form. Suppose that the metric

$$ds^2 = V^2 dt^2 + V^{-2}\gamma_{\alpha\beta} dx^\alpha dx^\beta, \quad (28)$$

with V and $\gamma_{\alpha\beta}$ functions of x^1, x^2, and x^3, satisfies the field equations with incoherent matter as source. Then the metric

$$ds^2 = \frac{4\lambda^2 V^2}{(V^2+1)^2} dt^2 + \frac{(V^2+1)^2}{4\lambda^2 V^2} \gamma_{\alpha\beta} dx^\alpha dx^\beta \quad (29)$$

and the potential

$$\phi = \lambda(V^2-1)/(V^2+1) \quad (30)$$

satisfy the Einstein-Maxwell equations.

III. CONCLUDING REMARKS

The immediate use of the present result is in the generation of electromagnetic metrics from known solutions of the field equations corresponding to nonempty spaces filled with dust. Further, from the correspondence of the metrics (28) and (29), we note that any singularity in $\gamma_{\alpha\beta}$ of one metric is also a singularity of the second metric. Also, if $V^2 \ll 1$, we observe that the singularities $V=0$ in Eq. (28) give the same singularities as in (29). However, if V is large, $V^2/(V^2+1)^2$ tends to zero and the two metrics behave differently with regard to their singularities. Thus, there is a one-to-one correspondence between the singularities of both metrics except when $|V| \to \infty$ in the nonempty space. This correspondence between the singularities of metrics (28) and (29) may have some relevance in the study of the problem of gravitational collapse.

In conclusion, we hope that the results of the present paper will lead to deeper understanding of gravitoelectrodynamics in nonempty spaces.

PHYSICAL REVIEW D VOLUME 2, NUMBER 10 15 NOVEMBER 1970

Relativistic Center-of-Mass Motion and the Electromagnetic Interaction of Systems of Charged Particles

FRANCIS E. CLOSE*
Department of Theoretical Physics, 12 Parks Road, Oxford, England

AND

HUGH OSBORN
Department of Natural Philosophy, The University, Glasgow W.2., Scotland

(Received 5 May 1970)

Effects arising from relativistic corrections to the c.m. motion for particles interacting with an external electromagnetic field and also for their interparticle interaction are considered. The Hamiltonian discribing the electromagnetic interaction to order v^2/c^2, including Coulomb, magnetic, and spin-orbit effects, between two slowly moving charged particles is derived and the correct choice of c.m. dynamical variables is shown to be different from the usual nonrelativistic form. With the modified treatment of the c.m. motion introduced by considerations of relativistic invariance, the Hamiltonian is put in a form which exhibits, except for one term, a clean separation of over-all c.m. motion and internal dynamics and, but for this term, is of the required relativistic form to order v^2/c^2. The extra term, which spoils the above results, is demonstrated to be removed by correct treatment of the Thomas precession of the internal orbital angular momentum. Alternatively, the desired c.m. separation can be achieved by a modification of the free c.m. relativistic variables induced by the interaction. The relativistic variables for an N-body system with arbitrary interaction are briefly considered.

I. INTRODUCTION

IN order to discuss the dynamics of composite systems, it is essential to distinguish the kinematics of the motion of the system as a whole from the dynamics arising from forces between the constituent particles. Nonrelativistically, this is just the problem of the separation of c.m. motion, which is well understood for particles interacting through local potentials.

For slowly moving charged particles, the forces arising from electromagnetic interaction can be distinguished according to the order in v^2/c^2 (where v is the velocity of the moving particles) in which they arise. After the Coulomb forces the next most important effect, of first order in v^2/c^2, is the interaction due to the magnetic fields of each particle acting on the others.[1] While

* Present address: Stanford Linear Accelerator Center, Stanford, Calif. 94305.

[1] A lucid discussion of the magnetic interaction between two spinless particles is given in E. Breitenberger, Am. J. Phys. **36**, 505 (1968). This paper also provides a useful selection of background references on many aspects on electromagnetic forces and slowly moving particles. For readers who appreciate a rather more discursive approach, we would strongly recommend digesting this paper first. The present article has been written so as to avoid unnecessary overlap.

for atomic energy levels this effect is at most barely noticeable, the magnetic forces due to charged particles moving in macroscopic conductors are, of course, the foundation of whole industries.

The point of departure of this paper is the well-known result that for a system of two particles with a magnetic interaction it is not possible, according to the conventional treatment, to separate cleanly the over-all motion of the c.m. and internal motion which should contain the real dynamics of the system. Thus it would appear, for example, that the position of the c.m. itself does not move uniformly or that the internal angular momentum is not conserved. These results are obtained by applying the usual, nonrelativistic decomposition of the single-particle dynamical variables into parts describing the over-all c.m. motion and parts pertaining to the internal dynamics. For a nonrelativistic system this is sufficient to produce the required separation of internal and c.m. motion. However, electromagnetism is the relativistic theory par excellence and, since the magnetic interaction is a v^2/c^2 effect, we are not dealing with a purely nonrelativistic situation.

A similar problem arises in the treatment of the interaction of a system of particles with an external electromagnetic field. It is conventional to write the interaction of such a system, to given order in v/c, as the sum of the Foldy-Woutheysen interactions of the individual particles with the external field to that order. However, when a nonrelativistic separation of c.m. and internal motion is made, certain general theorems on the low-energy behavior of the amplitudes are not satisfied.

Recently we have given a unified derivation of the internal and over-all c.m. variables for a quantized relativistic system of particles with spin.[2,3] These variables, which reduce to the well-known expressions in the nonrelativistic limit, allow for a relativistic separation of the internal and over-all c.m. motion. Furthermore, it is straightforwardly apparent how to introduce an interaction while maintaining relativistic invariance, and the equally necessary condition that at large separations the particles should be free.[4] For particles without spin, the form of these c.m. variables has been known for some time[5] but, probably due to their extremely complicated form, have not appeared outside their original home of general discussions on relativistic two-body problems. However, for applications where relativistic kinematical effects are significant but the phenomena entailed by covariant local field theory, such as pair creation, are not, an expansion in powers of v^2/c^2 should be permissible. To first order in v^2/c^2 we have demonstrated how these modifications are necessary for the correct treatment of the electromagnetic spin-orbit interaction in N-body bound states[3,6] and a Hamiltonian was derived[3,6,7] with which it was possible to satisfy the low-energy theorems.

Encouraged by the above success, we here apply the relativistic c.m. variables, to first order in v^2/c^2, to the quantized Hamiltonian describing the Coulomb-plus-magnetic interaction for two charged particles. We shall also briefly discuss the more general case of N particles interacting through arbitrary two-body potentials.

In the electromagnetic case there are three insights to be gained. First there is the very delicate interplay between Coulomb, magnetic, and also spin-orbit terms so as to achieve a final form for the Hamiltonian which is mainifestly relativistic to v^2/c^2. This demonstrates the subtle consistency of the electromagnetic interaction: how the Coulomb force entails the existence of

[2] H. Osborn, Phys. Rev. **176**, 1514 (1968).

[3] F. E. Close and L. A. Copley, Nucl. Phys. **B19**, 477 (1970).

[4] The literature on relativistic two-body equations in quantum mechanics (by which we mean equations not directly derived from relativistic field theory) is rather involved, so only an incomplete resumé is possible. A general discussion on the relativistic treatment of internal variables, with suggestions for formulating the dynamics in the c.m. frame, can be found in A. S. Eddington, Proc. Cambridge Phil. Soc. **35**, 196 (1939); P. A. M. Dirac, R. Peierls, and M. H. L. Pryce, *ibid.* **39**, 193 (1942); and A. S. Eddington, *ibid.* **39**, 201 (1942). The possible forms which relativistic dynamics might take, and various alternative methods of introducing an interaction, are elucidated in P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949). The actual relativistic two-body single-time Hamiltonian used here was first introduced by B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953); see also L. H. Thomas, *ibid.* **85**, 868 (1952); and B. Bakamjian, *ibid.* **121**, 1849 (1961). A very good account of the problems posed by a relativistic treatment of two-body interactions is in L. L. Foldy, Phys. Rev. **122**, 289 (1961). It is here that the asymptotic separability requirement, viz., particles should be freely propagating at large separations, is strongly emphasized. An approximate, to low orders of v^2/c^2, solution of these constraints is obtained, closely similar to the Bakamjian-Thomas expressions for the Hamiltonian and other generators of the inhomogeneous Lorentz group. Alternative relativistic two-body theories have been given by E. C. G. Sudarshan, in *Lectures in Theoretical Physics* (Benjamin, New York, 1962), Vol. 2; and by T. F. Jordan, A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. **133**, B487 (1964). Formal two-particle scattering theory based on the Bakamjian-Thomas Hamiltonian is amply treated by R. Fong and J. Sucher, J. Math. Phys. **5**, 456 (1964); and G. Schierholz, Nucl. Phys. **B7**, 432 (1968). Fong and Sucher introduce the internal momentum in the form used here and are then able to satisfy Foldy's separability condition without approximation. Further discussion of scattering theory and generalization to a relativistic description of three particles is contained in T. F. Jordan, J. Math. Phys. **5**, 1345 (1964); and F. Coester, Helv. Phys. Acta **38**, 7 (1965). The above papers are all rather formal and/or mathematical in nature. The only calculations we know of that have been performed in the Bakamjian-Thomas framework are in N. D. Son and J. Sucher, Phys. Rev. **153**, 1496 (1967); **161**, 1694(E) (1967); D. Avison *ibid.* **154**, 1583 (1967); and M. R. Wallace, J. Phys. **A3**, 505 (1970), where a surprising numerical result in a bootstrap calculation is obtained.

[5] B. Barsella and E. Fabri, Phys. Rev. **126**, 1561 (1962); **128**, 451 (1962). It is the correct definition of c.m. variables that allows the solution of Foldy's separability condition, mentioned in Ref. 4.

[6] H. Osborn, Phys. Rev. **176**, 1523 (1968). The same problems were also solved by S. J. Brodsky and J. R. Primack, Ann. Phys. (N. Y.) **52**, 315 (1969). These authors have derived a modified two-particle electromagnetic interaction Hamiltonian to take account of relativistic modifications of the spin-orbit term; see S. J. Brodsky and J. R. Primack, Phys. Rev. **174**, 2071 (1968). The same modified Hamiltonian is also given in the first article cited in this reference.

[7] R. A. Krajcik and L. L. Foldy, Phys. Rev. Letters **24**, 545 (1970).

magnetic and spin-orbit interactions so as to achieve relativistic invariance. Secondly, once the Hamiltonian is in this form, the separation of internal and over-all motion has been achieved. In particular, contrary to other treatments,[1] the position operator for the over-all c.m. has a uniform rate of change and the internal angular momentum is conserved. Finally, the above mellifluous results cannot be achieved without the introduction of an additional term in the two-body electromagnetic interaction Hamiltonian. This has an immediate interpretation; it corresponds to the Thomas precession of the internal orbital angular momentum of the two-particle system.

In Sec. II we discuss relativistic kinematics for quantized theories and the introduction of c.m. variables with the usual desired properties. In Sec. III we discuss the Hamiltonian derived in Refs. 3 and 7 which described, to $O(v^2/c^2)$, the interaction of a weakly bound composite system with an electromagnetic field. This Hamiltonian has terms present in addition to the conventional sum of individual particle-field Foldy-Wouthuysen Hamiltonians in order that a correct separation of the c.m. and internal motion of the system results when conventional (Galilean) definitions of the c.m. and internal dynamical variables are used. In Sec. IV the form of the two-particle Hamiltonian describing Coulomb and magnetic interactions is obtained from the Hamiltonian describing the interaction of a single particle with the radiation field. The result, with the correct ordering of the operators, is not new, but the derivation, of slightly novel form, is perhaps worth including here for completeness and for demonstrating where the various terms arise. Section V sees the combination of this Hamiltonian and the modified c.m. variables so the desired final form is obtained, provided internal orbital-momentum Thomas precession is introduced in the original Hamiltonian. The origin of this extra term as related to the relativistic invariance of the system is examined by introducing the operator for Lorentz boosts. Some technical matters and the direct proof of approximate relativistic invariance for an arbitrary potential interaction between particle charge densities is relegated to the two Appendices.

