Possible Binding of a Magnetic Monopole to a Particle with Electric Charge and a Magnetic Dipole Moment

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The author argues that Dirac monopoles, if they exist, could be strongly bound to those naturally occurring free nuclei with magnetic dipole moments, and discusses the effect this binding would have on the interpretation of experimental monopole searches.

I. INTRODUCTION

THIS subject was first discussed by Malkus,[1] who concluded that there cannot be bound states between Dirac monopoles and naturally occurring free nuclei. I disagree with Malkus's interpretation of his results, and believe that bound states of this type with binding energies up to the GeV range are a definite possibility if monopoles exist at all.

Because they involve assumptions about the properties of an unobserved particle, the arguments in this paper, based on simple nonrelativistic calculations, are not conclusive. We cannot be sure that we are not neglecting certain crucial features of the problem. What these calculations do indicate is that the possibility of magnetically charged nuclei must be considered in the design of magnetic monopole searches.[2]

The plan of this paper is as follows. In Sec. II we discuss the formulation of a nonrelativistic, quantum-mechanical Hamiltonian for the problem. In Sec. III we obtain the eigenvalues for the angular operator for a spin-0 monopole and a spin-$\frac{1}{2}$ nucleus. In Sec. IV we examine the radial equation in two separate cases in which binding is possible. The first case occurs when the monopole has an electric charge and the problem is similar to that of the hydrogen atom. In the second, the monopole is electrically neutral and there is a hard-core repulsion at small radius. In each case, we calculate typical values for the binding energy. Finally, in Sec. V we briefly discuss the validity of the calculations and the effect the results would have on the conclusions of various types of experimental monopole searches.

II. FORMULATION OF PROBLEM AND CHARGE QUANTIZATION

We work in units with $\hbar = c = 1$ and, since we are primarily interested in binding to nuclei, choose the basic unit of energy by taking the mass of the nucleon, $M_N = 1$. In these units the nuclear magneton is given by $\frac{1}{2}e$. The properties of the charged particle and the monopole are summarized in Table I. Note the possibility that the monopole carries electric charge, as emphasized by Schwinger.[3] We do not necessarily adopt Schwinger's suggestion that the monopole be given fractional electric charge and identified with the quark.

The magnetic field of a monopole of strength νg located at $\mathbf{r} = 0$ is given by

$$\mathbf{B} = (\nu g / r^2)\hat{\imath}_r. \qquad (2.1)$$

In order to construct a nonrelativistic, quantum-mechanical Hamiltonian, we need to construct a suitable vector potential.[4] This is awkward when $\nabla \cdot \mathbf{B} \neq 0$, as in (2.1), since we cannot define $\mathbf{A}$ by

$$\mathbf{B}_0 = \nabla \times \mathbf{A} \qquad (2.2)$$

together with the gauge condition

$$\nabla \cdot \mathbf{A} = 0. \qquad (2.3)$$

Dirac's solution to this problem was to find an expression for $\mathbf{A}$ which satisfies (2.3), such that the $\mathbf{B}$ field given by (2.2) agrees with (2.1) except on certain singularity lines. Two possibilities for the vector potential are

$$A_r^{(1)} = A_\theta^{(1)} = 0, \quad A_\phi^{(1)} = \nu g (1 - \cos\theta)/\sin\theta, \quad (2.4)$$

$$A_r^{(2)} = A_\theta^{(2)} = 0, \quad A_\phi^{(2)} = -\nu g \cos\theta/\sin\theta, \quad (2.5)$$

where $\mathbf{A}^{(1)}$ is singular along the negative z axis ($r = -z$,

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[1] W. V. R. Malkus, Phys. Rev. **83**, 899 (1951). For related discussions, see also P. P. Bandaret, Helv. Phys. Acta **19**, 503 (1946); C. J. Eliezer and S. K. Roy, Proc. Cambridge Phil. Soc. **58**, 401 (1961).

[2] A complete review of monopole searches can be found in E. Amaldi, in *Old and New Problems in Elementary Particles*, edited by G. Puppi (Academic, New York, 1968), p. 20. More recent experiments that rely on magnetic fields to extract monopoles are R. L. Fleischer, H. R. Hart, I. S. Jacobs, P. B. Price, W. M. Schwarz, and F. Aumento, Phys. Rev. **184**, 1393 (1969); R. L. Fleischer, P. B. Price, and R. T. Woods, *ibid.* **184**, 1398

(1969). An example of an experiment that does not depend on extracting monopoles is L. W. Alvarez, P. H. Eberhard, R. R. Ross, and R. D. Watt, Science **167**, 701 (1970).

[3] J. Schwinger, Phys. Rev. **144**, 1087 (1966); **173**, 1536 (1968). Giving the magnetic monopole an electric charge is permitted provided one uses a vector potential with a two-sided singularity line, and the appropriate quantization. We do not consider the extra factor of 2 that Schwinger obtains by considering surfaces which intersect a gauge line on their boundary.

