\Theta^{(1)}\{G(\hat{a},\hat{a}^\dagger)\}=\pi\int d^2z\,\Delta^{(\bar{\Omega}^{(2)})}(z-\hat{a},\,z^*-\hat{a}^\dagger) \times\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\Delta^{(\tilde{\Omega}^{(1)})}(z-\hat{a},\,z^*-\hat{a}^\dagger)]. \quad \text{(D9)}$$

Let us express $\Delta^{(\bar{\Omega}^{(2)})}$ and $\Delta^{(\tilde{\Omega}^{(1)})}$, on the right-hand side of (D9), in the integral form (4.1). We then obtain

$$\bar{\Omega}^{(2)}\Theta^{(1)}\{G(\hat{a},\hat{a}^\dagger)\} =\frac{1}{\pi^3}\int\bar{\Omega}^{(2)}(\alpha,\alpha^*)\tilde{\Omega}^{(1)}(\beta,\beta^*)\hat{D}(\alpha)\,\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\hat{D}(\beta)] \times\exp\{-[(\alpha+\beta)z^*-(\alpha^*+\beta^*)z]\}d^2\alpha d^2\beta d^2z. \quad \text{(D10)}$$

The integration with respect to z gives $\pi^2\delta^{(2)}(\alpha+\beta)$, so that (D10) reduces to

$$\bar{\Omega}^{(2)}\Theta^{(1)}\{G(\hat{a},\hat{a}^\dagger)\} =\frac{1}{\pi}\int\bar{\Omega}^{(2)}(\alpha,\alpha^*)\tilde{\Omega}^{(1)}(-\alpha,\,-\alpha^*)\hat{D}(\alpha) \times\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\hat{D}(-\alpha)]d^2\alpha =\frac{1}{\pi}\int\bar{\Omega}^{(2)}(\alpha,\alpha^*)\bar{\Omega}^{(1)}(\alpha,\alpha^*)\hat{D}(\alpha) \times\mathrm{Tr}[\hat{G}\hat{D}(-\alpha)]d^2\alpha, \quad \text{(D11)}$$

where we have used the relation $\tilde{\Omega}(-\alpha,\,-\alpha^*)=\bar{\Omega}(\alpha,\alpha^*)$. It is seen that the right-hand side of (D11) remains unchanged if the superscripts (1) and (2) are interchanged. The same must, therefore, be true of the left-hand side of (D11) and this result implies the second reciprocity theorem (D2).

APPENDIX E: MAPPING OF POLYNOMIALS

The class of Ω mappings that we have considered in this paper is characterized by the property (3.17), viz.,

$$\Omega\,\{\exp(\alpha z^*-\alpha^* z)\}=\Omega(\alpha,\alpha^*)\,\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})\,, \quad \text{(E1)}$$

subject to certain restrictions on the functions $\Omega(\alpha,\alpha^*)$. To obtain a fuller understanding of such mappings, we will now investigate the mapping of polynomials.

Let us begin by determining the operator function $G_{mn}{}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$, onto which the monomial $z^{*m}z^n$ is mapped via the Ω association:

$$G_{m,n}{}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\Omega\{z^{*m}z^n\}\,. \quad \text{(E2)}$$

According to (E2) and Theorem II [Eq. (3.13)],

$$G_{m,n}{}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\int z^{*m}z^n\Delta^{(\Omega)}(z-\hat{a},\,z^*-\hat{a}^\dagger)d^2z\,. \quad \text{(E3)}$$

On substituting for $\hat{\Delta}^{(\Omega)}$ the integral representation (4.1), interchanging the order of integration, and evaluating the integral over the z domain, we find that

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\int \Omega(\alpha,\alpha^*)(-1)^m\left\{\frac{\partial^{m+n}}{\partial\alpha^m\partial\alpha^{*n}}\delta^{(2)}(\alpha)\right\} \times\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})d^2\alpha. \quad \text{(E4)}$$

