Application of a Finite Dispersion Approach to the Calculation of the $\omega \to \pi\pi\gamma$ Decay*

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The contribution of high-mass states in the dispersion relation for decay amplitudes is considered. A finite dispersion relation (FDR) implemented by duality and finite-energy sum rules (FESR), is suggested as a useful tool for such calculations, and an application to the decay $\omega \to \pi\pi\gamma$ is done in some detail.

INTRODUCTION

MANY estimates of hadron decays amount to simple pole models. Quite detailed experimental information on such processes is available at present, and calculations using new theoretical concepts and methods are called for.

In this paper we apply duality and finite-energy sum rules (FESR)[1,2] to the calculation of $\omega \to \pi\pi\gamma$ decay. Duality is used here simply as a device for estimating the contribution of the tail of the fixed-t dispersion integral when direct information on the couplings of the exchanged Regge pole is not available. We do not use a specific dual-resonance Veneziano model[3–5] for the amplitude since no relatively simple form exists for the process of interest and also since the difference between the simple polynomial expansion used by us and the "exact form" is probably quite negligible over the Dalitz region.

The choice of decay process was motivated by the experimental value for $\Gamma(\omega \to \pi\pi\gamma)$ quoted recently[6,7] which exceeds earlier theoretical estimates[8–10]. The same methods are applicable to a wide class of decays and low-energy scattering amplitudes.

The plan of the paper is the following: After describing briefly in Sec. I the method [the finite dispersion relation (FDR) approach], we describe the calculation in Sec. II. Section III contains a few more comments and speculations.

I. FINITE DISPERSION RELATION (FDR)

Let $A(\nu,t)$ be one of the invariant Mandelstam amplitudes for the process $a+b \to c+d$, where

$$s=(p_a+p_b)^2,\quad t=(p_a-p_c)^2,$$
$$u=(p_a-p_d)^2,\quad \text{and}\quad \nu=\tfrac{1}{2}(s-u).$$

Let us assume that for t, $A(\nu,t)$ is analytic in ν, so that using the Cauchy theorem we have[11]

$$A(\nu,t)=\frac{1}{2\pi i}\oint\frac{A(\nu',t)}{\nu'-\nu}d\nu'=\frac{1}{\pi}\int_{-N}^{N}\frac{\mathrm{Im}A(\nu',t)}{\nu'-\nu}d\nu' +\frac{1}{2\pi i}\int_{C_N}\frac{A(\nu',t)}{\nu'-\nu}d\nu',\quad (1)$$

where the contour used is shown in Fig. 1. In this FDR, the contribution from the two semicircles (C_N)

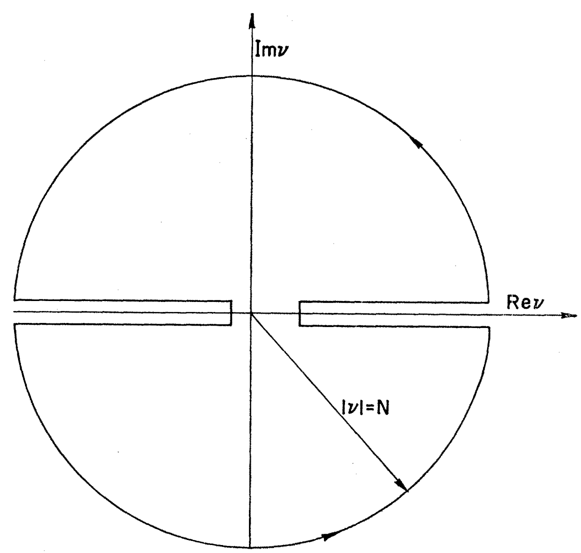

FIG. 1. Contour used in Eq. (1).

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[2] K. Igi and S. Matsuda, Phys. Rev. Letters **18**, 625 (1967); **18**, 822 (1967).

[3] G. Veneziano, Nuovo Cimento **57A**, 190 (1968).

[4] C. Lovelace, Phys. Letters **28B**, 264 (1968).

[5] Veneziano amplitudes for $PV \to PV$ processes which are relevant for our discussion were considered, e.g., by A. Capella *et al.* [Nuovo Cimento **64A**, 361 (1969)]; G. P. Canning [Nucl. Phys. **B14**, 432 (1969)]; and P. Carruthers and E. Lasley [Phys. Rev. D (to be published)].

[6] Z. S. Strugalski *et al.*, Phys. Letters **29B**, 532 (1969).

