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Static Axially Symmetric Gravitational Fields

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One of the major problems in understanding solutions of the gravitational field equations is to determine the physical meaning of the coordinates involved. In particular, for the Weyl solutions if the standard coordinate descriptions are used, then physical interpretation becomes difficult. A means of interpreting the Weyl coordinates for static axially symmetric metrics is presented which allows a reasonable physical interpretation of any given Weyl metric.

INTRODUCTION

THE initial work on static axially symmetric solutions of the Einstein field equations was done by Weyl[1] and Levi-Civita[2] shortly after the development of general relativity, and in principle all of the Weyl (i.e., static, axially symmetric) metrics are known.

However, although more than fifty years have passed since these solutions were discovered, there is still no consensus as to their true physical interpretation. This situation has been illustrated by Zipoy[3] for a restricted class of Weyl metrics. It is unpleasant for two reasons. First, solutions of the gravitational field equations are hard to come by, and it would be nice to at least be able to understand those which are known. Second, if there is to be any hope of doing actual physics, it is not enough to have a mathematical solution of a problem; there must also be some means of relating that solution to physical reality.

Before continuing, it is necessary to make precise the meaning of the terms "solution" and "source." In referring to a solution of the field equations, we mean a metric g_{ab} which satisfies the static axially symmetric vacuum field equations. This metric is then to be thought of as the exterior solution for some configuration of matter referred to as the source. For example, the Schwarzschild metric can be considered as the exterior gravitational field of a spherically symmetric shell of matter, which would then be referred to as the source.

It is assumed that coordinates (x^a) are chosen; these are adapted to the symmetries of the source in the sense that the exterior solutions obtained will be valid outside of some hypersurface $x^1=\text{const}$, and this hypersurface will be said to correspond to the (surface of the) source.

In a coordinate system (η,ξ,ϕ,t) the metric has the canonical form

$$ds^2=e^{-2\gamma}[e^{2\nu}(d\eta^2+d\xi^2)+\rho^2d\phi^2]-e^{2\gamma}dt^2, \qquad (1)$$

where γ, ν, and ρ depend only on the coordinates (η,ξ). If Δ is the Laplacian $D=\partial/\partial\eta+i\partial/\partial\xi$, then the field equations reduce to[4]

$$\begin{gathered}D\bar{D}\rho=0\,,\quad \Delta\gamma=0\,,\\ 2D\nu D\rho=D^2\rho+2\rho(D\gamma)^2\,,\end{gathered} \qquad (2)$$

where the bar means complex conjugation.

Thus, $\rho(\eta,\xi)$ is an arbitrary harmonic function, while γ is a solution of the Laplace equation, and ν is determined uniquely by ρ and γ. This exhausts the set of Weyl solutions.

WEYL SOLUTIONS

Consider some specific solution ρ, ν, γ. Since $g_{00}\sim 1+2\phi$ asymptotically, ϕ being the Newtonian potential associated with this solution, it is necessary that $\gamma=\phi$. Thus if some suitable mass distribution is given, having a Newtonian potential ϕ, then it is necessary to choose $\gamma=\phi$ in computing the corresponding relativistic solution.

The prescription for calculating a Weyl solution is then as follows: (a) Choose coordinates (η,ξ) adapted to the source; (b) choose a function ρ, harmonic in (η,ξ); and (c) let γ be the Newtonian potential and solve for ν.

[1] H. Weyl, Ann. Physik **54**, 117 (1917).

[2] T. Levi-Civita, Atti della Accad. dei Lincei Rendiconti **27**, 2 (1918).

[3] D. M. Zipoy, J. Math. Phys. **7**, 1137 (1966).

[4] J. L. Synge, *Relativity, The General Theory* (North-Holland, Amsterdam, 1966), p. 312.

There is an ambiguity in this procedure, however, caused by the arbitrary character of the function ρ. If z is the conjugate function to ρ, then for $\eta=\rho$, $\xi=z$ the Laplacian resembles the cylindrical Laplacian; hence, (ρ,z) are called the *canonical cylindrical* coordinates, and (η,ξ) are taken as defined in terms of (ρ,z). Since ρ is initially an arbitrary harmonic function of (η,ξ), there will clearly be no intrinsic means of determining what the coordinates (η,ξ) represent physically.

