Field-Theoretical Nucleon-Nucleon Potential*

M. Hossein Partovi† and Earle L. Lomon
Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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A formulation is presented for the derivation of a Schrödinger-equation potential from a field-theoretical model. The relativistic two-body equation of Bethe and Salpeter is reduced using a generalization of the Blankenbecler-Sugar method. The resulting equation is shown to be identical with the nonrelativistic Lippmann-Schwinger equation upon a unitarity-preserving identification of the amplitudes. An equivalent potential is thereby defined and expressed as a solution of an integral equation. The second- and fourth-order potentials are calculated, and their energy dependence and nonlocality are studied. An approximation scheme is developed for expanding the configuration-space potentials in the powers of the momentum operator. Terms up to and including the first power are retained, giving rise to a potential composed of central, spin-orbit, tensor, and spin-spin parts. The contributions of the meson resonances η, ρ, and ω are included to second order. The complete potential is numerically calculated using masses and coupling parameters taken from meson experiments; no parameter of the potential is searched upon. The resulting potential is remarkably similar to that of Hamada and Johnston (outside half a pion Compton wavelength), particularly for the parts that are relatively well determined by nucleon-nucleon scattering data. Further extensions of the program, including the treatment of the nucleon resonances and pair suppression, are discussed, and an outline of such extensions is given.

I. INTRODUCTION

THE prime concern of this paper is an unambiguous formulation and a calculation of the two-pion exchange potential (TPEP) of the nucleon-nucleon interaction.[1] The one-boson exchange potentials (OBEP) are derived and calculated by the same formalism. A realistic result is expected *a priori* for the internucleon distance $r \gtrsim 0.5$.[2] No free parameters are involved; all masses and most coupling strengths are taken from mesonic measurements, and the remaining coupling strengths from higher-symmetry postulates. The potential is reduced to a local form, with tensor and spin-orbit components. In this form it agrees remarkably well with phenomenological potentials for the above-stated internucleon distances.

We turn briefly to the question of the use of potentials and the need for such a formulation. The concept of a potential describing the interaction between nucleons has played an important role in nuclear physics. In the early days of nuclear theory, this was inspired primarily by the success of electromagnetic potentials in describing atomic phenomena. In the light of the present-day knowledge of elementary particles, however, the concept of a potential as such is clearly inadequate for many purposes. Nonetheless, the use of potentials, albeit in modified, more sophisticated forms, continues to the present. This circumstance is mainly due to the lack of a workable relativistic field theory of nuclear interactions either at short distances or, most importantly, in the presence of more than two nucleons. On the other hand, there are theoretical indications such as those of this paper supporting the assumption that a potential representation may still be a valid one within appropriate restrictions.

One of the objectives in attempting to represent the nucleon-nucleon interaction by a potential is the application of the latter to the problem of nuclear structure. Such an extension must, of course, be justified. At any rate, the potentials or similar objects used in the theory of nuclear structure (with the exception of the deuteron, triton, and nuclear-matter problems) have not until recently been realistic forms. In the present work, we shall be exclusively concerned with the two-nucleon system.

The use of potentials, phenomenological, theoretical, and mixtures thereof, has undergone many changes and improvements since the original Yukawa hypothesis.[3] Here we will briefly review some of the important developments bearing on our work. One of the useful guidelines in the progress of nuclear potentials has been the so-called Taketani program, originally proposed by Taketani, Nakamura, and Sasaki in 1951.[3,4] This program divides the range of interaction into classical ($r \gtrsim 1.5$), dynamical ($0.7 \lesssim r \lesssim 1.5$), and core ($r \lesssim 0.7$) regions. In the classical region the long-range part of the potential, namely, the one-pion exchange potential (OPEP), is expected to dominate. In the dynamical region the TPEP and (as it is nowadays recognized) the meson and baryon resonance effects enter the picture. Finally, in the core region a multitude of complex phenomena contribute to the potential, rendering it extremely nonlocal in that region. Accordingly, in the core region one seeks to represent the inter-

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† Present address: Arya-Mehr University of Technology, Teheran, Iran.
[1] M. H. Partovi and E. L. Lomon, Phys. Rev. Letters 22, 438 (1969); M. H. Partovi, Ph.D. thesis, MIT, 1969 (unpublished).
[2] Throughout this paper we shall use natural units, setting $\hbar = c = \mu = 1$, where μ is the mass of the charged pion.
[3] For a review of the history and a guide to the very extensive literature on the subject, we mention: Progr. Theoret. Phys. (Kyoto) Suppl. 3 (1956); M. J. Moravcsik and H. P. Noyes, Ann. Nucl. Sci. 11, 95 (1961); Progr. Theoret. Phys. (Kyoto) Suppl. 39 (1967).
[4] M. Taketani, S. Nakamura, and M. Sasaki, Progr. Theoret. Phys. (Kyoto) 6, 581 (1951).

action by a means other than the theoretically derived potentials. Phenomenological hard and soft cores and separable potentials, as well as boundary-condition models, are examples of such representations.

The Taketani program described above is a useful one in that it permits an approximate separation of the potential into parts that require different treatments. Thus the long-range part of the potential is based on the simple mechanism of the exchange of one pion. The dynamical (or intermediate) range stands in marked contrast to this, as evidenced by the many derivations of TPEP that have produced a variety of potentials, many of which differ importantly, and none of which is very convincing.

The one-pion exchange potential was established experimentally as the long-range part in the fifties and early sixties.[3,5] The evidence came largely from the analysis of the high-angular-momentum scattering states. The success of OPEP (which, incidentally, is the the only well-established meson-theoretic potential) prompted many attempts to devise a similar potential for the intermediate range.[6,7] Already a few derivations of TPEP had appeared in the literature, of which the best known are the so-called TMO[8] and BW[9] potentials. However, these derivations were beset by a number of difficulties and ambiguities, and on the whole they contributed little in the way of producing a potential that successfully compares with experiments.

The recognition of the importance of heavy-boson exchange[6] led to the next important stage in the development of nuclear potentials: the introduction of pure one-boson-exchange (OBE) models.[10] This model is essentially based on the hypothesis that the nucleon-nucleon force is meson mediated, and that the exchanged systems may be adequately represented by the meson resonances observed experimentally. It is thus hoped that the uncorrelated (many-pion) contributions of the exchanged systems may be neglected, or, as it is sometimes argued in the case of the two-pion continuum, they may be approximated by equivalent resonances. The model has been partially successful in accounting for experimental data (although unobserved "equivalent" resonances are required), and attempts are being made at extending it.[11] One of our main conclusions is that the TPEP continuum contribution cannot be ignored,[7] except at very long range, where only OPEP is of importance.

Some of the notions referred to in the above form the basis of the present work. Our fundamental concept is the Yukawa hypothesis of a boson-mediated nuclear force together with the relationship between the mass of the exchanged system and the range of the interaction. Thus we assume that the nucleon-nucleon interaction is caused by the exchange of mesons, and that the inclusion of a suitable part of the meson mass spectrum is sufficient for describing the interaction outside a certain internucleon separation. In the absence of a dynamical theory of meson interactions, we shall be forced to utilize the existing experimental information for a segment of the boson mass spectrum. Thus we shall rely on the multimeson resonances to represent the effect of the exchange of high-mass boson systems, as it is expected that the contributions of the uncorrelated multimeson exchanges (i.e., those not resonating as a single meson), *aside from* TPEP, are unimportant. Some justification of this expectation is given in Sec. VIII. Radiative corrections are of uncertain importance and have some formal ambiguity; they are discussed in Sec. VII.

The contents of this paper are divided into nine sections and two appendices. Section II presents the general basis and approach of this work. That basis is used in Sec. III in the definition of a potential from a field-theoretical model. Section IV examines the passage to the configuration representation and presents an approximation scheme for obtaining a Schrödinger potential from the results of Sec. III. The TPEP is calculated in Secs. IV B and IV C and Appendix A. Section V is devoted to the calculation of one-boson exchange potentials. The nucleon-nucleon potential and various numerical results are presented and discussed in Sec. VI. In Sec. VII further extensions of this work are considered. Section VIII contains a discussion of remaining limitations. Conclusions are presented in Sec. IX. The details of the numerical computations are described in Appendix B.

II. IMPLICATIONS OF POTENTIAL CONCEPT

In this section the general properties and limitations of the potential representation will be examined, and the basis for our definition of the potential will be presented. Generally speaking, the nonrelativistic potential representation is complicated in the relativistic region, where by the latter is meant the region in which the dynamical process very significantly depends on the existence of the negative-energy states characteristic of relativistic kinematics. (Here we are discussing only the basic structure of the relativistic mode of particle propagation and not the other related complications such as particle production.) The essential difficulty is the difference in the analytic structure of the kinematics, namely, of the Green's functions or propagators. The

[5] G. Breit, M. H. Hull, Jr., K. Lassila, and H. M. Ruppel, Phys. Rev. Letters **5**, 274 (1960); P. Cziffra, M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **114**, 880 (1959).

[6] G. Breit, Phys. Rev. **120**, 287 (1960); J. J. Sakurai, Ann. Phys. (N. Y.) **11**, 1 (1960).

[7] G. Breit, K. I. Lassila, H. M. Ruppel, and M. H. Hull, Jr., Phys. Rev. Letters **6**, 138 (1961).

[8] M. Taketani, S. Machida, and S. Ohnuma, Progr. Theoret. Phys. (Kyoto) **7**, 45 (1952), hereafter referred to as TMO.

[9] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953), hereafter referred to as BW.

[10] N. Hoshizaki *et al.*, Progr. Theoret. Phys. (Kyoto) **27**, 1199 (1962); S. Sawada *et al.*, *ibid.* **28**, 991 (1962).

[11] R. A. Bryan and B. L. Scott, Phys. Rev. **135**, B434 (1964); **177**, 1435 (1969); T. Ueda and A. E. S. Green, *ibid.* **174**, 1304 (1968).

simple case of the reduction of the free Dirac equation to a Pauli-Schrödinger representation and the resulting square-root operator illustrates the point. We therefore anticipate that the potential representation of a field-theoretical S matrix, or even that of the simpler case of a Bethe-Salpeter equation, will be a complicated one in certain kinematical regions, if it exists there at all. However, because of the reasons mentioned in the Introduction, and encouraged by the fact that potential representation reproduces the assumed analytic properties of the field-theoretical S matrix (the assumed properties having completely been proven only for potential scattering), one hopes to devise a potential representation that is manageable and useful outside of the extreme relativistic region referred to above.

The question of the domain of validity of a potential representation has been discussed extensively in the literature.[3,12–15] This question has two related aspects. The first is the adequacy of the approximation procedure used to obtain information from a field-theoretical model. (This, of course, is additional to the question of the validity of the model itself.) The second aspect is the validity of the approximation procedure used in reducing the above information to a potential. In this section, we will be mainly concerned with the first aspect.

There exist several methods for deriving a potential from a field-theoretical model.[3,13] Among these, the method of Charap, Fubini, and Tausner[15] (CFT) is closest to ours. The essential notion in this approach is the definition of a potential, which, when used in a Schrödinger equation, generates the field-theoretical scattering amplitude to some order of approximation. This, of course, is an old and oft-used concept for the derivation of potentials. In exploiting this idea, we shall have to resort to covariant perturbation theory. Our justification for the use of perturbation theory with a strong coupling is threefold. First, as we shall actually reduce, with an accuracy to be discussed, a Bethe-Salpeter equation which represents a summation of the "laddered" Feynman graphs, we probably achieve better convergence than the straightforward perturbation expansion in the coupling constant. The low-mass (two-nucleon) intermediate state between steps of the ladder tends to make the contribution of these diagrams larger than others of the same order. Exact solutions of OBEP and TPEP nuclear potentials differ substantially from Born approximations to them, confirming that at least some of the iterations are important. In any case some unitarization must be made, and laddering in a Schrödinger equation relates the unitarization to definite dynamical processes. Second, the relationship between the mass of the exchanged mesons and the range of the potential for Yukawa-type couplings indicates that the dynamical and classical regions are mainly (but certainly not entirely) dominated by the irreducible diagrams of the second and fourth orders. Third, dispersion theory indicates that the main effects of higher-order interactions and corrections will be accounted for by the inclusion of the experimentally known meson and baryon resonances, which bring singularities of the amplitude close to the physical region. Concerning the other methods referred to above,[8,9] we remark that the noncovariant schemes based on the number of quanta exchanged include, in practice, the set (or a subset deleting a so-called ladder diagram) of time-ordered graphs corresponding to the ones mentioned above. Moreover, these methods are practically useful only in the static limit, which limit we shall be at pains to avoid. Finally, the method of reducing the Bethe-Salpeter equation to an "equal-times" form, as employed in the past, also suffers from a noncovariant and somewhat arbitrary elimination of the "negative-energy components" of the Bethe-Salpeter wave function.[16]

Our principal motivation in relying on the Bethe-Salpeter equation (or, effectively, the covariant perturbation theory) is the desire to eliminate the approximation known as the static limit. This approximation, which is presumed to effect the passage to the double limit of nonrelativistic nucleons and vanishing meson-to-nucleon mass ratio, in practice introduces serious errors and distortions, as previously noted.[14,15,17] This circumstance arises primarily in treating the fourth-order graphs, as well as the iteration of the second-order graph, where an integration over intermediate momenta is involved. Clearly, such an integration is over the entire momentum range, and therefore the static conditions are violated for a sizable part of this range. Moreover, the contribution from this part (i.e., the high-momentum part) to the iteration of OPEP is quite important for the spatial range where TPEP is supposed to dominate ($\sim$1 F), a fact which makes the application of the static limit inappropriate. The method of CFT largely surmounts this difficulty by avoiding the static approximation. However, CFT found it necessary to use the approximation of nonrelativistic *external* momenta. Such an approximation, adequate though it may be for a configuration-space potential in a certain range, is not justified for the one-pion-exchange kernel (OPEK).[18] The reason is that TPEP involves the iteration of OPEK (where now the external legs of OPEK become internal legs) over all momenta, and therefore a nonrelativistic approximation to OPEK

[12] H. Feshbach and E. L. Lomon, Ann. Phys. (N. Y.) **29**, 19 (1964).