[7] W. Deinet *et al.*, Phys. Letters **30B**, 426 (1969). This work also contains information on $\Gamma_{\omega\to\pi\gamma}$ and further references on neutral ω decays.

[8] P. Singer, Phys. Rev. **128**, 2789 (1962); **130**, 2441 (1963).

[9] J. Yellin, Phys. Rev. **147**, 1080 (1966). This work contains a detailed exposition at various applications of pole models.

[10] S. M. Renard, Nuovo Cimento **62A**, 475 (1969).

[11] In an application for decay processes we consider $A(\nu,t)$ at points (ν,t) inside the Dalitz region. The analytic properties of $A(\nu,t)$ may be more complicated than that implied by Eq. (1). Our present discussion ignores these singularities and also involves a simple resonance approximation to the first integral in (1). In the particular case of $\omega \to \pi\pi\gamma$, rescattering corrections which are of higher order in e^2 are unimportant.

replaces the "tails" from N to ∞ and $-\infty$ to $-N$ of the ordinary dispersion relations.

As in the derivation of the FESR, which starts with $\oint A(\nu',t)d\nu'=0$ over the same contour, we assume that N is chosen sufficiently large so that an asymptotic Regge form is adequate on the semicircle. Assuming A is crossing even,[12]

$$A(\nu,t)=A(-\nu,\,t)=\tfrac{1}{2}[A(\nu,t)+A(-\nu,\,t)],$$

Eq. (1) takes the form

$$A(\nu,t)=\frac{1}{\pi}\int_{-N}^{+N}\frac{\nu'\,\mathrm{Im}A(\nu',t)}{\nu'^2-\nu^2}d\nu' + \frac{1}{2\pi i}\int_{C_N}\frac{\nu' A(\nu')}{\nu'^2-\nu^2}d\nu'\equiv L+R. \quad (2)$$

In practical application the first term L will be approximated by the contribution of the low resonances and Born terms. Also N is larger than ν values occurring in the Dalitz region which is relevant to the calculation of decay amplitudes. We can therefore expand the denominator in the second term in powers of (ν^2/ν'^2). Substituting the asymptotic Regge form,

$$A(\nu',t)\,|_{\,|\nu'|=N}\approx\sum_j \frac{\pi\gamma_j(t)}{\sin\pi\alpha_j(t)}[(\nu')^{\alpha_j(t)}+(-\nu')^{\alpha_j(t)}], \quad (3)$$

we can explicitly perform the ν' integration:

$$A(\nu,t)=\sum[\text{resonances}] +2\sum_{n=0}\left(\frac{\nu}{N}\right)^{2n}\left(\sum_j \gamma_j(t)\frac{N^{\alpha_j(t)}}{\alpha_j(t)-2n}\right). \quad (4)$$

The last term clearly displays all the poles which lie (in the narrow-resonance approximation) on the even-signatured trajectories contributing to A. If t values close to some of these poles occur in the decay problem of interest, the last term will be large and a simple $s+u$ resonance saturation will obviously be inadequate.

Independently of the occurrence of poles the contribution of the Regge terms may be quite important—and this is indeed the case for the reaction at hand. In some of the earlier works, such t-channel contributions were included by adding a Feynman diagram corresponding to the σ exchange.[9,10] The present approach avoids possible double counting and allows a more sophisticated structure for the t-channel exchanges suggested by the Regge-pole phenomenology. This is quite important, and indeed in the $\omega\to\pi\pi\gamma$ calculation quite different results were obtained when just one effective t-channel trajectory was included rather than the f^0 and ϵ trajectories. Finally, our work differs from the earlier calculation in that we use FESR's in order to fix the residues of the Regge trajectories—rather than resorting to universality arguments.

We write the lowest moment FESR

$$-\frac{1}{\pi}\int_0^N \nu\,\mathrm{Im}A(\nu,t)d\nu=\sum_j \gamma_j(t)\frac{N^{\alpha_j(t)+2}}{\alpha_j(t)+2}. \quad (5)$$

Equation (5), like Eq. (4), will be considered in practice only in the region of small t so that we will use the conventional parametrization $\gamma_j(t)=\beta_j/\Gamma(\alpha_j(t)+m)$ with no extra t dependence beyond that of the "ghost-nonsense eliminating" Γ function. We will also assume linear trajectories $\alpha_j(t)$. Provided only a few trajectories need be considered, the β_j can be determined by expanding both sides of (5) in powers of t, and keeping the lowest few terms.

