possible deviations from a universal curve. It is an open and intriguing question whether the statistical character of the distributions continues to prevail at higher energies or whether it undergoes some systematic changes.

Note added in proof. The distribution $P^{(0)}$ has been suggested by H. A. Kastrup, Nucl. Phys. **B1**, 309(1967). We thank Dr. Kastrup for bringing this to our attention.

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Evaluation of the $\pi\pi$ Scattering Lengths Using On-Mass-Shell Pions*

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We have evaluated the $\pi\pi$ scattering lengths, using current algebra but without the use of a power-series expansion or extrapolation of the scattering amplitude. We have used the usual Lehmann-Symanzik-Zimmermann reduction in terms of an axial-vector current $J=A+c\partial\phi$, where A is the usual axial-vector current, c is the Goldberger-Treiman constant, and ϕ is the pion field. The scattering amplitude is decomposed into four terms, three of which are due to the equal-time commutators; two of these three are evaluated using the σ model in order to show that the usual current algebra of the current A holds for the current J. The other terms are evaluated in terms of single-particle intermediate states; we show that among these states, only s waves contribute at threshold. Assuming that the ϵ resonance is the only s wave dominating the low-energy ($\pi\pi$) scattering, we find a relation connecting the form factors arising from the equal-time commutators of the current J to the ϵ-pion coupling constant. Finally, we obtain the scattering lengths corresponding to isospins 0 and 2, and ϵ resonance width 200 MeV, as $a_0=0.278m^{-1}$ and $a_2=-0.044m^{-1}$, where m is the pion mass.

I. INTRODUCTION

BOTH experimental data and theoretical arguments have been extensively used in the discussion of the $\pi\pi$ scattering lengths.[1–7] Low-energy $\pi\pi$ scattering has been studied by many authors, using dispersion relations, phase-shift analysis, and the current algebra. Owing to the lack of adequate, accurate experimental data, the results of these calculations cannot be compared with well-established experimental numbers. Even so, the smallness of the scattering lengths as obtained by Weinberg[3] from a soft-pion treatment seems valid.

However, some plausible arguments raise questions concerning this calculation. The main objections to the application of the soft-pion treatment,[8] which have been pointed out by some of the previous authors, may be summarized as follows: First, there is the well-known partial conservation of axial-vector current (PCAC) assumption of the scattering amplitude with respect to k^2, for $0\leq k^2\leq m^2$, where m and k are the pion mass and four-momentum,[9] respectively. Yet this assumption, as pointed out by Sucher and Woo,[10] contradicts the result of the power-series expansion of the amplitude which is involved in applying the soft-pion limit in $\pi\pi$ scattering. Also, it is known that the results of Weinberg's soft-pion treatment do not satisfy the Adler sum rule[11] without some additional assumption concerning the $\pi\pi$ scattering ranges.[5]

As a contribution to the clarification of these points,

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[1] Since 1960 many authors have studied the ($\pi\pi$) scattering lengths. We start with the work of J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962), and the the references given therein.

[2] See, e.g., Ref. 1, and Heinz J. Rothe, Phys. Rev. **140**, B1421 (1965), where the scattering lengths (in units of m^{-1}) for isospins 0, 1, and 2 are claimed to be $a_0=1.7$, $a_1=0.4$, and $a_2=-0.4$. See also S. H. Patil, *ibid.* **179**, 1405 (1969), where it is claimed that $a_0=-1.2$ and $a_2=-0.4$.

[3] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).

[4] N. N. Khuri, Phys. Rev. **153**, 1477 (1967).

[5] F. T. Meiere and M. Sugawara, Phys. Rev. **153**, 1702 (1967); **153**, 1709 (1967). In this reference the authors have a special prescription for the use of Adler sum rule. Their results which satisfy this sum rule are very close to those obtained in the present paper, and differ by about 25% with those obtained in Ref. 4.

[6] Haruichi Yabuki, Progr. Theoret. Phys. (Kyoto) **39**, 118 (1968).

[7] J. R. Fulco and D. Y. Wong, Phys. Rev. Letters **19**, 1399 (1967).

[8] By soft-pion treatment or technique we mean the complete procedure and the treatment used in Ref. 3.

[9] Y. Nambu, Phys. Rev. Letters **4**, 380 (1959); M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[10] J. Sucher and Ching-Hung Woo, Phys. Rev. Letters **18**, 723 (1967).