[13] A. Klein, Progr. Theoret. Phys. (Kyoto) **20**, 257 (1958).

[14] N. Hoshizaki and S. Machida, Progr. Theoret. Phys. (Kyoto) **27**, 288 (1962).

[15] J. M. Charap and S. P. Fubini, Nuovo Cimento **14**, 540 (1959); **15**, 73 (1960); J. M. Charap and M. J. Tausner, *ibid.* **18**, 316 (1960).

[16] R. Arnowitt and S. Gasiorowicz, Phys. Rev. **94**, 1057 (1954).

[17] S. N. Gupta, Phys. Rev. **117**, 1146 (1960).

[18] The designation "kernel" is applied to the operator that plays the role of a potential in the Bethe-Salpeter equation. The similar operator for the Schrödinger equation is called a potential as usual.

amounts to a partial static approximation. That such an approximation is invalid is amply demonstrated by a divergence in the iterated term of CFT for pseudoscalar coupling. The prescription of CFT for removing this divergence, namely, the subtraction of a potential of zero range and infinite strength, is clearly inadequate, as the zero-range character of the subtracted term is a result of their approximation. Indeed, the anomalously large isosinglet central potential of Cottingham and Vinh Mau,[19] who used the CFT method for the calculation of TPEP, can be traced to this same subtraction. The sensitivity of the iteration procedure to the choice of OPEK can also be seen within the context of the earlier static calculations. As clearly recognized by Klein,[13] the TMO-BW discrepancy can be attributed to a corresponding difference in the choice of OPEK's. It should be noted that the divergence mentioned above does not explicitly occur in the earlier static calculations based on the S matrix.[20] The reason is that an over-all application of the static limit in the latter calculations renders the iterated term identical with a part of the fourth-order ladder diagram, and it thus eliminates the need for an explicit iteration of OPEK.

In view of the above considerations, we insist that the definition of our potential from the field-theoretical model be independent of whatever approximations we may choose to exploit in reducing it to a usable form. In particular, our definition will involve the iteration of an unapproximated OPEK, and it will thereby avoid the divergence encountered in CFT. We shall formulate our method in the language of the Bethe-Salpeter equation, although it may also be described within the formulation of CFT.

III. DEFINITION OF POTENTIAL

This section defines a potential function for the nonrelativistic Schrödinger equation that is designed to represent the Bethe-Salpeter equation of a field-theoretical model to some order of approximation. The Bethe-Salpeter equation[21] for the (elastic) scattering of two distinguishable nucleons in the center-of-momentum system from a state of relative four-momentum p to one of p' is

$$\mathfrak{M}(p',p|W)=\mathfrak{K}(p',p|W)+\int d^4k \times\mathfrak{K}(p',k|W)G(k|W)\mathfrak{M}(k,p|W), \quad (3.1)$$

where $\mathfrak{M}$ is the invariant amplitude, $W=(W,0,0,0)$, half the total four-momentum, $\mathfrak{K}$ is the interaction kernel consisting (in principle) of all irreducible diagrams, and G is the two-particle (free) Green's function

$$G(k|W)=\frac{i}{2\pi}\left[\frac{1}{\gamma^\mu(W+k)_\mu-m+i\epsilon}\right]^{(1)} \times\left[\frac{1}{\gamma^\mu(W-k)_\mu-m+i\epsilon}\right]^{(2)}, \quad (3.2)$$

where m is the nucleon mass. The superscripts (1) and (2) refer to the two nucleons, and the spin and isospin indices have been suppressed.[22] From our point of view, the essential difference between (3.1) and a Lippmann-Schwinger equation is the difference between G and the corresponding nonrelativistisc propagator g. Our goal, therefore, is to construct another equation for $\mathfrak{M}$ that involves g for the Green's function. Following Blankenbecler and Sugar[23] and others,[24] we write G as a sum of two terms:

$$G=g+(G-g), \quad (3.3)$$

where g is to be an appropriate nonrelativistic propagator corresponding to G. As mentioned before, the outstanding characteristic of G as a relativistic propagator is its ability to produce negative-energy states; stated in other words, it can produce two-particle cuts in the unphysical region. It is therefore natural to define g as a function that possesses the cut structure of G only in the physical region. This, of course, is the prescription of Blankenbecler and Sugar, which we modify slightly and extend to the spin-$\frac{1}{2}$ case. We require that g, considered as a function of $\mathbf{q}^2=W^2-m^2$, have the singularity structure of G when both legs of the latter are on the mass shell:

$$g(k|W)=\int_0^\infty\frac{d\mathbf{q}'^2}{\mathbf{q}^2-\mathbf{q}'^2+i\epsilon}\delta((W'+k)^2-m^2) \times\delta((W'-k)^2-m^2)[\gamma^\mu(W'+k)_\mu+m]^{(1)} \times[\gamma^\mu(W'-k)_\mu+m]^{(2)}, \quad (3.4)$$

where $\mathbf{q}'^2=W'^2-m^2$. The integration is easily carried out, and the result is

$$g(k|W)=\frac{[\gamma^0E(\mathbf{k})-\boldsymbol{\gamma}\cdot\mathbf{k}+m]^{(1)}[\gamma^0E(\mathbf{k})+\boldsymbol{\gamma}\cdot\mathbf{k}+m]^{(2)}}{4E(\mathbf{k})[\mathbf{q}^2-\mathbf{k}^2+i\epsilon]} \times\delta(k^0), \quad (3.5)$$

where we have used the notations $E(\mathbf{k})=(m^2+\mathbf{k}^2)^{1/2}$ and $k=(k^0,\mathbf{k})$. The function g has several important properties. It contains the basic Schrödinger propagator with the correct imaginary part. The additional factors, including $E(\mathbf{k})$ in the denominator, are necessary for a

[19] W. N. Cottingham and R. Vinh Mau, Phys. Rev. **130**, 735 (1963).

[20] A. Klein and B. H. McCormick, Phys. Rev. **104**, 1747 (1956).

[21] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); J. Schwinger, Proc. Natl. Acad. Sci. (U. S.) **37**, 452 (1951); **37**, 455 (1951).

[22] Our relativistic notation conforms to that of J. D. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), unless otherwise stated.

[23] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

[24] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963).

correct reduction to a Lippmann-Schwinger form. Thus the positive-energy projection operators will serve to reduce the original equation to one that contains only "positive-energy" components, while the δ function accomplishes the reduction to a three-dimensional form. The appearance of the factor $E(\mathbf{k})$ is connected with unitarity as we shall see below. Finally, the function g, as pointed out in Ref. 23, is not unique, since one can add to it a part that vanishes at $E(\mathbf{k})=W$. Indeed, as pointed out by Saenger,[25] (and recognized also by Schwartz and Zemach[26] in the configuration representation), the limit of G for nonrelativistic energies (i.e., $\mathbf{q}/m\rightarrow 0$) is precisely the expression given by (3.5) with $E(\mathbf{k})$ everywhere replaced by W.[27] Our choice of g, therefore, is indicated by the requirement of a correct reduction to a Lippmann-Schwinger form.

Having defined g, we follow Ref. 23 in writing a new equation for $\mathfrak{M}$. Reverting to a symbolic operator notation, we write

$$\mathfrak{M}=\mathbf{U}+\mathbf{U}g\mathfrak{M}, \tag{3.6}$$

where $\mathbf{U}$ is an effective interaction kernel whose connection with $\mathfrak{K}$ is easily ascertained by comparing (3.1) with (3.6):

$$\mathbf{U}=\mathfrak{K}+\mathfrak{K}(G-g)\mathbf{U}. \tag{3.7}$$

Equation (3.7) may be interpreted to mean that $\mathbf{U}$ is evolved from $\mathfrak{K}$ by means of the propagator $G-g$. Since the two-particle cut (with threshold at $W=m$) is canceled out in the propagator $G-g$, it is expected that $\mathbf{U}$ will have a weak energy dependence near threshold. This is indeed the case, as will be shown in Secs. IV A and IV B. Precisely this cancellation occurs within the "adiabatic" approximation of CFT for the fourth-order potential, and it is made the basis of their definition of an energy-independent potential.

Returning to our derivation, we rewrite Eq. (3.6) in detailed form:

$$\mathfrak{M}(p',p\,|\,W)=\mathbf{U}(p',p\,|\,W)+\int d^4k\;\mathbf{U}(p',k\,|\,W)\times\frac{[\gamma^0E(\mathbf{k})-\boldsymbol{\gamma}\cdot\mathbf{k}+m]^{(1)}[\gamma^0E(\mathbf{k})+\boldsymbol{\gamma}\cdot\mathbf{k}+m]^{(2)}\delta(k^0)}{4E(\mathbf{k})[\mathbf{q}^2-\mathbf{k}^2+i\epsilon]}\times\mathfrak{M}(k,p\,|\,W). \tag{3.8}$$

Next we restrict our consideration to the elements of $\mathfrak{M}$ connecting positive-energy spinors, and also change the basis of representation from Dirac spinors to Pauli spinors. To this end, we observe that

$$\sum_s|u(\mathbf{k},s)\rangle\langle\bar{u}(\mathbf{k},s)|=\frac{\gamma^0E(\mathbf{k})-\boldsymbol{\gamma}\cdot\mathbf{k}+m}{2m}, \tag{3.9}$$

where $u(\mathbf{k},s)$ denotes a positive-energy spinor of momentum $\mathbf{k}$ and spin s, $\bar{u}(\mathbf{k},s)$ denotes the adjoint spinor, and the content of the equation is the familiar expression for the positive-energy spinor projection operator. We now define $\mathfrak{\bar{M}}$ by

$$\mathfrak{\bar{M}}=\langle\bar{u}^{(1)}(\mathbf{p}',s_1')\bar{u}^{(2)}(-\mathbf{p}',s_2')\,|\,\mathfrak{M}(\mathbf{p}',p'^0=0;\mathbf{p},p^0=0\,|\,W)\times|\,u^{(1)}(\mathbf{p},s_1)u^{(2)}(-\mathbf{p},s_2)\rangle, \tag{3.10}$$

and similarly for $\bar{\mathbf{U}}$. The subscripts and superscripts (1) and (2) refer to the two nucleons as before. Using Eqs. (3.9) and (3.10), we may write

$$\mathfrak{\bar{M}}(\mathbf{p}',\mathbf{p}\,|\,W)=\bar{\mathbf{U}}(\mathbf{p}',\mathbf{p}\,|\,W)+\int d\mathbf{k}\;\bar{\mathbf{U}}(\mathbf{p}',\mathbf{k}\,|\,W)\times\frac{1}{\mathbf{q}^2/m-\mathbf{k}^2/m+i\epsilon}\frac{m}{E(\mathbf{k})}\mathfrak{\bar{M}}(\mathbf{k},\mathbf{p}\,|\,W), \tag{3.11}$$

where we have suppressed the spin indices. (We shall continue suppressing spin and isospin indices until we need to exhibit them explicitly.) Note that the matrix multiplication implied in the product $\bar{\mathbf{U}}\mathfrak{\bar{M}}$ ranges over positive-energy indices only. In other words, $\mathfrak{\bar{M}}$ and $\bar{\mathbf{U}}$ are now 4×4 matrices, and may therefore be considered as acting on the product space of two Pauli spinors. The explicit Pauli-Schrödinger representation will be developed in Sec. IV B.