II. KINEMATICS OF $\omega\to\pi\pi\gamma$

In specializing to the decay $\omega\to\pi\pi\gamma$, we denote by (p,ω_μ) the monentum and polarization vector of the vector meson. Similarly, (k,ϵ_μ) specify the photon state. The pions' momenta are q_i.

Defining

$$P=\tfrac{1}{2}(p+k)\,,\quad Q=\tfrac{1}{2}(q_2-q_1)\,,$$

we write the Feynman-invariant amplitude

$$F=\omega^\nu\epsilon^\mu T_{\mu\nu}. \quad (6)$$

The covariant decomposition of the gauge-invariant tensor is

$$\begin{aligned}T_{\mu\nu}=A(\nu,t)[(P\cdot k)g_{\mu\nu}-P_\mu k_\nu]&\\ +B(\nu,t)[(Q\cdot k)g_{\mu\nu}-Q_\mu k_\nu]&\\ +C(\nu,t)[(Q\cdot k)P_\mu Q_\nu-(P\cdot k)Q_\mu Q_\nu]&, \quad (7)\end{aligned}$$

where $s=(p-q_1)^2$, $t=(q_1+q_2)^2$, and $\nu=\frac{1}{2}(s-u)$.

The contributions of the two low-lying ρ and B mesons to the various amplitudes are readily evaluated:

$$\begin{aligned}A:\ &[g_{\omega\rho\pi}g_{\rho\pi\gamma}/(s-m_\rho^2)]\times\tfrac{1}{4}(t-4s-4m_\pi^2+m_\omega^2)\\ &+[g_{B\omega\pi}{}^T g_{B\pi\gamma}/(s-m_B^2)]\\ &\times[\tfrac{1}{4}(m_\omega^2-2s-t+2m_\pi^2)(1+\xi)\\ &-\tfrac{1}{2}(m_\omega^2+m_B^2-m_\pi^2)]+(s\rightleftarrows u)\,; \quad (8a)\end{aligned}$$

$$\begin{aligned}B:\ &[g_{\omega\rho\pi}g_{\rho\pi\gamma}/(s-m_\rho^2)]\times\tfrac{1}{4}(t+2s+m_\omega^2-2m_\pi^2)\\ &+[g_{B\omega\pi}{}^T g_{B\pi\gamma}/(s-m_B^2)]\times[\tfrac{1}{4}(m_\omega^2-t)(1+\xi)\\ &-\tfrac{1}{2}(m_\omega^2+s-m_\pi^2)]-(s\rightleftarrows u)\,; \quad (8b)\end{aligned}$$

$$\begin{aligned}C:\ &-2\times g_{\omega\rho\pi}g_{\rho\pi\gamma}/(s-m_\rho^2)+g_{B\omega\pi}{}^T g_{B\pi\gamma}\\ &\times 2(1+\xi)/(s-m_B^2)+(s\rightleftarrows u)\,, \quad (8c)\end{aligned}$$

where we used the couplings

$$g_{\omega\rho\pi}\epsilon_{\tau\nu\lambda\sigma}P_{(\omega)}{}^\tau\omega^\nu P_{(\rho)}{}^\lambda\rho^\sigma\phi\,, \quad (9a)$$

$$[g_{B\omega\pi}{}^T(\omega^\mu P_{(\omega)}{}^\nu-\omega^\nu P_{(\omega)}{}^\mu) +g_{B\omega\pi}{}^L\omega^\mu P_{(\omega)}{}^\nu]P_{(B)\mu}B_\nu\phi\,, \quad (9b)$$

and the corresponding γ couplings (only the transversal part occurs in $B-\pi-\gamma$). The parameter ξ is defined as $\xi=g_{B\omega\pi}{}^L/g_{B\omega\pi}{}^T$.

[12] An essentially analogous treatment can be given to the crossing-odd amplitudes.

The leading trajectories exchanged in the t channel are the P, P', and ϵ. Adopting the Harari-Freund[13] conjecture, we omit the P (Pomeranchon) contribution to the FESR and also neglect its presumably small contribution in the low-energy Dalitz region.[14]

The A amplitude is even and gets an ϵ-pole contribution at $t=m_\epsilon^2$. According to the remark following Eq. (5), we parametrize its Regge expansion as follows[15]:

$$A(\nu,t)\to\sum_{j=1}^{2}\frac{\pi\gamma_j{}^A(t)}{\sin\pi\alpha_j(t)}[(\nu)^{\alpha_j(t)}+(-\nu)^{\alpha_j(t)}],\quad(10)$$

$$\gamma_1{}^A(t)=\frac{\beta_{P'}{}^A}{\Gamma(\alpha_{P'}(t))},\quad\gamma_2{}^A(t)=\frac{\beta_\epsilon{}^A}{\Gamma(\alpha_\epsilon(t)+1)}.\quad(11)$$