[4] P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948). The use of the vector potential in quantum mechanics is obligatory because of the necessity of obtaining a gauge-independent translation operator—J. Zak, *ibid.* **134**, A1602 (1964). It is easily shown that the correct generator of the translation group in the presence of a magnetic field is $\mathbf{P} = -i\nabla - e\mathbf{A} - \frac{1}{2}e\mathbf{r} \times \mathbf{B}$. See the discussion of B. Zumino, in *1966 International School of Physics "Ettore Majorana,"* edited by Z. Zichichi (Academic, New York, 1966), about the interpretation of the singularity lines and alternative forms for the vector potential.

TABLE I. Assumed properties of the magnetic monopole and the nucleus (Refs. 3 and 5). The unit of magnetic charge is chosen to be $g=1/2e$, from (2.16). The mass of the monopole is, of course, undetermined. In making sample calculations we have taken $M_Y=10$, 100, but the author has found no theoretical arguments to indicate that this is the right order of magnitude.

	Magnetic monopole	Nucleus
Magnetic charge	$\nu g, \nu=0, \pm1, \pm2, \ldots$	0
Electric charge	$Ye, Y=0, \pm1, \ldots$	$Ze, Z=0, 1, 2, \ldots$
Mass	M_Y	M_Z
Spin	0	$\Sigma=0, \frac{1}{2}, 1, \ldots$
Magnetic dipole mom.	0	$\frac{1}{2}\mu e$
Electric dipole mom.	0	0

$\theta=\pi$) and $\mathbf{A}^{(2)}$ is singular along the entire z axis. The correspondence between (2.1) and (2.2) is then given by

$$(\nu g/r^2)\hat{\imath}_r = \nabla \times \mathbf{A}^{(1,2)} - \mathbf{B}_f^{(1,2)}, \qquad (2.6)$$

where

$$\mathbf{B}_f^{(1)} = \hat{\imath}_z \begin{cases} 4\pi\nu g\delta(x)\delta(y), & z<0 \quad (\theta=\pi) \\ 0, & z>0 \quad (\theta=0) \end{cases} \qquad (2.7)$$

$$\mathbf{B}_f^{(2)} = \hat{\imath}_z \begin{cases} 2\pi\nu g\delta(x)\delta(y), & z<0 \quad (\theta=\pi) \\ -2\pi\nu g\delta(x)\delta(y), & z>0 \quad (\theta=0). \end{cases} \qquad (2.8)$$

It is, of course, also possible to take alternative orientations for the singularity line or to find expressions for the vector potential involving curved singularity lines. In order to have rotation invariance in the theory, all physical observables must be independent of the singularity line.

Within the approximations of nonrelativistic quantum mechanics, it is acceptable to use either vector potential, (2.4) or (2.5), directly in the Hamiltonian, although we must subtract the appropriate fictitious field, (2.7) or (2.8), if we intend to calculate the correct stress-energy tensor.[5] We therefore have the Hamiltonian[6]

$$H^{(1)} = -\frac{1}{2Tr^2}\left\{\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\right.$$

$$+\frac{1}{\sin^2\theta}\left[\frac{\partial}{\partial\phi} - iZ\nu eg(1-\cos\theta)\right]^2$$

$$\left.+\nu\mu egT\boldsymbol{\Sigma}\cdot\hat{\imath}_r\right\} + \frac{YZe^2}{r} + U(r), \qquad (2.9)$$

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[5] E. Amaldi, Ref. 2, points out that in order to have the correct form for the stress-energy tensor we must subtract the fictitious field, (2.7) or (2.8). The stress-energy tensor is invariant under a rotation in "charge space" of the form

$$\begin{pmatrix} g' \\ e' \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix}\begin{pmatrix} g \\ e \end{pmatrix}.$$

If a suitable rotation is chosen, our calculation can be valid for a spin-0 nucleus and a magnetic monopole with an electric dipole moment.

[6] The Hamiltonian is in the Pauli approximation, neglecting the spin-orbit interaction. Relativistic corrections are assumed to be approximated by the undetermined potential $U(r)$.

where $T=M_YM_Z/(M_Y+M_Z)$, $\boldsymbol{\varepsilon}$ is the Pauli spin operator, and $U(r)$ is an undetermined potential which is assumed to be appreciable only at small distances. The necessity for including this undetermined potential is discussed later, but at this point it reflects our ignorance of hadronic and form-factor effects at small distances.