Our final task is to comprehend the factor $m/E(\mathbf{k})$. Having treated the kinematics of the equation carefully up to this juncture, we turn to unitarity for the interpretation of this factor. Our basic observation is that Eq. (3.11), which is in a near Lippmann-Schwinger form, is an equation for $\mathfrak{\bar{M}}$, the invariant scattering amplitude in field theory. On the other hand, the invariant amplitude $\mathfrak{\bar{M}}$ and the corresponding Lippmann-Schwinger amplitude $\mathbf{T}$ are not the same objects in that the physical observables (e.g., scattering cross section) are obtained from them by means of different phase-space factors. Indeed, on the energy shell, taking account of the appropriate phase-space factors, one arrives at the correspondence (note that $\mathbf{T}$ is not the field-theoretical T matrix)

$$|\,\mathbf{T}\,|^2=(m^2/W^2)\,|\,\mathfrak{\bar{M}}\,|^2. \tag{3.12}$$

Off the energy shell we generalize (3.12) in the following manner:

$$\mathbf{T}(\mathbf{p}',\mathbf{p}\,|\,W)=\left[\frac{m}{E(\mathbf{p}')}\right]^{1/2}\mathfrak{\bar{M}}(\mathbf{p}',\mathbf{p}\,|\,W)\left[\frac{m}{E(\mathbf{p})}\right]^{1/2}. \tag{3.13}$$

Inserting (3.13) in (3.11), we find that $\mathbf{T}$ does indeed satisfy a Lippmann-Schwinger equation:

$$\mathbf{T}(\mathbf{p}',\mathbf{p}\,|\,W)=\mathbf{V}(\mathbf{p}',\mathbf{p}\,|\,W)+\int d\mathbf{k}\times\mathbf{V}(\mathbf{p}',\mathbf{k}\,|\,W)\frac{1}{\mathbf{q}^2/m-\mathbf{k}^2/m+i\epsilon}\mathbf{T}(\mathbf{k},\mathbf{p}\,|\,W), \tag{3.14}$$

[25] R. Saenger, J. Math. Phys. **8**, 2366 (1967).
[26] C. Schwartz and C. Zemach, Phys. Rev. **141**, 1454 (1966).
[27] Actually, these authors treat the case of two scalar particles. The verification for the present case follows straightforwardly.

where the potential $\mathbf{V}$ is defined by

$$\mathbf{V}(\mathbf{p}',\mathbf{p}|W)=\left[\frac{m}{E(\mathbf{p}')}\right]^{1/2}\tilde{\mathbf{U}}(\mathbf{p}',\mathbf{p}|W)\left[\frac{m}{E(\mathbf{p})}\right]^{1/2}. \quad (3.15)$$

It should be noted that the reduction achieved in (3.14) depends crucially upon the way (3.12) was generalized to (3.13). An alternative route for arriving at Eq. (3.13) is provided by the (elastic) unitarity condition satisfied by $\mathfrak{M}$. That condition is

$$\mathfrak{M}^{\dagger}(\mathbf{p}',\mathbf{p}|W)-\mathfrak{M}(\mathbf{p}',\mathbf{p}|W)$$
$$=2\pi i\int d\mathbf{k}\,\mathfrak{M}^{\dagger}(\mathbf{p}',\mathbf{k}|W)\delta(2E(\mathbf{k})-2W)\mathfrak{M}(\mathbf{k},\mathbf{p}|W). \quad (3.16)$$

Recalling that $W^2=\mathbf{q}^2+m^2$, we rewrite the δ function as follows:

$$\delta(2E(\mathbf{k})-2W)=\frac{E(\mathbf{k})}{m}\delta\left(\frac{\mathbf{k}^2}{m}-\frac{\mathbf{q}^2}{m}\right),$$

and use this in (3.16) to get

$$(\mathfrak{M}^{\dagger}-\mathfrak{M})(\mathbf{p}',\mathbf{p}|W)=2\pi i\int d\mathbf{k}\,\mathfrak{M}^{\dagger}(\mathbf{p}',\mathbf{k}|W)$$
$$\times\frac{E(\mathbf{k})}{m}\delta\left(\frac{\mathbf{k}^2}{m}-\frac{\mathbf{q}^2}{m}\right)\mathfrak{M}(\mathbf{k},\mathbf{p}|W). \quad (3.17)$$

It is now easily verified that the identification of Eq. (3.13) reduces (3.17) to the nonrelativistic unitarity condition for $\mathbf{T}$. One can easily see that the consistency achieved here is owing to the identical two-particle cut structures of G and g in the physical region.

The potential $\tilde{\mathbf{U}}$ that appears in Eq. (3.14) is ultimately obtained from Eq. (3.7). The various orders of the kernel $\mathbf{U}$ can be easily written using the latter equation:

$$\mathbf{U}^{(2)}=\mathcal{K}^{(2)},$$
$$\mathbf{U}^{(4)}=\mathcal{K}^{(4)}+\mathcal{K}^{(2)}(G-g)\mathcal{K}^{(2)}, \quad (3.18)$$

and so on. The superscripts refer to the order of the terms in the perturbation expansion.

The potential we have defined above is both nonlocal (in the configuration representation) and energy dependent (the meaning of these terms will be made precise in Sec. IV), and hence unsuitable for use with the many-body Schrödinger equation. In Sec. IV, we shall introduce an expansion in the powers of the momentum operator that will extract the useful information contained in the potential. Meanwhile, in this section we have achieved our primary goal of defining a potential operator with which the Schrödinger equation generates an amplitude identical to that of the field-theoretical Bethe-Salpeter equation.

We conclude this section by showing that the method of CFT, when so modified as to conform to our requirement of an unapproximated OPEK and the identification of Eq. (3.13), yields a potential that is identical on the energy shell with the one defined above. The CFT method is ambiguous off the energy shell; however, according to CFT, we define a potential that gives rise to a scattering matrix that agrees in each order of the perturbation expansion with the corresponding field-theoretical one. [This correspondence, made precise in Eq. (3.13), is actually ambiguous within the adiabatic approximation of CFT, since factors of E/m are set equal to unity in the latter.] Up to fourth order, we have from the Lippmann-Schwinger equation

$$\mathbf{T}^{(2)}=\mathbf{V}^{(2)},$$
$$\mathbf{T}^{(4)}=\mathbf{V}^{(4)}+\mathbf{V}^{(2)}\frac{1}{E-H_0+i\epsilon}\mathbf{V}^{(2)}$$
$$=\mathbf{V}^{(4)}+\mathbf{T}^{(2)}\frac{1}{\mathbf{q}^2/m-H_0+i\epsilon}\mathbf{T}^{(2)}, \quad (3.19)$$

where $\mathbf{q}$ is the c.m. 3-momentum of a particle. (Here, to achieve a compact notation, we demonstrate this for scalar particles.) If we now connect the nonrelativistic amplitude $\mathbf{T}$ with the field-theoretical one $\mathfrak{M}$ according to (3.13), we will have

$$\langle\mathbf{p}'|\mathbf{V}^{(2)}|\mathbf{p}\rangle=\left[\frac{m}{E(\mathbf{p}')}\right]^{1/2}\langle\mathbf{p}'|\mathfrak{M}^{(2)}|\mathbf{p}\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{1/2},$$
$$\langle\mathbf{p}'|\mathbf{V}^{(4)}|\mathbf{p}\rangle=\left[\frac{m}{E(\mathbf{p}')}\right]^{1/2}\langle\mathbf{p}'|\mathfrak{M}^{(4)}|\mathbf{p}\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{1/2}$$
$$-\int d\mathbf{k}\left[\frac{m}{E(\mathbf{k})}\right]\left[\frac{m}{E(\mathbf{p}')}\right]^{1/2}\langle\mathbf{p}'|\mathfrak{M}^{(2)}|\mathbf{k}\rangle$$
$$\times\langle\mathbf{k}|\frac{1}{\mathbf{q}^2/m-\mathbf{k}^2/m+i\epsilon}\mathfrak{M}^{(2)}|\mathbf{p}\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{1/2}. \quad (3.20)$$

We now observe that

$$\frac{m}{E(\mathbf{k})}\frac{1}{\mathbf{q}^2/m-\mathbf{k}^2/m+i\epsilon},$$

aside from unimportant constants, is g. Equations (3.20) can now be rearranged in the following form:

$$\left[\frac{m}{E(\mathbf{p}')}\right]^{-1/2}\langle\mathbf{p}'|\mathbf{V}^{(2)}|\mathbf{p}\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{-1/2}=\langle\mathbf{p}'|\mathfrak{M}^{(2)}|\mathbf{p}\rangle,$$
$$\left[\frac{m}{E(\mathbf{p}')}\right]^{-1/2}\langle\mathbf{p}'|\mathbf{V}^{(4)}|\mathbf{p}\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{-1/2}$$
$$=\langle\mathbf{p}'|\mathfrak{M}^{(4)}-\mathfrak{M}^{(2)}g\mathfrak{M}^{(2)}|\mathbf{p}\rangle. \quad (3.21)$$

The amplitudes $\mathfrak{M}^{(2)}$ and $\mathfrak{M}^{(4)}$ must now be identified. Clearly, $\mathfrak{M}^{(2)}$ is the one-pion exchange amplitude, and $\mathfrak{M}^{(4)}$ is the sum of the crossed and fourth-order ladder

diagrams. Thus, in terms of the irreducible kernel $\mathcal{K}$, we have

$$\mathfrak{M}^{(2)}=\mathcal{K}^{(2)},$$
$$\mathfrak{M}^{(4)}=\mathcal{K}^{(4)}+\mathcal{K}^{(2)}G\mathcal{K}^{(2)}.$$

Inserting these expressions in Eqs. (3.21), we get

$$\left[\frac{m}{E(\mathbf{p}')}\right]^{-1/2}\langle\mathbf{p}'|\mathbf{V}^{(2)}|\mathbf{p}\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{-1/2}=\langle\mathbf{p}'|\mathcal{K}^{(2)}|\mathbf{p}\rangle,$$
$$\left[\frac{m}{E(\mathbf{p}')}\right]^{-1/2}\langle\mathbf{p}'|\mathbf{V}^{(4)}|\mathbf{p}\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{-1/2}$$
$$=\langle\mathbf{p}'|\mathcal{K}^{(4)}+\mathcal{K}^{(2)}(G-g)\mathcal{K}^{(2)}|\mathbf{p}\rangle. \quad (3.22)$$

These expressions are identical with Eqs. (3.15) and (3.18).

IV. TWO-PION EXCHANGE POTENTIAL

A. Configuration-Space Potential

The content of this section is the derivation of TPEP and its transformation, by means of an approximation, to a useful form. By TPEP, of course, we mean the fourth-order potential originating in $\mathbf{U}^{(4)}$ of Eqs. (3.18). Explicitly,

$$\mathbf{V}^{(4)}(\mathbf{p}',s_1',s_2';\mathbf{p},s_1,s_2|W)$$
$$=\left[\frac{m}{E(\mathbf{p}')}\right]^{1/2}\langle\bar{u}^{(1)}(\mathbf{p}',s_1')\bar{u}^{(2)}(-\mathbf{p}',s_2')|$$
$$\times\mathbf{U}^{(4)}(\mathbf{p}',p^{0\prime}=0,\mathbf{p},p^0=0|W)$$
$$\times|u^{(1)}(\mathbf{p},s_1)u^{(2)}(-\mathbf{p},s_2)\rangle\left[\frac{m}{E(\mathbf{p})}\right]^{1/2}. \quad (4.1)$$

We shall begin by recording the connection between the momentum- and configuration-space representations. Denoting the potential in configuration space by the same symbol V, and suppressing spin indices, we have

$$V(\mathbf{r}',\mathbf{r})=(2\pi)^{-3}\int d\mathbf{p}'d\mathbf{p}\; e^{-i\mathbf{r}'\cdot\mathbf{p}'}\mathbf{V}(\mathbf{p}',\mathbf{p}|W)e^{i\mathbf{r}\cdot\mathbf{p}}. \quad (4.2)$$

In this section the dependence on W may occasionally be suppressed. The mode of operation for the operator $\mathbf{V}$ on the Schrödinger state vector ψ is given by

$$\langle\mathbf{r}'|\mathbf{V}|\psi\rangle=\int d\mathbf{r}\langle\mathbf{r}'|\mathbf{V}|\mathbf{r}\rangle\langle\mathbf{r}|\psi\rangle=\int d\mathbf{r}\; V(\mathbf{r}',\mathbf{r})\psi(\mathbf{r}). \quad (4.3)$$

With the definitions

$$\mathbf{p}'-\mathbf{p}=\boldsymbol{\Delta},\quad \mathbf{p}'+\mathbf{p}=\mathbf{Q}, \quad (4.4)$$

we can rewrite (4.2) as follows:

$$V(\mathbf{r}',\mathbf{r})=(4\pi)^{-3}\int d\mathbf{Q}d\boldsymbol{\Delta}\exp\left(i\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}'}{2}\right)$$
$$\times\exp\left(-i\boldsymbol{\Delta}\cdot\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\mathbf{V}(\tfrac{1}{2}(\mathbf{Q}+\boldsymbol{\Delta}),\tfrac{1}{2}(\mathbf{Q}-\boldsymbol{\Delta})|W). \quad (4.5)$$

In the event that the dependence of $\mathbf{V}$ on $\mathbf{p}$ and $\mathbf{p}'$ is through the combination $\mathbf{p}'-\mathbf{p}$, we get the familiar connection,

$$V(\mathbf{r}',\mathbf{r})=\delta(\mathbf{r}'-\mathbf{r})\int d\boldsymbol{\Delta}\exp(-i\boldsymbol{\Delta}\cdot\mathbf{r}')V(\boldsymbol{\Delta}|W)$$
$$\equiv\delta(\mathbf{r}-\mathbf{r}')V(\mathbf{r}'|W), \quad (4.6)$$
$$\langle\mathbf{r}'|\mathbf{V}|\psi\rangle=V(\mathbf{r}'|W)\psi(\mathbf{r}').$$