[11] S. Adler, Phys. Rev. **140**, B736 (1965).

we evaluate the $\pi\pi$ scattering lengths, using a technique described in a previous work,[12] which allows all the pions to remain on the mass shell. This method avoids using the power-series expansion of the amplitude, the Adler consistency condition,[13] and the extrapolation. We use the standard Lehmann-Symanzik-Zimmermann (LSZ) formalism[14] to reduce partially the $\pi\pi$ scattering amplitude. The matrix element involving the pion fields is then reworked into one involving the current $J=A+c\partial\phi$, where A is the usual axial-vector current, c is the Goldberger-Treiman coefficient, and ϕ is the pion interacting field. Note that $\partial_\mu J^\mu=c(\partial^2+m^2)\phi$. The current J contains no pion pole. This and the fact that the action of the Klein-Gordon operator on the pion field is included in this equation will allow us to find manageable expressions with the pions on the mass shell.

Our expression for the scattering amplitude consists of four terms: three terms, W^{00}, W^0, and W^1, due to the equal-time commutators, and one term, W, due to an unequal-time commutator of the current J. For the evaluation of W^0 and W^1 terms we shall use the result of an application of the σ model[15,16] from which we find that the commutation rules of the current J are similar in form, and in fact are related, to those normally used for the weak current A.[17]

For the evaluation of the other two terms W^{00} and W, we assume that the main contributions can be gotten by inserting single-particle intermediate states. We show, however, that among these states only the spin-zero particles contribute to the threshold values of these terms. We then further assume that there is only one spin-0 particle significant for low-energy ($\pi\pi$) scattering, Namely, the ϵ resonance with $m_\epsilon=720$ MeV, $\Gamma_\epsilon=200$ MeV.[18,19]

For the calculation of the scattering lengths, we take advantage of the transformation properties of the above W terms under the interchange of the pion states. In doing this, we also obtain a relation between the form factors of the scalar and vector terms W^0 and W^1, and the ϵ-pion coupling constant.

[12] A. A. Golestaneh and V. P. Gautam, Phys. Rev. **179**, 1449 (1969). In this reference it is shown that at $k^2=0$, the commutation rules of both currents J and A, used in the LSZ formalism, lead to the same results.

[13] S. L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).

[14] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1965).

[15] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[16] One of us (A.A.G.) has studied the equal-time commutation relations of the current $J=A+c\phi$, Eq. (3d) of the text, on the basis of the Lagrangian of the strong interaction, and the σ model combined with the usual canonical commutation rules. This work is under preparation for publication.

[17] The commutation rules of the current A are those suggested by M. Gell-Mann, Physics **1**, 63 (1964).

[18] See, e.g., B. Dutta-Roy and I. R. Lapidus, Phys. Rev. **169**, 1357 (1968), and references in this paper to C. Lovelace in *Proceedings of the Heidelberg Conference on High-Energy Physics, 1967*, edited by H. F. Filthuth (North-Holland, Amsterdam, 1968); W. D. Walker *et al.*, Phys. Rev. Letters **18**, 630 (1967).

[19] For the latest data on the ϵ resonance, see Particle Data Group, Rev. Mod. Phys. **42**, 128 (1970).

Our results for the scattering lengths a_0 and a_2 (corresponding to the isospins 0 and 2) are $a_2=-0.044\, m^{-1}$ and $a_0=0.278m^{-1}$ for $\Gamma_\epsilon=200$ MeV, where m is the pion mass. If the ϵ resonance width is 400 MeV, then we find $a_2=0.03m^{-1}$ and $a_0=0.383m^{-1}$.

II. EVALUATION OF SCATTERING AMPLITUDE

A. General Formula

The two-particle scattering amplitudes given by the LSZ formalism may be expressed as

$$T_{\beta b,\alpha a}=\frac{-i(2\pi)^4\delta^4(p'+k'-p-k)}{(2\pi)^3(4k_0k_0')^{1/2}c^2}\times(W^{00}+W^0+W'+W)\,, \quad (1a)$$

where

$$W^0=i\int d^4z\, e^{-ikz}\langle p'\beta|\,\delta(z_0)[J_b{}^0(0),\partial_\nu J_a{}^\nu(z)]\,|p\alpha\rangle\,, \quad (1b)$$

$$W^1=k_\mu'\int d^4z\, e^{-ikz}\langle p'\beta|\,\delta(z_0)[J_a{}^0(z),J_b{}^\mu(0)]\,|p\alpha\rangle\,, \quad (1c)$$

$$W=-ik_\mu'k_\nu\int d^4z\, e^{-ikz}\langle p'\beta|\,T\{J_b{}^\mu(0)J_a{}^\nu(z)\}\,|p\alpha\rangle\,, \quad (1d)$$

and

$$W^{00}=c\int d^4z\, e^{-ikz}\delta(z_0)\left\{(p_0-p_0')-i\frac{\partial}{\partial z_0}\right\}\times\langle p'\beta|\,[\partial_\mu J_b{}^\mu(z),\phi_a(0)]\,|p\alpha\rangle\,. \quad (1e)$$