We will convert the integral on the right-hand side of (E4) into an expansion in powers of $\hat{a}$ and $\hat{a}^\dagger$. Such an expansion may be expressed in many different ways. It will be convenient to express it first as a normally ordered form. For this purpose we make use of the Baker-Hausdorff identity[40] to rewrite the exponential term in (E4) in the form

$$\exp(\alpha\hat{a}-\alpha^*\hat{a})=\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})\exp(-\tfrac{1}{2}\alpha\alpha^*),$$

and carry out the integration. We then obtain the following expression for $G_{m,n}^{(\Omega)}$:

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=(-1)^n\frac{\partial^{m+n}}{\partial\alpha^m\partial\alpha^{*n}}\{\Omega(\alpha,\alpha^*)\exp(-\tfrac{1}{2}\alpha\alpha^*) \times\exp(\alpha\hat{a}^\dagger)\exp(-\alpha^*\hat{a})\}\big|_{\alpha=\alpha^*=0}. \quad \text{(E5)}$$

Next let us expand $\Omega(\alpha,\alpha^*)$ into a power series:

$$\Omega(\alpha,\alpha^*)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\omega_{ij}\alpha^{*i}\alpha^j. \quad \text{(E6)}$$

Since $\Omega(0,0)=1$, $\omega_{00}=1$. We also expand each of the exponentials on the right-hand side of (E5) into a series and obtain the formulas

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=(-1)^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\omega_{ij}\frac{(-\frac{1}{2})^k}{k!}\frac{(\hat{a}^\dagger)^r}{r!} \times\frac{(-\hat{a})^s}{s!}\frac{\partial^{m+n}}{\partial\alpha^m\partial\alpha^{*n}}\left[\alpha^{*i}\alpha^j\alpha^{*k}\alpha^k\alpha^r\alpha^{*s}\right]\Big|_{\alpha=\alpha^*=0}. \quad \text{(E7)}$$

Now

$$\frac{\partial^{m+n}}{\partial\alpha^m\partial\alpha^{*n}}\left[(\alpha^*)^{i+k+s}(\alpha)^{j+k+r}\right]\Big|_{\alpha=\alpha^*=0} =m!n!\delta_{m,j+k+r}\delta_{n,i+k+s}. \quad \text{(E8)}$$

Hence

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=(-1)^n m!n!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\omega_{ij}\frac{(-\frac{1}{2})^k}{k!} \times\frac{(\hat{a}^\dagger)^r}{r!}\frac{(-\hat{a})^s}{s!}\delta_{m,j+k+r}\delta_{n,i+k+s}. \quad \text{(E9)}$$

Now since

$$\Omega^{(N)}\{z^{*r}z^s\}=\hat{a}^{\dagger r}\hat{a}^s, \quad \text{(E10)}$$

where $\Omega^{(N)}$ is the mapping operator for the normal rule of association, we may evidently express (E9) in the form

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\Omega\{z^{*m}z^n\}=\Omega^{(N)}\{F_{m,n}^{(N)}(z,z^*;\Omega)\}, \quad \text{(E11)}$$

where the c-number function $F_{m,n}^{(N)}(z,z^*;\,\Omega)$, which isevidently the normally ordered equivalent of $G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)$, is given by

$$F_{m,n}^{(N)}(z,z^*;\Omega)=(-1)^n m!n!\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\omega_{ij}\frac{(-\frac{1}{2})^k}{k!} \times\frac{(z^*)^r}{r!}\frac{(-z)^s}{s!}\delta_{m,j+k+r}\delta_{n,i+k+s}$$

$$=(-1)^n m!n!\sum_i\sum_j\sum_k\omega_{ij}\frac{(-\frac{1}{2})^k}{k!} \times\frac{(z^*)^{m-j-k}}{(m-j-k)!}\frac{(-z)^{n-i-k}}{(n-i-k)!}. \quad \text{(E12)}$$

In the last expression the summation extends over all the non-negative integers i, j, k such that

$$i+k\leq n,\quad j+k\leq m. \quad \text{(E13)}$$

Let us now introduce the set of polynomials $\mathcal{H}_{MN}(z,z^*)$, defined by the formula

$$\mathcal{H}_{MN}(z,z^*)=M!N!\sum_{k=0}^{\min(M,N)}\frac{(-\frac{1}{2})^k}{k!}\frac{(z^*)^{M-k}}{(M-k)!} \times\frac{(z)^{N-k}}{(N-k)!}. \quad \text{(E14)}$$