Schrödinger's equation is separable:

$$H^{(1)}R^{(1)}(r)\psi^{(1)}(\theta,\phi) = ER^{(1)}(r)\psi^{(1)}(\theta,\phi) \qquad (2.10)$$

becomes

$$\left\{\frac{1}{\sin^2\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \left(\frac{\partial}{\partial\phi} - iZ\nu eg(1-\cos\theta)\right)^2\right]\right.$$

$$\left.+\nu\mu egT\boldsymbol{\Sigma}\cdot\hat{\imath}_r\right\}\psi^{(1)}(\theta,\phi) = -\beta^{(1)}\psi^{(1)}(\theta,\phi) \qquad (2.11)$$

and

$$\left[-\frac{1}{r^2}\left(\frac{\partial}{\partial r}\right)r^2\left(\frac{\partial}{\partial r}\right) + \frac{\beta^{(1)}}{r^2} + 2TU(r)\right]R^{(1)}(r)$$

$$= 2TER^{(1)}(r). \qquad (2.12)$$

There are completely analogous expressions for the Hamiltonian obtained by inserting the vector potential (2.5) instead of (2.4). As yet, we have no way of knowing that the results depend on whether we choose a vector potential with a one-sided or a two-sided singularity line.

To investigate this point, we look at the solution of the angular eigenvalue equation for $\mu=0$,

$$\Lambda^{(1)}\psi_{lm}^{(1)}(\theta)e^{im\phi} = -\alpha^{(1)}(l,m,Z\nu)\psi_{lm}^{(1)}(\theta)e^{im\phi}, \qquad (2.13)$$

with

$$\Lambda^{(1)} = \frac{1}{\sin^2\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \left[\frac{\partial}{\partial\phi}\left(\frac{1+\cos\theta}{2}\right)\right.\right.$$

$$\left.\left.+\left(\frac{\partial}{\partial\phi} - 2Z\nu eg\right)\left(\frac{1-\cos\theta}{2}\right)\right]^2\right\}. \qquad (2.14)$$

Equation (2.13) has the solution[7]

$$\psi_{lm}^{(1)}(\theta) = d_{m-Z\nu eg, -Z\nu eg}^{l+s}(\theta) \quad (l=0, 1, 2, \ldots), \qquad (2.15)$$

where $s=\max(|m-Z\nu eg|, |Z\nu eg|)$ and the $d_{mn}^J(\theta)$ are the familiar representations of the rotation group.[8] In order to have rotation invariance, we must therefore

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[7] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966), p. 209.

[8] M. E. Rose, *Elementary Theory of Angular Momentum* (Wiley, New York, 1957), p. 48.

have the quantization condition[9]

$$Z\nu eg = \tfrac{1}{2}n \quad (n=1, 2, \ldots). \qquad (2.16)$$

Proceeding in the same way for the vector potential with the two-sided singularity line, we find the angular eigenfunctions

$$\psi_{lm}^{(2)} = d_{m,-Z\nu eg}{}^{l+s}(\theta), \qquad (2.17)$$

and requiring rotation invariance in this instance gives the stronger quantization condition,[9]

$$Z\nu eg = n \quad (n=1, 2, \ldots). \qquad (2.18)$$

The familiar quantization conditions (2.16) and (2.18) have been obtained in many independent ways.[10] Here they are necessary in order that the angular eigenfunctions of the quantum-mechanical Hamiltonian correspond to irreducible representations of the rotation group. The eigenvalues corresponding to (2.15) and (2.17), are, respectively,

$$\alpha^{(1)}(l,m,Z\nu) = [l + \tfrac{1}{2}(|m| + |m - 2Z\nu eg|)]$$
$$\times [l + 1 + \tfrac{1}{2}(|m| + |m - 2Z\nu eg|)] - Z^2\nu^2 e^2 g^2 \qquad (2.19)$$

and

$$\alpha^{(2)}(l,m,Z\nu) = [l + \tfrac{1}{2}(|m + Z\nu eg| + |m - Z\nu eg|)]$$
$$\times [l + 1 + \tfrac{1}{2}(|m + Z\nu eg| + |m - Z\nu eg|)]$$
$$- Z^2\nu^2 e^2 g^2. \qquad (2.20)$$

From now on, we take

$$eg = \tfrac{1}{2}, \qquad (2.21)$$

and note that the stronger two-sided quantization condition, (2.18), is equivalent to requiring ν to be an even integer in (2.16). We then have

$$\alpha^{(2)}(l,m,Z\nu) = \alpha^{(1)}(l, m + \tfrac{1}{2}Z\nu, Z\nu),$$
$$\nu = 0, \pm 2, \pm 4, \ldots, \qquad (2.22)$$

and all results can be obtained by using the $\psi_{lm}^{(1)}(\theta)$ and (2.22). Thus, from examining Schrödinger's equation with $\mu = 0$, the only difference between a two-sided and a one-sided vector potential are the different quantization conditions, (2.16) and (2.18), and we cannot distinguish between the two possibilities unless we observe experimentally a magnetic monopole with $\nu = 1$. Note that the eigenvalues (2.19) and (2.20) are positive definite.