Since the potential which is not diagonal in the configuration representation is usually referred to as nonlocal, we shall also apply the term "nonlocality" to a dependency on $\mathbf{Q}$ in the momentum representation. We now examine the configuration representation of a potential which has a linear dependence on $\mathbf{Q}$, e.g., $\mathbf{Q}\cdot\mathbf{f}(\boldsymbol{\Delta})$. According to (4.5), we have

$$V_Q(r',r)=\frac{1}{8(2\pi)^3}\int d\mathbf{Q}d\boldsymbol{\Delta}\exp\left(i\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}'}{2}\right)$$
$$\times\exp\left(-i\boldsymbol{\Delta}\cdot\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\mathbf{f}(\boldsymbol{\Delta})\cdot\mathbf{Q}$$
$$=\frac{1}{8(2\pi)^3}\int d\mathbf{Q}d\boldsymbol{\Delta}\exp\left(-i\boldsymbol{\Delta}\cdot\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\mathbf{f}(\boldsymbol{\Delta})$$
$$\cdot(-2i\nabla_r)\exp\left(i\mathbf{Q}\cdot\frac{\mathbf{r}-\mathbf{r}'}{2}\right)$$
$$=\mathbf{f}\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\cdot(-2i\nabla_r)\delta(\mathbf{r}-\mathbf{r}'),$$

where

$$\mathbf{f}(r)=\int d\boldsymbol{\Delta}\exp(-i\boldsymbol{\Delta}\cdot\mathbf{r})\mathbf{f}(\boldsymbol{\Delta}).$$

Using (4.3), we get

$$\langle\mathbf{r}'|\mathbf{V}|\psi\rangle=\int d\mathbf{r}\,\psi(\mathbf{r})\mathbf{f}\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\cdot(-2i\nabla_r)\delta(\mathbf{r}-\mathbf{r}')$$
$$=\int d\mathbf{r}\,\delta(\mathbf{r}-\mathbf{r}')(2i\nabla_r)\cdot\left[\psi(\mathbf{r})\mathbf{f}\left(\frac{\mathbf{r}+\mathbf{r}'}{2}\right)\right]$$
$$=\mathbf{f}(\mathbf{r}')\cdot(2i\nabla_{r'})\psi(\mathbf{r}')+\psi(\mathbf{r}')(i\nabla_{r'})\cdot\mathbf{f}(\mathbf{r}')$$
$$=-[\mathbf{P}\cdot\mathbf{f}(\mathbf{r}')+\mathbf{f}(\mathbf{r}')\cdot\mathbf{P}]\psi(\mathbf{r}'), \quad (4.7)$$

where $\mathbf{P}=-i\nabla$ is the momentum operator in the configuration representation. We may symbolically repre-

sent this connection as follows:

Mom. rep.		Conf. rep.
$\mathbf{f}(\boldsymbol{\Delta})$	$\rightarrow$	$\mathbf{f}(\mathbf{r})$
$-\mathbf{Q}\cdot\mathbf{f}(\boldsymbol{\Delta})$	$\rightarrow$	$\mathbf{P}\cdot\mathbf{f}(\mathbf{r})+\mathbf{f}(\mathbf{r})\cdot\mathbf{P}$

Note that a dependence on $\mathbf{Q}$ in momentum space is equivalent to a configuration-space potential linear in the momentum operator $\mathbf{P}$. The latter is sometimes referred to as an energy-dependent potential. According to our definition above, however, a dependency of the potential on $\mathbf{P}$ is a nonlocality, and we reserve the designation "energy dependence" specifically for the appearance of the center-of-mass energy W in the potential. [Note that $W^2-m^2=\mathbf{q}^2$, $\mathbf{p}^2$, and $\mathbf{p}'^2$ are all different in $\mathbf{V}(\mathbf{p}',\mathbf{p}|W)$. They are, of course, identical on the energy shell.] To complete the connection between the momentum and configuration representations, we consider the expansion of a nonlocal potential $V(\mathbf{r}',\mathbf{r})$ in terms of the momentum operator, and show the equivalence of this expansion with that of $\mathbf{V}(\mathbf{Q},\boldsymbol{\Delta})$ about $\mathbf{Q}=0$. Consider

$$\langle\phi|\mathbf{V}|\psi\rangle=\int d\mathbf{r}d\mathbf{r}'\,\phi^*(\mathbf{r}')V(\mathbf{r}',\mathbf{r})\psi(\mathbf{r})$$

$$=\int d\mathbf{z}d\mathbf{Z}\,\phi^*(\mathbf{Z}+\tfrac{1}{2}\mathbf{z})V(\mathbf{Z}+\tfrac{1}{2}\mathbf{z},\,\mathbf{Z}-\tfrac{1}{2}\mathbf{z})\times\psi(\mathbf{Z}-\tfrac{1}{2}\mathbf{z}). \quad (4.8)$$

Expanding ϕ^* and ψ about $\mathbf{z}=0$, we get

$$\langle\phi|\mathbf{V}|\psi\rangle=\int d\mathbf{z}d\mathbf{Z}\;V(\mathbf{Z}+\tfrac{1}{2}\mathbf{z},\,\mathbf{Z}-\tfrac{1}{2}\mathbf{z})[\phi^*(\mathbf{Z})\psi(\mathbf{z})$$

$$+\tfrac{1}{2}\psi(\mathbf{Z})\mathbf{z}\cdot\boldsymbol{\nabla}\phi^*(\mathbf{Z})-\tfrac{1}{2}\phi^*(\mathbf{Z})\mathbf{z}\cdot\boldsymbol{\nabla}\psi(\mathbf{Z})+\cdots].$$

Integration by parts on $\mathbf{z}$ leads to

$$\langle\phi|\mathbf{V}|\psi\rangle=\int d\mathbf{z}d\mathbf{Z}\,\phi^*(\mathbf{Z})[V(\mathbf{Z}+\tfrac{1}{2}\mathbf{z},\,\mathbf{Z}-\tfrac{1}{2}\mathbf{z})$$

$$-\tfrac{1}{2}iV(\mathbf{Z}+\tfrac{1}{2}\mathbf{z},\,\mathbf{Z}-\tfrac{1}{2}\mathbf{z})\mathbf{z}\cdot\mathbf{P}-\tfrac{1}{2}i\mathbf{P}\cdot\mathbf{z}V(\mathbf{Z}+\tfrac{1}{2}\mathbf{z},\,\mathbf{Z}-\tfrac{1}{2}\mathbf{z})+\cdots]\psi(\mathbf{Z}), \quad (4.9)$$

where

$$\mathbf{P}=-i\boldsymbol{\nabla}_Z.$$

The operator V can thus be written as follows:

$$V(r',r)=\delta(\mathbf{r}-\mathbf{r}')\int d\mathbf{z}[V(\mathbf{r}+\tfrac{1}{2}\mathbf{z},\,\mathbf{r}-\tfrac{1}{2}\mathbf{z})$$

$$-\tfrac{1}{2}iV(\mathbf{r}+\tfrac{1}{2}\mathbf{z},\,\mathbf{r}-\tfrac{1}{2}\mathbf{z})\mathbf{z}\cdot\mathbf{P}-\tfrac{1}{2}i\mathbf{P}\cdot\mathbf{z}V(\mathbf{r}+\tfrac{1}{2}\mathbf{z},\,\mathbf{r}-\tfrac{1}{2}\mathbf{z})+\cdots]. \quad (4.10)$$

On the other hand, from Eq. (4.5) we have

$$\int d\mathbf{z}\;V(\mathbf{r}+\tfrac{1}{2}\mathbf{z},\,\mathbf{r}-\tfrac{1}{2}\mathbf{z})=\int d\boldsymbol{\Delta}\,\exp(-i\boldsymbol{\Delta}\cdot\mathbf{r})V(\boldsymbol{\Delta},\,\mathbf{Q}=0),$$

$$\int d\mathbf{z}\,(-i\tfrac{1}{2}\mathbf{z})V(\mathbf{r}+\tfrac{1}{2}\mathbf{z},\,\mathbf{r}-\tfrac{1}{2}\mathbf{z})=\int d\boldsymbol{\Delta}\,\exp(-i\boldsymbol{\Delta}\cdot\mathbf{r})[\boldsymbol{\nabla}_Q V(\boldsymbol{\Delta},\mathbf{Q})]_{Q=0},$$

and so on. It is thus seen that the expansion of a nonlocal potential in terms of the momentum operator is equivalent to an expansion in momentum space about $\mathbf{Q}=0$. These connections between momentum and configuration representations are of course old material, and are recorded here for later reference.

It is evident from the above considerations that, to obtain a useful potential, one must expand it in the powers of the momentum operator and retain a suitable number of terms. It is therefore important to examine the nature of this expansion. The behavior of the potential in momentum space has been studied by Hoshizaki and Machida[14] in detail. These authors use a formalism different from ours for defining their potential. However, the general characteristics of the potential, including nonlocality, are substantially the same as ours. They confirm the well-known result[15] that, whereas the expansion in μ/m (pion-to-nucleon mass ratio) is inappropriate, the expansion with respect to $\mathbf{Q}/m$ is valid for not too large $\mathbf{Q}$, i.e., $|\mathbf{Q}|\leq 8\mu$. As is well known, the dependence of the potential on $\mathbf{Q}$ is through $\mathbf{Q}/m$, and to the order $O(\mathbf{Q}/m)$, all nonlocalities except for a spin-orbit term vanish. As regards the configuration-space potential, the essential point is that expansions in $\mathbf{Q}$ or $\boldsymbol{\Delta}$ are asymptotic, and they break down for sufficiently small distances. To illustrate this point, we shall consider a typical term in the potential to the order $O(\mathbf{Q}/m)$:

$$f(\mathbf{r})=\int\frac{d\boldsymbol{\Delta}\,\exp(-i\boldsymbol{\Delta}\cdot\mathbf{r})}{(m^2+\frac{1}{4}\boldsymbol{\Delta}^2)^{1/2}}\frac{1}{\boldsymbol{\Delta}^2+4\mu^2}.$$

Clearly, for $r\gg 1/2\mu$, the expansion of the square root about $\boldsymbol{\Delta}^2/4m^2=0$ is a very good approximation. However, this expansion leads to an asymptotic series in r (Watson's lemma), with the consequence that, for any fixed r, the series diverges and there is an optimal point for cutting off the series. It is also evident that for small r, i.e., $r\lesssim 1/m$, the approximation is quite poor for any number of terms. This is of course connected with the familiar fact that a particle may not be localized beyond its Compton wavelength. On the analytic side, the cause of the difficulty is of course the branch singularities of the square root at $\frac{1}{4}\boldsymbol{\Delta}^2+m^2=0$ and ∞, which in turn originates in relativistic kinematics.

Overlooking the operator nature of $\mathbf{Q}$ for the moment, one can say, as in the above example, that expansions in $\mathbf{Q}/m$ and $\boldsymbol{\Delta}/m$ are reliable for $r\gg 1/2\mu$ and

definitely unreliable for $r \lesssim 1/m$. The operator nature of $\mathbf{Q}$, however, plays a crucial role here, since it is the expectation value of this operator in a process that must be regarded as its magnitude in assessing the validity of the $\mathbf{Q}/m$ expansion. Hence the validity of such an expansion must be decided in the context of the Schrödinger equation in which it will be used. Let us therefore consider a nonlocal potential V that depends on $(\mathbf{Q}/m)^2$, and can thus be expanded as follows:

$$V = V^0 + v^0 \mathbf{Q}^2/m^2 + \cdots, \tag{4.11a}$$

where V^0 and v^0 are independent of $\mathbf{Q}/m$, and higher-order terms are regarded as unimportant. We are going to assume the validity of the expansion in (4.11a), and examine the consequences of such an assumption. In configuration space, the nonlocal component will be of the order of $(2v^0/m^2)\mathbf{P}^2$, and the Schrödinger equation will read

$$\left(\mathbf{P}^2 - \mathbf{q}^2 + mV^0 + \frac{2v^0}{m}\mathbf{P}^2 + \cdots\right)\psi = 0.$$

The validity of (4.11a) implies that the contribution of the nonlocal component is a small fraction of the total potential and therefore can be estimated as a perturbation. Thus the total potential in the Schrödinger equation is found to be

$$V = V^0\left(1 - \frac{v^0}{\frac{1}{2}m} + \frac{v^0}{V^0}\frac{\mathbf{q}^2}{\frac{1}{2}m^2} + \cdots\right). \tag{4.11b}$$

Assuming $\mathbf{q}^2$ to be sufficiently small for the moment (i.e., for low energies), we see that the validity of the $\mathbf{Q}^2/m^2$ expansion assumed in (4.11a) requires that v^0, which is of the order of V^0, be a small fraction of the reduced mass of the system. This is an intuitively obvious result; recoil effects can be regarded as small only when the potential energy is small relative to the (reduced) mass of the system. We therefore conclude that the expansion in the powers of $\mathbf{P}^2$ can be valid only for internucleon distances for which the potential is a small fraction of the nucleon mass.