II. RELATIVISTIC CENTER-OF-MASS MOTION

At this point is is necessary to be more concrete about the basic dynamical variables out of which the Hamiltonian and other operators describing the quantized theory are to be constructed. As mentioned in the Introduction, we wish to develop a relativistically invariant theory with a fixed number of particles, and we take as the basic dynamical variables the individual particle "position," momentum, and spin operators, with the usual commutation relations. This far is just as conventional in nonrelativistic physics. The reason for the quotes around the word "position," however, is that the operator $\mathbf{r}$, which we use and which has the canonical commutation relations with the momentum operator $\mathbf{p}$, does not transform covariantly.[8] Thus the Lorentz transformation properties of $\mathbf{r}$ are not the same as the transformation of position in going between inertial reference frames. For a quantized theory this lack of immediate observable interpretation for $\mathbf{r}$ is unimportant. It is only necessary to regard $\mathbf{r}$ as an operator out of which observables can be constructed, in the same way as in quantum field theory the observables are constructed out of the unobservable (certainly for the spin-$\frac{1}{2}$ case) basic fields.

The requirement of relativistic invariance for a quantized theory is achieved by exhibiting the generators of time and space displacements (Hamiltonian and momentum), rotations (angular momentum), and boosts, which induce transformations between different inertial reference frames, in a form which obeys the Lie algebra of the Poincaré group.[9] For a single particle of mass m the generators constructed out of $\mathbf{r}$, $\mathbf{p}$, and the spin $\mathbf{s}$, taking $\hbar=c=1$, are

$$\begin{aligned} &H=(m^2+\mathbf{p}^2)^{1/2}\equiv E, \quad \mathbf{P}=\mathbf{p},\\ &\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{s},\\ &\mathbf{K}=t\mathbf{p}-\tfrac{1}{2}(\mathbf{r}E+E\mathbf{r})-\mathbf{p}\times\mathbf{s}/(E+m), \end{aligned} \tag{2.1}$$

where $\mathbf{s}^2=s(s+1)$, s being the spin of the particle.

For two free particles 1 and 2, the Hamiltonian and other generators are obviously formed by adding the respective individual particle expressions given by (2.1) (with subscripts 1 and 2 now appended to the dynamical variables $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{s}$ and also E and m). To separate the over-all c.m. motion from the internal dynamics, the crucial requirement is that the motion of the c.m. be uniform, effectively as for a single free particle. This can be realized by introducing a total momentum $\mathbf{P}$ and over-all "position" operator $\mathbf{R}$, and ensuring that the Hamiltonian, momentum, angular momentum, and boost generators for the two-free-particle system should have exactly the same form expressed in terms of $\mathbf{P}$ and $\mathbf{R}$ as for a single particle.

$$\begin{aligned} &H=E_1+E_2=(\hat{M}^2+\mathbf{P}^2)^{1/2}\equiv E, \quad \mathbf{P}=\mathbf{p}_1+\mathbf{p}_2,\\ &\mathbf{J}=\mathbf{r}_1\times\mathbf{p}_1+\mathbf{r}_2\times\mathbf{p}_2+\mathbf{s}_1+\mathbf{s}_2=\mathbf{R}\times\mathbf{P}+\hat{\mathbf{S}},\\ &\mathbf{K}=\mathbf{K}_1+\mathbf{K}_2=t\mathbf{P}-\tfrac{1}{2}(\mathbf{R}E+E\mathbf{R})\\ &\qquad -\mathbf{P}\times\hat{\mathbf{S}}/(E+\hat{M}). \end{aligned} \tag{2.2}$$

In (2.2) a mass operator $\hat{M}$ and spin operator $\hat{\mathbf{S}}$ have been introduced for the two-particle system. Along with $\mathbf{P}$ and $\mathbf{R}$, expressions for these dynamical variables in terms of the basic dynamical variables $\mathbf{r}_1$, $\mathbf{r}_2$,

[8] It is impossible for a many-particle theory with interaction to have a position operator that is both canonical with respect to the momentum operator and also transforms covariantly; see D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963); D. G. Currie, J. Math. Phys. **4**, 1470 (1963); J. T. Cannon and T. F. Jordan, *ibid.* **5**, 299 (1964); and H. Leutwyler, Nuovo Cimento **37**, 556 (1965). For a demonstration of the transformation properties of the "position" operator used here, see Appendix A of Ref. 2.

[9] Throughout this section, see Ref. 2 for further details.

$\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{s}_1$, and $\mathbf{s}_2$ can be found by solving (2.2). By virtue of their implicit definition (2.2), automatically the expected commutation relations are obeyed:

$$[\mathbf{R},\mathbf{P}]=i\mathbf{1}\,,\quad \hat{\mathbf{S}}\times\hat{\mathbf{S}}=i\hat{\mathbf{S}}\,,$$
$$[\mathbf{R},\hat{M}]=[\mathbf{P},\hat{M}]=[\hat{\mathbf{S}},\hat{M}]=[\mathbf{R},\hat{\mathbf{S}}]=[\mathbf{P},\hat{\mathbf{S}}]=0\,. \tag{2.3}$$

These definitions are such that uniform motion of the two-particle c.m. and conservation of the internal angular momentum are guaranteed:

$$d\mathbf{R}/dt=i[H,\mathbf{R}]=\mathbf{P}/E\,,\quad d^2\mathbf{R}/dt^2=0\,,$$
$$d\hat{\mathbf{S}}/dt=i[H,\hat{\mathbf{S}}]=0\,. \tag{2.4}$$

Since the over-all c.m. motion is described by the operators $\mathbf{R}$ and $\mathbf{P}$, the internal dynamics is governed by operators which commute with $\mathbf{R}$ and $\mathbf{P}$. To find the relative momentum $\hat{\mathbf{p}}$ and position operator $\hat{\mathbf{q}}$ and spin operators $\hat{\mathbf{s}}_1$, $\hat{\mathbf{s}}_2$ with the canonical commutation relations is a straightforward mathematical problem, albeit algebraically complicated (we are introducing the caret to mean operators which describe the internal dynamics, i.e., commute with $\mathbf{R}$ and $\mathbf{P}$). This has been solved for the general case when the particles have spin in Ref. 2. $\hat{\mathbf{p}}$ results from $\mathbf{p}_1$ or $-\mathbf{p}_2$ merely by Lorentz transformation to the instantaneous c.m. rest frame $\mathbf{P}=\mathbf{0}$, and, unlike the nonrelativistic case, the spin operators are also transformed by the Wigner rotation necessitated in going to this frame. In terms of these operators,[10]

$$\hat{M}=(m_1^2+\hat{\mathbf{p}}^2)^{1/2}+(m_2^2+\hat{\mathbf{p}}^2)^{1/2}\,,$$
$$\hat{\mathbf{S}}=\hat{\mathbf{q}}\times\hat{\mathbf{p}}+\hat{\mathbf{s}}_1+\hat{\mathbf{s}}_2\,. \tag{2.5}$$

There is not too much to be gained by displaying explicit expressions for $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}$, $\hat{\mathbf{s}}_1$, or $\hat{\mathbf{s}}_2$; they reduce to the well-known forms in the nonrelativistic limit (note that nonrelativistically, it is always implied, even if not stated, that $\hat{\mathbf{s}}_1=\mathbf{s}_1$, $\hat{\mathbf{s}}_2=\mathbf{s}_2$). To order v^2/c^2, or, what amounts to the same, $O(M^{-2})$,[11] the single-particle operators in terms of c.m. variables are[2]

$$\mathbf{p}_1=\frac{m_1}{M}\mathbf{P}+\hat{\mathbf{p}}+\left(\frac{m_2-m_1}{2m_1m_2M}\hat{\mathbf{p}}^2+\frac{1}{2M^2}\mathbf{P}\cdot\hat{\mathbf{p}}\right)\mathbf{P}\,,$$

$$\begin{aligned}\mathbf{r}_1=\mathbf{R}&+(m_2/M)\hat{\mathbf{q}}+\tfrac{1}{2}\hat{\mathbf{q}}\left(\frac{m_1-m_2}{2m_1m_2M}\hat{\mathbf{p}}^2-\frac{1}{2M^2}\mathbf{P}\cdot\hat{\mathbf{p}}\right)+\text{H.c.}\\ &+\tfrac{1}{2}\mathbf{q}\cdot\mathbf{P}\frac{1}{M^2}\left(\frac{1}{2}\hat{\mathbf{p}}-\frac{m_2}{m_1}\hat{\mathbf{p}}-\frac{m_2}{2M}\mathbf{P}\right)+\text{H.c.}\\ &-\frac{1}{2Mm_1}\hat{\mathbf{p}}\times\hat{\mathbf{s}}_1+\frac{1}{2Mm_2}\hat{\mathbf{p}}\times\hat{\mathbf{s}}_2\\ &\qquad+\frac{m_2}{2M^2m_1}\mathbf{P}\times\hat{\mathbf{s}}_1-\frac{1}{2M^2}\mathbf{P}\times\hat{\mathbf{s}}_2\,,\end{aligned} \tag{2.6}$$

$$\mathbf{s}_1=\hat{\mathbf{s}}_1+(1/2m_1M)(\hat{\mathbf{p}}\times\mathbf{P})\times\hat{\mathbf{s}}_1\,,$$
$$M=m_1+m_2\,,$$

and also $1\leftrightarrow 2$, when $\hat{\mathbf{p}}\to-\hat{\mathbf{p}}$, $\hat{\mathbf{q}}\to-\hat{\mathbf{q}}$. For a general system of N noninteracting particles the analogous relations are given in Ref. 3.

To introduce an interaction while maintaining relativistic invariance, it is merely necessary to modify the mass operator $\hat{M}$ keeping the conditions (2.3); i.e., $\hat{M}$ is constructed out of the internal operators $\hat{\mathbf{p}}$, $\hat{\mathbf{q}}$, $\hat{\mathbf{s}}_1$, and $\hat{\mathbf{s}}_2$ so as to commute with $\hat{\mathbf{S}}$. If $\hat{M}$ reduces to the free form (2.5) for large particle separations, then the physically necessary condition of asymptotically freely propagating particles is satisfied. Hence we add for an interaction extra terms to the free form of $\hat{M}$ which commute with $\hat{\mathbf{S}}$ and vanish for large $|\hat{\mathbf{q}}|$. This assumes a weak condition on the physical interpretation of $\hat{\mathbf{q}}$, or $\mathbf{r}_1-\mathbf{r}_2$, on which $\hat{\mathbf{q}}$ depends linearly, namely, that as particle separations increase indefinitely, then so does the expectation value of $|\hat{\mathbf{q}}|$ or $|\mathbf{r}_1-\mathbf{r}_2|$.

It is our aim in Secs. IV and V to show that the electromagnetic interaction between two particles, to order v^2/c^2, gives rise to a Hamiltonian H and a mass operator $\hat{M}$ which is consistent with $H=(\hat{M}^2+\mathbf{P}^2)^{1/2}$ and $\hat{M}$ satisfying the above conditions, to the extent appropriate for the order of approximation in v^2/c^2. The result provides an *a posteriori* justification for the above assumption on $|\hat{\mathbf{q}}|$.

III. INTERACTION HAMILTONIAN FOR COMPOSITE SYSTEM IN ELECTROMAGNETIC FIELD

When a composite system interacts with an external field and is not excited, the over-all motion of the system is identical to that of a single particle, whose mass, total momentum, spin, and static electromagnetic properties (electric and magnetic moments and form factors) are the same as those of the system in its ground state. If additivity is postulated, the sum of the Foldy-Wouthuysen Hamiltonians of the individual constituents $[H_{\text{FW}}(\mathbf{r}_i)]$ must separate into two parts. One part is dependent *only* upon the c.m. variables and has the same form as that of a single particle $H_{\text{FW}}(\mathbf{R})$ when H_{FW} is written as a function of the total mass, charge, spin, and momentum of the system; the other part (H_{FW}') consists of the remaining terms which are called "higher-moment" interactions, depends upon the internal variables, and leads to excitations of the system.