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[9] E. P. Wigner, *Gruppentheorie* (Freidrich Vieweg und Sohn, Braunschweig, 1931) has shown the $d_{mn}{}^J$ are the wave functions of a symmetric top. We can get a qualitative feel for why we have eigenfunctions of the form $d_{m,Z\nu/2}{}^J$ instead of $d_{m0}{}^J = Y_m{}^J$ by noting that applying a rotation to (2.10) also rotates the direction of the singularity line of the vector potential and we must apply the gauge transformation to restore the form, (2.4) or (2.5). The theory would therefore *not* be rotation invariant without gauge transformations, that is, without massless photons. The quantization condition (2.16) comes from requiring m and n in $d_{mn}{}^J$ to be half integral, while (2.18) comes from requiring $m - n$ to be integral.

[10] Perhaps the cleanest derivation of the quantization condition (2.16) is that of A. Goldhaber, Phys. Rev. 140, B1407 (1965), which does not depend on a singularity line.

III. ANGULAR OPERATOR WHEN $\mu \neq 0$

Now, consider the solution of the angular eigenvalue equation (2.11) for a spin-$\tfrac{1}{2}$ nucleus with $\mu \neq 0$. Let $\gamma = \tfrac{1}{2}\mu T$ and $x = \cos\theta$; then (2.11) becomes

$$\begin{pmatrix} \Lambda^{(1)} + \nu\gamma x & \nu\cdot\gamma(1-x)^{1/2}(1+x)^{1/2}e^{-i\phi} \\ \nu\gamma(1-x)^{1/2}(1+x)^{1/2}e^{i\phi} & \Lambda^{(1)} - \nu\gamma x \end{pmatrix}$$
$$\times \begin{pmatrix} \psi_A e^{i(m-1)\phi} \\ \psi_B e^{im\phi} \end{pmatrix} = -\beta \begin{pmatrix} \psi_A e^{i(m-1)\phi} \\ \psi_B e^{im\phi} \end{pmatrix}. \qquad (3.1)$$

The polynomials $\psi_{lm}^{(1)}(\theta)$, (2.15), are orthogonal, and this suggests we make the expansion

$$\psi_A = \sum_{l=0}^{N_A} c_l \psi_{l,m-1}^{(1)}(x), \qquad (3.2)$$

$$\psi_B = \sum_{j=0}^{N_B} d_j \psi_{jm}^{(1)}(x). \qquad (3.3)$$

We have to consider separately two cases of (3.1),

$$\text{Case 1:} \quad m \geqslant 1, \quad (m - Z\nu) \geqslant 1, \qquad (3.4a)$$

$$\text{Case 2:} \quad m \geqslant 1, \quad (m - Z\nu) \leqslant 0. \qquad (3.4b)$$

Other values of m can be reduced to one of these two cases by using the identity[8]

$$d_{m,n}{}^J(x) = d_{-m,-n}{}^J(x). \qquad (3.5)$$

In order to relate $\psi_{l,m}$ and $\psi_{l,m-1}$ it is convenient to express the $d_{mn}{}^J$ in terms of Jacobi polynomials,[7]

$$\psi_{lm}(x) = (1-x)^{m/2}(1+x)^{|m-Z\nu|/2} P_l^{(m,|m-Z\nu|)}(x). \qquad (3.6)$$

After the replacement (3.6), we see that Eq. (3.1) has a different form in each of the two cases (3.4).

Case 1. Absorbing the normalization of (3.6) into the coefficients of (3.2) and (3.3), we have the equations

$$\sum_{k=0}^{N_A} c_k[-\alpha(k, m-1, Z\nu) + \beta + \nu\gamma x]P_k^{(m-1,m-1-Z\nu)}(x)$$

$$+ \sum_{l=0}^{N_B} \nu\gamma d_l(1-x^2)P_l^{(m,m-Z\nu)}(x) = 0, \qquad (3.7a)$$

$$\sum_{k=0}^{N_A} c_k \nu\gamma P_k^{(m-1,m-1-Z\nu)}(x)$$

$$+ \sum_{l=0}^{N_B} d_l[-\alpha(k,m,Z\nu) + \beta - \nu\gamma x]$$

$$\times P_l^{(m,m-Z\nu)}(x) = 0. \qquad (3.7b)$$

If the series terminates, then $N_A = N_B + 1$. Expanding the Jacobi polynomials in power series and matching powers of x, we find the eigenvalues

$$\beta_1^{\pm}(N_B,m,Z\nu,\nu\gamma) = (N_B + m - \tfrac{1}{2}Z\nu)^2 - \tfrac{1}{4}Z^2\nu^2$$
$$\pm [(N_B + m - \tfrac{1}{2}Z\nu)^2 - \tfrac{1}{4}Z^2\nu^2 + (\tfrac{1}{2}Z\nu - \nu\gamma)^2]^{1/2}. \qquad (3.8)$$