The ϵ trajectory decouples from the t-channel helicity-flip amplitudes B and C at $\alpha_\epsilon(t)=0$. Introducing the nonsense factor for the $\alpha=0$ state, we get for the antisymmetric B

$$B(\nu,t)\to\sum_{j=1}^{2}\frac{\pi\gamma_j{}^B(t)}{\sin\pi\alpha_j(t)}[\nu^{\alpha_j-1}-(-\nu)^{\alpha_j-1}]\quad(12)$$

and for the symmetric C

$$C(\nu,t)\to\sum_{j=1}^{2}\frac{\pi\gamma_j{}^C(t)}{\sin\pi\alpha_j(t)}[\nu^{\alpha_j-2}+(-\nu)^{\alpha_j-2}],\quad(13)$$

with

$$\gamma_1{}^B(t)=\frac{\beta_{P'}{}^C}{\Gamma(\alpha_{P'}(t))},\quad\gamma_2{}^C(t)=\frac{\beta_\epsilon{}^B}{\Gamma(\alpha_\epsilon(t))},\quad(14)$$

$$\gamma_1{}^C(t)=\frac{\beta_{P'}{}^C}{\Gamma(\alpha_{P'}(t))},\quad\gamma_2{}^C(t)=\frac{\beta_\epsilon}{\Gamma(\alpha_\epsilon(t))}.\quad(15)$$

The finite dispersion formula described in Sec. I then gives

$$A(\nu,t)=\sum[\text{resonances}]+2\sum_{n=0}\left(\frac{\nu}{N}\right)^{2n}\sum_{j=1}^{2}\gamma_j{}^A\frac{N^{\alpha_j}}{\alpha_j(t)-2n},\quad(16\text{a})$$

$$B(\nu,t)=\sum[\text{resonances}]+2\sum_{n=1}^{2}\left(\frac{\nu}{N}\right)^{2n-1}\sum_{j=1}^{2}\gamma_j{}^B(t)\frac{N^{\alpha_j-1}}{\alpha_j(t)-2n},\quad(16\text{b})$$

$$C(\nu,t)=\sum[\text{resonances}]+2\sum_{n=1}^{2}\left(\frac{\nu}{N}\right)^{2n-2}\sum_{j=1}^{2}\gamma_j{}^C(t)\frac{N^{\alpha_j-2}}{\alpha_j-2n},\quad(16\text{c})$$

where "[resonances]" is given by Eqs. (8), with numerators calculated at the poles.

The lowest-moment FESR[16] is applied as a convenient tool for the calculation of the t-channel couplings β_i:

$$-\frac{1}{\pi}\int_0^N\nu\,\text{Im}A(\nu,t)d\nu=\gamma_{P'}{}^A(t)\frac{N^{\alpha_{P'}(t)+2}}{\alpha_{P'}(t)+2}+\gamma_\epsilon{}^A(t)\frac{N^{\alpha_\epsilon(t)+2}}{\alpha_\epsilon(t)+2},$$

$$-\frac{1}{\pi}\int_0^N\text{Im}B(\nu,t)d\nu=\gamma_{P'}{}^B(t)\frac{N^{\alpha_{P'}(t)}}{\alpha_{P'}(t)}+\gamma_\epsilon{}^B(t)\frac{N^{\alpha_\epsilon(t)}}{\alpha_\epsilon(t)},\quad(17)$$

$$-\frac{1}{\pi}\int_0^N\nu\,\text{Im}C(\nu,t)d\nu=\gamma_{P'}{}^C(t)\frac{N^{\alpha_{P'}(t)}}{\alpha_{P'}(t)}+\gamma_\epsilon{}^C(t)\frac{N^{\alpha_\epsilon(t)}}{\alpha_\epsilon(t)}.$$

We will use these equations for $t\leq0$ where real linear trajectories can be assumed.

III. RESULTS AND DISCUSSION

The contribution of the ρ pole to the various amplitudes (which turn out to exceed considerably that of the B meson) depends on $g_{\rho\pi\gamma}g_{\omega\rho\pi}$. The linear procedure of solving the FESR's tends to endow a similar dependence to the Regge residues—and hence to the whole amplitude.