As an example, suppose that γ is the Newtonian potential of an oblate spheroidal homoloid, $\gamma=(m/\kappa)\cot^{-1}u$. Then according to prescription, the oblate spheroidal coordinates defined by[5]

$$\begin{aligned}\rho&=\kappa(u^2+1)^{1/2}(1-v^2)^{1/2},\\ z&=\kappa uv\end{aligned}\tag{3}$$

are to be chosen and the resulting solution is[6]

$$\begin{aligned}ds^2=\kappa^2\exp\left(\frac{-m}{\kappa}\cot^{-1}u\right)&\left[(u^2+1)(1-v^2)d\phi^2\right.\\ &\left.+(u^2+v^2)\left(\frac{u^2+v^2}{u^2+1}\right)^{(m/\kappa)^2}\left(\frac{du^2}{u^2+1}+\frac{dv^2}{1-v^2}\right)\right]\\ &-\exp\left(\frac{m}{\kappa}\cot^{-1}u\right)dt^2.\end{aligned}\tag{4}$$

Now, if this is actually to represent an oblate spheroid in the expected manner, then in the limit $\kappa\to 0$, it should reduce to the metric obtained by setting $\gamma=-m(\rho^2+z^2)^{-1/2}$, and this should be the Schwarzschild metric.

When this limit is actually taken, the result is the Curzon metric

$$\begin{aligned}ds^2=\exp\left[\frac{m}{(\rho^2+z^2)^{1/2}}\right]&\{\exp[-m^2\rho^2/(\rho^2+z^2)^2](d\rho^2+dz^2)\\ &+\rho^2d\phi^2\}-\exp[-m/(\rho^2+z^2)^{1/2}]dt^2.\end{aligned}\tag{5}$$

Since this is clearly not the Schwarzschild solution, the true source configuration of the metric (4) remains uncertain.

In light of this example, it is natural to ask what form the Schwarzschild metric takes in Weyl coordinates. It is a well-known result that if the prolate spheroidal coordinates defined by[7]

$$\begin{aligned}\rho&=\kappa(u^2-1)^{1/2}(1-v^2)^{1/2},\\ z&=\kappa uv\end{aligned}\tag{6}$$

are used, the solution for a "rod" of length 2κ is obtained, and in the special case $\kappa=m$, this may be reduced to the Schwarzschild metric by the transformation[8]

$$u=r/m-1,\quad v=\cos\theta.\tag{7}$$

Since the Schwarzschild solution is the only static spherically symmetric vacuum solution of Einstein's field equations, the following assumption is suggested: *The coordinate system defined by (6) for which $\kappa=m$, is, in the sense of (7), a spherical coordinate system.*

A result of this assumption is that in the coordinates (6) the surfaces $u=$const will be elongated spheroids if $\kappa>m$ and flattened spheroids if $\kappa<m$.

Let $\delta=m/\kappa$, (u,v) be the special prolate spheroidal coordinate system for which $\delta=1$, and (x,y) be another prolate spheroidal system adapted to the source. The metric is then

$$\begin{aligned}ds^2=\kappa^2\left(\frac{x+1}{x-1}\right)^{\delta}&\left[(x^2-y^2)\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)\right.\\ &\left.+(x^2-1)(1-y^2)d\phi^2\right]-\left(\frac{x-1}{x+1}\right)dt^2.\end{aligned}\tag{8}$$

It is simple to find (x,y) in terms of (u,v):

$$\begin{aligned}x&=\frac{\delta}{\sqrt{2}}\left\{u^2+v^2+\frac{1-\delta^2}{\sqrt{2}}+\left[\left(u^2+v^2+\frac{1-\delta^2}{\delta^2}\right)^2-\frac{4u^2v^2}{\delta^2}\right]^{1/2}\right\}^{1/2},\\ y&=\frac{\delta}{\sqrt{2}}\left\{u^2+v^2+\frac{1-\delta^2}{\sqrt{2}}-\left[\left(u^2+v^2+\frac{1-\delta^2}{\delta^2}\right)^2-\frac{4u^2v^2}{\delta^2}\right]^{1/2}\right\}^{1/2},\end{aligned}\tag{9}$$

and the metric (8) is given in spherical coordinates by substitution of (9) and then (7). As expected from the oblate spheroidal example, (8) reduces to the Curzon metric in the limit $\kappa\to 0$.