Clearly, for sufficiently small internucleon distances, the potential will exceed $\frac{1}{2}m$, and the above expansion [(4.11a) and (4.11b)] will break down. This result again demonstrates the impossibility of a simple potential representation of a field-theoretical model at small distances. To find out the nature of the transition region where the $\mathbf{Q}/m$ expansion is expected to break down, we note that the fourth-order potential is a superposition of Yukawa potentials with a mass spectrum starting at 2μ. Thus the second term in (4.11b), being a product of two such potentials, has a mass spectrum starting at 4μ. This indicates a sharp rise over a distance of about $1/4\mu$, implying that the transition between the (simple) potential representation and extremely nonlocal regions is rather abrupt. Anticipating later results, we observe that our potential strengths (and indeed those of phenomenological ones as well) are well below $\frac{1}{2}m$ outside of $r = 1/2\mu$, and they begin to become comparable to $\frac{1}{2}m$ at about this distance. Thus a transition at about $r = 1/2\mu$ is indicated.

It is appropriate at this point to record a typical term in the potential, and discuss the dependence on W. Consider

$$f(\mathbf{p}',\mathbf{p}\,|\,W) = \frac{1}{[E(\mathbf{p}')E(\mathbf{p})]^{1/2}}\int_0^1 d\beta \int_0^{1-\beta} \frac{\alpha d\alpha}{\boldsymbol{\Delta}^2(\beta-\beta^2)+\alpha(2\beta\boldsymbol{\Delta}\cdot\mathbf{p}+m^2)+(W^2-\mathbf{p}^2)(\alpha^2-\alpha)+\mu^2(1-\alpha)}, \tag{4.12}$$

where we have employed a mixed notation using $\mathbf{p}'$, $\mathbf{p}$, and $\boldsymbol{\Delta} = \mathbf{p}' - \mathbf{p}$. As claimed before, it is seen that in the limit $\mathbf{Q}/m \to 0$ ($\mathbf{Q} = \mathbf{p}' + \mathbf{p}$), all $\mathbf{Q}$ dependences vanish. In this limit, and indeed for $|\mathbf{p}|$, $|\mathbf{p}'| < m$, one can verify by inspection that the threshold of singularities for $\boldsymbol{\Delta}^2$ occurs at $\alpha = 0$, $\beta = \frac{1}{2}$, and $\boldsymbol{\Delta}^2 = -4\mu^2$. To study the role of W, we first observe that on the energy shell, the factor $W^2 - \mathbf{p}^2$ is equal to m^2, eliminating W dependence. Off the energy shell, however, if one were to pass to the limit $\mathbf{Q}/m \to 0$, and consider the potential as dependent on W, for sufficiently large W, the threshold for $\boldsymbol{\Delta}^2$ would reach zero, where the potential would degenerate into an inverse-power function. The condition for the vanishing of the denominator in the limit $\mathbf{Q}/m = 0$ is

$$\boldsymbol{\Delta}^2(\beta - \beta^2 - \alpha\beta - (\alpha^2-\alpha)/4) + \mu^2(1-\alpha) + \alpha m^2 + W^2(\alpha^2-\alpha) = 0, \tag{4.13}$$

from which we get the condition for a $\boldsymbol{\Delta}^2 = 0$ singularity:

$$W^2 = \frac{\mu^2(1-\alpha)+\alpha m^2}{\alpha - \alpha^2}.$$

We now seek the minimum value of W^2 for which this equation is satisfied, subject to the constraint $0 \leq \alpha \leq 1$. The end points will not do, so we set

$$dW^2/d\alpha = 0, \quad (m^2-\mu^2)(\alpha-\alpha^2) - (1-2\alpha) \times [(1-\alpha)\mu^2 + \alpha m^2] = 0,$$

which yields the solution

$$\alpha = \mu/(m+\mu), \quad W = \mu + m.$$

Thus the minimum value of W for which the potential dengerates is $m+\mu$, half a pion mass above the inelastic threshold. At such high energies, the present approximation is in doubt, and therefore this is not a very disturbing fact. However, numerical calculations on sample cases showed that there is a strong dependence on W even for low energies. The reason is that any value of W larger than m allows the threshold for $-\boldsymbol{\Delta}^2$ singu-

larity to drop below $4\mu^2$, and it thus fundamentally alters the spectral composition of the potential. Moreover, this alteration persists for large r, making it all the more unsatisfactory. On the other hand, for sufficiently large r, the potential energy is negligible compared to the kinetic energy, and the energy-shell condition is satisfied, i.e., $\mathbf{p}^2=\mathbf{p}'^2=W^2-m^2$, and as noted before, the W dependence must vanish, a fact that contradicts the behavior actually observed in the sample case above. The difficulty lies in taking the $\mathbf{Q}/m\to 0$ limit first, and then considering the W dependence. More explicitly, for large r, where the energy-shell condition is satisfied, an expansion in $\mathbf{Q}^2/m^2$ is equivalent to that in $(W^2-m^2)/m^2$ (and in fact $\mathbf{p}^2/m^2$ and $\mathbf{p}'^2/m^2$), so that effecting one without the other leads to erroneous results. Thus, if one insists on retaining terms to the order $\mathbf{Q}^2/m^2$, one must also retain the dependence on $(W^2-m^2)/m^2$.

We now summarize the important conclusions of the foregoing paragraphs. The nonlocality of the potential in configuration space is of an exponential character, and it is mainly confined to the core region, where it is quite strong. The expansion of the potential in the powers of the momentum operator is an asymptotic one, and it is definitely unreliable in the core region. The breakdown region for this expansion is around $r=1/2\mu$, and the transition is rather sharp (on the scale of wavelengths of typical energies of elastic nucleon-nucleon scattering). On the other hand, the proper long-range behavior of the potential demands that the quantities $\mathbf{Q}^2/m^2$ and $(W^2-m^2)/m^2$ be treated in a like manner as far as expansions are concerned, since these two quantities must cancel each other for large distances. Consequently, the inclusion of terms of the order of $\mathbf{P}^2$ and higher requires (a) an expansion of the momentum-space potential about $\mathbf{Q}/m=0$, and the retention of terms that are small for distances where they are reliable, and (b) a simultaneous inclusion of terms of the order of $(W^2-m^2)/m^2$ and higher, hence an energy-dependent potential. Therefore the cost of including $\mathbf{P}^2$ terms is not only their unreliability at about $r=1/2\mu$ and shorter distances, but also the loss of a useful property of a potential, namely, its energy independence. The advantage, on the other hand, is a gain in accuracy outside the core radius, where that gain is of the order of the ratio of the potential strength to the reduced mass of the nucleon-nucleon system.

Accordingly, we choose to retain terms up to and including $\mathbf{Q}/m$, thus obtaining central, spin-orbit, tensor, and spin-spin potentials. We further remove all W dependence in each of these potentials by consistently neglecting terms of the order of $(W^2-m^2)/m^2$, $\mathbf{p}^2/m^2$, and $\mathbf{p}'^2/m^2$ relative to 1. One would expect these approximations to break down for energies and/or distances where the typical energies (kinetic or potential) of the scattering process become comparable to the nucleon mass. Accordingly, below the pion-production threshold, and for distances $r\gtrsim 1/2\mu$ (anticipating later results), we expect these approximations to hold.

B. Integral Representations

In Sec. IV A we defined our approximation scheme to be applied to the energy-dependent and nonlocal potential formally defined in Sec. III. Here we shall formulate the appropriate amplitudes, apply the approximations, and obtain integral representations for the parts comprising the two-pion exchange potential. Before delving into the mass of formulas, it is appropriate to establish our notation for the various components of the potential. All potentials will be designated by the letter V followed by two letters specifying the exchanged boson, a number (either 0 or 1) designating the total isotopic spin, and a second symbol (+, 0, or −) specifying the third component of the total isotopic spin in the case of OPEP and TPEP. The specification of the third component of the total isotopic spin for OPEP and TPEP anticipates the incorporation of the pion mass differences into our potentials. In addition, every potential is indexed from 1 to 4 corresponding to the central, spin-orbit, tensor, and spin-spin parts, respectively. The two letters specifying the exchanged bosons are PI (pion), PP (two pions), ET (eta), RH (rho), and OM (omega). Thus, for example, V(PP1+(2)) denotes the spin-orbit part of TPEP in the isotriplet state corresponding to proton-proton scattering. Analogously, V(RH0(3)) denotes the tensor part of the ρ exchange potential in the isosinglet state. For the present, we shall ignore the pion mass differences and use only the charged pion mass $\mu=1$. Occasionally, we shall find it expedient to use new symbols for groups of potentials.

We begin by calculating $\mathbf{U}^{(4)}$. From Eqs. (3.18), we have

$$\mathbf{U}^{(4)}=\mathcal{K}^{(4)}+\mathcal{K}^{(2)}G\mathcal{K}^{(2)}-\mathcal{K}^{(2)}g\mathcal{K}^{(2)}.$$

The first two terms correspond to the familiar fourth-order crossed and ladder diagrams, respectively, and the last term is the iteration of the exact OPEK. We will formulate these kernels and the resulting potentials for the case of two distinguishable nucleons, each assigned an isotopic spin τ. As is well known, the necessary antisymmetrization can be effected when computing Schrödinger amplitudes. Using the familiar Feynman rules for pseudoscalar coupling, we obtain

$$\mathcal{K}^{(4)}=\frac{(G_\pi)^4}{4\pi^4}(3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{i}{2\pi}\int d^4k \times\frac{[\gamma\cdot(W+p-k)-m]^{(1)}[\gamma\cdot(W-p'-k)-m]^{(2)}}{[(W+p-k)^2-m^2+i\epsilon][(W-p'-k)^2-m^2+i\epsilon][k^2-\mu^2+i\epsilon][(k-p+p')^2-\mu^2+i\epsilon]},$$

$$\mathcal{K}^{(2)}G\mathcal{K}^{(2)}=\frac{(G_\pi)^4}{4\pi^4}(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\frac{i}{2\pi}\int d^4k \times\frac{[\gamma\cdot(W+p+k)-m]^{(1)}[\gamma\cdot(W-p-k)-m]^{(2)}}{[k^2-\mu^2+i\epsilon][(k+p-p')^2-\mu^2+i\epsilon][(W-p-k)^2-m^2+i\epsilon][(W+p+k)^2-m^2+i\epsilon]}, \quad (4.14)$$

$$\mathcal{K}^{(2)}g\mathcal{K}^{(2)}=\frac{(G_\pi)^4}{4\pi^4}(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\tfrac{1}{4}\int d\mathbf{k}\frac{[\gamma^0E(\mathbf{k})-\boldsymbol{\gamma}\cdot\mathbf{k}-m]^{(1)}[\gamma^0E(\mathbf{k})+\boldsymbol{\gamma}\cdot\mathbf{k}-m]^{(2)}}{[(\mathbf{k}-\mathbf{p}')^2+\mu^2][W^2-E^2(\mathbf{k})+i\epsilon]E(\mathbf{k})[(\mathbf{k}-\mathbf{p})^2+\mu^2]}.$$

The symbols W, p, p', and $E(\mathbf{k})$ have the same meaning as before. The rationalized pion-nucleon coupling constant has been denoted by G_π. Our first task is to demonstrate the cancellation between $\mathcal{K}^{(2)}GK^{(2)}$ and $\mathcal{K}^{(2)}gK^{(2)}$ that removes the strong energy dependence present in both. To accomplish this, we must recast the integrals of (4.14) into more suitable forms. This is done in Appendix A, using mostly the familiar Feynman techniques. There, we consider the quantities $\mathcal{J}$ and $\mathcal{I}$ defined by

$$\mathcal{K}^{(2)}\ g\mathcal{K}^{(2)}|_{p^0=p'^0=0}=\frac{(G_\pi)^4}{4\pi^4}(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\mathcal{I},$$

$$\mathcal{K}^{(2)}G\mathcal{K}^{(2)}|_{p^0=p'^0=0}=\frac{(G_\pi)^4}{4\pi^4}(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\mathcal{J}. \quad (4.15)$$