These eigenvalues need not all be positive. The minimum occurs when $N_B=0$, $m=Z+1$, or $m=1$,

$$\min_{(N,m)} (\beta_1{}^-)=1+|Z\nu|-[1+|Z\nu|+(\tfrac{1}{2}Z\nu-\nu\gamma)^2]^{1/2}, \quad (3.9)$$

which can be negative when $\tfrac{1}{4}\nu^2(Z-\mu T)^2$ is large enough. Inserting (3.5) into (3.7), we get the eigenvalues for the case when $m\leqslant 0$ and $(m-Z\nu)\leqslant 0$,

$$\beta_3{}^\pm(N_B,m,Z\nu,\gamma\nu)=\beta_1{}^\pm(N_B, -m+1, -Z\nu, -\gamma\nu),$$
$$m=0, -1, -2, \ldots. \quad (3.10)$$

Case 2. In this instance, the eigenvalues (2.19) are independent of m and all equal to

$$\alpha^{(1)}(l,0,Z\nu)=l(l+1)+(l+\tfrac{1}{2})|Z\nu|. \quad (3.11)$$

Equation (3.1) becomes

$$\sum_{k=0}^{N_A} c_k[-\alpha(k,0,Z\nu)+\beta+\nu\gamma x]P_k{}^{(m-1,Z\nu-m+1)}(x)$$

$$+\sum_{j=0}^{N_B} d_j\nu\gamma(1-x)P_j{}^{(m,Z\nu-m)}(x), \quad (3.12a)$$

$$\sum_{k=0}^{N_A} \nu\gamma c_k(1+x)P_k{}^{(m-1,Z\nu-m+1)}(x)$$

$$+\sum_{j=0}^{N_B} d_j[-\alpha(j,0,Z\nu)+\beta-\nu\gamma x]P_j{}^{(m,Z\nu-m)}(x), \quad (3.12b)$$

where $N_A=N_B$. Using the identity[7]

$$(1-x)[P_N{}^{(b,a)}-P_N{}^{(a-1,b+1)}]$$

$$=\frac{a+N-1}{\tfrac{1}{2}(a+b)+N}P_{N-1}{}^{(a-1,b+1)}$$

$$-\frac{N}{\tfrac{1}{2}(a+b)+N}P_N{}^{(a-1,b+1)}, \quad (3.13)$$

we can combine the sums in (3.12) and obtain the eigenvalues[11]

$$\beta_2(N_B,Z\nu,\gamma\nu)=N_B(N_B+1)$$

$$+(N_B+\tfrac{1}{2})|Z\nu|-\gamma\nu\left(1-\frac{N_B}{N_B+\tfrac{1}{2}|Z\nu|}\right). \quad (3.14)$$

When $N_B=0$ this reduces to

$$\beta_2(0,Z\nu,\gamma\nu)=\tfrac{1}{2}Z\nu-\gamma\nu=\tfrac{1}{2}\nu(Z-\mu T). \quad (3.15)$$

Inserting (3.5) into (3.12), we get the eigenvalues for the case when $m\leqslant 0$, $m-Z\nu\geqslant 1$,

$$\beta_4(N_B,Z\nu,\gamma\nu)=\beta_2(N_B, -Z\nu, -\gamma\nu). \quad (3.16)$$

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[11] The eigenvalues (3.8) were obtained by Malkus in Ref. 1. The eigenvalues (3.14) were omitted except for the special case $N_B=0$, which gives the minimum value of β_2.
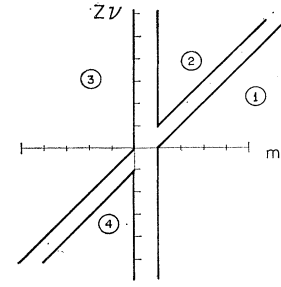
FIG. 1. Plot of $Z\nu$ versus m shows the regions where different expressions for the angular eigenvalues are obtained. In region 1, use Eq. (3.8); in 2, Eq. (3.14); in 3, Eq. (3.10); and in 4, Eq. (3.16).

Figure 1 shows a plot of $Z\nu$ versus m and indicates where the various expressions for the eigenvalues are valid. Tables II and III give a tabulation of $\beta_{\min}$ for naturally occurring nuclei with various assumptions about the magnetic monopole mass and charge. As can be seen, negative eigenvalues are in abundance and, as is discussed in Sec. IV, this opens the possibility of bound states between an electrically neutral monopole and nuclei if there is a hard-core repulsion. Numerical calculations of eigenvalues for spin-1 and spin-$\tfrac{3}{2}$ nuclei also indicate the possibility of negative values for the angular eigenvalue. The methods used for the spin-$\tfrac{1}{2}$ case can lead to cubic or quartic equations involving β for spin 1 and spin $\tfrac{3}{2}$, respectively, so no attempt was

TABLE II. Typical values of β_2 for spin-$\tfrac{1}{2}$ nuclei. The value of monopole mass which gives $\beta_2=-\tfrac{1}{4}$ is also shown.