In these amplitudes, (k,a) and (p,α) are the momenta and isospin indices of the incident pions, while (k',b) and (p',β) are the corresponding items for the final pions, and ϕ is the pion interacting field. The current operator J represents the source of the pion interacting field which we have chosen as[20]

$$(\partial_\mu{}^2+m^2)\phi(z)=c^{-1}\partial_\mu J_\mu(z)\,, \quad (2)$$

where c is the Goldberger-Treiman coefficient. We shall give further information on this source-current relation (2) and the relation of the current J with the usual weak axial-vector current of hadrons at the beginning of Sec. II B. Here we continue our remarks on Eq. (1a) as follows.

We note that the amplitude (1e) is due to the last term in the expression of the time-ordered product[21]

$$(\partial_x{}^2+m_\pi{}^2)(\partial_y{}^2+m_\pi{}^2)T\{\phi_a(x)\phi_b(y)\}=c^{-2}T\{\partial_\mu J_a{}^\mu(x),\partial_\nu J_b{}^\nu(y)\}+c^{-1}\delta(x_0-y_0)[\partial_\mu J_a{}^\mu(x),\partial_0\phi(y)]\,,$$

[20] The expression (2) has been also used by other authors for the source of the pion field. In addition to Ref. 12, see, e.g., G. Halliday *et al.*, Phys. Rev. **164**, 1834 (1967); S. Gasiorowicz, *Elementary Particle Theory* (Wiley, New York, 1966), p. 379.

[21] See S. S. Schweber, *Relativistic Quantum Field Theory* (Row, Peterson, Evanston, Ill., 1961), p. 696. See also Ref. 12 for further information on the term W^{00}, Eq. (1e).

which is used in writing Eq. (1a). The terms (1e) do not appear in the expression of the amplitude which is usually used for the soft-pion treatment. We also note from relation (2a) that there are no pion poles in the amplitude (1a).[22] Finally, as we shall see below, under the interchange of the pion states, while the sum of W's in Eq. (1a) is invariant, each of the W's is not. This fact allows us to derive two independent expressions for the $(\pi\pi)$ scattering amplitude in Sec. III. From these relations we shall obtain the scattering lengths for isospin 0 and 2, as well as a relation between the form factor of the scalar and vector terms W^0 and W^1, and the ϵ-pion coupling constant.

B. Terms W^0 and W^1

To evaluate these terms, we need the current algebra of the current J. Previously, in Ref. 12, it has been assumed that the equal-time commutation rules of the current J and the weak axial-vector current of hadrons have the same form as suggested by Gell-Mann.[17] Recently, one of us (A.A.G.)[16] reached this result by choosing the original Lagrangian which is used in the σ model,[15] and the usual canonical commutation rules. On the basis of this model, we find that our source-current relation (2) represents the equation of motion for the pion field variable, where

$$J_\mu{}^a=\bar{N}\tau^a\gamma_\mu\gamma_5N+2i(\sigma\overleftrightarrow{\partial}_\mu\phi). \tag{3a}$$

Here N, σ, and ϕ denote the coupled fields of the nucleon, the scalar meson, and the pion which appear in the σ model based on the $SU(2)\otimes SU(2)$ scheme. Also, τ represents the isospin operator of the nucleon field. Now we recall that in the σ model the usual weak axial-vector current of hadrons comes out as

$$A_\mu{}^a=\bar{N}\tau^a\gamma_\mu\gamma_5N+2i(\sigma'\overleftrightarrow{\partial}\phi), \tag{3b}$$

where $\sigma'=\sigma-\frac{1}{2}ic$ is the shifted scalar-meson field, and we get

$$\partial_\mu A_\mu=cm^2\phi, \tag{3c}$$

which is the mathematical expression of the PCAC hypotheses.[9] Combining either Eqs. (3a) and (3b), or (2) and (3c), we find

$$\begin{aligned}J_\mu{}^a&=A_\mu{}^a+c\partial_\mu\phi^a\\&=\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\right)A_\nu{}^a.\end{aligned} \tag{3d}$$

Two interesting facts about this current operator J, given by relations (3), are that it has a closed algebra, and that its equal-time commutators are similar in form with those of the weak current A. That is, following the above σ model, and expressing our result within the $SU(3)$ scheme, we have

$$\delta(z_0)[J_b{}^0(0),\partial_\nu J_a{}^\nu(z)]=id_{abc}\Sigma_c(z)\delta^4(z), \tag{3e}$$

$$\delta(z_0)[J_b{}^0(0),J_a{}^\mu(z)]=\delta(z_0)[A_b{}^0(0),A_a{}^\mu(z)]+(1-\delta_{0\mu})S_{ab}{}^{\text{tr}}, \tag{3f}$$

$$\delta(z_0)[A_b{}^0(0),A_a{}^\mu(z)]=2if_{abc}V_c{}^\mu(z)\delta^4(z)+(1-\delta_{0\mu})S'_{ab}{}^{\text{tr}}. \tag{3g}$$