If $(\bar{x},\bar{y})$ are oblate spheroidal coordinates, the equivalent of (9) is obtained by taking $\omega=i\delta$ to obtain

$$\begin{aligned}\bar{x}&=\frac{\omega}{\sqrt{2}}\left\{\left[\left(u^2+v^2-\frac{1+\omega^2}{\omega^2}\right)^2+\frac{4u^2v^2}{\omega^2}\right]^{1/2}+u^2+v^2-\frac{1+\omega^2}{\omega^2}\right\}^{1/2}\\ \bar{y}&=\frac{\omega}{\sqrt{2}}\left\{\left[\left(u^2+v^2-\frac{1+\omega^2}{\omega^2}\right)^2+\frac{4u^2v^2}{\omega^2}\right]^{1/2}-\left(u^2+v^2-\frac{1+\omega^2}{\omega^2}\right)\right\}^{1/2},\end{aligned}\tag{10}$$

[5] To put the metric in the canonical form, set $u=\sinh\eta$, $v=\sin\xi$.
[6] M. Misra, Proc. Natl. Inst. Sci. India A26, 673 (1960).
[7] The canonical form of the metric is achieved by taking $u=\cosh\eta$, $v=\sin\xi$.
[8] F. J. Ernst, Phys. Rev. **167**, 1175 (1968).

which when coupled with (7) allows (4) to be written in spherical coordinates.

Note that the above represents only two specific examples of a general procedure for interpretation of the Weyl coordinates.

So far the only basis for the proposed interpretation of the Weyl solutions has been a rather intuitive one. This can be made more plausible by considering the intrinsic geometry of the coordinate surfaces $\gamma=\text{const}$. Inspection of (4) and (8) shows that $\gamma=\text{const}$ corresponds to $\bar{x}=\text{const}$ or $x=\text{const}$, respectively. Thus, the surfaces $x=a$, $\bar{x}=(a^2-1)^{1/2}$ will be examined.[9]

The corresponding surfaces in spherical coordinates are given by

$$r=m\left[1+\frac{a}{\delta}\left(\frac{a^2-1+\delta^2\sin^2\theta}{a^2-\cos^2\theta}\right)^{1/2}\right]\quad\text{(prolate case)},\tag{11}$$

$$r=m\left[1+\frac{(a^2-1)^{1/2}}{\omega}\left(\frac{a^2+\omega^2\sin^2\theta}{a^2-\sin^2\theta}\right)^{1/2}\right]\quad\text{(oblate case)},\tag{12}$$

so that once a and δ (or ω) are given, the source configuration is specified.

Consider the canonical coordinates (η,ξ) such that $(x,y)\to(\cosh\eta,\sin\xi)$ and $(\bar{x},\bar{y})\to(\sinh\eta,\sin\xi)$. The metrics of the $\gamma=\text{const}$ surfaces are, respectively,

$$d\Omega^2=\kappa^2\left(\frac{a+1}{a-1}\right)^{\delta}(a^2-1)\times\left[\left(\frac{a^2}{a^2-1}\right)^{1-\delta^2}\left(1-\frac{1}{a^2}\sin^2\xi\right)^{1-\delta^2}d\xi^2+\cos^2\xi d\phi^2\right],$$
$$d\bar{\Omega}^2=a^2\kappa^2\exp[-\omega\cot^{-1}(a^2-1)^{1/2}]\times\{a^{2(1+\omega^2)}[1-(1/a^2)\cos^2\xi]^{1+\omega^2}d\xi^2+\cos^2\xi d\phi^2\},\tag{13}$$

where $x=a$, $\bar{x}=(a^2-1)^{1/2}$ as before.

The metric for a two-surface of revolution has the canonical form

$$d\Omega^2=R^2[d\psi^2+f^2(\psi)d\phi^2],\tag{14}$$

in which R is some constant. For general values of δ or ω, it is difficult to cast the metrics (13) into this form, although it is possible to do so in terms of elliptic functions. One particularly simple case is the prolate spheroidal metric with $\delta=\sqrt{2}$. Now the metric of the $\gamma=\text{const}$ surface is

$$d\Omega^2=R(d\psi^2+\text{sn}^2(\psi/\alpha)d\phi^2),\tag{15}$$

with $\alpha=(1-1/a^2)^{1/2}$, and

$$R=\frac{m^2}{2}\left(\frac{a+1}{a-1}\right)^{\sqrt{2}}(a^2-1).$$

Since $\alpha\to1$ as $a\to\infty$ and the modulus of the elliptic sine is $1/a$, this becomes the sphere metric asymptotically.