Vector dominance suggests[17]

$$g_{\rho\pi\gamma}=(g_\rho/g_\omega)g_{\omega\pi\gamma},\quad(18\text{a})$$

$$g_{\omega\rho\pi}=(g_\rho/e)g_{\omega\pi\gamma}.\quad(18\text{b})$$

$g_{\omega\pi\gamma}$ can be obtained directly from the width $\Gamma_{\omega\to\pi\gamma}$ (which we took as 1 MeV). We used the values $g_\omega{}^2/4\pi=14.8$ and $g_\rho{}^2/4\pi=2.1$ obtained by the recent analysis[18] of the colliding-beam experiments. This yields $g_{\omega\rho\pi}=14$ BeV^{-1}. To the extent that the extrapolations of the $\omega\rho\pi$ vertex to zero-mass ω—or to zero-mass ρ, are similar, Eq. (18a) may be more accurate than (18b). Nonetheless, the combined error in determining $g_{\omega\rho\pi}g_{\rho\pi\gamma}$ may well be $\approx30\%$. This will affect the computed $\Gamma_{\omega\to\pi\pi\gamma}$ by roughly a factor of 2 in either direction. $g_{B\omega\pi}{}^T$ depends on the little-known ξ. It turns out that the final results for $\Gamma_{\omega\to\pi\pi\gamma}$ are almost independent of ξ. We have used $\xi=0$ ($g^L=0$), $g_{B\omega\pi}{}^T=2.9$ BeV^{-1}, and by the vector-dominance model $g_{B\pi\gamma}=eg_{B\omega\pi}/f_\omega$.

In evaluating the right-hand side of the FESR's (17), we used $\alpha_{P'}(t)=0.5+t$, $\alpha_\epsilon(t)=-0.5+t$, a value

[13] P. G. O. Freund, Phys. Rev. Letters **20**, 235 (1968); H. Harari, *ibid.* **20**, 1395 (1968).

[14] The consequences of assuming that the Pomeranchukon does not contribute in the low-energy region were discussed in detail recently by M. Kugler, Phys. Letters **31B**, 379 (1970). It must be stressed, however, that in the cases considered by him there was at least one exotic channel, which is not the case for the $\omega\to\pi\pi\gamma$, considered here.

[15] The usual scale factor was taken equal to 1 BeV. In general, all dimensional quantities will be given by powers of BeV.

[16] In principle, we could use two FESR's—the first and third moment sum rules—at $t=0$. This would be, however, completely unjustified in view of the low cutoff, as previous experience with the πN has taught us.

[17] M. Gell-Mann, D. Sharp, and W. Wagner, Phys. Rev. Letters **8**, 261 (1962).

[18] J. Augustin *et al.*, Phys. Letters **28B**, 503 (1969).

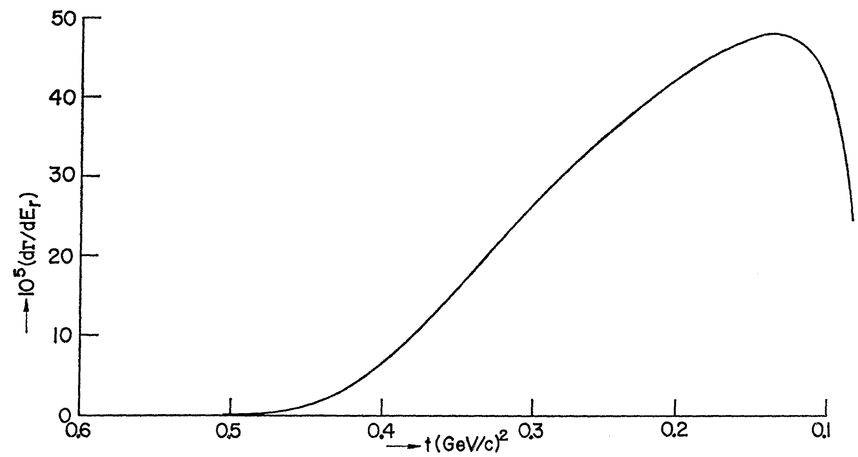

FIG. 2. Calculated photon spectrum plotted with respect to $t=(q_1+q_2)^2 = m_\omega^2-2m_\omega E_\gamma$.

of the cutoff[19]

$$N=\tfrac{1}{2}(m_B^2+m_{\rho'}^2)+\tfrac{1}{2}t-\tfrac{1}{2}(m_\omega^2+2m_\pi^2)\,, \qquad (19)$$

and the parametrization of the residues indicated in Sec. I [see remark following Eq. (5)].