Then (E12) may be expressed in the form

$$F_{m,n}^{(N)}(z,z^*;\Omega)=\sum_{i=0}^{n}\sum_{j=0}^{m}\omega_{ij}(-1)^n\frac{m!n!}{(m-j)!(n-i)!} \times\mathcal{H}_{m-j,n-i}(-z,z^*). \quad \text{(E15)}$$

The polynomial $\mathcal{H}_{M,N}$ defined by (E14) is evidently a polynomial of degree M in z^* and degree N in z. These polynomials are closely related to the Hermite polynomial of two variables[65] and may be generated by means of the formula

$$\mathcal{H}_{M,N}(z,z^*)=\exp\left(-\frac{1}{2}\frac{\partial^2}{\partial z\partial z^*}\right)z^{*M}z^N. \quad \text{(E16)}$$

[65] *Higher Transcendental Functions* (Bateman Manuscript Project), edited A. Erdélyi, W. Magnus, F. Oberhettinger, and F. C. Tricomi (McGraw-Hill, New York, 1953), Vol. II, p. 283.

The expression on the right-hand side of (E11), together with (E15), expresses $\hat{G}_{m,n}^{(\Omega)}$ as a normally ordered polynomial of degree m in $\hat{a}^\dagger$ and degree n in $\hat{a}$, i.e., in the form

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\sum_{p=0}^{m}\sum_{q=0}^{n} c_{pq}(\hat{a}^\dagger)^p(\hat{a})^q, \quad \text{(E17)}$$

where the c_{pq} are constants.

We may, of course, express $\hat{G}_{m,n}^{(\Omega)}$ also as an antinormally ordered polynomial. It is readily verified that this polynomial is also of degree m in $\hat{a}^\dagger$ and degree n in $\hat{a}$, i.e., it is of the form

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\sum_{p=0}^{m}\sum_{q=0}^{n} c_{pq}'(\hat{a})^q(\hat{a}^\dagger)^p, \quad \text{(E18)}$$

where the c_{pq}' are constants. This fact may be readily verified with the help of the commutation relation $[\hat{a},\hat{a}^\dagger]=1$. More explicitly, calculations strictly similar to those that led to (E11) and (E15) show that $\hat{G}_{m,n}^{(\Omega)}$ may be expressed in the form

$$G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\Omega\{z^{*m}z^n\}=\Omega^{(A)}\{F_{m,n}^{(A)}(z,z^*;\Omega)\}, \quad \text{(E19)}$$

where

$$F_{m,n}^{(A)}(z,z^*;\Omega)=\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij}(-1)^i\frac{m!n!}{(m-j)!(n-i)!}\times\mathcal{H}_{m-j,n-i}(z,z^*). \quad \text{(E20)}$$

In (E19), $\Omega^{(A)}$ is, of course, the mapping operator for the antinormal rule of association,

$$\Omega^{(A)}\{z^{*r}z^s\}=\hat{a}^s\hat{a}^{\dagger r}, \quad \text{(E21)}$$

and $F_{mn}^{(A)}(z,z^*;\Omega)$, given by (E20), is the antinormally ordered equivalent of $\hat{G}_{mn}^{(\Omega)}$.

From the linearity of the Ω mapping and from the results that we have just established, it follows that a c-number polynomial

$$F(z,z^*)=\sum_{m=0}^{M}\sum_{n=0}^{N} d_{m,n}z^{*m}z^n, \quad \text{(E22)}$$

where the d_{mn} are constants, is mapped onto the operator function

$$G^{(\Omega)}(\hat{a},\hat{a}^\dagger)=\Omega\{F(z,z^*)\}=\sum_{m=0}^{M}\sum_{n=0}^{N} d_{m,n}G_{m,n}^{(\Omega)}(\hat{a},\hat{a}^\dagger), \quad \text{(E23)}$$

where $\hat{G}_{m,n}^{(\Omega)}$ is given in normally ordered form by (E11) and (E15) and in antinormally ordered form by (E19) and (E20). Thus we have now established the following result: *Each mapping Ω maps a polynomial of degree M in z^* and degree N in z onto a polynomial of degree M in $\hat{a}^\dagger$ and degree N in $\hat{a}$.*