Here Σ is a scalar operator[23] which differs from that appearing in the equal-time commutator $[A_0{}^a,\partial_\mu A_\mu{}^b]$, while the vector-current operator V_μ is the same for both currents J and A, as seen in Eqs. (3f) and (3g). Also, d_{abc} and f_{abc} are the structure constants of $SU(3)$. Note that the Schwinger terms $S_{ab}{}^{\text{tr}}$ in the above commutators are proportional to $(1-\delta_{0\mu})$, so they vanish at the threshold $\mathbf{k}=\mathbf{p}=0$, where we shall make use of these commutators. The expression (2), or (3d), permits us to identify

$$J_{\pi^\pm}=(1/\sqrt{2})(J_1\mp iJ_2),\quad J_{\pi^0}=J_3. \tag{4}$$

Using (4) in (3e),

$$d_{abc}\Sigma_c=\delta_{ab}[(\sqrt{\tfrac{1}{3}}\Sigma_8+(\sqrt{\tfrac{2}{3}})\Sigma_0]\equiv\delta_{ab}\tilde{\Sigma}, \tag{5}$$

where

$$\langle p'\beta|\tilde{\Sigma}(0)|p\alpha\rangle=[(2\pi)^3(4p_0p_0')^{1/2}]^{-1}\delta_{\alpha\beta}f(t,p^2,p'^2), \tag{6}$$

in which

$$t=(p-p')^2.$$

also f is the scalar form factor which we discuss in Sec. III.

Combining Eqs. (5), (3e), and (1b) we find, for $\mathbf{k}=\mathbf{p}=0$,

$$W^0=-[2m(2\pi)^3]^{-1}\delta_{\alpha\beta}\delta_{ab}f(0,m^2). \tag{7}$$

To evaluate the amplitude W^1, we write the matrix element of the vector current $V_c{}^\mu$, in (3g), as

$$\langle p'\beta|V_c{}^\mu|p\alpha\rangle=+i\epsilon_{\beta c\alpha}[(2\pi)^3(4p_0p_0')^{1/2}]^{-1}\times(p+p')^\mu g(t). \tag{8}$$

Since $V_c{}^\mu$ is the usual electromagnetic current operator, the CVC (conservation of vector current) hypothesis and normalization of the pion charge to unity give

$$g(t=0)=1. \tag{9}$$

In general, combining Eqs. (8), (3f), and (1c), we have, for $\mathbf{p}=\mathbf{k}=0$,

$$W^1=2m(2\pi)^{-3}(\delta_{\alpha b}\delta_{a\beta}-\delta_{\alpha a}\delta_{b\beta})g(0). \tag{10}$$

[22] In some hard-pion treatments we note that the axial-vector current also has a part which does not have singularities on the mass shell. See, for example, the work on the current algebra and Ward identities by I. S. Gerstein and H. J. Schnitzer, Phys. Rev. **170**, 1638 (1968), and particularly the material after Eq. (11c).

[23] Generally in Eq. (32) there is also an isospin-symmetric tensor $\phi^a(0)\phi^b(z)\delta^4(z)$. Because of the symmetry, however, one can show that the matrix elements of this term (at least between the pion states) vanish unless $a=b$. Thus, effectively, we have included this term in Σ.

C. Term W

To evaluate the terms W, we use Eq. (1d) in the following form:

$$W=W_1+W_2, \tag{11a}$$

$$W_1=-(2\pi)^3\sum_n \langle p'\beta|k_\mu' J_b{}^\mu(0)\frac{|p_n\rangle\langle p_n|}{p_{n0}-p_0-k_0}k_\nu J_a{}^\nu(0)|p,\alpha\rangle \times\delta^{(3)}(\mathbf{p}_n-\mathbf{p}-\mathbf{k}), \tag{11b}$$

$$W_2=-(2\pi)^3\sum_n \langle \beta p'|k_\nu J_a{}^\nu(0)\frac{|p_n\rangle\langle p_n|}{p_{n0}-p_0'+k}k_\mu' J_b{}^\mu(0)|p\alpha\rangle \times\delta^{(3)}(\mathbf{p}_n-\mathbf{p}'+\mathbf{k}), \tag{11c}$$

where we have introduced a sum over a complete set of intermediate states. For the summation over spin n, we introduce the projection operators