Linearizing the equations and solving for the β_i's, we find

$$\beta_{P'}{}^A=2.35\,,\quad \beta_{P'}{}^B=1.19\,,\quad \beta_{P'}{}^C=0.05\,,$$
$$\beta_\epsilon{}^A=-3.36\,,\quad \beta_{\epsilon B}=0.11\,,\quad \beta_\epsilon{}^C=-4.11\,.$$

These values were then used in Eq. (16) where, in order to take account of the finite width of the ϵ meson which occurs at the physical Dalitz region, we added an imaginary part to $\alpha_\epsilon(t)$ and used the parametrization suggested by Lovelace[4]:

$$\alpha_\epsilon(t)=-0.5+0.28(t-4m_\pi^2)^{1/2}i+t\,. \qquad (20)$$

The calculated photon spectrum is shown in Fig. 2, and the total width for $\Gamma_{\omega^0\to\pi^0\pi^0\gamma}$ was computed to be 0.1 MeV. We made several "standard" checks for the stability of the predicted $\Gamma_{\omega\to\pi\pi\gamma}$ with respect to variation of the cut-off N and the ϵ width in Eq. (20). We also tried varying the parametrization of $\beta_i(t)$ by introducing an additional nonsense factor at $\alpha(t)=1$ in the $B-C$ amplitude. This had a negligible effect on $\Gamma_{\omega\to\pi\pi\gamma}$. We found that decreasing the cutoff to $\frac{1}{2}(m_\rho^2+m_B^2)+\frac{1}{2}t-(m_\omega^2+2m_\pi^2)$ (which may perhaps be justified by the dominance of the ρ contribution) roughly doubled Γ. Bearing in mind also the ambiguity in $|g_{\omega\pi\gamma}g_{\rho\pi\gamma}|^2$, we can only give the prediction

$$0.05\text{ MeV}\leq\Gamma_{\omega\to\pi\pi\gamma}{}^{\text{theor}}\leq 0.4\text{ MeV}\,. \qquad (21)$$

This prediction is consistent with the upper bound $\Gamma_{\omega\to\pi^0\pi^0\gamma}\leq 0.2$ MeV (90% confidence level) quoted in Ref. 7. Reliable lower bounds will be necessary, however, in order to choose between the present calculation and the much smaller values ($\Gamma_\omega\leq 0.004$ MeV) predicted in earlier calculations.[20]

The prominent feature of the present calculation is the large enhancement (by roughly a factor of 5 in amplitude) caused by the FESR-FDR cycle over the direct low-energy contributions. This enhancement does not reflect *only* the ϵ-pole contribution (which was included in some way in Refs. 9 and 10). The computed photon spectrum reflects no ϵ resonance in the low-energy region,[21] and also the Γ computed from the A amplitude was (recall that A is the only amplitude that has the "Feynman" ϵ pole) smaller by a factor of 4.

Indeed, the detailed structure of a leading P' and "secondary" ϵ trajectory seems to be very important in our calculation. The attempt to dispense with the ϵ trajectory altogether, using an effective leading trajectory with $\alpha(0)=0.5$, leads again to much smaller values of $\Gamma_{\omega\to\pi\pi\gamma}$.[22]

CONCLUDING REMARKS

We have shown above that the Regge-tail corrections to amplitudes may be quite important in processes at low energy. FESR are suggested and applied in the $\omega\to\pi\pi\gamma$ case as a tool in estimating these contributions.

It would be interesting to apply the same approach to other decay processes. It should be emphasized, however, that in decay processes leading to three hadrons in the final state (such as $K\to 3\pi$), rescattering corrections are present (unlike our present particular case[11]. Our present analysis, which assumes a very simple analytic pole structure, may be less adequate in such cases.

[19] This corresponds to $s=\frac{1}{2}(m_B^2+m_{\rho'}^2)$, halfway between the last resonance included and the next one. Our final result, Eq. (21), includes also the effects of some variations of the cutoff.

[20] It may be that the charged decay mode $\omega\to\pi^+\pi^-\gamma$, when related to $\omega^0\to\pi^0\pi^0\gamma$ by isospin, would be more convenient for this purpose.

[21] We note that phase space is very small in the ϵ-pole region, which strongly suppress its contribution to $\Gamma_{\omega\to\pi\pi\gamma}$.

[22] The importance of the secondary trajectory structure in the low-energy region is illustrated by the special Veneziano-type $\pi\pi$ amplitude of Ref. 4.