It is well known that for a Galilean system interacting with the electromagnetic field through its Coulomb, convection current, magnetic, and spin-orbit interactions to $O(M^{-1})$,[11] the Galilean definitions of the individual particle operators in terms of the c.m. variables yield the above-desired separation of the c.m. motion and give also the dipole and higher multipole moment interactions. However, if these Galilean

---

[10] For typographical reasons we are forced to use $\hat{\mathbf{q}}$ for the internal position operator.

[11] Reckoning interchangeably in inverse powers of m_1, m_2 or $M=m_1+m_2$.

definitions are used in the Thomas-precession terms of $H_{\rm FW}(\mathbf{r}_i)$, the above-desired separation is not obtained and for this reason previous calculations[12] gave incorrect results for the Thomas precession.[13] The definitions of the single-particle operators in terms of c.m. variables [Eq. (2.6)] reduce to the familiar Galilean results to lowest order in the masses. To $O(M^{-2})$ there are seen to be terms present in addition to the Galilean results. Therefore, the interaction of a composite system in an electromagnetic field, when written to $O(M^{-2})$ as the additive sum of individual Foldy-Wouthuysen interactions to that order, needs the $O(M^{-2})$ definitions in Eq. (2.6) in order to yield a consistent separation of the c.m. motion of the system from the internal dynamics. We shall first give a detailed discussion of Thomas precession because an understanding of this effect is crucial both here and later in this paper.

In transforming between relatively moving inertial frames, the spin of a particle undergoes Wigner rotation.[2] The origin of this rotation may be found in the commutation relations obeyed by the Lorentz group generators, in particular, that the commutator of two different boost generators is a rotation generator:

$$[K_i,K_j]=-i\epsilon_{ijk}J_k/c^2. \tag{3.1}$$

The velocity of light c has been explicitly included so that the contraction to the Galilean group is seen by sending $c\to\infty$. The Wigner rotation of the spin is a relativistic effect because for the Galilean group boost generators are seen to commute.

For any operator $\hat{O}$ in a reference frame A,

$$\hat{O}'=\exp(i\theta\hat{v}\cdot K)\hat{O}\exp(-i\theta\hat{v}\cdot K)\,,\quad \tanh\theta=|\mathbf{v}| \tag{3.2}$$

represents the same dynamical quantity referred to a reference frame B moving with velocity $-\hat{\mathbf{v}}$ with respect to A. Therefore, if in frame A there is a particle with spin angular momentum $\mathbf{s}$ and linear momentum then $\mathbf{p}$, then an observer in frame B will see the particle with momentum $\mathbf{p}'$ [given by Eq. (3.2)] and for the spin he will find that

$$\frac{d\mathbf{s}'}{d\theta}=\frac{1}{E'+m}(\mathbf{p}'\times\hat{v})\times\mathbf{s}'\,,$$

which describes a rotation of the spin about an axis $(\mathbf{p}\times\hat{v})/|\mathbf{p}\times\hat{v}|$. For an accelerating particle the spin is not a constant of the motion, but rotates, or rather precesses, at a rate proportional to the acceleration $\mathbf{a}$. To see this, write $\delta\theta\hat{v}=\delta\hat{v}=\mathbf{a}\delta t$ in Eq. (3.2), neglecting v^2/c^2 corrections, so that we have introduced an acceleration. Then simply $\theta\to t$ and $\hat{v}\to\mathbf{a}$. For a single particle with charge e, at position $\mathbf{r}$, and interacting with a field $\mathbf{E}(\mathbf{r})$, we have $\mathbf{a}=e\mathbf{E}(\mathbf{r})/m$; and so for this case

$$d\mathbf{s}/dt=(e/2m^2)(\mathbf{p}\times\mathbf{E})\times\mathbf{s}.$$

This kinematical time dependence, in the framework usually employed, is ascribed to an additional term in the interaction Hamiltonian for the particle and the electromagnetic field. This is the origin of the Thomas-precession term in the interaction.[14]

The argument above did not depend upon whether the particle was simple or a composite system of "subparticles." In the c.m. frame, the system has a well-defined angular momentum (spin) $\hat{\mathbf{S}}$ and mass M. Therefore, when a composite system interacts with an external electromagnetic field, there will be in the Hamiltonian a term of the form

$$(-e/2M^2)(\mathbf{P}\times\mathbf{E})\cdot\hat{\mathbf{S}}\,, \tag{3.3}$$

in order to yield the correct time dependence of the system's total spin $\hat{\mathbf{S}}$.

As remarked to Eq. (3.1), the Wigner rotation is a relativistic effect and hence the same is true of the Thomas precession. For this reason the Galilean variables were unable to separate the c.m. Thomas precession satisfactorily[12] and, therefore, in order to obtain the desired separation, the relativistic $O(v^2/c^2)$ results (2.6) are required. For an interaction

$$V=\sum_{i=1}^{N}H_{\rm FW}(\mathbf{r}_i)\,,$$

if one wishes to use the conventional Galilean, nonrelativistic, c.m. definitions for $\mathbf{r}_i$, one must include in V a set of terms $H_{\Delta\rm FW}$ which take account of the $O(M^{-2})$ corrections of (2.6):

$$V=\sum_{i=1}^{N}H_{\rm FW}(\mathbf{r}_i^{\rm NR})+H_{\Delta\rm FW}\,,$$
$$\mathbf{r}_i^{\rm NR}=\mathbf{R}+\sum_{j=1}^{N}\frac{m_j}{M}\hat{\mathbf{q}}_{ij}\,,\quad \hat{\mathbf{q}}_{ij}=\mathbf{r}_i^{\rm NR}-\mathbf{r}_j^{\rm NR}. \tag{3.4}$$

Expressions for $H_{\Delta\rm FW}$ have been derived for both two-body and N-body systems.[3,7] Explicitly, as in Refs. 3 and 7,

$$H_{\Delta\rm FW}=-\tfrac{1}{2}\sum_{i=1}^{N}(\boldsymbol{\varrho}_i\cdot\mathbf{E}_i+\mathbf{E}_i\cdot\boldsymbol{\varrho}_i)\,, \tag{3.5}$$

where $\mathbf{E}_i$ is the electric field at $\mathbf{r}_i^{\rm NR}$. The forms for $\boldsymbol{\varrho}_i$ in Refs. 3 and 7 are shown to be identical in Appendix

[12] In electron-nucleus scattering, see, e.g., K. McVoy and L. Van-Hove, Phys. Rev. **125**, 1304 (1962); in muon capture, see J. Friar, Nucl. Phys. **87**, 407 (1966). Both of these papers work to $O(m^{-2})$ and use as interaction Hamiltonian for the system the sum of the constituent Foldy-Woutheysen interactions. The failure of such an approach to satisfy fundamental low-energy theorems for Compton scattering is shown in G. Barton, University of Sussex report (unpublished).

[13] L. H. Thomas, Nature **117**, 514 (1926); also J. Frenkel, Z. Physik **37**, 243 (1926).

[14] For a modern derivation, see J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962), Chap. 11, p. 364; or D. Shelupsky, Am. J. Phys. **35**, 650 (1967).

A, and the individual terms in $H_{\Delta FW}$ can be written

$$\varrho_i\cdot\mathbf{E}_i=-\frac{\mathbf{j}_i\times\mathbf{P}\cdot\mathbf{E}_i}{2m_iM}+\sum_i\frac{\mathbf{j}_j\times\hat{\mathbf{p}}_j\cdot\mathbf{E}_i}{2m_jM}+\frac{1}{2M^2}\hat{\mathbf{S}}\times\mathbf{P}\cdot\mathbf{E}_i$$
$$-\sum_j\frac{m_j}{2m_iM^2}[(\hat{\mathbf{q}}_{ij}\ \mathbf{P})\hat{\mathbf{p}}_i+\hat{\mathbf{q}}_{ij}(\hat{\mathbf{p}}_i\cdot\mathbf{P})]\cdot\mathbf{E}_i$$
$$-\sum_j\frac{m_j}{2M^3}\hat{\mathbf{q}}_{ij}\cdot\mathbf{P}\mathbf{P}\cdot\mathbf{E}_i$$
$$-\sum_j\sum_k\frac{m_k}{2m_jM^2}(\hat{\mathbf{q}}_{jk}\cdot\hat{\mathbf{p}}_j)\hat{\mathbf{p}}_j\cdot\mathbf{E}_i,\quad(3.6)$$

where

$$\mathbf{j}_i=\hat{\mathbf{s}}_i+(\mathbf{r}_i^{\mathrm{NR}}-\mathbf{R})\times\hat{\mathbf{p}}_i,\quad \hat{\mathbf{S}}=\sum_i\mathbf{j}_i.$$

The first three terms can also be written (compare with Ref. 6)

$$\sum_j\left(\frac{\mathbf{j}_i}{2m_iM}-\frac{\mathbf{j}_j}{2m_jM}\right)\times\mathbf{p}_j^{\mathrm{NR}}\cdot\mathbf{E}_i,$$

and if a Galilean separation is made in the $H_{\mathrm{FW}}(\mathbf{r}_i)$, we will obtain the desired form of the c.m. interaction, $H_{\mathrm{FW}}(\mathbf{R})$, when the $H_{\Delta\mathrm{FW}}$ is included. In particular, we note that the correct c.m. Thomas term is present at the third term in (3.6).

The methods used for deriving $H_{\Delta\mathrm{FW}}$ in Refs. 3 and 7 are quite different. The form of $H_{\Delta\mathrm{FW}}$ at (3.5) is perhaps more readily appreciated by the approach of Ref 3, which was to write $\mathbf{r}_i=\mathbf{r}_i^{\mathrm{NR}}+\varrho_i$ in $H_{\mathrm{FW}}(\mathbf{r}_i)$ [where NR denotes nonrelativistic, and $\mathbf{r}_i^{\mathrm{NR}}$ are those terms of leading order in the masses; see Eq. (3.4)] which are then expressed in terms of $\mathbf{r}_i^{\mathrm{NR}}$ with $-\nabla_i\Phi(\mathbf{r}_i)$ being replaced by $\mathbf{E}(\mathbf{r}_i)$ and where $\Phi(\mathbf{r}_i)$ and $\mathbf{E}(\mathbf{r}_i)$ are the Coulomb potential and electric field at position $\mathbf{r}_i$. Then we see that $H_{\Delta\mathrm{FW}}$ is formally an electric dipole interaction whose origin is the noncoincidence of $\mathbf{r}_i$ and $\mathbf{r}_i^{\mathrm{NR}}$. Thus the $H_{\Delta\mathrm{FW}}$ is the relativistic correction to the electric dipole interaction caused by the Lorentz transformation [to $O(v^2/c^2)$] of of the position operators, and its presence is required due to the description of the internal structure of the system by wave functions which are invariant under Lorentz boosts.[3,6] It is interesting to compare this approach with that of Brodsky and Primack.[6] They obtain a Hamiltonian whose spin structure is identical with the two-body version of our (3.5) with (3.6). The wave function $\varphi(\mathbf{x}_1',\mathbf{x}_2')$ to be used for evaluating matrix elements of their Hamiltonian must include the Lorentz contraction $\mathbf{x}'=\Lambda\mathbf{x}$. They point out that this is important for evaluating the low-energy theorems for bound states with orbital angular momentum $l\geq 1$ (see Ref. 3 and the first article of Ref. 6.)

The fact that a position operator does not transform covariantly[8] yields the spin-dependent terms in $H_{\Delta\mathrm{FW}}$. It may at first sight seem surprising that such spin dependence will occur and, moreover, give birth to the desired c.m. Thomas precession. The reason for this happy event is that the spin operator undergoes a momentum-dependent transformation (Wigner rotation and Thomas precession) when transforming between relatively moving frames. Then because $\mathbf{R}$ and the internal position variables are required to commute with the transformed spin, there will arise spin dependence in the position operators, with origin in the spin rotation, and so the correct Thomas precession results in a consistent fashion.