Let us now consider the inverse problem, of mapping polynomials in $\hat{a}$ and $\hat{a}^\dagger$ onto c-number functions. We first determine the Ω equivalent of the normally ordered product $a^{\dagger m}a^n$, i.e., the c-number function

$$F_{m,n}^{(\Omega)}(z,z^*)=\Theta\{\hat{a}^{\dagger m}\hat{a}^n\}. \quad \text{(E24)}$$

The function $F_{m,n}^{(\Omega)}$ may readily be determined with the help of the reciprocity relation (3.29) and some of the results that we just established.

According to the reciprocity relation, we have, for any two rules of association [superscripts (1) and (2)],

$$\Omega^{(2)}\Theta^{(1)}=\bar{\Omega}^{(1)}\bar{\Theta}^{(2)}, \quad \text{(E25)}$$

where the bar again denotes reciprocal mappings. Now we have, from (E19) and the relation

$$z^{*m}z^n=\Theta^{(N)}\{\hat{a}^{\dagger m}\hat{a}^n\},$$
$$\Omega\{\Theta^{(N)}\{\hat{a}^{\dagger m}\hat{a}^n\}\}=\Omega^{(A)}\{F_{m,n}^{(A)}(z,z^*;\Omega)\}. \quad \text{(E26)}$$

Moreover, the normal and the antinormal rules of associations are mutually reciprocal, i.e., $\Omega^{(A)}=\bar{\Omega}^{(N)}$; if we make use of this result on the right-hand side of (E26) and compare the resulting expression with (E25), we may immediately conclude that

$$\bar{\Theta}\{\hat{a}^{\dagger m}\hat{a}^n\}=F_{m,n}^{(A)}(z,z^*;\Omega).$$

Finally, letting $\Omega\to\bar{\Omega}$, we obtain the result

$$\Theta\{\hat{a}^{\dagger m}\hat{a}^n\}=F_{m,n}^{(A)}(z,z^*;\bar{\Omega}). \quad \text{(E27)}$$

$F_{m,n}^{(A)}(z,z^*;\bar{\Omega})$ is, of course ,given by (E20), with Ω replaced by $\bar{\Omega}$, i.e., with the coefficients ω_{ij} [see (E6)] replaced by coefficients ω_{ij}, where

$$\bar{\Omega}(\alpha,\alpha^*)=[\Omega(\alpha,\alpha^*)]^{-1}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\bar{\omega}_{ij}\alpha^{*i}\alpha^j. \quad \text{(E28)}$$

It follows from (E27) and the linearity of the Θ mappings that a normally ordered polynomial in $\hat{a}$ and $\hat{a}^\dagger$, i.e., the operator function

$$G(\hat{a},\hat{a}^\dagger)=\sum_{m=0}^{M}\sum_{n=0}^{N} b_{mn}\hat{a}^{\dagger m}\hat{a}^n \quad \text{(E29)}$$

(where the b_{mn} are constants), is mapped onto the c-number polynomial

$$F^{(\Omega)}(z,z^*)=\Theta\{\hat{G}\}=\sum_{m=0}^{M}\sum_{n=0}^{N} b_{mn}F_{mn}^{(A)}(z,z^*;\bar{\Omega}). \quad \text{(E30)}$$

In a strictly similar way as led to the derivation of (E27), we find that the Ω equivalent of the antinormally ordered product $\hat{a}^n\hat{a}^{\dagger m}$ is the c-number function

$$\Theta\{\hat{a}^n\hat{a}^{\dagger m}\}=F_{m,n}^{(N)}(z,z^*;\bar{\Omega}), \quad \text{(E31)}$$

where $F_{mn}^{(N)}(z,z^*;\bar{\Omega})$ is given by (E15), with Ω being replaced by $\bar{\Omega}$ in accordance with (E28). This result implies, in view of the linearity of the Θ mappings, that an antinormally ordered ploynomial in $\hat{a}$ and $\hat{a}^\dagger$, i.e., the operator function

$$G(\hat{a},\hat{a}^\dagger)=\sum_{n=0}^{N}\sum_{m=0}^{M} b_{mn}'\hat{a}^n\hat{a}^{\dagger m} \tag{E32}$$