$$O_0{}^{\alpha\beta}=m_n{}^{-2}p_n{}^\alpha p_n{}^\beta, \tag{12a}$$

$$O_1{}^{\alpha\beta}=g_{\alpha\beta}-m_n{}^{-2}p_n{}^\alpha p_n{}^\beta, \tag{12b}$$

$$O_2{}^{\alpha\beta,\sigma\rho}=\tfrac{1}{2}O_1{}^{\alpha\sigma}O_1{}^{\beta\rho}+\tfrac{1}{2}O_1{}^{\alpha\rho}O_1{}^{\beta\sigma}-\tfrac{1}{3}O_1{}^{\alpha\beta}O_1{}^{\sigma\rho}, \tag{12c}$$

where m_n is the mass of the particle $|n\rangle$, and O_0, O_1, and O_2 are the operators for spin $n=0$, 1, and 2, respectively. For the higher spin states the operators $O_n{}^{\alpha\beta,\sigma\rho,\cdots}$ can be expressed in terms of the spin-1 projection operators.

We now note that because of the δ^3 functions in Eq. (11), the states with spin higher than zero do not contribute to the threshold value of the amplitude W. To see this we consider the matrix eelment of the current J, viz.,

$$\langle p|J^\mu(0)|p_n,n\rangle=(f_1A^\mu A^\alpha+f_2A^\mu B^\alpha+f_3B^\mu A^\alpha +f_4B^\mu B^\alpha+f_5g_{\mu\alpha})\epsilon_\alpha(n) \tag{13a}$$

for spin-0 or spin-1 states,

$$\langle p|J^\mu(0)|p_n,n\rangle =(f_1A^\mu A^\alpha p^\beta+f_2A^\mu B^\alpha p^\beta+\cdots)\epsilon_{\alpha\beta}(n) \tag{13b}$$

for spin 2, and so on. Here the f's are form factors, $A^\mu=(p+p_n)^\mu$, $B^\mu=(p-p_n)^\mu$, and the ϵ's are the polarization vectors satisfying

$$\sum_{\text{spin}}\epsilon^\alpha\epsilon^\beta=O_{n=0,1}{}^{\alpha\beta},$$

$$\sum_{\text{spin}}\epsilon^{\alpha\beta}\epsilon^{\sigma\rho}=O_{n=2}{}^{\alpha\beta,\sigma\rho}, \tag{13c}$$

and similar expressions for states with spin $n>2$. Using expressions such as (13), and projection operators (12) in (11), we encounter terms such as $A_\alpha O_1{}^{\alpha\beta}$, $A_\alpha O_1{}^{\alpha\beta}B_n$, $B_\alpha O_1{}^{\alpha\beta}A_\beta$, etc., multiplied by $\delta^3(\mathbf{p}_n-\mathbf{p}\pm\mathbf{k})$. The δ^3 function and the fact that we are at threshold guarantee that the momenta have only time components; excepting spin 0, all the projection operators will have only pure space components. Hence only spin-0 intermediate states contribute at threshold.

Considering this point, Eqs. (11) and (A5) of the Appendix yield

$$W=-\frac{c^2}{24\pi^2}\left(\frac{g_{\epsilon\pi\pi}{}^2}{4\pi}\right)\frac{1}{mm_\epsilon}\left(\frac{m}{m_\epsilon-m}\right)^2 \times[(m_\epsilon-2m)^{-1}\delta_{a\alpha}\delta_{b\beta}+m_\epsilon{}^{-1}\delta_{a\beta}\delta_{\alpha b}], \tag{14}$$

where m_ϵ and $g_{\epsilon\pi\pi}$ are the mass and coupling constant of the ϵ resonance, the only $I=0$ single-particle intermediate state known in the $\pi\pi$ interaction.

D. W^{00} Term

For the evaluation of this term, we may write Eq. (1e) as

$$W^{00}=-(2\pi)^3\sum_n B_n\left[\frac{(p_{n0}-p_0)}{t_n-m^2}\delta_{\alpha a}\delta_{\beta b}\delta^{(3)}(\mathbf{p}_n-\mathbf{p}-\mathbf{k}) +\frac{(p_{n0}-p_0')}{t_n'-m^2}\delta_{\alpha b}\delta_{\beta a}\delta^{(3)}(\mathbf{p}_n-\mathbf{p}+\mathbf{k})\right], \tag{15a}$$

with $t_n=(p_n-p)^2$, $t_n'=(p_n-p')^2$, and

$$B_n=(p'-p_n)_\mu\langle p'\beta|J^\mu(0)|p_nn\rangle \times\langle np_n|J^\nu(0)|p\alpha\rangle(p-p_n)_\nu. \tag{15b}$$