Nucleus (% abundance)	Z	μ	Monopole mass ($M_p=1$)	ν	$\beta_2(0,Z\nu,\mu T)$
n	0	−1.91	0.35	−1	−0.25
			10	−1	−0.87
			100	−1	−0.95
			0.15	−2	−0.25
			10	−2	−1.74
			100	−2	−1.89
p, H	1	2.79	1.16	1	−0.25
			10	1	−0.77
			100	1	−0.88
			0.812	2	−0.25
			10	2	−1.54
			100	2	−1.77
C^{13} (1.11×10⁻²)	6	0.702	10	1	1.01
			32.2	1	−0.25
			100	1	−1.04
			10	2	2.03
			23.3	2	−0.25
			100	2	−2.09
F^{19} (100)	9	2.63	4.46	1	−0.25
			10	1	−4.12
			100	1	−16.6
			4.32	2	−0.25
			10	2	−8.24
			100	2	−33.1
P^{31} (100)	15	1.13	10	1	3.25
			24.6	1	−0.25
			100	1	−5.9
			10	2	6.50
			23.9	2	−0.25
			100	2	−11.8

TABLE III. Typical values of β_1 for spin-$\frac{1}{2}$ nuclei.

Nucleus (% abundance)	Z	μ	Monopole mass ($M_p=1$)	ν	$\beta_1{}^{\min}(N,m,Z\nu,\mu T)$
n	0	-1.91	10	-1	-0.32
			100	-1	-0.37
			10	-2	-1.01
			100	-2	-1.14
He^3 (1.3×10^{-6})	2	-2.12	10	1	-0.86
			100	1	-1.45
			10	2	-2.20
			100	2	-3.55
F^{19} (100)	9	2.63	10	1	4.81
			100	1	-6.80
			10	2	9.70
			100	2	-14.32

made to obtain the analytic form for the eigenvalues in these cases.

As can be seen in (3.8) and (3.14), the angular eigenvalues depend on the combination μT, where μ is the strength of the dipole moment in nuclear magnetons and T is the reduced mass. The value of β therefore depends on the monopole mass. In Table I we have therefore included the monopole mass that gives $\beta=-\frac{1}{4}$. The significance of this value is apparent in Sec. IV. Malkus[1] did not calculate β_2 for the heavy nuclei or for values of the monopole mass larger than the "canonical" mass

$$M_Y = (g/e)^2 M_e \cong 2.56 .\qquad (3.17)$$

He therefore neglected all the negative eigenvalues in Table II except those of the proton and neutron. Current monopole searches[2] are probing for monopoles with masses much larger than (3.17).

IV. RADIAL EQUATION

Hydrogen-Atom Type of Solution

First consider the case when the Coulomb potential is attractive and the $1/r^2$ "potential" is repulsive. For this case the form of $U(r)$ is unimportant if it is appreciable only for small radii. We then have the familiar form of the radial equation for the hydrogen atom. Let $k^2=-2TE$, $TYZe^2/k=p$, and $\rho=2kr$; then (2.12) becomes

$$\frac{1}{\rho^2}\frac{\partial}{\partial\rho}\rho^2\frac{\partial}{\partial\rho}R-\left(\frac{1}{4}+\frac{\beta}{\rho^2}-\frac{p}{\rho}\right)R=0 .\qquad (4.1)$$

Equation (4.1) has the solution[12]

$$R(\rho)=e^{-\rho/2}\rho^{-(s+1)/2}\,{}_1F_1(-K;1-s;\rho),\qquad (4.2)$$

where

$$s=\pm(4\beta+1)^{1/2},\qquad (4.3)$$

$$+2p=2K-s+1=2K+(4\beta+1)^{1/2}+1 .\qquad (4.4)$$

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[12] W. Magnus et al., Ref. 7, p. 239.

The positive root of (4.3) gives a divergence at the origin, and in order for s to be real we must have $\beta\geqslant-\frac{1}{4}$. The confluent hypergeometric function, ${}_1F_1(-K;c;z)$, reduces to a polynomial when K is a positive integer, and this gives the quantization condition

$$E_K=-2T\frac{Y^2Z^2e^4}{[2K+(4\beta+1)^{1/2}+1]^2} .\qquad (4.5)$$

Typical values of these binding energies are given in Table IV. As can be seen there, these binding energies are quite large for a wide range of values for the mass and the strength of the monopole. We therefore justify a posteriori the neglect of the atomic electrons in the problem. Because of the factor e^{-kr} in (4.2), the wave function is small at the radius of a typical orbital electron. Including the effect of electrons in the calculation gives a perturbation of order M_e/M_Z. This type of binding can occur, of course, when $\mu=0$ so that the nucleus can have spin 0.