(where the b_{nm}' are constants), is mapped onto the c-number polynomial

$$F^{(\Omega)}(z,z^*)=\Theta\{G(\hat{a},\hat{a}^\dagger)\}$$
$$=\sum_{n=0}^{N}\sum_{m=0}^{M} b_{nm}'F_{nm}^{(N)}(z,z^*;\bar{\Omega}). \tag{E33}$$

Since $F_{mn}^{(A)}$ and $F_{mn}^{(N)}$ are each a polynomial of degree m in z^* and n in z, Eqs. (E29), (E30), (E32), and (E33) imply that *each Ω maps a polynomial (irrespective of whether ordered normally or antinormally) of degree M in $\hat{a}^\dagger$ and degree N in $\hat{a}$ onto a polynomial of degree M in z^* and degree N in z.*

It is remarkable that the two functions $F_{mn}^{(N)}$ and $F_{mn}^{(A)}$ defined by (E15) and (E20), respectively, play such an important role in the Ω mapping of polynomials in z and z^*, as well as in the inverse Θ mapping of polynomials in $\hat{a}$ and $\hat{a}^\dagger$.

APPENDIX F: DERIVATION OF (3.44) FOR WEYL EQUIVALENT $F^{(W)}(z,z^*)$

The mapping Δ operator $\Delta^{(W)}(z-\hat{a}, z^*-\hat{a}^\dagger)$ is, according to (3.14) and Table III, given by

$$\Delta^{(W)}(z-\hat{a}, z^*-\hat{a}^\dagger)=\frac{1}{\pi^2}\int \exp[(-\alpha z^*-\alpha^* z)]$$
$$\times\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})d^2\alpha. \tag{F1}$$

According to Theorem III [Eq. (3.25)] and (F1), the c-number equivalent, for the Weyl rule of association, of an operator function $G(\hat{a},\hat{a}^\dagger)$ is given by

$$F^{(W)}(z,z^*)=\frac{1}{\pi}\int \text{Tr}[\hat{G}\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})]$$
$$\times\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha, \tag{F2}$$

i.e., it is the Fourier transform of the function $(1/\pi)\times\text{Tr}[\hat{G}\exp(\alpha\hat{a}^\dagger-\alpha^*\hat{a})]$. We will also make use of the identity

$$\exp(-2|z|^2)=\frac{1}{2\pi}\int \exp(-\tfrac{1}{2}|\alpha|^2)$$
$$\times\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha. \tag{F3}$$

It follows from (F2), (F3), and the convolution theorem on Fourier transforms that

$$F^{(W)}(z,z^*)\exp(-2|z|^2)$$
$$=\int C(\alpha,\alpha^*)\exp[-(\alpha z^*-\alpha^* z)]d^2\alpha, \tag{F4}$$

where $C(\alpha,\alpha^*)$ is the convolution product

$$C(\alpha,\alpha^*)=\frac{1}{2\pi^2}\int \text{Tr}\{\hat{G}\exp[(\alpha-\beta)\hat{a}^\dagger-(\alpha^*-\beta^*)\hat{a}]\}$$
$$\times\exp(-\tfrac{1}{2}\beta\beta^*)d^2\beta$$
$$=\frac{1}{2\pi^2}\int \text{Tr}\{\hat{G}\exp[(\alpha-\beta)\hat{a}^\dagger]$$
$$\times\exp[-(\alpha^*-\beta^*)\hat{a}]\}$$
$$\times\exp(-\tfrac{1}{2}|\alpha-\beta|^2-\tfrac{1}{2}|\beta|^2)d^2\beta. \tag{F5}$$

In passing from the first to the second expression on the right-hand side of (F5), we have used the Baker-Hausdorff identity.[40]