IV. DERIVATION OF MAGNETIC INTERACTION HAMILTONIAN

The Hamiltonian for the electromagnetic interaction of two charged spin-$\frac{1}{2}$ particles with charges e_1, e_2 and magnetic moments μ_1, μ_2 is[15]

$$H=H_{\mathrm{rad}}+H_{\mathrm{Coul}}+H_{1\,\mathrm{KE}}+H_{2\,\mathrm{KE}}+H_{1\,\mathrm{int}}+H_{2\,\mathrm{int}},\quad(4.1\mathrm{a})$$

$$H_{\mathrm{rad}}=\frac{1}{2}\int d\mathbf{x}[\mathbf{E}_T(\mathbf{x})^2+\mathbf{B}(\mathbf{x})^2],\quad(4.1\mathrm{b})$$

$$H_{\mathrm{Coul}}=\frac{1}{2}\int d\mathbf{x}\ \nabla\Phi(\mathbf{x})\cdot\nabla\Phi(\mathbf{x})\rightarrow\frac{e_1e_2}{4\pi|\mathbf{r}_1-\mathbf{r}_2|},\quad(4.1\mathrm{c})$$

$$H_{1\,\mathrm{KE}}=m_1+\mathbf{p}_1^2/2m_1-\mathbf{p}_1^4/8m_1^3,\quad(4.1\mathrm{d})$$

$$H_{1\,\mathrm{int}}=-\frac{e_1}{2m_1}[\mathbf{A}(\mathbf{r}_1)\cdot\mathbf{p}_1+\mathbf{p}_1\cdot\mathbf{A}(\mathbf{r}_1)]-\mu_1\boldsymbol{\sigma}_1\cdot\mathbf{B}(\mathbf{r}_1)-\frac{1}{4m_1}\left(2\mu_1-\frac{e_1}{2m_1}\right)\times[\nabla\cdot\mathbf{E}(\mathbf{r}_1)+\boldsymbol{\sigma}_1\cdot\mathbf{E}(\mathbf{r}_1)\times\mathbf{p}_1-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\times\mathbf{E}(\mathbf{r}_1)],\quad(4.1\mathrm{e})$$

where

$$\begin{aligned}&\mathbf{B}(\mathbf{x})=\nabla\times\mathbf{A}(\mathbf{x}),\quad \mathbf{E}(\mathbf{x})=\mathbf{E}_T(x)-\nabla\Phi(\mathbf{x}),\\&-\nabla^2\Phi(\mathbf{x})=e_1\delta(\mathbf{x}-\mathbf{r}_1)+e_2\delta(\mathbf{x}-\mathbf{r}_2)\\&\qquad=\rho(\mathbf{x}),\\&\nabla\cdot\mathbf{A}(\mathbf{x})=\nabla\cdot\mathbf{E}_T(\mathbf{x})=0,\quad \boldsymbol{\sigma}_{1,2}=2\mathbf{s}_{1,2}.\end{aligned}\quad(4.2)$$

The following points should be noted.

(i) The Hamiltonian, particularly $H_{1\,\mathrm{int}}+H_{2\,\mathrm{int}}$, has been written in a manifestly Hermitian form.

(ii) We are using the Coulomb or radiation gauge, which is certainly the best gauge to use for electro-

[15] We are using Heaviside-Lorentz units, if e is the electron charge $e^2/4\pi\approx 1/137$. We assume in terms of order of magnitude $e_1\sim e_2\sim e$, $\mu_1\sim e_1/m_1$, $\mu_2\sim e_2/m_2$, and Ref. 11 applies. When it is convenient we use dyadic notation, and if an equation applies both to particles 1 and 2, it is written for particle 1 only. The final Hamiltonian is an approximation for the electromagnetic interaction to order v^2/c^2, and throughout equations are written consistent with this approximation. The Hamiltonian (4.1) is not really restricted to particles of spin $\frac{1}{2}$ but should be valid for particles of arbitrary spin s_1, s_2 provided the replacements $\boldsymbol{\sigma}_1=\mathbf{s}_1/s_1$ and $(2\mu_1-e_1/2m_1)\rightarrow(2\mu_1-s_1e_1/m_1)$ are made.

magnetic bound-state problems, since the instantaneous Coulomb potential gives the dominant contribution to the binding energy. Further, the radiation field is quantized without any indefinite metric:

$$[\mathbf{A}(\mathbf{x}),\mathbf{E}_T(\mathbf{y})]_{\text{equal times}}=-i(1-\nabla\nabla/\nabla^2)\delta(\mathbf{x}-\mathbf{y}).$$

(iii) In (4.1c) the Coulomb self-interaction is neglected. This infinite self-interaction term is strictly necessary in the formal proof of relativistic invariance but, since we are dealing with a relativistically invariant theory and self-interaction is irrelevant to our discussion in this paper, we shall, without further comment, throughout neglect self-interaction and assume that observed values are to be inserted for parameters in equations.

(iv) The relativistic v^2/c^2 correction to the kinetic energy is retained in (4.1d). For an electromagnetically bound state,

$$\langle|\mathbf{r}_1-\mathbf{r}_2|\rangle\sim 1/m_R\alpha,\quad \alpha=|e_1e_2|/4\pi,\quad m_R=m_1m_2/M,$$
$$\langle\mathbf{p}_1^2/m_R\rangle=\langle\mathbf{p}_2^2/m_R\rangle\sim m_R\alpha^2\quad\text{when}\quad \mathbf{P}=0,$$

so that this kinetic-energy correction is of the same order, $m_R\alpha^4$, as the electromagnetic interaction effects due to (4.1e). (This becomes more apparent later when the radiation field is eliminated.)

(v) The interaction (4.1e) is exactly that obtained by the Foldy-Wouthuysen transformation of the Dirac equation with an anomalous magnetic moment.[16] The Foldy-Wouthuysen transformation applied to the Dirac-equation representation of the generators of the inhomogeneous Lorentz group gives exactly (2.1), when factors β are disregarded. Hence the discussion of the c.m. motion which is given in Sec. II is appropriate for the Hamiltonian (4.1).

(vi) It is not directly possible to write $\mathbf{E}_T(x)=-\dot{\mathbf{A}}(x)$, except in the interaction picture, since the interaction itself depends on $\mathbf{E}_T(x)$. For the problems of interest here this is quite unimportant.

The interpretation of the various terms in the electromagnetic interaction (4.1a) is well known.[17] The first two terms are here referred to as the electric and magnetic interaction; the last, which also contains the $\nabla\cdot\mathbf{E}$ term resulting from smearing of the charge due to zitterbewegung, is called the spin-orbit interaction. That part of the spin-orbit interaction proportional to the magnetic moment appears since a particle moving with a velocity $\mathbf{v}=\mathbf{p}/m$ in an electric field $\mathbf{E}$ sees a magnetic field $\mathbf{v}\times\mathbf{E}$; the remaining part arises from Thomas precession[13] which was discussed in Sec. III.

After these digressions it is now possible to derive a field equation from the Hamiltonian (4.1) where, using

[16] H. Neuer and P. Urban, Acta Phys. Austriaca **15**, 380 (1962).

[17] See standard textbooks such as J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), Chap. 4.

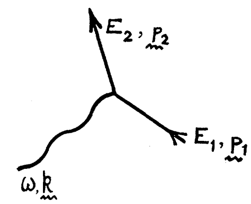

FIG. 1. Particle absorption of a photon.

current conservation,

$$\begin{aligned}\ddot{\mathbf{A}}(\mathbf{x})-\nabla^2\mathbf{A}(\mathbf{x})&=\mathbf{j}(\mathbf{x})-\nabla\dot{\Phi}(\mathbf{x})\\&=(1-\nabla\nabla/\nabla^2)\cdot\mathbf{j}(\mathbf{x}),\\ \mathbf{j}(\mathbf{x})=(e_1/2m_1)\{\mathbf{p}_1,\,\delta(\mathbf{r}_1-\mathbf{x})\}&-\mu_1\boldsymbol{\sigma}_1\times\nabla_x\delta(\mathbf{r}_1-\mathbf{x})\\&+1\leftrightarrow 2+O(v^2/c^2).\end{aligned}\tag{4.3}$$

From (4.3) the current $\mathbf{j}(x)$ is of order v/c, so when the transverse photon field $\mathbf{A}(x)$ couples to both particles 1 and 2 it gives rise to an interaction of order v^2/c^2. Suppose now, as in Fig. 1, a photon is absorbed by a particle, changing the energy and momentum from E_1, $\mathbf{p}_1$, to E_2, $\mathbf{p}_2$. The photon energy and momentum are obviously given by

$$\mathbf{k}=\mathbf{p}_2-\mathbf{p}_1,\quad \omega=E_2-E_1\approx\mathbf{p}_2^2/2m-\mathbf{p}_1^2/2m,\tag{4.4}$$

where the last part of (4.4) holds, of course, only if the particle is slowly moving. From (4.4) it is at once apparent that, in this case, ω is smaller than $|\mathbf{k}|$ by a factor v/c. So, in (4.3), when the transverse photon field $A(x)$ couples to a slowly moving particle, $\ddot{\mathbf{A}}(x)$ is smaller than $\nabla^2\mathbf{A}(x)$ in the ratio $\omega^2/k^2\approx v^2/c^2$. Hence it is valid to neglect $\ddot{\mathbf{A}}(x)$ in (4.3), if only effects due to the transverse photon field of lowest order in v/c are required. To the extent that effectively $\omega=0$, $|\mathbf{k}|\neq 0$, the transverse photon field propagates instantaneously and so retardation is being neglected. Thus, if a Hamiltonian correct only to first order-effects in v^2/c^2 is required, to be consistent (4.1b) is modified:

$$H_{\text{rad}}=\frac{1}{2}\int d\mathbf{x}\,\mathbf{B}(\mathbf{x})^2=\frac{1}{2}\int d\mathbf{x}\,\mathbf{A}(\mathbf{x})\cdot\mathbf{j}(\mathbf{x}),\tag{4.5}$$

where the modified field equation

$$-\nabla^2\mathbf{A}(\mathbf{x})=(1-\nabla\nabla/\nabla^2)\cdot\mathbf{j}(\mathbf{x})\tag{4.6}$$

is used. Combining (4.5) with (4.1e), we can now write

$$H_{\text{rad}}+H_{1\,\text{int}}+H_{2\,\text{int}}=H_{1\,\text{int}}'+H_{2\,\text{int}}',$$

$$\begin{aligned}H_{1\,\text{int}}'=&-\frac{1}{2}\left\{\frac{e_1}{2m_1}[\mathbf{A}(\mathbf{r}_1)\cdot\mathbf{p}_1+\mathbf{p}_1\cdot\mathbf{A}(\mathbf{r}_1)]+\mu\boldsymbol{\sigma}_1\cdot\mathbf{B}(\mathbf{r}_1)\right\}\\&+\frac{1}{4m_1}\left(2\mu_1-\frac{e_1}{2m_1}\right)[\nabla^2\Phi(\mu_1)\\&\qquad+\boldsymbol{\sigma}_1\cdot\nabla\Phi(\mathbf{r}_1)\times\mathbf{p}_1-\boldsymbol{\sigma}_1\cdot\mathbf{p}_1\times\nabla\Phi(\mathbf{r}_1)],\end{aligned}\tag{4.7}$$

where Φ is given by (4.2) and $\mathbf{A}$ by (4.6). Equation (4.6) is no longer a dynamical equation but an equation of constraint, and it may be used to eliminate the photon field $\mathbf{A}$ from the Hamiltonian in the same manner as the scalar potential Φ may be exactly eliminated in the Coulomb gauge.