According to Eqs. (A2) and (A5), we have

$$\mathcal{J}=-\tfrac{1}{4}\pi\int_0^1 d\beta\int_0^{1-\beta}\zeta d\zeta\frac{\gamma^{\mu(1)}\gamma_\mu{}^{(2)}}{R+W^2(\zeta^2-\zeta)+\zeta m^2+\mu^2(1-\zeta)}-\tfrac{1}{2}\pi\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_0^\zeta d\xi \times\frac{-\xi^2W^2\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}+\{\gamma^0W-\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(1)}\{\gamma^0W+\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(2)}}{[R+W^2(\xi^2-\zeta)+\zeta m^2+\mu^2(1-\zeta)-i\epsilon]^2}, \quad (4.16)$$

$$\mathcal{I}=-\tfrac{1}{2}W\int_{-\infty}^{+\infty}dZ\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_0^\infty d\xi\frac{\gamma^{0(1)}\gamma^{0(2)}}{[R+W^2\xi^2+(1-\zeta)\mu^2+\zeta(Z^2+m^2)]^2}+\tfrac{1}{2}W\int_{-\infty}^{+\infty}dZ\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_0^\infty d\xi \times\frac{-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}W^2\xi^2+\{\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(1)}\{-\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(2)}+W^2\gamma^{0(1)}\gamma^{0(2)}}{(Z^2+m^2)[R+W^2\xi^2+(1-\zeta)\mu^2+\zeta^2(Z^2+m^2)]^2}$$
$$-\tfrac{1}{2}\pi\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_0^\infty d\xi \times\frac{-\xi^2W^2\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}+\{\gamma^0W-\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(1)}\{\gamma^0W+\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(2)}}{[R+W^2(\xi^2-\zeta)+\zeta m^2+\mu^2(1-\zeta)-i\epsilon]^2}, \quad (4.17)$$

$$R=\beta(\mathbf{p}-\mathbf{p}')^2+\zeta\mathbf{p}^2-[\beta(\mathbf{p}'-\mathbf{p})-\zeta\mathbf{p}]^2$$
$$=(1-\beta-\zeta)(\beta\boldsymbol{\Delta}^2+\zeta\mathbf{p}^2)+\beta\zeta\mathbf{p}'^2.$$

We now proceed to examine the analytic structure of the integrals as a function of W. Let the distinct denominators appearing in (4.16) and (4.17) be denoted by D_i, i denoting the order in which they are written. Thus we have

$$D_1=R+W^2(\zeta^2-\zeta)+\zeta m^2+\mu^2(1-\zeta),$$
$$D_2=R+W^2(\xi^2-\zeta)+\zeta m^2+\mu^2(1-\zeta), \quad (4.18)$$
$$D_3=R+W^2\xi^2+\zeta(Z^2+m^2)+\mu^2(1-\zeta).$$

Starting with the threshold behavior, we observe that only D_2 can vanish for $W=m$, at $\xi=0$, $\zeta=1$, $\beta=0$. Therefore the usual energy cut starting at $W=m$ is contained in the second and third integrals, respectively, of (4.16) and (4.17). However, when these two terms are combined in $\mathcal{K}^{(2)}(G-g)\mathcal{K}^{(2)}$, the range of integration for ξ is restricted to $[\zeta,\infty]$. With ξ restricted to this range, D_2 cannot vanish at $W=m$, i.e., the energy cut has canceled out, at least in the neighborhood of the threshold. This is, of course, the cancellation first pointed out and emphasized by CFT within their adiabatic approximation. Here we have demonstrated it without making any approximation. As discussed in

Sec. III, the cancellation is a direct consequence of the identical cut structures of G and g in the physical region. Moreover, we observe that after the cancellation, the lowest threshold occurs for D_1 at $R=0$, $\zeta=\mu/(\mu+m)$, $W=m+\mu$ (cf. Sec. IV A). The energy cut has thus moved to the two-pion production threshold. Finally, we must examine the behavior of the integrals originating in $\mathcal{K}^{(4)}$. With the notation

$$\mathcal{K}^{(4)}=\frac{(G_\pi)^4}{4\pi^4}(3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\mathfrak{L}, \tag{4.19}$$

it turns out that $\mathfrak{L}$ involves precisely the denominator D_1 occurring in $\mathcal{J}$. The above analysis therefore holds for $\mathfrak{L}$ as well. The procedure carried out for $\mathcal{I}$ and $\mathcal{J}$ in Appendix A can be duplicated for $\mathfrak{L}$, and the steps are therefore omitted here.

We are now in a position to apply the approximations set forth in Sec. IV A. This will entail an expansion in $\mathbf{Q}/m$, neglecting terms of the order of $\mathbf{Q}^2/m^2$, and the limits $(W^2-m^2)/m^2$, $\mathbf{p}^2/m^2$, $\mathbf{p}'^2/m^2\to 0$. These approximations will be applied in two steps. First, we shall apply them to the integrals leaving the γ matrices unevaluated. Next, we shall proceed to reduce these matrices according to the same approximations. Carrying out the first step, we obtain

$$\begin{aligned}
\mathcal{J}-\mathcal{I}=&-\tfrac{1}{4}\pi\int_0^1 d\beta\int_0^{1-\beta}\zeta d\zeta\,\frac{\gamma^{\lambda(1)}\gamma_\lambda{}^{(2)}}{\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\zeta^2m^2+(1-\zeta)\mu^2}\\
&+\tfrac{1}{2}\pi\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_\zeta^\infty d\xi\,\frac{-\xi^2m^2\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}}{[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\xi^2m^2+(1-\zeta)\mu^2]^2}\\
&+\tfrac{1}{2}m\int_{-\infty}^{+\infty}dZ\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_0^\infty d\xi\,\frac{\gamma^{0(1)}\gamma^{0(2)}}{[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\xi^2m^2+\zeta(Z^2+m^2)+(1-\zeta)\mu^2]^2}\\
&-\tfrac{1}{2}m\int_{-\infty}^{+\infty}dZ\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_0^\infty d\xi\,\frac{-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}m^2\xi^2+\{\gamma^0m-\zeta\boldsymbol{\gamma}\cdot\mathbf{p}\}^{(1)}\{\gamma^0m+\zeta\boldsymbol{\gamma}\cdot\mathbf{p}\}^{(2)}+m^2\gamma^{0(1)}\gamma^{0(2)}}{(Z^2+m^2)[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\xi^2m^2+\zeta(Z^2+m^2)+(1-\zeta)\mu^2]^2},
\end{aligned}\tag{4.20}$$

$$\begin{aligned}
\mathfrak{L}=&\tfrac{1}{4}\pi\int_0^1 d\beta\int_0^{1-\beta}\zeta d\zeta\,\frac{\gamma^{\lambda(1)}\gamma_\lambda{}^{(2)}}{\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\zeta^2m^2+(1-\zeta)\mu^2}\\
&-\tfrac{1}{2}\pi m^2\int_0^1 d\beta\int_0^{1-\beta}\zeta^3d\zeta\,\frac{\gamma^{0(1)}\gamma^{0(2)}}{[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\zeta^2m^2+(1-\zeta)\mu^2]^2}.
\end{aligned}$$

The third and fourth integrals in $\mathcal{J}-\mathcal{I}$ can be further reduced. Let their sum be $\mathcal{S}$. Then changing variables according to

$$Z=t\cos\varphi,\quad \xi=\frac{\sqrt{\zeta}}{m}t\sin\varphi,$$

and carrying out the angular integration over φ, we get

$$\begin{aligned}
\mathcal{S}=&\tfrac{1}{2}\pi\int_0^1 d\beta\int_0^{1-\beta}d\zeta\zeta(\sqrt{\zeta})\int_0^\infty tdt\left[\frac{(t^2+m^2)^{1/2}}{m}-1\right]\frac{\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}}{[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\zeta t^2+\zeta m^2+\mu^2(1-\zeta)]^2}\\
&-\tfrac{1}{2}\pi\int_0^1 d\beta\int_0^{1-\beta}(\sqrt{\zeta})d\zeta\int_0^\infty tdt\frac{1}{m(t^2+m^2)^{1/2}}\,\frac{(\gamma^0m-\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(1)}(\gamma^0m+\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(2)}}{[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\zeta t^2+\zeta m^2+(1-\zeta)\mu^2]^2}\\
&+\tfrac{1}{2}\pi\int_0^1 d\beta\int_0^{1-\beta}(\sqrt{\zeta})d\zeta\int_0^\infty tdt\left[1-\frac{m}{(t^2+m^2)^{1/2}}\right]\frac{\gamma^{0(1)}\gamma^{0(2)}}{[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\zeta t^2+\zeta m^2+(1-\zeta)\mu^2]^2}.
\end{aligned}$$

Another change of variable $(t^2+m^2)^{1/2}=(m/\sqrt{\zeta})\xi$ and some rearrangement yield

$$\begin{aligned}
\mathcal{S}=&\tfrac{1}{4}\pi\int_0^1 d\beta\int_0^{1-\beta}\frac{d\zeta}{\sqrt{\zeta}}\,\frac{\gamma^{0(1)}\gamma^{0(2)}}{\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\zeta m^2+(1-\zeta)\mu^2}+\tfrac{1}{4}\pi\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_{\sqrt{\zeta}}^\infty d\xi\,\frac{\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}}{\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\xi^2m^2+(1-\zeta)\mu^2}\\
&-\tfrac{1}{2}\pi\int_0^1 d\beta\int_0^{1-\beta}d\zeta\int_{\sqrt{\zeta}}^\infty d\xi\,\frac{(\gamma^0m-\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(1)}(\gamma^0m+\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(2)}+m^2\gamma^{0(1)}\gamma^{0(2)}}{[\beta(1-\beta-\zeta)\boldsymbol{\Delta}^2+\xi^2m^2+(1-\zeta)\mu^2]^2}.
\end{aligned}\tag{4.21}$$

Finally, considering all the integrals, we make the change $\bar{\beta}=2\beta(1-\zeta)^{-1}-1$, rescale ξ, and rearrange them in the following formulas:

$$\mathcal{J}-\mathcal{I}=-\tfrac{1}{4}\pi\int_0^1\int_0^1 d\zeta d\beta\,\zeta\gamma^{0(1)}\gamma^{0(2)}X^{-1}-\tfrac{1}{4}\pi m^{-2}\int_0^1\int_0^1 d\zeta d\beta\,\zeta^{-1/2}(\gamma^0 m-\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(1)}(\gamma^0 m+\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(2)}Y^{-1}$$
$$-\tfrac{1}{4}\pi\int_0^1\int_0^1 d\zeta d\beta\,\zeta\int_1^\infty d\xi\,\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}W^{-1}+\tfrac{1}{4}\pi\int_0^1\int_0^1 d\zeta d\beta\,\zeta^{1/2}\int_1^\infty d\xi\,\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}Z^{-1}$$
$$+\tfrac{1}{4}\pi m^{-2}\int_0^1\int_0^1 d\zeta d\beta\,\zeta^{-1/2}\int_1^\infty d\xi^{-2}[(\gamma^0 m-\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(1)}(\gamma^0 m+\zeta\boldsymbol{\gamma}\cdot\mathbf{p})^{(2)}+m^2\gamma^{0(1)}\gamma^{0(2)}]Z^{-1}, \quad (4.22)$$
$$\mathcal{L}=\tfrac{1}{4}\pi\int_0^1\int_0^1 d\zeta d\beta\,\zeta(\gamma^{0(1)}\gamma^{0(2)}-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)})X^{-1}-\tfrac{1}{2}\pi m^2\int_0^1\int_0^1 d\zeta d\beta\,\zeta^3(1-\zeta)^{-1}\gamma^{0(1)}\gamma^{0(2)}X^{-2},$$

where the bar in $\bar{\beta}$ has been dropped and

$$[X,Y,W,Z]=\tfrac{1}{4}\boldsymbol{\Delta}^2(1-\beta^2)(1-\zeta)+\mu^2+m^2(1-\zeta)^{-1}\times[\zeta^2,\zeta,\zeta^2\xi^2,\zeta\xi^2]. \quad (4.23)$$

The second step in the reduction procedure involves computing $\mathcal{J}-\mathcal{I}$ and $\mathcal{L}$ as functions of $(s_1's_2';s_1,s_2)$ according to (4.1), and rewriting them as matrix operators in the Pauli-Schrödinger representation. This is achieved by writing

$$u(\mathbf{p},s)=\left[\frac{E(\mathbf{p})+m}{2m}\right]^{1/2}\begin{bmatrix}\chi_s\\ \{\boldsymbol{\sigma}\cdot\mathbf{p}/[E(\mathbf{p})+m]\}\chi_s\end{bmatrix}, \quad (4.24)$$

where χ_s is the Pauli spinor of spin s, and reducing the matrix elements of the γ matrices to those of Pauli operators between the χ's. We shall explicitly do this for a typical case, and record the result for the rest. Consider, for example, $\gamma^{0(1)}\gamma^{0(2)}$. We may consider the two matrices separately, as they operate on a product space. Then

$$\bar{u}(\mathbf{p}',s')\gamma^0 u(\mathbf{p},s)=\frac{\{[E(\mathbf{p}')+m][E(\mathbf{p})+m]\}^{1/2}}{2m}\times\langle\chi_{s'}|1+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'}{E(\mathbf{p}')+m}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E(\mathbf{p})+m}|\chi_s\rangle.$$

Therefore the passage from the Dirac to the Pauli-Schrödinger representation is in this instance given by the replacement

$$\gamma^{0(1)}\gamma^{0(2)}\to\frac{E(\mathbf{p}')+m}{2m}\frac{E(\mathbf{p})+m}{2m}\times\left(1+\frac{\boldsymbol{\sigma}^{(1)}\cdot\mathbf{p}'\boldsymbol{\sigma}^{(1)}\cdot\mathbf{p}}{[E(\mathbf{p}')+m][E(\mathbf{p})+m]}\right)\times\left(1+\frac{\boldsymbol{\sigma}^{(2)}\cdot\mathbf{p}'\boldsymbol{\sigma}^{(2)}\cdot\mathbf{p}}{[E(\mathbf{p}')+m][E(\mathbf{p})+m]}\right).$$