The integral may be simplified by inserting the resolution of the identity in terms of the coherent states [Eq. (B3)] between the first and the second term, and also between the second and the third term in the trace in (F5). We then obtain the following expression for $C(\alpha,\alpha^*)$:

$$C(\alpha,\alpha^*)=\frac{1}{2\pi^4}\int\int\int \exp[-\tfrac{1}{2}|\alpha-\beta|^2-\tfrac{1}{2}|\beta|^2]$$
$$\times\langle z_2|\hat{G}|z_1\rangle\langle z_1|z_2\rangle\exp[(\alpha-\beta)z_1^*-(\alpha^*-\beta^*)z_2]$$
$$\times d^2z_1d^2z_2d^2\beta. \tag{F6}$$

On integrating over β and on using Eq. (B2) for the scalar product $\langle z_1|z_2\rangle$, (F6) reduces to

$$C(\alpha,\alpha^*)=\frac{1}{2\pi^3}\int\int\langle z_2|\hat{G}|z_1\rangle$$
$$\times\exp(-\tfrac{1}{2}|z_1|^2-\tfrac{1}{2}|z_2|^2-\tfrac{1}{4}|\alpha|^2)$$
$$\times\exp(-\tfrac{1}{2}\alpha^*z_2+\tfrac{1}{2}\alpha z_1^*)d^2z_1d^2z_2$$
$$=\frac{1}{2\pi^3}\int\int\langle-\tfrac{1}{2}\alpha|z_2\rangle\langle z_2|\hat{G}|z_1\rangle$$
$$\times\langle z_1|\tfrac{1}{2}\alpha\rangle d^2z_1d^2z_2$$
$$=(1/2\pi)\langle-\tfrac{1}{2}\alpha|\hat{G}|\tfrac{1}{2}\alpha\rangle. \tag{F7}$$

Finally, on substituting (F7) into (F4) and changing the variables of integration from α, α^* to 2α, $2\alpha^*$, we obtain formula (3.44) for the Weyl equivalent $F^{(W)}(z,z^*)$ of an operator function $G(\hat{a},\hat{a}^\dagger)$:

$$F^{(W)}(z,z^*)=\frac{2}{\pi}\exp(2|z|^2)\int\langle-\alpha|\hat{G}|\alpha\rangle \times\exp[-2(\alpha z^*-\alpha^*z)]d^2\alpha. \quad (F8)$$

This simple expression for $F^{(W)}(z,z^*)$ in terms of the coherent-state matrix elements is very useful in computing the Weyl equivalent of many commonly occurring operators.

APPENDIX G: DERIVATION OF (7.16)

In this appendix we will derive the c-number equivalent of the operator $\hat{G}=\hat{q}^n\hat{p}^m$, corresponding to the antistandard rule of association. From relation (7.11a) it is clear that $F^{(AS)}(q,p)$, corresponding to $\hat{G}=\hat{q}^n\hat{p}^m$, is given by

$$F^{(AS)}(q,p)=\hbar\int\langle q-\hbar\xi|\hat{p}^m|q\rangle(q-\hbar\xi)^n\exp(i\xi p)d\xi. \quad (G1)$$

It should be noted that (G1) may be written in the form

$$F^{(AS)}(q,p) =\text{coefficient of } \frac{(i\lambda)^m}{m!} \text{ in } \hbar\int\langle q-\hbar\xi|\exp(i\lambda\hat{p})|q\rangle \times(q-\hbar\xi)^n\exp(i\xi p)d\xi. \quad (G2)$$

If we use the displacement property of the operator $\exp(i\lambda\hat{p})$ and the orthogonality of the $|q\rangle$ states, it follows that

$$\langle q-\hbar\xi|\exp(i\lambda\hat{p})|q\rangle=\langle q-\hbar\xi|q-\hbar\lambda\rangle =\delta(\hbar\xi-\hbar\lambda)=(1/\hbar)\delta(\xi-\lambda). \quad (G3)$$

On substituting from (G3) into (G2), we finally obtain the formula

$$F^{(AS)}(q,p)=\text{coefficient of } \frac{(i\lambda)^m}{m!} \text{ in } [\exp(i\lambda p)(q-\hbar\lambda)^m] =\sum_{r=0}^{\infty}\frac{(i\hbar)^r}{r!}\frac{n!m!}{(n-r)!(m-r)!}q^{n-r}p^{m-r}, \quad (G4)$$

which is the desired result.