The coordinate ψ is defined by

$$\psi=\int_0^{\xi}\left(1-\frac{1}{a^2}\sin^2\xi'\right)^{-1/2}d\xi',\tag{16}$$

and the real period of $\text{sn}^2(\psi/\alpha)$ is given by

$$\tau=2\pi F(\tfrac{1}{2},\tfrac{1}{2};1;1/a^2).\tag{17}$$

Thus, we expect to find a cusp or node at the poles of the surface $x=a$. This is reasonable since in the coordinate picture being used the surface $x=1$ corresponds to a rod with length 2κ.

If the metrics (4) and (8) are studied asymptotically using our proposed coordinate interpretation to cast them into the "Schwarzschild" coordinates (r,θ,ϕ,t), then to first order in $1/r$ both of these reduce to

$$ds^2=\frac{dr^2}{1-2m/r}+r^2(d\theta^2+\sin^2\theta d\phi^2)-\left(1-\frac{2m}{r}\right)dt^2,\tag{18}$$

which is immediately recognizable as the standard form of the Schwarzschild metric. Also it is possible to calculate the quadrupole moments of the source configurations[10] and, as expected, they correspond to prolate objects for $\delta>1$ and to oblate objects for $0<\delta<1$ and all values of ω. This can be compared with the Newtonian theory by expanding $e^{2\gamma}$ in the "Schwarzschild" coordinates and comparing the asymptotic form to the asymptotic form of the Newtonian potential expressed in the normal spherical coordinates. To order $(1/r)^3$, we obtain

$$\phi\sim-\frac{m}{r}+\frac{1}{3}\left(1-\frac{1}{\delta^2}\right)\frac{m^3}{r^3}(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2})+O\left(\frac{1}{r^4}\right)\quad\text{(prolate case)},$$
$$\phi\sim-\frac{m}{r}+\frac{1}{3}\left(1+\frac{1}{\omega^2}\right)\frac{m^3}{r^3}(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2})+O\left(\frac{1}{r^4}\right)\quad\text{(oblate case)}.\tag{19}$$

These have Newtonian analogs as the fields of disks (if the coefficient of r^{-3} is positive) or rods (if this coefficient is negative). Clearly the case $\delta=1$ corresponds to a point mass.

As a brief digression, we observe that the Newtonian potential for a disk of radius R is asymptotically given by

$$\phi\sim-\frac{m}{r}+\frac{m^3}{3\epsilon^2r^3}(\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2})+O\left(\frac{1}{r^4}\right),\tag{20}$$

[9] The form of the constants involved is chosen for convenience and results from the identification $x=\cosh\eta$, $\bar{x}=\sinh\eta$, which can be made. Observe that $a\geq1$.

[10] B. Godfrey, Ph.D. thesis, Princeton University, 1970 (unpublished).

and if ω (or δ) is allowed to become infinite in Eq. (19) (corresponding to the Curzon solution), a comparison with (20) suggests that the Curzon solution represents the exterior metric for a disk of radius m. In this case such a disk would be hidden behind the horizon possessed by the Curzon solution.

It is also worth noting that our coordinate interpretation removes the problem of directional singularities that has been discussed by several authors (e.g., Stachel,[11] Gautreau and Anderson,[12] and Godfrey[10]). This can be seen by studying the surfaces on which some curvature invariant becomes singular.