As usual, the scalar potential is

$$\Phi(\mathbf{x})=e_1/4\pi|\mathbf{x}-\mathbf{r}_1|+e_2/4\pi|\mathbf{x}-\mathbf{r}_2|\,. \tag{4.8}$$

Using

$$\frac{\nabla_x\nabla_x}{\nabla^2}\frac{1}{|\mathbf{x}-\mathbf{r}_1|}=\frac{1}{2|\mathbf{x}-\mathbf{r}_1|}\left[1-\frac{(\mathbf{x}-\mathbf{r}_1)(\mathbf{x}-\mathbf{r}_1)}{|\mathbf{x}-\mathbf{r}_1|^2}\right],$$

$$(\nabla_x\nabla_x-\tfrac{1}{3}1\nabla^2)\frac{1}{|\mathbf{x}-\mathbf{r}_1|}=\frac{1}{|\mathbf{x}-\mathbf{r}_1|^3}\left[\frac{3(\mathbf{x}-\mathbf{r}_1)(\mathbf{x}-\mathbf{r}_1)}{|\mathbf{x}-\mathbf{r}_1|^2}-1\right], \tag{4.9}$$

(4.6) can be solved:

$$\mathbf{A}(\mathbf{x})=\frac{e_1}{16\pi m_1}\left[\left\{\mathbf{p}_1,\frac{1}{|\mathbf{x}-\mathbf{r}_1|}\right\}+\mathbf{p}_1\cdot(\mathbf{x}-\mathbf{r}_1)\frac{(\mathbf{x}-\mathbf{r}_1)}{|\mathbf{x}-\mathbf{r}_1|^3}+\frac{(\mathbf{x}-\mathbf{r}_1)}{|\mathbf{x}-\mathbf{r}_1|^3}(\mathbf{x}-\mathbf{r}_1)\cdot\mathbf{p}_1\right]-\mu_1\frac{(\mathbf{x}-\mathbf{r}_1)\times\boldsymbol{\sigma}_1}{4\pi|\mathbf{x}-\mathbf{r}_1|^3},$$
$$\mathbf{B}(\mathbf{x})=\frac{e_1}{4\pi m_1}\frac{\mathbf{p}_1\times(\mathbf{x}-\mathbf{r}_1)}{|\mathbf{x}-\mathbf{r}_1|^3}+\tfrac{2}{3}\mu_1\boldsymbol{\sigma}_1\delta(\mathbf{x}-\mathbf{r}_1)-\frac{\mu_1\boldsymbol{\sigma}_1}{4\pi|\mathbf{x}-\mathbf{r}_1|^3}+3\mu_1\frac{(\mathbf{x}-\mathbf{r}_1)\boldsymbol{\sigma}_1\cdot(\mathbf{x}-\mathbf{r}_1)}{4\pi|\mathbf{x}-\mathbf{r}_1|^5}. \tag{4.10}$$

Inserting (4.8) and (4.10) into (4.7), forgetting self-interaction as discussed earlier, and indulging in some simplifying albegra, in particular, using at one point

$$(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_2)=0\,,$$

we obtain for the Hamiltonian

$$H=H_{1\,\mathrm{KE}}+H_{2\,\mathrm{KE}}+H_{\mathrm{Coul}}+H_{\mathrm{int}}\,,$$
$$H_{\mathrm{int}}=H'_{1\,\mathrm{int}}+H'_{2\,\mathrm{int}}$$
$$=-\frac{e_1e_2}{16\pi m_1m_2}\left[\mathbf{p}_1\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\cdot\mathbf{p}_2+\mathbf{p}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2)\cdot\mathbf{p}_2+\mathrm{H.c.}\right]$$
$$-\frac{e_1\mu_2}{4\pi m_1}\frac{\boldsymbol{\sigma}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_1}{|\mathbf{r}_1-\mathbf{r}_2|^3}+\frac{e_2\mu_1}{4\pi m_2}\frac{\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}$$
$$-\tfrac{2}{3}\mu_1\mu_2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\delta(\mathbf{r}_1-\mathbf{r}_2)+\frac{\mu_1\mu_2}{4\pi}\left[\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}-3\frac{\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\boldsymbol{\sigma}_2\cdot(\mathbf{r}_1-\mathbf{r}_2)}{|\mathbf{r}_1-\mathbf{r}_2|^5}\right]$$
$$+\left[\frac{e_1e_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)-\frac{e_2\mu_1}{2m_1}-\frac{e_1\mu_2}{2m_2}\right]\delta(\mathbf{r}_1-\mathbf{r}_2)$$
$$-\frac{e_2}{8\pi m_1}\left(2\mu_1-\frac{e_1}{2m_1}\right)\frac{\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_1}{|\mathbf{r}_1-\mathbf{r}_2|^3}+\frac{e_1}{8\pi m_2}\left(2\mu_2-\frac{e_2}{2m_2}\right)\frac{\boldsymbol{\sigma}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}. \tag{4.11}$$

This is the conventional Hamiltonian which is supposed to describe the electromagnetic interaction of two spin-$\frac{1}{2}$ particles to order v^2/c^2. For an electromagnetically bound state each term in H_{int} gives corrections to the binding energy of order $m_R\alpha^4$. The interpretation of the various terms in H_{int} is, according to the various lines in (4.11), (1) straightforward Darwin or Breit interaction[1,18] resulting from unretarded one-photon exchange; (2) interaction between magnetic moment of one particle and the field produced by the motion of the other particle; (3) conventional interaction between two stationary magnetic dipoles plus a point term representing the effect of interpenetration of the two dipoles; (4) effect of zitterbewegung, this results from the Darwin term in the Hamiltonian (4.1e); (5) conventional spin-orbit interaction including Thomas precession.

If one of the particles becomes very massive, as in hydrogenic atoms, and its magnetic moment neglected, the only terms which survive are part of the zitterbewegung effect and one of the spin-orbit interaction terms.

V. SEPARATION OF CENTER-OF-MASS MOTION

The expression (4.11) is certainly not in any form which displays manifest relativistic invariance to

[18] C. G. Darwin, Phil. Mag. 39, 537 (1920); G. Breit, Phys. Rev. 34, 553 (1929); 51, 248 (1937); 53, 153 (1938); and B. Leaf, Physica 28, 206 (1962).

order v^2/c^2 or which shows the separation of the over-all c.m. motion from the internal dynamics. Conventionally, the c.m. frame $\mathbf{P}=0$ is used to derive a Hamiltonian to calculate, for example, the various corrections arising from H_{int} in (4.11) to the binding energy of an atomic state. While this is certainly correct (in our formalism the Hamiltonian in the c.m. frame reduces to the mass operator), the above problems have been swept under the carpet.

As promised, we apply the relativistic c.m. variables to order v^2/c^2. Since H_{int} and the relativistic kinetic-energy corrections in $H_{1\,\text{KE}}+H_{2\,\text{KE}}$ are already of this order, we can here apply the conventional nonrelativistic formulas

$$\hat{\mathbf{q}}=\mathbf{r}_1-\mathbf{r}_2,\quad \mathbf{p}_1=\frac{m_1}{M}\mathbf{P}+\hat{\mathbf{p}},\quad \mathbf{p}_2=\frac{m_2}{M}\mathbf{P}-\hat{\mathbf{p}},$$
$$\boldsymbol{\sigma}_1=\boldsymbol{\sigma}_1,\quad \boldsymbol{\sigma}_2=\boldsymbol{\sigma}_2. \tag{5.1}$$

In H_{Coul} and the nonrelativistic kinetic energy $\mathbf{p}_1^2/2m_1+\mathbf{p}_2^2/2m_2$, the relativistic modifications (2.6) must be used. Treating the kinetic energy first,

$$H_{1\,\text{KE}}+H_{2\,\text{KE}}=M+\frac{\mathbf{P}^2}{2M}-\frac{\mathbf{P}^4}{8M^3}+\hat{\mathbf{p}}^2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)\left(1-\frac{\mathbf{P}^2}{2M^2}\right)-\hat{\mathbf{p}}^4\left(\frac{1}{8m_1^3}+\frac{1}{8m_2^3}\right), \tag{5.2}$$

which is already consistent with the form

$$(\hat{M}_{\text{KE}}^2+\mathbf{P}^2)^{1/2},$$
$$\hat{M}_{\text{KE}}=(m_1^2+\hat{\mathbf{p}}^2)^{1/2}+(m_2^2+\hat{\mathbf{p}}^2)^{1/2}, \tag{5.3}$$

to order v^2/c^2. Of course, this is not surprising since our c.m. variables were constructed to transform the two-free-particle Hamiltonian to the canonical expression (5.3) and the result (5.2) serves only to check our calculations. To demonstrate that the electromagnetic interaction can also be consistently expressed is less trivial. From (2.6), to order v^2/c^2,

$$\mathbf{r}_1-\mathbf{r}_2=\hat{\mathbf{q}}-\frac{\hat{\mathbf{q}}\cdot\mathbf{P}}{2M^2}\mathbf{P}-\frac{1}{2M}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)(\hat{\mathbf{q}}\cdot\mathbf{P}\hat{\mathbf{p}}+\text{H.c.})+\frac{1}{4Mm_1}\mathbf{P}\times\boldsymbol{\sigma}_1-\frac{1}{4Mm_2}\mathbf{P}\times\boldsymbol{\sigma}_2,$$
$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}=\frac{1}{|\hat{\mathbf{q}}|}+\frac{1}{2M^2}\frac{(\hat{\mathbf{q}}\cdot\mathbf{P})^2}{|\hat{\mathbf{q}}|^3}+\frac{1}{2M}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\left(\frac{\hat{\mathbf{q}}\cdot\mathbf{P}}{|\hat{\mathbf{q}}|^3}\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}+\text{H.c.}\right)-\frac{1}{4M}\left(\frac{\boldsymbol{\sigma}_1}{m_1}-\frac{\boldsymbol{\sigma}_2}{m_2}\right)\cdot\frac{\hat{\mathbf{q}}\times\mathbf{P}}{|\hat{\mathbf{q}}|^3}. \tag{5.4}$$

It is now apparent that the Coulomb interaction, by itself, contains parts dependent on the total momentum $\mathbf{P}$. Since the mass operator, which we wish to contain the real dynamics, must be independent of $\mathbf{P}$, we require that these $\mathbf{P}$-dependent terms in the Coulomb interaction should completely cancel with the $\mathbf{P}$-dependent parts of H_{int}, except for terms corresponding to expressions arising from the consistent expansion of $(\hat{M}^2+\mathbf{P}^2)^{1/2}$. Let us first combine the Coulomb and Breit interactions:

$$\frac{e_1e_2}{4\pi}\left[\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}-\frac{1}{4m_1m_2}\left(\mathbf{p}_1\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|}\cdot\mathbf{p}_2+\mathbf{p}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|^3}(\mathbf{r}_1-\mathbf{r}_2)\cdot\mathbf{p}_2+\text{H.c.}\right)\right]$$
$$=\frac{e_1e_2}{4\pi}\left[\frac{1}{|\hat{\mathbf{q}}|}\left(1-\frac{\mathbf{P}^2}{2M^2}\right)+\frac{1}{2m_1m_2}\times\left(\hat{\mathbf{p}}\frac{1}{|\hat{\mathbf{q}}|}\cdot\hat{\mathbf{p}}+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\cdot\frac{1}{|\hat{\mathbf{q}}|^3}\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}\right)\right]-\frac{e_1e_2}{16\pi M}\times\left\{\frac{\hat{\mathbf{q}}\times\mathbf{P}}{|\hat{\mathbf{q}}|^3}\cdot\left[\frac{1}{m_1}(\tfrac{1}{2}\boldsymbol{\sigma}_1+\hat{\mathbf{q}}\times\hat{\mathbf{p}})-\frac{1}{m_2}(\tfrac{1}{2}\boldsymbol{\sigma}_2+\hat{\mathbf{q}}\times\hat{\mathbf{p}})\right]+\text{H.c.}\right\}. \tag{5.5}$$

The first line on the right-hand side of (5.5) is of the desired form. Looking at (4.11), then it is easy to see that the magnetic dipole interaction and zitterbewegung terms depend only on internal variables, to zero order in v/c as required here. Further, the interaction of the "magnetic dipole with the field due to the other charge" combines with the magnetic moment part of the spin-orbit interaction to eliminate dependence on $\mathbf{P}$. There remains in (4.11) the Thomas-precession term, whose dependence on $\mathbf{P}$ is canceled by the spin-dependent terms on the right-hand side of (5.5). We are now left with the parts on the right-hand side of (5.5) proportional to the internal orbital angular momentum $\hat{\mathbf{l}}=\hat{\mathbf{q}}\times\hat{\mathbf{p}}$. If the arguments so far were complete, these terms would flout our desired final form for the Hamiltonian expressed in c.m. variables. However, the fact that the spin-dependent terms were canceled by part of the Thomas-precession interaction in the original Hamiltonian provides the necessary clue. From our discussion in Sec. III it is evident that the internal orbital angular momentum $\hat{\mathbf{l}}$ will suffer Thomas precession when the over-all state of motion is changed, just as does the spin of a single particle undergoing acceleration. As in (3.3), the effective interaction is

$$V_T^{(2)}=-\tfrac{1}{2}\mathbf{l}\cdot\mathbf{v}\times\mathbf{a}. \tag{5.6}$$

In this case the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ are given by

$$\mathbf{v}=\mathbf{P}/M\,,\quad \mathbf{a}=e_1\mathbf{E}(\mathbf{r}_1)/m_1+e_2\mathbf{E}(\mathbf{r}_2)/m_2\,. \tag{5.7}$$