According to our approximation procedure, $E(\mathbf{p}')=E(\mathbf{p})=m$ when occuring symmetrically. Then

$$\gamma^{0(1)}\gamma^{0(2)}\to\left(1+\frac{\mathbf{p}\cdot\mathbf{p}'+i\boldsymbol{\sigma}^{(1)}\cdot(\mathbf{p}'\times\mathbf{p})}{4m^2}\right)\times\left(1+\frac{\mathbf{p}\cdot\mathbf{p}'+i\boldsymbol{\sigma}^{(2)}\cdot(\mathbf{p}'\times\mathbf{p})}{4m^2}\right).$$

Again, we neglect $\mathbf{p}\cdot\mathbf{p}'/m^2$ relative to 1 in the central part of this expression, and retain

$$1+i\frac{(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{p}'\times\mathbf{p})}{4m^2},$$

where we have dropped the term quadratic in $\mathbf{p}'\times\mathbf{p}/m^2$, since the latter expression is already of first order in $\mathbf{Q}/m$. An analogous procedure may be carried out for the rest of the operators. We record the results in the following formulas[19]:

$$I^{(1)}I^{(2)}\to 1-\frac{\boldsymbol{\Omega}_{\mathrm{SO}}}{2m^2},\quad \gamma^{0(1)}\gamma^{0(2)}\to 1+\frac{\boldsymbol{\Omega}_{\mathrm{SO}}}{2m^2},$$
$$\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}\to\frac{\boldsymbol{\Omega}_{\mathrm{T}}}{12m^2}+\frac{\boldsymbol{\Omega}_{\mathrm{SS}}}{6m^2}-\frac{\boldsymbol{\Omega}_{\mathrm{SO}}}{m^2},\quad (4.25)$$
$$\gamma^{0(1)}I^{(2)}+I^{(1)}\gamma^{0(2)}\to 2,$$

where

$$\boldsymbol{\Omega}_{\mathrm{SO}}=\tfrac{1}{2}i(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})\cdot(\mathbf{p}'\times\mathbf{p}),$$
$$\boldsymbol{\Omega}_{\mathrm{T}}=\boldsymbol{\Delta}^2\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}-3\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\Delta}\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\Delta}, \quad (4.26)$$
$$\boldsymbol{\Omega}_{\mathrm{SS}}=\boldsymbol{\Delta}^2\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}.$$

The last three operators in Eqs. (4.26) correspond to the spin-orbit, tensor, and spin-spin operators in configuration space,[19] as we shall presently verify. We first observe that $\mathbf{p}'\times\mathbf{p}=\frac{1}{2}\boldsymbol{\Delta}\times\mathbf{Q}$. So we have

$$\boldsymbol{\Omega}_{\mathrm{SO}}=\tfrac{1}{2}i[\tfrac{1}{2}(\boldsymbol{\sigma}^{(1)}+\boldsymbol{\sigma}^{(2)})]\cdot(\boldsymbol{\Delta}\times\mathbf{Q})\equiv\tfrac{1}{2}i\boldsymbol{\Sigma}\cdot(\boldsymbol{\Delta}\times\mathbf{Q}),$$

where the last equation defines $\boldsymbol{\Sigma}$. We have already seen in (4.7) that a potential of the form $\mathbf{Q}\cdot\mathbf{f}(\boldsymbol{\Delta})$ trans-

forms into $-[\mathbf{P}\cdot\mathbf{f}(\mathbf{r})+\mathbf{f}(\mathbf{r})\cdot\mathbf{P}]$ in the configuration representation. Thus remembering that $\boldsymbol{\Delta}f(\boldsymbol{\Delta})$ transforms into $i\boldsymbol{\nabla}f(\mathbf{r})$, we see that $\boldsymbol{\Omega}_{\mathrm{SO}}f(\boldsymbol{\Delta})$ will transform into

$$\tfrac{1}{2}\{-\mathbf{P}\times[\boldsymbol{\nabla}f(\mathbf{r})]+[\boldsymbol{\nabla}f(\mathbf{r})]\times\mathbf{P}\}\cdot\boldsymbol{\Sigma}.$$

As $f(\mathbf{r})$ will always be a function of r only, we may write

$$\boldsymbol{\nabla}f(r)=\mathbf{r}f'(r)/r,$$

where the prime denotes differentiation with respect to the argument. Since $-\mathbf{P}\times\mathbf{r}=\mathbf{r}\times\mathbf{P}$, and $[\mathbf{r}\times\mathbf{P},f'(r)/r]=0$, we finally get

$$\boldsymbol{\Omega}_{\mathrm{SO}}f(\boldsymbol{\Delta})\rightarrow\frac{f'(r)}{r}(\mathbf{r}\times\mathbf{P})\cdot\boldsymbol{\Sigma}=\frac{f'(r)}{r}\Omega_{\mathrm{SO}},$$

where now Ω_{SO} denotes the spin-orbit operator in the configuration representation. An analogous analysis yields the following connections:

$$\begin{aligned}
&f(\boldsymbol{\Delta})\rightarrow f(r),\\
&f(\boldsymbol{\Delta})\boldsymbol{\Omega}_{\mathrm{SO}}\rightarrow\frac{f'(r)}{r}\Omega_{\mathrm{SO}},\\
&f(\boldsymbol{\Delta})\boldsymbol{\Omega}_{\mathrm{T}}\rightarrow\left[f''(r)-\frac{f'(r)}{r}\right]\Omega_{\mathrm{T}},\\
&f(\boldsymbol{\Delta})\boldsymbol{\Omega}_{\mathrm{SS}}\rightarrow-\left[f''(r)+\frac{2}{r}f'(r)\right]\Omega_{\mathrm{SS}},
\end{aligned}\tag{4.27}$$

where the configuration-space operators are defined as usual:

$$\begin{aligned}
&\Omega_{\mathrm{SO}}=\boldsymbol{\Sigma}\cdot(\mathbf{r}\times\mathbf{P}),\\
&\Omega_{\mathrm{T}}=3\boldsymbol{\sigma}^{(1)}\cdot\hat{r}\boldsymbol{\sigma}^{(2)}\cdot\hat{r}-\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)},\\
&\Omega_{\mathrm{SS}}=\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}.
\end{aligned}\tag{4.28}$$

We have now assembled all the information needed to give explicit integral representations for the parts occurring in TPEP. We shall designate the integrals occurring in (4.22) (in the order they are written) by X_A, Y_A, W_A, Z_A, Z_B, X_B, and X_C. In transforming to the configuration representation, each of these integrals may give rise to different types of potentials. We shall express this by indexing the above symbols according to the notation established at the beginning of this section. Thus, for example,

$$X_A=X_A(1)+X_A(2)\Omega_{\mathrm{SO}}.$$

We shall explicitly work out the transformation to configuration space for X_A, and record the result for the rest, since they are obtained in an entirely analogous fashion. We have

$$X_A=-\tfrac{1}{4}\pi\int_0^1\int_0^1 d\zeta d\beta\,\zeta X^{-1}(1+\tfrac{1}{2}m^{-2}\boldsymbol{\Omega}_{\mathrm{SO}}),$$

and in configuration space, using the auxiliary functions

$$\begin{aligned}
&D\equiv(1-\zeta)^{-1}(1-\beta^2)^{-1},\\
&E(\alpha)\equiv 2D^{1/2}[\mu^2+\alpha m^2(1-\zeta)^{-1}]^{1/2},
\end{aligned}\tag{4.29}$$

we have

$$\begin{aligned}
X_A&=-\tfrac{1}{4}\pi\int_{-\infty}^{\infty}d\boldsymbol{\Delta}\exp(-i\boldsymbol{\Delta}\cdot\mathbf{r})\int_0^1\int_0^1 d\zeta d\beta\\
&\quad\times\zeta X^{-1}(1+\tfrac{1}{2}m^{-2}\boldsymbol{\Omega}_{\mathrm{SO}})\\
&=-\pi\left(1+\tfrac{1}{2}m^{-2}\Omega_{\mathrm{SO}}r^{-1}\frac{\partial}{\partial r}\right)\int_0^1\int_0^1\int_{-\infty}^{\infty}d\zeta d\beta\\
&\quad\times\zeta D\frac{d\boldsymbol{\Delta}\exp(-i\boldsymbol{\Delta}\cdot\mathbf{r})}{\boldsymbol{\Delta}^2+[E(\zeta^2)]^2}\\
&=-2\pi^3\left(1+\tfrac{1}{2}m^{-2}\Omega_{\mathrm{SO}}r^{-1}\frac{\partial}{\partial r}\right)\int_0^1\int_0^1 d\zeta d\beta\,\zeta D r^{-1}\\
&\quad\times\exp[-rE(\zeta^2)],
\end{aligned}$$

$$X_A(1)=-2\pi^3 r^{-1}\int_0^1\int_0^1 d\zeta d\beta\,\zeta D\exp[-rE(\zeta^2)],$$

$$\begin{aligned}
X_A(2)&=\pi^3 m^{-2}r^{-3}\int_0^1\int_0^1 d\zeta d\beta\,\zeta D[1+rE(\zeta^2)]\\
&\quad\times\exp[-rE(\zeta^2)].
\end{aligned}$$

To record the configuration-space potentials, we make the further auxiliary definitions

$$\begin{aligned}
&g_1[E(\alpha),r]\equiv 2\pi^3 r^{-1}\zeta D\exp[-E(\alpha)r],\\
&g_2[E(\alpha),r]\equiv-\tfrac{1}{2}(mr)^{-2}[E(\alpha)r+1]g_1[E(\alpha),r],\\
&g_3[E(\alpha),r]\equiv-\tfrac{1}{6}(2mr)^{-2}\{[E(\alpha)]^2r^2+3E(\alpha)r+3\}\\
&\qquad\times g_1[E(\alpha),r],\\
&g_4[E(\alpha),r]\equiv\tfrac{1}{12}m^{-2}[E(\alpha)]^2 g_1[E(\alpha),r].
\end{aligned}\tag{4.30}$$

The integral representations follow.

$$X_A(j)=-\int_0^1\int_0^1 d\zeta d\beta\,g_j[E(\zeta^2),r],\quad j=1,2$$

$$\begin{aligned}
Y_A(j)&=-\int_0^1\int_0^1 d\zeta d\beta\,g_j[E(\zeta),r]\\
&\quad\times\zeta^{-3/2}[1-2(j-1)\zeta],\quad j=1,2
\end{aligned}$$

$$W_A(j)=2\int_0^1\int_0^1\int_1^{\infty}d\zeta d\beta d\xi\,g_j[E(\zeta^2\xi^2),r],\quad j=2,3,4$$

$$Z_A(j)=-2\int_0^1\int_0^1\int_1^{\infty}d\zeta d\beta d\xi\,g_j[E(\zeta\xi^2),r]\zeta^{-1/2},\quad j=2,3,4$$

$$Z_B(j)=2\int_0^1\int_0^1\int_1^\infty d\zeta d\beta d\xi\, g_j[E(\zeta\xi^2),r]$$
$$\times\xi^{-2}\zeta^{-3/2}[1-(j-1)\zeta],\quad j=1,2$$
$$X_B(1)=-X_A(1),X_B(2)=-3X_A(1),$$
$$X_B(j)=2\int_0^1\int_0^1 d\zeta d\beta\, g_j[E(\zeta^2),r],\quad j=3,4$$
$$X_C(1)=-8\pi^3m^2\int_0^1\int_0^1 d\zeta d\beta\,\zeta^3(1-\zeta)^{-1}$$
$$\times D^2[E(\zeta^2)]^{-1}\exp[-E(\zeta^2)r],$$
$$X_C(2)=4\pi^3\int_0^1\int_0^1 d\zeta d\beta\,\zeta^3(1-\zeta)^{-1}D^2r^{-1}$$
$$\times\exp[-E(\zeta^2)r]. \quad (4.31)$$

We conclude this section by recording spectral representations for the integrals occurring above. The technique for obtaining these can be found in Ref. 15, where some of the integrals listed below may also be found. Not all the integrals listed in (4.31) need to be recorded. The following set contains all the basically different representations. Let

$$I_j=\tfrac{1}{4}\pi\int_0^1\int_0^1 d\zeta d\beta\, h_j,$$
$$h_1=-\zeta X^{-1},$$
$$h_2=-\zeta\int_1^\infty d\xi\, W^{-1},$$
$$h_3=-2m^2\zeta^3(1-\zeta)^{-1}X^{-2},$$
$$h_4=(\zeta)^{-1/2}Y^{-1},\quad (4.32)$$
$$h_5=(\zeta)^{1/2}\int_1^\infty d\xi\, Z^{-1},$$
$$h_6=2m^2(\zeta)^{1/2}(1-\zeta)^{-1}\int_1^\infty d\xi\, Z^{-2},$$

and

$$\tilde{I}_j(r)=\tfrac{1}{2}\pi^{-2}\int e^{-i\Delta\cdot r}I_j d\boldsymbol{\Delta}.$$