It is simple to compute the curvature tensor for the metrics (4) and (8). The only independent components of $R^a{}_{bcd}$ for a Weyl solution are $R^0{}_{101}$, $R^0{}_{102}$, $R^0{}_{202}$, and $R^0{}_{303}$,[13] and these are given by[14]

$$\begin{aligned}
R^0{}_{101}&=-\frac{\partial^2\gamma}{\partial u^2}+\frac{\partial\gamma}{\partial u}\frac{\partial\nu}{\partial u}-2\left(\frac{\partial\gamma}{\partial u}\right)^2,\\
R^0{}_{102}&=\frac{\partial\gamma}{\partial u}\frac{\partial\nu}{\partial v},\\
R^0{}_{202}&=-\frac{\partial\gamma}{\partial u}\frac{\partial\nu}{\partial u}+\left(\frac{\partial\gamma}{\partial u}\right)^2,\\
R^0{}_{303}&=-\rho^2e^{-2\nu}\left[\frac{1}{\rho}\frac{\partial\rho}{\partial u}\frac{\partial\gamma}{\partial u}-\left(\frac{\partial\gamma}{\partial u}\right)^2\right].
\end{aligned}\tag{21}$$

In terms of these, the scalar $k=R_{abcd}R^{abcd}$ is given by

$$\begin{aligned}
k=4(g^{11})^2\{(7/4)[(R^0{}_{101})^2+(R^0{}_{202})^2+(g_{11})^2(g^{33})^2(R^0{}_{303})^2]\\
+4(R^0{}_{102})^2-\tfrac{1}{2}[R^0{}_{101}R^0{}_{202}+g_{11}g^{33}R^0{}_{101}R^0{}_{303}\\
+g_{11}g^{33}R^0{}_{202}R^0{}_{303}]\}\,.
\end{aligned}\tag{22}$$

A long but straightforward calculation shows that for $\delta\neq 1$ this is singular on the horizon $\eta=0$ and at the points $\eta=0$, $\xi=\frac{1}{2}\pi$ in the prolate case, while for the oblate case, it is singular on the ring $\eta=\xi=0$. The last two of these singularities correspond to the classical singularities found at the ends of a rod or at the edge of a disk, respectively. The singularity at the horizon has no classical analog.

In Fig. 1 the horizons and singular surfaces are sketched in the Schwarzschild coordinates defined by (7) with δ and ω as parameters. In particular, note their close relation to the Schwarzschild horizon.

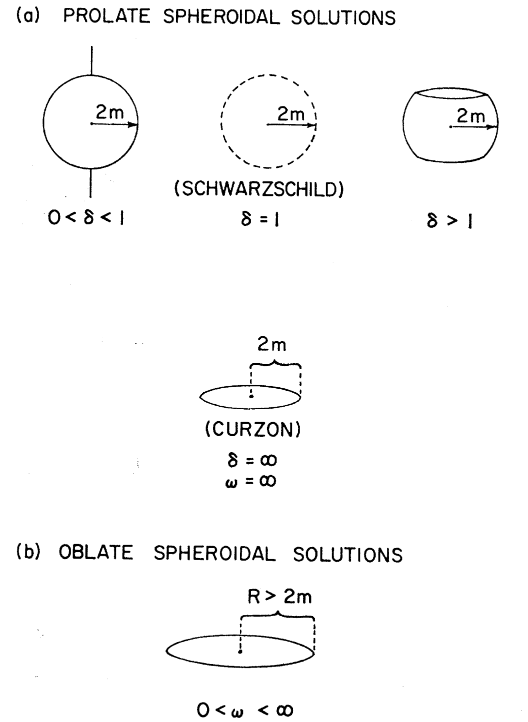

FIG. 1. Singularities of spheroidal metrics. Singular surfaces are solid; nonsingular surfaces are dashed.

Observe that for $\delta>1$ in the prolate spheroidal solutions the singular region does not cover the entire $r=2m$ surface, and if one approaches this surface along the symmetry axis, no singularity is encountered. The usual interpretation of the singular region as having directional properties arises when the canonical cylindrical coordinates (ρ,z) are used since in these coordinates the surface $r=2m$ corresponds to $\rho=0$ or for $\theta=\frac{1}{2}\pi$ to $z=0$.

CONCLUSION

A simple and intuitive means of relating the Weyl solutions to physical source configurations has been proposed and several of the consequences studied.

Further results of interest should come from an analysis of other vacuum solutions generated by the use of different axially symmetric coordinate systems. It would also be interesting to attempt to find a corresponding set of interior solutions.

[11] J. Stachel, Phys. Letters **27A**, 60 (1968).
[12] R. Gautreau and J. L. Anderson, Phys. Letters **25A**, 291 (1967).
[13] They are the only independent components in the sense that the remainder of the nonzero components of $R^a{}_{bcd}$ can be obtained in terms of these four by use of the vacuum field equations.
[14] M. Walker (private communication).