Hence for two particles interacting with the electromagnetic field we should add an additional term to (4.1) in order to be correct to order v^2/c^2:

$$V_T{}^{(2)}=-\frac{1}{4M}\hat{\mathbf{l}}\cdot\mathbf{P}\times\left[\frac{e_1\mathbf{E}(\mathbf{r}_1)}{m_1}+\frac{e_2\mathbf{E}(\mathbf{r}_1)}{m_2}\right]+\text{H.c.}, \tag{5.8}$$

where we have given an explicitly Hermitian form. The contribution to the two-particle interaction is

$$-\frac{e_1e_2}{16\pi M}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\hat{\mathbf{l}}\cdot\mathbf{P}\times\frac{\mathbf{r}_1-\mathbf{r}_2}{(\mathbf{r}_1-\mathbf{r}_2)^3}+\text{H.c.}$$

$$=\frac{e_1e_2}{16\pi M}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\left(\frac{\hat{\mathbf{q}}\times\mathbf{P}}{|\hat{\mathbf{q}}|^3}\cdot\hat{\mathbf{l}}+\text{H.c.}\right), \tag{5.9}$$

to lowest order in v^2/c^2, when we may legitimately, if desired, take $\hat{\mathbf{l}}=M^{-1}(\mathbf{r}_1-\mathbf{r}_2)\times(m_2\mathbf{p}_1-m_1\mathbf{p}_2)$. As hoped for, this necessary extra contribution finally eliminates the $\hat{\mathbf{l}}$-dependent terms on the right-hand side of (5.5). We may now combine results to obtain for the two-particle Hamiltonian

$$\begin{aligned}H=M&+\frac{\mathbf{P}^2}{2M}-\frac{\mathbf{P}^4}{8M^3}+\left[\hat{\mathbf{p}}^2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)+\frac{e_1e_2}{4\pi|\hat{\mathbf{q}}|}\right]\left(1-\frac{\mathbf{P}^2}{2M^2}\right)-\hat{\mathbf{p}}^4\left(\frac{1}{8m_1^3}+\frac{1}{8m_2^3}\right)\\&+\frac{e_1e_2}{8\pi m_1m_2}\left(\hat{\mathbf{p}}\frac{1}{|\hat{\mathbf{q}}|}\cdot\hat{\mathbf{p}}+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\frac{1}{|\hat{\mathbf{q}}|^3}\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}\right)-\frac{M}{4\pi m_1m_2}(e_1\mu_2\boldsymbol{\sigma}_2+e_2\mu_1\boldsymbol{\sigma}_1)\cdot\frac{\hat{\mathbf{q}}\times\hat{\mathbf{p}}}{|\hat{\mathbf{q}}|^3}+\frac{e_1e_2}{16\pi}\left(\frac{\boldsymbol{\sigma}_1}{m_1^2}+\frac{\boldsymbol{\sigma}_2}{m_2^2}\right)\cdot\frac{\hat{\mathbf{q}}\times\hat{\mathbf{p}}}{|\hat{\mathbf{q}}|^3}-\tfrac{2}{3}\mu_1\mu_2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\delta(\hat{\mathbf{q}})\\&+\frac{\mu_1\mu_2}{4\pi}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{|\hat{\mathbf{q}}|^3}-\frac{3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{q}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{q}}}{|\hat{\mathbf{q}}|^3}\right)+\left[\frac{e_1e_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)-\frac{e_2\mu_1}{2m_1}-\frac{e_1\mu_2}{2m_1}\right]\delta(\hat{\mathbf{q}}),\end{aligned} \tag{5.10}$$

which is consistent, to order v^2/c^2, with

$$\begin{aligned}H&=(\hat{M}^2+\mathbf{P}^2)^{1/2},\\\hat{M}&=(m_1^2+\hat{\mathbf{p}}^2)^{1/2}+(m_2^2+\hat{\mathbf{p}}^2)^{1/2}+\frac{e_1e_2}{4\pi|\hat{\mathbf{q}}|}+\frac{e_1e_2}{8\pi m_1m_2}\left(\hat{\mathbf{p}}\frac{1}{|\hat{\mathbf{q}}|}\cdot\hat{\mathbf{p}}+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}\frac{1}{|\hat{\mathbf{q}}|^3}\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}\right)\\&-\frac{M}{4\pi m_1m_2}(e_1\mu_2\boldsymbol{\sigma}_2+e_2\mu_1\boldsymbol{\sigma}_1)\cdot\frac{\hat{\mathbf{q}}\times\hat{\mathbf{p}}}{|\hat{\mathbf{q}}|^3}+\frac{e_1e_2}{16\pi}\left(\frac{\boldsymbol{\sigma}_1}{m_1^2}+\frac{\boldsymbol{\sigma}_2}{m_2^2}\right)\cdot\frac{\hat{\mathbf{q}}\times\hat{\mathbf{p}}}{|\hat{\mathbf{q}}|^3}-\tfrac{2}{3}\mu_1\mu_2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\delta(\hat{\mathbf{q}})+\frac{\mu_1\mu_2}{4\pi}\left(\frac{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2}{|\hat{\mathbf{q}}|^3}-\frac{3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{q}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{q}}}{|\hat{\mathbf{q}}|^5}\right)\\&+\left[\frac{e_1e_2}{8}\left(\frac{1}{m_1^2}+\frac{1}{m_2^2}\right)-\frac{e_2\mu_1}{2m_1}-\frac{e_1\mu_2}{2m_2}\right]\delta(\hat{\mathbf{q}}).\end{aligned} \tag{5.11}$$

This final expression for the mass operator, or Hamiltonian in the c.m. frame, is well known. It can be derived in a variety of different fashions[19] but the basic ingredient is neglect of retardation of the electromagnetic interaction.

VI. MODIFICATION OF CENTER-OF-MASS VARIABLES

The arguments given in Sec. V for the presence of the additional term, (5.8) or (5.9), are essentially heuristic, so in order to obtain a clearer understanding of why the extra term produces the consistent relativistic Hamiltonians (5.10), it is necessary to inquire more deeply into the approximate relativistic invariance of this system. In this approximation of keeping only v^2/c^2 or[11] $O(M^{-2})$ corrections beyond the nonrelativistic result, the Hamiltonian H in (4.11) was derived consistently from a formally relativistic theory. Hence to this order of approximation it must necessarily be an element of generators H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ obeying the Lie algebra of the Poincaré group, as explicitly demonstrated in Appendix B. With $\mathbf{E}_T=\mathbf{0}$, the momentum $\mathbf{P}$ and angular momentum $\mathbf{J}$ still have the simple two-free-particle form (2.2), but the generator of boosts $\mathbf{K}$ in the same approximation is

$$\mathbf{K}=\mathbf{K}_{1\,\text{free}}+\mathbf{K}_{2\,\text{free}}+\mathbf{K}_{\text{Coulomb}}\,, \tag{6.1a}$$

$$\mathbf{K}_{1\,\text{free}}=t\mathbf{p}_1-m_1\mathbf{r}_1-\tfrac{1}{2}\{\mathbf{r}_1,\mathbf{p}_1^2/2m_1\}\,, \tag{6.1b}$$

$$\mathbf{K}_{\text{Coul}}=-\frac{1}{2}\int d\mathbf{x}\,\mathbf{x}\nabla\Phi(\mathbf{x})\cdot\nabla\Phi(\mathbf{x})\to-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\frac{e_1e_2}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}. \tag{6.1c}$$

[19] W. A. Barker and F. N. Glover, Phys. Rev. 99, 317 (1955); see also T. Itoh, Rev. Mod. Phys. 37, 159 (1965).

By employing the c.m. variables (2.6), the sum of the free-particle boosts (6.1b) becomes

$$\mathbf{K}_{1\,\mathrm{free}}+\mathbf{K}_{2\,\mathrm{free}}=t\mathbf{P}-M\mathbf{R}$$
$$-\frac{1}{2}\left\{\mathbf{R},\frac{\mathbf{P}^2}{2M}+\tfrac{1}{2}\hat{\mathbf{p}}^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right\}-\frac{1}{2M}\mathbf{P}\times\hat{\mathbf{S}}. \quad (6.2)$$

This is consistent with the canonical forms (2.2) for free particles, as required by the construction of the c.m. variables. However, the expected term required by the presence of the Coulomb interaction in the mass operator (5.11) is

$$-\mathbf{R}\frac{e_1e_2}{4\pi|\hat{\mathbf{q}}|}=-\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{M}\frac{e_1e_2}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}, \quad (6.3)$$

to our approximation. For unequal masses this is not the same as (6.1c). The discrepancies between $\mathbf{K}$, H and their canonical forms (2.2) are related since they can both be restored to the canonical expressions by the same unitary transformations which does not affect $\mathbf{P}$ or $\mathbf{J}$. The algebra of H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ is obviously the same as the unitarily transformed canonical generators, which is that of the Poincaré group, so this is consistent with our relativistic starting point. For verification let[20]

$$X=\frac{m_1-m_2}{2M^2}\frac{e_1e_2}{4\pi|\hat{\mathbf{q}}|}\mathbf{P}\cdot\hat{\mathbf{q}}. \quad (6.4)$$

Then

$$-i[X,M\mathbf{R}]$$
$$=-\frac{e_1e_2}{4\pi|\mathbf{r}_1-\mathbf{r}_2|}\left[\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{M}-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\right], \quad (6.5a)$$

$$i\left[X,\tfrac{1}{2}\hat{\mathbf{p}}^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right]$$
$$=\frac{e_1e_2}{16\pi M}\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\left(\frac{\hat{\mathbf{q}}\times\mathbf{P}}{|\hat{\mathbf{q}}|^3}\cdot\hat{\mathbf{l}}+\mathrm{H.c.}\right). \quad (6.5b)$$

So $e^{iX}\cdots e^{-iX}$ is the unitary transformation which to $O(M^{-2})$ achieves the above-mentioned desired results; compare (6.5b) with (5.9). Alternatively, this can be viewed as a modification in the definition of our c.m. variables induced by the interaction. Suppose

$$\begin{aligned}\mathbf{R}'&=e^{-iX}\mathbf{R}e^{iX}=\mathbf{R}-\frac{m_1-m_2}{2M^2}\frac{e_1e_2}{4\pi|\hat{\mathbf{q}}|}\hat{\mathbf{q}},\\ \mathbf{P}'&=e^{-iX}\mathbf{P}e^{-iX}=\mathbf{P},\\ \hat{\mathbf{q}}'&=e^{-iX}\hat{\mathbf{q}}e^{iX}=\hat{\mathbf{q}},\\ \hat{\mathbf{p}}'&=e^{-iX}\hat{\mathbf{p}}e^{iX}=\hat{\mathbf{p}}-\frac{m_1-m_2}{2M^2}\frac{e_1e_2}{4\pi|\hat{\mathbf{q}}|^3}\hat{\mathbf{q}}\times(\hat{\mathbf{q}}\times\mathbf{P}),\end{aligned} \quad (6.6)$$

where $\mathbf{R}$, $\mathbf{P}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{p}}$ are given in terms of the basic single-particle variables by our normal equations, to this approximation, implicit in (2.6). Then $\mathbf{R}'$, $\mathbf{P}'$, $\hat{\mathbf{q}}'$, and $\hat{\mathbf{p}}'$ are new c.m. variables such that H, $\mathbf{K}$ and also $\mathbf{P}$, $\mathbf{J}$ defined by reductions from conventional electromagnetic expressions are reduced to the canonical form (2.2), to order v^2/c^2, without any extra terms or unitary transformations. The important property of these new variables is that the division of the angular momentum $\mathbf{J}$ into an over-all orbital part and a spin $\mathbf{S}'$ constructed from particle spins and internal orbital angular momentum,

$$\begin{aligned}\mathbf{J}&=\mathbf{R}'\times\mathbf{P}'+\hat{\mathbf{S}}',\\ \hat{\mathbf{S}}'&=\hat{\mathbf{q}}'\times\hat{\mathbf{p}}'+\hat{\mathbf{s}}_1'+\hat{\mathbf{s}}_2',\end{aligned} \quad (6.7)$$

depends on the interaction. Now $\hat{\mathbf{S}}$, as defined by the canonical representation (2.2), is such that under Lorentz boosts it undergoes just Wigner rotation,[2] and hence $\hat{\mathbf{S}}$ is the dynamical variable which undergoes Thomas precession. With the free-particle form of the relativistic c.m. variables (2.6) used in Sec. II, $\hat{\mathbf{S}}$ did not allow for Thomas-precession effects arising from the interaction, so the theory could not immediately be cast into the canonical relativistic forms. But when an effective interaction for this effect was introduced, $\hat{\mathbf{S}}$ then had the correct Thomas precession and so the desired form was obtained.