Since Schwinger has emphasized the possibility of a magnetic monopole's having electric charge,[3] this type of binding is potentially important.

Electrically Neutral Monopole with Repulsive Core

Let $Y=0$, so that we have a $1/r^2$ potential and there is an infinite repulsive core of radius r_0,

$$U(r)=\begin{cases}\infty, & 0<r<r_0\\0, & r_0<r<\infty .\end{cases}\qquad (4.6)$$

Equation (2.12) then reduces to Bessel's equation subject to the boundary conditions

$$R(r_0)=0,\qquad (4.7)$$

$$\lim_{x\to\infty}R(x)=0 .\qquad (4.8)$$

Let $k^2=-2TE$, then the most general negative-energy solution of (2.12) is given by

$$R(r)=r^{-1/2}[c_1I_p(kr)+c_2K_p(kr)],\qquad (4.9)$$

where I_p and K_p are Bessel functions of order

$$p=(\tfrac{1}{4}+\beta)^{1/2} .\qquad (4.10)$$

The asymptotic behavior of the Bessel functions is

TABLE IV. Typical values of the binding energy E_0 given by Eq. (4.5) for naturally occurring nuclei.

Nucleus	β	M_Y ($M_p=1$)	Binding energy (eV)
p, H	0.52	100	1.9×10^4
C^{12}	6	100	5.8×10^5
N^{14}	7	100	9.3×10^5
P^{31}	12.8	100	8.4×10^6

given by

$$K_p(x) \underset{x\to\infty}{\sim} (\pi/2x)^{1/2}e^{-x},$$ (4.11)

$$I_p(x) \underset{x\to\infty}{\sim} (1/2\pi x)^{1/2}e^{+x},$$ (4.12)

so that in order to satisfy the boundary condition, Eq. (4.8), we must have $c_1=0$. As for the boundary condition (4.7) for p and x real, $K_p(x)$ has no zeros, as can be seen by examining the integral representation

$$K_p(x) = \int_0^\infty \exp(-x\cosh t)\cosh(pt)dt.$$ (4.13)

Therefore, from (4.10), we cannot satisfy (4.7) when $\beta \geqslant -\frac{1}{4}$. When $\beta < -\frac{1}{4}$, p is purely imaginary and we have the possibility of zeros. The location of the zeros can be approximately determined by the asymptotic expansion[13]

$$K_{ip}(x) \sim \tfrac{1}{3}\pi e^{-\pi p/2}\left[\frac{2(p-x)}{x}\right]^{1/2}$$

$$\times [J_{1/3}+J_{-1/3}]\left(\frac{1}{2}\left[\frac{2(x-p)}{x^{1/2}}\right]^{3/2}\right).$$ (4.14)

The first zero on the right-hand side of (4.14) is at $p=x$, and this gives an approximate value for the binding energy

$$\frac{\frac{1}{4}+\beta}{2Tr_0{}^2} \cong E_0 \quad for \quad \beta < -\frac{1}{4}.$$ (4.15)

Typical values of this binding energy for different nuclei are given in Table V. The importance of the repulsive core is now evident. Without it, the wave function would "fall" to the origin and the energies (4.15) can become arbitrarily large in absolute value.[14] We emphasize that the use of a repulsive core is not necessarily unphysical, since hadronic effects can give rise to a repulsive Yukawa potential and since relativistic effects can give corrections of undetermined nature at small radii. Even if the infinite repulsive core is replaced by a finite potential, the wave function is required to be small at some typical nuclear radius r_0. This leads to binding energies the same order of magnitude as those given by (4.15), since the energy depends roughly on the smallest radius on which the wave function is allowed to fall. Table V gives values of the binding energy for typical values of nuclear radi.i

The use of the infinite repulsive core attempts to circumvent the limitations of Schrödinger's equation at

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[13] Equation (4.15) is an asymptotic representation of the "Nicholsen" type valid for x/p near unity and $x-p$ large, and therefore gives only a rough estimate for the location of the zeros. See W. Magnus *et al.*, Ref. 7, p. 142.

[14] L. D. Landau and E. M. Lifschitz, *Quantum Mechanics*, 2nd ed. (Addison-Wesley, Reading, Mass., 1965). See especially the discussion in Secs. 22 and 35.

TABLE V. Typical values of the binding energy, Eq. (4.15), for common nuclei under the assumption that the monopole wave function cannot penetrate the nucleus. The cutoff is therefore chosen to be the nuclear radius given by $r_0=bA^{1/3}$, where $b=31.5M_p{}^{-1}$.