Here again, because of the $\delta^{(3)}$ functions in (16a), the argument given for the term W, Eq. (11), shows that the intermediate states with spin higher than zero do not contribute to the threshold value of W^{00}. Hence, the only single-particle state in Eq. (16a) which contributes at $\mathbf{k}=\mathbf{p}=0$ is the ϵ resonance. Considering this and combining Eqs. (16), (13a), and Eqs. (A1)–(A3) and (A6) of the Appendix, we find

$$W^{00}=-\frac{c^2}{24\pi^2}\left(\frac{g_{\epsilon\pi\pi}{}^2}{4\pi}\right)\frac{1}{mm_\epsilon{}^2}\frac{m_\epsilon-m}{m_\epsilon-2m} \times[\delta_{\alpha b}\delta_{a\beta}+\delta_{\alpha a}\delta_{b\beta}]. \tag{16}$$

III. SCATTERING LENGTHS

A. Amplitudes at Threshold

The well-known expression of the $(\pi\pi)$ scattering amplitude is given as

$$T_{\beta b,a\alpha}=A\delta_{\alpha a}\delta_{\beta b}+B\delta_{\alpha\beta}\delta_{ab}+C\delta_{\alpha b}\delta_{a\beta}, \tag{17a}$$

with A, B, and C being functions of the usual independent variables s, u, and t.[24] These coefficients are related to the amplitudes A_0, A_1, and A_2 which are the amplitudes corresponding to isospins 0, 1, and 2, respectively. That is,

$$A_0=3A+B+C,\quad A_1=B-C,\quad A_2=B+C. \tag{17b}$$

[24] G. V. Chew and S. Mandelstam, Phys. Rev. 119, 497 (1960).

We may calculate these amplitudes by evaluating the expression (16a) from Eq. (1a) and our treatment in Sec. II. Thus, inserting Eqs. (16a), (14), (10), and (7) into Eq. (1a) and omitting, conveniently, the $\delta^{(4)}$ function in the latter equation, we find

$$T_{\beta b,\alpha a}=iL\left\{\left[G_\epsilon\frac{\gamma^3-3\gamma^2+4\gamma-1}{\gamma^2(\gamma-1)^2(\gamma-2)}+\frac{g(0)}{\pi}\right]\delta_{\alpha a}\delta_{\beta b}\right.$$
$$\left.-F\delta_{\alpha\beta}\delta_{ab}+\left[G_\epsilon\frac{\gamma^3-3\gamma^2+4\gamma-3}{\gamma^2(\gamma-1)^2(\gamma-2)}-\frac{g(0)}{\pi}\right]\delta_{\alpha b}\delta_{a\beta}\right\}. \tag{18a}$$

Here $\gamma=m_\epsilon/m$,

$$F(0)=4\pi m^{-2}f(t=0)\,, \tag{18b}$$

$$G_\epsilon=(2.76\pi)^{-1}m^{-2}(g_{\epsilon\pi\pi}{}^2/4\pi)\,. \tag{18c}$$

Also L is a convenient length used in Ref. 3, as

$$L=g_r{}^2m/8\pi g_A{}^2M^2\simeq0.115m^{-1}\,, \tag{18d}$$

where g_r is the pion coupling constant, $(g_r{}^2/4\pi)=14.6$, g_A is the axial-vector coupling constant, $g_A=-1.23$, and M is the nucleon mass. In writing our equations we have taken the Goldberger-Treiman coefficient as

$$c\simeq-g_AM/g_r\,.$$

Combining Eqs. (16)–(18), we find for $\mathbf{k}=\mathbf{p}=0$

$$A_2=-iL\left[F(0)+\frac{g(0)}{\pi}-G_\epsilon\frac{\gamma^3-3\gamma^2+4\gamma-3}{\gamma^2(\gamma-1)^2(\gamma-2)}\right], \tag{19a}$$

$$A_0=i3L\left[\frac{g(0)}{\pi}+G_\epsilon\frac{\gamma^3-\gamma^2+4\gamma-1}{\gamma^2(\gamma-1)^2(\gamma-2)}\right]+A_2\,. \tag{19b}$$

In order to obtain the scattering lengths from these results, we shall make use of the transformation properties of the W terms in Eq. (1a), or those of the coefficients A, B, and C in (16a), under the interchanges of the pions. We consider the interchange $(k',b)\rightleftharpoons(p',\beta)$ which corresponds to the amplitude

$$T_{b\beta,\alpha a}=A'\delta_{\alpha a}\delta_{b\beta}+B'\delta_{\alpha b}\delta_{a\beta}+C'\delta_{\alpha\beta}\delta_{ab}\,. \tag{20}$$

Since the amplitude (1a) is invariant under different reductions of the pion states, it follows that for a given set of invariants s, t, and u, the two amplitudes (19) and (16a) are equal, and the crossing symmetry gives

$$A'=A\,,\quad B'=C\,,\quad C'=B\,. \tag{21}$$

Thus A_0 and A_2, Eqs. (19), are invariant under the above transformation.