In conclusion we briefly discuss the modifications to the definitions of the c.m. and internal varibles $\mathbf{R}$, $\mathbf{P}$, $\hat{\mathbf{q}}$, and $\hat{\mathbf{p}}$ in terms of the basic single-particle variables for an arbitrary N-body system when interaction is present. The definitions for these variables, when the N particles were free, have been derived in Ref. 3. The system consists of particles of mass m_i and charge e_i, with the position $\mathbf{r}_i$, momentum $\mathbf{p}_i$, and spin $\mathbf{s}_i$ as the basic dynamical variables, where $i=1,\ldots,N$. If the interaction takes place instantaneously between the conserved charge current densities for each individual particle,[6]

$$\begin{aligned}\rho_i(\mathbf{x})&=e_i\delta(\mathbf{x}-\mathbf{r}_i)-(1/4m_i)(2\mu_i-e_i/2m_i)\\ &\quad\times[-\nabla^2\delta(\mathbf{x}-\mathbf{r}_i)-\boldsymbol{\sigma}_i\cdot\mathbf{p}_i\times\boldsymbol{\nabla}_x\delta(\mathbf{x}-\mathbf{r}_i)\\ &\qquad+\boldsymbol{\sigma}_i\cdot\boldsymbol{\nabla}_x\delta(\mathbf{x}-\mathbf{r}_i)\times\mathbf{p}_i], \quad (6.8)\\ \mathbf{j}_i(\mathbf{x})&=(e_i/2m_i)\{\mathbf{p}_i,\delta(\mathbf{x}-\mathbf{r}_i)\}-\mu_i\boldsymbol{\sigma}_i\times\boldsymbol{\nabla}_x\delta(\mathbf{x}-\mathbf{r}_i),\end{aligned}$$

then, as explicitly verified in Appendix B, a consistent, approximately relativistic, form for the Hamiltonian and the boosts is obtained where the leading interaction-

[20] Alternatively, one can solve for $\mathbf{R}$ (and $\mathbf{R}'$) in terms of the generators H_0 (and $H=H_0+H'$), $\mathbf{K}_0$ (and $\mathbf{K}=\mathbf{K}_0+\mathbf{K}'$), etc. (see, e.g., Refs. 2, 3, and 5). Then to our order of approximation the relation between $\mathbf{R}$ and $\mathbf{R}'$ is given formally by $\mathbf{R}'-\mathbf{R}=\mathbf{K}_0/H_0-\mathbf{K}/H$ and the result is (6.5a) and (6.11). One could, in principle, continue in this manner and solve algebraically for $\mathbf{q}'$, etc., following Refs. 2 and 3. Clearly the method of unitary transformation is much simpler and the required X is found uniquely from the calculated $\mathbf{R}'-\mathbf{R}$.

dependent terms are

$$H'=\frac{1}{2}\int d\mathbf{x}\,\tfrac{1}{2}\sum_{i\neq j}\rho_i(\mathbf{x})V(|\mathbf{x}-\mathbf{x}'|)\rho_j(\mathbf{x}')\to \sum_{i>j}e_ie_jV(|\mathbf{r}_i-\mathbf{r}_j|)$$

$$\mathbf{K}'=-\frac{1}{2}\int d\mathbf{x}\,\mathbf{x}\tfrac{1}{2}\sum_{i\neq j}\rho_i(\mathbf{x})V(|\mathbf{x}-\mathbf{x}'|)\rho_j(\mathbf{x}')\to -\sum_{i>j}e_ie_j\,\tfrac{1}{2}(\mathbf{r}_i+\mathbf{r}_j)V(|\mathbf{r}_i-\mathbf{r}_j|)\,. \tag{6.9}$$

In general, V may be an arbitrary potential but, as in the two-particle Coulomb case, $\mathbf{K}'$ differs from the canonical expression

$$\mathbf{K}'_{\text{can}}=-\mathbf{R}\sum_{i>j}e_ie_jV(|\mathbf{r}_i-\mathbf{r}_j|)\,,$$
$$M\mathbf{R}=\sum_i m_i\mathbf{r}_i\,,\quad M=\sum_i m_i\,. \tag{6.10}$$

The unitary transformation which achieves the same desired results as previously obtained is now generated by[20]

$$X=\frac{1}{2M}\sum_{i>j}e_ie_j(2\mathbf{R}-\mathbf{r}_i-\mathbf{r}_j)\cdot\mathbf{P}V(|\mathbf{r}_i-\mathbf{r}_j|)\,, \tag{6.11}$$

so that $e^{ix}\mathbf{K}'e^{-ix}=\mathbf{K}'$ canonical. Correspondingly, the new c.m. variables are such that $\mathbf{P}$ and $\hat{\mathbf{q}}_{ij}$ are unchanged but

$$\mathbf{R}'=e^{-iX}\mathbf{R}e^{iX}=\mathbf{R}-\frac{1}{2M}\sum_{i>j}e_ie_j(2\mathbf{R}-\mathbf{r}_i-\mathbf{r}_j)V(|\mathbf{r}_i-\mathbf{r}_j|)$$
$$=\mathbf{R}+\frac{1}{2M^2}\sum_{i\neq j}\sum_k m_k\hat{\mathbf{q}}_{ik}e_ie_jV(|\hat{\mathbf{q}}_{ij}|)\,,$$

$$\hat{\mathbf{p}}_i'=e^{-iX}\hat{\mathbf{p}}_ie^{iX}=\hat{\mathbf{p}}_i+\frac{\mathbf{P}}{2M}\left[\frac{m_i}{M}\sum_{j\neq k}e_je_kV(|\hat{\mathbf{q}}_{jk}|)-\sum_{j\neq i}e_ie_jV(|\hat{\mathbf{q}}_{ij}|)\right]$$
$$-\frac{1}{2M}\sum_{j\neq i}e_ie_j\hat{\mathbf{q}}_{ij}\frac{V'(|\mathbf{q}_{ij}|)}{|\mathbf{q}_{ij}|}(\mathbf{r}_i+\mathbf{r}_j-2\mathbf{R})\cdot\mathbf{P}\,. \tag{6.12}$$

When the modified internal momenta in (6.11) are employed in the kinetic-energy terms of the Hamiltonian, they generate Thomas-precession terms for the internal orbital angular momenta as induced by the interaction such that a consistent separation of c.m. motion to order v^2/c^2 is obtained.

Thus, in general, the form of relativistic c.m. variables is dependent on the interaction. Already the free-particle form has had interesting consequences regarding the effective electromagnetic interaction for composite systems of weakly bound particles, both with regard to spin-dependent effects[6] and also the internal orbital angular momentum.[3,7,21] Further effects may then be expected when the binding energy is not negligible.

ACKNOWLEDGMENTS

We wish to thank L. Van Hove, L. L. Foldy, and L. A. Copley for discussions and communications. This work was performed while one of us (H.O.) was at Queen Mary College, London, and the other (F.E.C.) was at the Department of Theoretical Physics, Oxford. The latter also wishes to thank Professor R. H. Dalitz for his encouragement, and Magdalen College for the Don Scholarship.

APPENDIX A

We show here that the Hamiltonian for the interaction of a system of N particles with an external electromagnetic field as derived in Ref. 3 is identical with that in Ref. 7. The system is ascribed a mass M, total momentum $\mathbf{P}$, and c.m. position $\mathbf{R}$. It is comprised of N particles with the ith particle having mass m_i, internal momentum $\hat{\mathbf{p}}_i$, spin angular momentum $\hat{\mathbf{s}}_i$, and position $\mathbf{r}_i$.

The interaction $H=H_{\text{FW}}+H_{\Delta\text{FW}}$, where H_{FW} is the sum of the Foldy-Wouthuysen interactions of the individual constituents while

$$H_{\Delta\text{FW}}=-\tfrac{1}{2}\sum_{i=1}^{N}(\boldsymbol{\varrho}_i\cdot\mathbf{E}_i+\mathbf{E}_i\cdot\boldsymbol{\varrho}_i)\,,$$

with $\mathbf{E}_i$ the electric field at position $\mathbf{r}_i$. Defining $\boldsymbol{\eta}_i=\mathbf{r}_i^{\text{NR}}-\mathbf{R}$, Krajcik and Foldy[7] have

$$\boldsymbol{\varrho}_\beta\equiv-\tfrac{1}{2}\boldsymbol{\eta}_\beta\cdot\frac{\mathbf{P}}{M}\left[\frac{\hat{\mathbf{p}}_\beta}{m_\beta}+\frac{\mathbf{P}}{2M}\right]-\tfrac{1}{2}\sum_\alpha\frac{\hat{\mathbf{p}}_\alpha^2\boldsymbol{\eta}_\alpha}{2m_\alpha M}+\tfrac{1}{2}\sum_\alpha\frac{(\boldsymbol{\eta}_\alpha\times\hat{\mathbf{p}}_\alpha)\times\mathbf{P}}{2M^2}+\text{H.c.}$$
$$-\frac{\hat{\mathbf{s}}_\beta\times\mathbf{P}}{4m_\beta M}+\sum_\alpha\frac{\hat{\mathbf{s}}_\alpha}{4M}\times\left(\frac{\hat{\mathbf{p}}_\alpha}{m_\alpha}+\frac{\mathbf{P}}{M}\right), \tag{A1}$$

whereas in Ref. 3 one finds

$$\boldsymbol{\varrho}_i\equiv-\tfrac{1}{2}\sum_j(\mathbf{r}_i-\mathbf{r}_j)^{\text{NR}}\cdot\mathbf{P}\frac{m_j}{M^2}\left(\frac{\hat{\mathbf{p}}_i^{\text{NR}}}{m_i}-\frac{\mathbf{P}}{2M}\right)$$
$$+\tfrac{1}{2}\sum_j\sum_k(\mathbf{r}_k-\mathbf{r}_j)^{\text{NR}}\cdot\mathbf{P}\frac{m_k}{2M^3}\mathbf{p}_j^{\text{NR}}$$
$$-\tfrac{1}{2}\sum_j\sum_k(\mathbf{r}_j-\mathbf{r}_k)^{\text{NR}}\frac{m_k}{2m_jM^2}\mathbf{p}_j^{\text{NR}}\cdot\hat{\mathbf{p}}_j+\text{H.c.}$$
$$+\sum_j\left(\frac{\hat{\mathbf{s}}_j}{m_j}-\frac{\hat{\mathbf{s}}_i}{m_i}\right)\times\frac{\mathbf{p}_j^{\text{NR}}}{4M}, \tag{A2}$$

[21] H. Osborn (unpublished).

where

$$\mathbf{r}_i^{\mathrm{NR}}=\mathbf{R}+\sum_j \frac{m_j}{M}\hat{\mathbf{q}}_{ij}, \tag{A3}$$

$$\mathbf{p}_j^{\mathrm{NR}}=\frac{m_j}{M}\mathbf{P}+\hat{\mathbf{p}}_j, \tag{A4}$$

and

$$\hat{\mathbf{p}}_j=\sum_n \frac{(m_j\mathbf{p}_n-m_n\mathbf{p}_j)}{M}.$$

The identity of the spin-dependent terms in (A1) and (A2) follows immediately by using (A4) in (A2) and noting that $\sum_j m_j=M$, $\sum_j \hat{\mathbf{p}}_j=0$. The identity of the first terms in (A1) and (A2),

$$\sum_j(\mathbf{r}_i-\mathbf{r}_j)^{\mathrm{NR}}\cdot\mathbf{P}\frac{m_j}{M^2}\left(\frac{\mathbf{p}_i^{\mathrm{NR}}}{m_i}-\frac{\mathbf{P}}{2M}\right)\equiv\boldsymbol{\eta}_i\cdot\frac{\mathbf{P}}{M}\left(\frac{\hat{\mathbf{p}}_i}{m_i}+\frac{\mathbf{P}}{2M}\right),$$

is seen by noting that since nonrelativistic variables are being used, then $\sum_j (\mathbf{r}_i-\mathbf{r}_j)m_j/M\equiv\mathbf{r}_i-\mathbf{R}\equiv\boldsymbol{\eta}_i$. Then with the use of (A4), the identity follows at once.