Nucleus	M_Y $(M_p=1)$	β	Binding energy (eV)
n	100	-0.95	3.5×10^5
p, H	100	-0.88	3.2×10^5
C^{13}	100	-1.04	6.2×10^3
F^{19}	100	-16.6	7.3×10^4
P^{31}	100	-5.9	1.5×10^4

small radii. Once these limitations are recognized, the conclusion that bound states exist seems inescapable although we cannot achieve more than an order-of-magnitude estimate on the energy spectrum. This approach diverges from that of Malkus,[1] who did not take seriously the radial equation (2.12) with $\beta < -\frac{1}{4}$ since, without cutoff, it leads to infinite binding energies.

V. DISCUSSION AND CONCLUSIONS

The possibility that magnetic monopoles can be captured and bound by common nuclei must be considered in the evaluation of monopole searches. For a complete classification and review of the experimental situation the reader is referred to the paper by Amaldi.[2] Here we discuss only those experiments which search for monopoles trapped in geological materials that are potential collectors. Many of these experiments rely on extracting the monopoles from the host material by use of pulsed magnetic fields. If the monopole is loosely held with a binding energy of a few eV, the magnetic fields can easily extract it. But the binding energies proposed in this paper are orders of magnitude greater than those envisioned by Malkus,[1] and cannot be overcome by conveniently available magnetic fields.

If the bound states discussed here exist, one must be prepared to show that the external field used in a particular experiment can remove a magnetically charged nucleus-monopole bound state from the surrounding chemical structure. If the magnetic charge is measured by ionization, one must consider the possibility that a large electric charge is also present. There is also the possibility that the total mass of the monopole and nucleus is considerably larger than the original monopole mass.[15]

Another consideration is the abundance of potential collecting material. Many experiments have searched for monopoles in ferromagnetic material that is scarce compared with the quantity of material represented by the nuclei in Table II. A rough figure of merit for experimental searches of this type is given by the area of the

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[15] As long as experiments do not detect any monopole signal, this criticism is academic. The problem is that these experiments could not determine monopole properties from a positive result.

material exposed to cosmic rays multiplied by the exposure time. If common materials as well as ferromagnetic materials are potential monopole collectors, experiments involving a larger surface area of material may be considered.

It should be emphasized that those experiments which do not extract monopoles from potential collectors in order to detect them are cleaner than those which do, since their detection apparatus depends only on the classical, large-radius properties of monopoles.

Their results are therefore less susceptible of possible modification from the quantum-mechanical properties of the monopole.

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Factorization Test for the Pomeranchukon in Three-Particle Final States

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A factorization test for the Pomeranchukon is suggested for reactions with three particles in the final state. It is applied to the available experimental data on $\pi N \to \pi \pi N$ and $NN \to \pi NN$ and is found to be satisfied resaonably well.

I. INTRODUCTION

THE nature of the "Pomeranchukon" singularity in the angular momentum plane has been under investigation for the last few years. In the beginning it was believed that diffraction scattering is dominated by a Regge pole ("Pomeranchukon" P) in the t channel with the quantum numbers of the vacuum. However, there are difficulties with this and it now appears likely that diffraction scattering is dominated by a branch cut in the J plane.[1] There are several ways whereby a branch point and a pole can be experimentally distinguished. One, suggested by Gribov,[2] utilizes the fact that a pole has good parity but a branch point is unlikely to have good parity. Another uses the well-known fact that a simple pole must have factorizable residues, whereas for a branch cut the factorization does not hold.[3] It is this test that we consider here.

Freund[4] has shown that factorization holds for the production of the N^* resonance with isospin $\frac{1}{2}$ where "P" can be exchanged. The longitudinal phase-space analysis[5] shows that, whenever allowed, single

Pomeranchukon exchange dominates the three-body reactions at high energy. We deduce the explicit derivation of the factorization test in case of three-particle final states, which, of course, includes the production of resonances. The case discussed by Freund is a special case of our result. Here we do not have any difficulty about the "background" which one has to subtract from experimental results in order to find the resonances.

Our factorization test is expressed in terms of a double differential cross section (at fixed momentum transfer and invariant mass of the two-particle subsystem in the final state) of the inelastic reactions. In applying it to the reactions $\pi N \to \pi \pi N$ and $NN \to NN\pi$, we find that experimental data at the same value of momentum transfer for the two reactions are not available. We have, therefore, averaged them over a range of momentum transfer $0.01 < |t| < 0.2$ (BeV/c)^2. It is obvious that Pomeranchukon exchange in the two reactions discussed requires that the (πN) subsystem in the final state have $I = \frac{1}{2}$. The $I = \frac{1}{2}$ contributions in our data have been estimated by making use of the results of Boggild et al.[6] who have shown that at the 19-GeV/c proton-proton interaction there is a predominance of 70% of the (πN) system in the isospin state $I = \frac{1}{2}$ and similarly in $\pi^+ p$ interactions (by the longitudinal phase-space considerations) there is a dominance of

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