On the other hand, we may calculate the amplitude $T_{b\beta,\alpha a}$ by interchanging (k',b) and (p',β) in our master amplitude (1a). If we do this, we find the same amplitude (17a) in which the last two coefficients are interchanged. From this result and Eqs. (19) and (20), we find $B=C$ in (16a), or

$$F(0)=\frac{g(0)}{\pi}-G_\epsilon\frac{\gamma^3-3\gamma^2+4\gamma-3}{\gamma^2(\gamma-1)^2(\gamma-2)} \tag{22}$$

in (17a), which is valid for the threshold $\mathbf{k}=\mathbf{p}=0$. Note that inserting (21) in (17a) gives $A_1=0$ at $\mathbf{k}=\mathbf{p}=0$, in agreement with the Bose statistic.

B. S-Wave Scattering Lengths

The scattering amplitudes corresponding to isospins 0 and 2 are obtained by writing $(-i\pi/2)$ times the scattering amplitudes A_0 and A_2 given by Eqs. (18). Hence using the relations (21), and taking the mass of the ϵ resonance as $m_\epsilon=720$ MeV,[19] we find

$$F(0)=\frac{g(0)}{\pi}-0.006m^{-2}g_{\epsilon\pi\pi}{}^2/4\pi\,, \tag{23a}$$

$$a_2=-L[g(0)-19.8\times10^{-3}m^{-2}g_{\epsilon\pi\pi}{}^2/4\pi]\,, \tag{23b}$$

$$a_0=\tfrac{3}{2}L[g(0)+19.1\times10^{-3}m^{-2}g_{\epsilon\pi\pi}{}^2/4\pi]\,. \tag{23c}$$

The scattering lengths (22) are expressed in terms of the ϵ-pion coupling constant given by Eq. (A3) of the Appendix. Experimentally, the width of the ϵ resonance is not well established and is estimated to be between 150 and 400 MeV, more likely 200 MeV.[25] Hence using $m_\epsilon=720$ MeV and $\Gamma_\epsilon=200$ MeV in Eq. (A3),

$$m^{-2}(g_{\epsilon\pi\pi}{}^2/4\pi)=32\,. \tag{24}$$

Using this and Eq. (9b) in Eqs. (22), we find the on-mass-shell values of the scattering lengths:

$$a_2=-0.044m^{-1}\,, \tag{25a}$$

$$a_0=0.278m^{-2}\,. \tag{25b}$$

Note that if the ϵ resonance width is $\Gamma_\epsilon=400$ MeV, then Eqs. (22) give

$$a_2=0.03m^{-1}\,, \tag{26a}$$

$$a_0=0.383m^{-1}\,. \tag{26b}$$

IV. COMMENTS AND CONCLUSION

The essential points in this on-mass-shell treatment of $\pi\pi$ scattering may be summarized as follows: We have calculated the scattering lengths, using the LSZ reduction formalism, the mathematical expression of the PCAC, and crossing-symmetry relations.

Our master amplitude (1a) decomposes into four terms, W^{00}, W^0, W^1, and W. For the evaluation of the scalar and vector terms, W^0 and W^1, we have the current algebra of the current J, which is shown[16] to be similar to the usual current algebra.[17] For the evaluation of

[25] See the survey article by R. Arnowitt, in Proceedings of Argonne Conference on $\pi\pi$ and $K\pi$ Interactions, 1969, p. 619 (unpublished).

the terms W^{00} and W, which do not appear in the soft-pion treatment, we have introduced a set of one-particle intermediate states. Among these intermediate states, only spin-0 states contribute to the threshold values of W^{00} and W; so we have assumed that only the s-wave ϵ resonance contributes appreciably.