The proof that the two remaining terms in (A2) are identical with the two remaining in (A1) follows straightforwardly by substituting (A4) in (A2), noting that $\sum_a (\mathbf{r}_a^{\mathrm{NR}}-\mathbf{R})m_a/M\equiv0$, and using the vector identity for $(\boldsymbol{\eta}\times\hat{\mathbf{p}})\times\mathbf{P}$.

APPENDIX B

We here give a direct proof of the Lorentz invariance of the magnetic interaction, as derived in the text, to order v^2/c^2 in accordance with the starting point of the conventional relativistic electromagnetism. Neglecting $d\mathbf{A}/dt$, such a framework gives

$$H=H_0+H',\quad \mathbf{K}=\mathbf{K}_0+\mathbf{K}',$$

$$H'=\frac{1}{2}\int d\mathbf{x}\ \nabla\Phi(\mathbf{x})\cdot\nabla\Phi(\mathbf{x})+\frac{1}{2}\int d\mathbf{x}\ \mathbf{B}(\mathbf{x})^2-\int d\mathbf{x}\ \mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x}), \tag{B1}$$

$$\mathbf{K}'=-\frac{1}{2}\int d\mathbf{x}\ \mathbf{x}\nabla\Phi(\mathbf{x})\cdot\nabla\Phi(\mathbf{x})-\frac{1}{2}\int d\mathbf{x}\ \mathbf{x}\mathbf{B}(\mathbf{x})^2+\int d\mathbf{x}\ \mathbf{x}\mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x}),$$

with the equations of constraint

$$\begin{aligned}&-\nabla^2\Phi(\mathbf{x})=\rho(\mathbf{x}),\\&-\nabla^2\mathbf{A}(\mathbf{x})=(1-\nabla\nabla/\nabla^2)\mathbf{j}(\mathbf{x}).\end{aligned} \tag{B2}$$

By manipulation of (B1) using (B2),

$$H'=\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\ \rho(\mathbf{x})\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x})'-\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\mathbf{j}(\mathbf{x})\frac{1}{8\pi|\mathbf{x}-\mathbf{x}'|}\cdot\left[1+\frac{(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right]\cdot\mathbf{j}(\mathbf{x}'),$$

$$\begin{aligned}\mathbf{K}'=-\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\ \mathbf{x}\rho(\mathbf{x})\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}')+\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\ \mathbf{x}\mathbf{j}(\mathbf{x})\frac{1}{8\pi|\mathbf{x}-\mathbf{x}'|}\\ \cdot\left[1+\frac{(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2}\right]\cdot\mathbf{j}(\mathbf{x}')-\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\ \mathbf{j}(\mathbf{x})\frac{1}{8\pi|\mathbf{x}-\mathbf{x}'|}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{j}(\mathbf{x}').\end{aligned} \tag{B3}$$

Instead of directly verifying the commutation relations of the generators of the Poincaré group with the forms of interaction (B3), it is more interesting to construct H' and $\mathbf{K}'$ out of ρ and $\mathbf{j}$ such that they have the required algebra and then verifying that the result is consistent with (B3). The basic assumption then is that H_0, and $\mathbf{K}_0$ along with $\mathbf{P}$ and $\mathbf{J}$, which are not modified by the interaction, obey the Poincaré algebra and $(\rho,\mathbf{j})$ is a conserved 4-vector, at least to order v^2/c^2:

$$\begin{aligned}&i[H_0,\rho(\mathbf{x})]+\nabla\cdot\mathbf{j}(\mathbf{x})=0,\\&i[\mathbf{K}_0,\rho(\mathbf{x})]=\mathbf{x}\nabla\cdot\mathbf{j}(\mathbf{x})+\mathbf{j}(\mathbf{x}),\\&i[\mathbf{K}_0,\mathbf{j}(\mathbf{x})]=-\mathbf{x}i[H_0,\mathbf{j}(\mathbf{x})]+\mathbf{1}\rho(\mathbf{x}),\end{aligned} \tag{B4}$$

with the conventional commutators with $\mathbf{P}$, and $\mathbf{J}$. Consistent with our v^2/c^2 approximation, $i[H_0,\mathbf{j}(\mathbf{x})]$ is neglected throughout. For an equal-time interaction between the charge and current densities, the general form for a scalar H' and vector $\mathbf{K}'$ satisfying

$$[H',\mathbf{P}]=0,\quad i[\mathbf{K}',\mathbf{P}]=H',$$

along with parity and time-reversal invariance is

$$H'=\int d\mathbf{x}d\mathbf{x}'\ \rho(\mathbf{x})\mathfrak{D}(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')+\int d\mathbf{x}d\mathbf{x}'\ \mathbf{j}(\mathbf{x})\cdot\mathbf{D}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{j}(\mathbf{x}'),$$

$$\begin{aligned}\mathbf{K}'=&-\int d\mathbf{x}d\mathbf{x}'\ \mathbf{x}\rho(\mathbf{x})\mathfrak{D}(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}')\\&-\int d\mathbf{x}d\mathbf{x}'\ \mathbf{x}\mathbf{j}(\mathbf{x})\cdot\mathbf{D}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{j}(\mathbf{x}')\\&-\int d\mathbf{x}d\mathbf{x}'\ \mathbf{j}(\mathbf{x})\mathfrak{D}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{j}(\mathbf{x}'),\end{aligned} \tag{B5}$$

where $\mathfrak{D}(\mathbf{x})$, $\mathsf{D}(\mathbf{x})$, and $\mathfrak{D}(\mathbf{x})$ are, respectively, a scalar, a symmetric dyadic, and a vector, the first two being invariant under $\mathbf{x}\leftrightarrow -\mathbf{x}$. For Poincaré invariance to hold, the two crucial commutators are

$$i[\mathbf{K},H]=i[\mathbf{K}_0,H_0]=\mathbf{P}\,,\qquad \mathbf{K}\times\mathbf{K}=\mathbf{K}_0\times\mathbf{K}_0=-i\mathbf{J}\,, \tag{B6}$$

so that to first order in the interaction, (B6) becomes

$$i[\mathbf{K}',H_0]+i[\mathbf{K}_0,H']=0\,,\qquad i\mathbf{K}'\times\mathbf{K}_0+i\mathbf{K}_0\times\mathbf{K}'=0\,. \tag{B7}$$

Verification of the commutators (B6) to second order in the interaction depends specifically on the commutation relations between $\rho(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ which are model dependent, so the results here are only valid to first order in the interaction. From (B4) and (B5), neglecting $\partial\mathbf{j}/\partial t$, then

$$i[\mathbf{K}',H_0]+i[\mathbf{K}_0,H']=\int d\mathbf{x}d\mathbf{x}'\,\mathbf{j}(\mathbf{x})\rho(\mathbf{x}')\cdot[\mathbf{1}\mathfrak{D}(\mathbf{x}-\mathbf{x}')-(\mathbf{x}-\mathbf{x}')\nabla_x\mathfrak{D}(\mathbf{x}-\mathbf{x}')+2\mathsf{D}(\mathbf{x}-\mathbf{x}')]\,, \tag{B8}$$

so that, writing $\mathfrak{D}(\mathbf{x}-\mathbf{x}')=\mathfrak{D}(|\mathbf{x}-\mathbf{x}'|)$, for (B7) to hold

$$\mathsf{D}(\mathbf{x}-\mathbf{x}')=-\frac{1}{2}\left[\mathbf{1}\mathfrak{D}(|\mathbf{x}-\mathbf{x}'|)-\frac{(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathfrak{D}'(|\mathbf{x}-\mathbf{x}'|)\right]. \tag{B9}$$

With the same approximation and using (B9),

$$\begin{aligned}i[\mathbf{K}_0,\mathbf{K}']=&\int d\mathbf{x}d\mathbf{x}'\,\rho(\mathbf{x}')\\&\times\{\mathfrak{D}(|\mathbf{x}-\mathbf{x}'|)[\mathbf{x}\mathbf{j}(\mathbf{x})+\tfrac{1}{2}\mathbf{j}(\mathbf{x})(\mathbf{x}+\mathbf{x}')]-\mathfrak{D}(\mathbf{x}-\mathbf{x}')\mathbf{j}(\mathbf{x})\}\\&+\int d\mathbf{x}d\mathbf{x}'\,\rho(\mathbf{x}')\frac{\mathfrak{D}'(|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\\&\times(\mathbf{x}-\mathbf{x}')\cdot\mathbf{j}(\mathbf{x})\tfrac{1}{2}(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')\\&+\int d\mathbf{x}d\mathbf{x}'\rho(x')\mathfrak{D}(\mathbf{x}-\mathbf{x}')\cdot\mathbf{j}(\mathbf{x})\mathbf{1}\,.\end{aligned} \tag{B10}$$

For (B7) to hold, (B10) must be a symmetric dyad, and this requires

$$\mathfrak{D}(\mathbf{x}-\mathbf{x}')=\tfrac{1}{2}(\mathbf{x}-\mathbf{x}')\mathfrak{D}(|\mathbf{x}-\mathbf{x}'|)\,. \tag{B11}$$

From (B5), (B9), and (B11), an interaction consistent with v^2/c^2 relativistic invariance is given by

$$\begin{aligned}H'=&\int d\mathbf{x}d\mathbf{x}'\,\rho(\mathbf{x})\tfrac{1}{2}V(|\mathbf{x}-\mathbf{x}'|)\rho(\mathbf{x}')\\&-\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\,\mathbf{j}(\mathbf{x})\frac{1}{2}\left[\mathbf{1}V(|\mathbf{x}-\mathbf{x}'|)-\frac{(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}V'(|\mathbf{x}-\mathbf{x}'|)\right]\mathbf{j}(\mathbf{x}')\,,\end{aligned}$$

$$\begin{aligned}\mathbf{K}'=&-\int d\mathbf{x}d\mathbf{x}'\,\mathbf{x}\rho(\mathbf{x})\tfrac{1}{2}V(|\mathbf{x}-\mathbf{x}'|)\rho(\mathbf{x}')\\&+\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\,\mathbf{x}\mathbf{j}(\mathbf{x})\cdot\frac{1}{2}\left[\mathbf{1}V(|\mathbf{x}-\mathbf{x}'|)-\frac{(\mathbf{x}-\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}V'(|\mathbf{x}-\mathbf{x}'|)\right]\mathbf{j}(\mathbf{x}')\\&-\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\,\mathbf{j}(\mathbf{x})\tfrac{1}{2}V(|\mathbf{x}-\mathbf{x}'|)(\mathbf{x}-\mathbf{x}')\cdot\mathbf{j}(\mathbf{x}')\,.\end{aligned} \tag{B12}$$

This is, of course, consistent with (B3) when V is just the normal Coulomb interaction. The proof still goes through if ρ and $\mathbf{j}$ are separated into individual-particle contributions and self-interaction terms excluded in (B3) or (B12), so that our treatment of c.m. motion can be extended to include general interparticle potentials.