The results may be summarized as follows: (a) Both the scattering length a_0 and a_2 are expressed in terms of the form factor $g(0)$, given by Eqs. (9), and the ϵ-pion coupling constant $g_{\epsilon\pi\pi}$. (b) The values $a_2=-0.044m^{-1}$ and $a_0=0.278m^{-1}$ are based on Eq. (9), which gives $g(0)=1$, the Goldberger-Treiman coefficient in which $g_A=-1.23$, and the ϵ mass and width, $m_\epsilon=720$ MeV and $\Gamma_\epsilon=200$ MeV. We see from Eqs. (22) that a_0 is always positive, while the sign of a_2 depends on the ϵ mass and width. For $\Gamma_\epsilon=400$ MeV, we have $a_2=0.024$ m^{-1} and $a_0=0.383m^{-1}$. (c) We have Eq. (21), the relation between the form factors $F(0)$ and $g(0)$ which belong to the scalar and vector terms W^0 and W^1, and the ϵ-pion coupling constant $g_{\epsilon\pi\pi}$. (d) Our results lie between $a_0\simeq(0.25\text{–}0.38)m^{-1}$ and $a_2=-(0.00\text{–}0.064)m^{-1}$, which are the ranges deduced from the work in Refs. 5, 6, and 26, where the calculation is made using the unitarized Veneziano formula. Our results, particularly those given by Eqs. (25), are small, in agreement with the prediction of Weinberg[3]; and those given by (25) are not substantially different from $a_0\cong0.23m^{-1}$ and $a_2\cong-0.06m^{-1}$ of the soft-pion treatment in Ref. 4.

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APPENDIX

Here we evaluate the amplitude W given by Eqs. (11), at the threshold $\mathbf{p}=\mathbf{k}=0$, for the single intermediate states $|n\rangle$ having isospin $I=0$. Also we shall give some relations which will be used in the evaulation of the term W^{00} given by Eq. (16a).

From the PCAC relation (2c) we find,

$$(p_n-p)_\nu\langle n,p_n|J_a{}^\nu(0)|p,\alpha\rangle =ic(m^2-t_n)\langle n,p_n|\phi_a(0)|p,\alpha\rangle\,, \quad \text{(A1)}$$

where $t_n=(m_n-m)^2$, m_n is the mass of the intermediate state $|n\rangle$, and a and α are the isospin indices of the pion. We may evaluate the matrix element of the pion interacting field ϕ by considering the decay of a state $|n\rangle$ into two pions, i.e.,

$$(n,p_n)\rightarrow(a,k)+(\beta,p)\,.$$

By writing the S matrix of this decay in terms of an effective Hamiltonian, and by the reduction formalism,[12] we find for an isoscalar state $|n\rangle$:

$$\langle n,p_n|\phi_a(0)|p,\alpha\rangle=\frac{-ig_{n\pi\pi}}{(2\pi)^3(4p_0p_{n0})^{1/2}}\frac{\delta_{a\alpha}}{m^2-t_n}\,, \quad \text{(A2)}$$

where m is the pion mass and

$$\frac{g_{n\pi\pi}{}^2}{4\pi}=\frac{4m_n{}^2}{(m_n{}^2-4m^2)^{1/2}}\Gamma_n \quad \text{(A3)}$$

is the decay coupling constant corresponding to the width Γ_n of an isoscalar state $|n\rangle$. Combining Eqs. (A1) and (A2) and letting $\mathbf{k}=\mathbf{p}=0$, we find

$$k_\mu\langle n,p_n|J_a{}^\mu(0)|p,\alpha\rangle =\frac{cg_{n\pi\pi}}{(2\pi)^3(4mm_n)^{1/2}}\left(\frac{k_0}{m_n-m}\right)\delta_{a\alpha}\,. \quad \text{(A4)}$$

Note that in the soft-pion case $k_0=0$, so (A4) vanishes. Finally, using this we can show that

$$\begin{bmatrix}W_1\\W_2\end{bmatrix}=\frac{-c^2}{24\pi^2}\sum_n\left(\frac{g_{n\pi\pi}{}^2}{4\pi}\right)\left(\frac{1}{mm_\epsilon}\right)\left(\frac{m}{m_n-m}\right)^2 \times\begin{bmatrix}(m_n-2m)^{-1}\delta_{\alpha a}\delta_{b\beta}\\ m_n{}^{-1}\delta_{\alpha b}\delta_{a\beta}\end{bmatrix}. \quad \text{(A5)}$$

In writing Eq. (A5), we have expressed the intermediate states $|n\rangle$ in Eqs. (11) as

$$|n\rangle=\sum_{I_n=0}^{2}C(I_\alpha,I_a;I_n)|I_n\rangle\,. \quad \text{(A6)}$$

Here I_α and I_a are the isospins of the two colliding particles, and I_n the isospin of the state $|I_n\rangle$. Also $C(I_\alpha,I_a;I_n)$ is the Clebsch-Gordan coefficient. Note that at $k=p=0$ the only known contributing single particle in Eq. (11) is ϵ with $I_\epsilon=0$ (see Sec. II C). Thus the sum in (A6) reduces to one term, namely, $|\epsilon\rangle\equiv|I=0\rangle$.

[26] C. Lovelace, in Ref. 25, p. 562.