With the above definitions, and explicitly setting $\mu=1$, we have

$$\tilde{I}_j(r)=r^{-1}\int_4^\infty dt\,\rho_j(t)\exp(-rt^{1/2}),$$
$$\rho_1(t)=-\tfrac{1}{2}\pi t^{-1/2}u^2[v-|u|(v^2+2)\Phi],$$
$$\rho_2(t)=-\tfrac{1}{2}\pi m^{-1}t^{-3/2}[(v^2+2)(\tfrac{1}{4}\pi-m|u|\Phi)-\Theta],$$
$$\rho_3(t)=\tfrac{1}{2}\pi m^2t^{-1/2}u^4[12|u|(v^2+2)\Phi-8v$$
$$-v(v^2+2)^2(1+m^2v^2)^{-1}],$$
$$\rho_4(t)=\pi t^{-1}\int_0^{\pi/2}d\theta R_1(\theta,t),$$
$$\rho_5(t)=-\tfrac{1}{2}\rho_4(t)+\tfrac{1}{8}m^{-1}t^{-1/2}\pi^2-\tfrac{1}{2}t^{-1}\pi$$
$$\times\int_0^{\pi/2}d\theta\, R_1(\theta,t)R_2(\theta,t)R_3(\theta,t),$$
$$\rho_6(t)=4\pi m^2t^{-2}\int_0^{\pi/2}d\theta\, R_1(\theta,t)R_2(\theta,t)R_3^{-1}(\theta,t). \quad (4.33)$$

The new symbols occurring above are defined as follows:

$$v^2=t-4,\quad u^{-2}=4m^2-t,\quad \Theta=\cot^{-1}mv,$$
$$\Phi=\begin{cases}\cos^{-1}[\tfrac{1}{2}(v^2+2)(1+m^2v^2)^{-1/2}], & u^{-2}\geqslant 0\\ -\tfrac{1}{2}\ln 4(1+m^2v^2)\\ \quad+\ln[(v^2+2)+v(-u^{-2})^{1/2}], & u^{-2}<0\end{cases}$$
$$R_1(\theta,t)=[(\cos\theta)^2-R_-+R_+(\sin\theta)^2]^{-1/2},$$
$$R_2(\theta,t)=(\sin\theta)^2[(\cos\theta)^2-R_-]^{-1},$$
$$R_3(\theta,t)=(\cos\theta)^2+R_+(\sin\theta)^2,$$
$$\tfrac{1}{2}tR_\mp=-m^2+1\mp[(m^2-1)^2+m^2t]^{1/2}.$$

C. Definition of TPEP

In Sec. IV B, we essentially completed the definition of TPEP. Considering the mass of the pions as degenerate for the moment, we have

$$V(\mathrm{PP})=\frac{(G_\pi)^4}{4\pi^4}[(3-2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(\mathcal{J}-\mathcal{I})$$
$$+(3+2\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\mathcal{L}]. \quad (4.34)$$

with $\mathcal{J}-\mathcal{I}$ and $\mathcal{L}$ defined by (4.22) and (4.31). We now proceed to take account of the mass differences between the pions. This will be done by considering all the contributions of the exact charge-independent coupling as before, separating the diagrams according to the mass of the pion (or pions) exchanged, and using the correct pion masses for each diagram. The appropriate (relative) coupling constants for the charge-independent theory are 1, $\sqrt{2}$, and -1 for $p\pi\bar{p}$, $n\pi\bar{p}$, and $n\pi\bar{n}$ vertices, respectively. As usual, T and T_3 are the total isotopic spin and the third component thereof. Note that the potential now depends on $|T_3|$ as well as on T (charge symmetry), whereas in the degenerate-mass limit it depends only on T (charge independence).

The case of OPEP will be treated first.[28] The appropriate graphs in this case are shown in Figs. 1(a)–1(c). Note that because of charge symmetry the n-n and

[28] M. H. Hull *et al.*, Phys. Rev. **122**, 1606 (1961); G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, *ibid.* **120**, 2227 (1960).

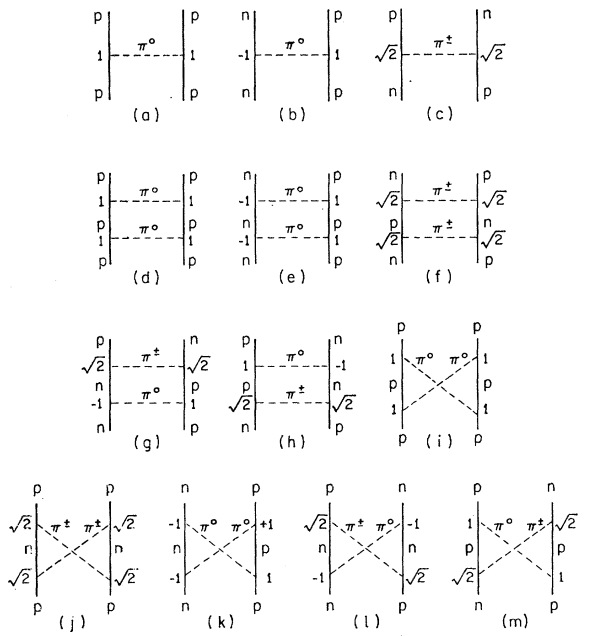

FIG. 1. Charge dependence of OPEP and TPEP. Graphs (a)–(c) are the three OPEP cases, graphs (d)–(h) represent the uncrossed TPEP diagrams, and graphs (i)–(m) represent the crossed TPEP diagrams. The isospin coefficients are given at each vertex.

p-p potentials are identical. The numbers next to the vertices represent the (relative) coupling constants. Let $C(0)(C(1))$ denote the contribution of the graph with a neutral (charged) pion exchange and with vertices of unit relative coupling constant. The contribution to each state T, T_3 is obtained by multiplying $C(0)$ or $C(1)$ by the coupling-constant weights indicated in each figure and then adding or subtracting them according to the symmetry of the isotopic wave function [add (subtract) for the symmetrical $T=1$ (antisymmetrical $T=0$) combination]. Using the above rules, we obtain

$$\begin{aligned} &C(0) && \text{for the } pp \text{ (or } nn\text{) case},\\ &-C(0)+2C(1) && \text{for the } T=1\ np \text{ case},\\ &-C(0)-2C(1) && \text{for the } T=0\ np \text{ case}. \end{aligned}$$

According to the notation established in Sec. IV B, we may write[28]

$$\begin{aligned} V(\mathrm{PI1}+)&=C(0),\\ V(\mathrm{PI10})&=-C(0)+2C(1),\\ V(\mathrm{PI0})&=-C(0)-2C(1). \end{aligned} \tag{4.35}$$

Next, we consider the two-pion ladder graphs[29] of Figs. 1(d)–1(h). The notation is as before with $C\rightarrow A$ and an added number [as in $A(00)$] specifying the charge of the second meson. The various potentials are deduced from the graphs. They are

$$\begin{aligned} V(\mathrm{PP1}+)&=A(00),\\ V(\mathrm{PP10})&=A(00)+4A(11)-2A(01)-2A(10),\\ V(\mathrm{PP0})&=A(00)+4A(11)+2A(01)+2A(10). \end{aligned} \tag{4.36}$$

[29] D. L. Lin, Nucl. Phys. 60, 192 (1969).

Finally, we consider the crossed graphs of Figs. 1(i)–1(m). The corresponding potentials are [with notation $B(ij)$]

$$\begin{aligned} V(\mathrm{PP1}+)&=B(00)+4B(11),\\ V(\mathrm{PP10})&=B(00)+4B(01),\\ V(\mathrm{PP0})&=B(00)-4B(01). \end{aligned} \tag{4.37}$$

The above potentials reduce to the appropriate charge-independent combinations when the mass differences are ignored.

We are now faced with the task of calculating fourth-order graphs with two distinct pion masses. Fortunately, it turns out that the graphs involving two pion masses μ_1 and μ_2 are quite insensitive to the parameter $\epsilon^2\equiv(\mu_2^2-\mu_1^2)/(\mu_2^2+\mu_1^2)\simeq 0.04$. Indeed, as we shall see below, the first nonvanishing contribution in an expansion about $\epsilon^2=0$ is of the order of ϵ^4. Consider, for example, Eqs. (4.20) for $\mathcal{J}-\mathcal{I}$. Had one used two different masses for the pion lines, the part of the denominator D that depends on these masses would have been $(1-\beta-\zeta)\mu_1^2+\beta\mu_2^2$ instead of $(1-\zeta)\mu^2$ [cf. Eq. (A1) of Appendix A]. If the variables ζ and β are then changed according to

$$\bar{\zeta}=\zeta,\quad \bar{\beta}=\beta/(1-\zeta)-\tfrac{1}{2},$$

the $\bar{\beta}$ dependence of all the denominators will be of the form

$$(\tfrac{1}{4}-\bar{\beta}^2)(1-\zeta)^2+(1-\zeta)\bar{\mu}^2(1+2\bar{\beta}\epsilon^2),$$

where

$$\bar{\mu}^2=\tfrac{1}{2}(\mu_1^2+\mu_2^2),\quad \epsilon^2=(\mu_2^2-\mu_1^2)/(\mu_2^2+\mu_1^2).$$

The range of integration for $\bar{\beta}$ is $[-\frac{1}{2},\frac{1}{2}]$, and since the only odd dependence on $\bar{\beta}$ is in the term $2\bar{\beta}\epsilon^2$, we see that in an expansion about $2\bar{\beta}\epsilon^2=0$, the first nonvanishing term is proportional to $4\bar{\beta}^2\epsilon^4\leq\epsilon^4$. Thus, neglecting the very small correction of the order of ϵ^4, we recover the original denominator with μ^2 replaced by $\bar{\mu}^2$. The same treatment applies to $\mathcal{L}$ in an identical manner. Equivalently, the contributions involving two distinct pion masses may be evaluated using a single "average mass."

We now proceed to give the complete definition of TPEP. Let the charged and neutral pion masses be m_{π^+} $(=1$ in our units) and m_{π^0}, respectively, and define

$$\bar{m}_\pi=\{\tfrac{1}{2}[(m_{\pi^+})^2+(m_{\pi^0})^2]\}^{1/2}.$$

Also, using the integrals of (4.31), let

$$\begin{aligned} A(\alpha)&=\frac{(G_\pi)^4}{4\pi^4}[X_A(\alpha)+Y_A(\alpha)+W_A(\alpha)\\ &\qquad +Z_A(\alpha)+Z_B(\alpha)],\\ B(\alpha)&=\frac{(G_\pi)^4}{4\pi^4}[X_B(\alpha)+X_C(\alpha)]. \end{aligned} \tag{4.38}$$

Note that some of the terms in the expressions for A and B may be missing. For example, $X_A(3)$ has not been defined; such terms are to be set equal to zero. The letters A and B shall be augmented by the symbol

1, 0, or M, specifying, respectively, which of the masses m_{π^+}, m_{π^0}, or $\bar{m}_\pi$ is substituted for the pion mass μ. For example, $A(1)$ (α) is the quantity A in Eq. (4.38) with $\mu=m_{\pi^+}$. Using this notation and Eqs. (4.36) and (4.37), we finally have

$$\begin{aligned} V(\mathrm{PP1}+(\alpha))&=[A(0)+B(0)+4B(1)](\alpha),\\ V(\mathrm{PP10}(\alpha))&=[A(0)+4A(1)-4A(M)\\ &\quad +B(0)+4B(M)](\alpha), \quad (4.39)\\ V(\mathrm{PP0}(\alpha))&=[A(0)+4A(1)+4A(M)\\ &\quad +B(0)-4B(M)](\alpha). \end{aligned}$$

This completes the definition of the TPEP.

V. ONE-BOSON EXCHANGE POTENTIALS

The task of this section is the derivation of the lowest-order potentials ($V^{(2)}$) for the exchange of various bosons. Specifically, we shall consider scalar bosons with scalar coupling, pseudoscalar bosons with pseudoscalar coupling, and vector bosons with both vector and tensor couplings. Moreover, each of these bosons may be considered as isoscalar or isovector particles. The isovector potentials are obtained from isoscalar ones by adjoining the usual $\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}$ factor. It therefore suffices to consider neutral bosons only in the course of calculations.

We begin by recording the interaction Hamiltonians for the various bosons. We shall refer to scalar, pseudoscalar, and vector bosons by means of the symbols S, P, and V. The interaction Hamiltonians are

$$\begin{aligned} H_S&=(4\pi)^{1/2}G_S\bar{\psi}\varphi\psi,\\ H_P&=i(4\pi)^{1/2}G_P\bar{\psi}\gamma^5\varphi\psi, \quad (5.1)\\ H_V&=(4\pi)^{1/2}\bar{\psi}[G_V\gamma^\mu\varphi_\mu+(F_V/2m)\sigma^{\mu\nu}(\varphi_{\mu,\nu}-\varphi_{\nu,\mu})]\psi, \end{aligned}$$

where

$$\sigma^{\mu\nu}=(1/2i)[\gamma^\mu,\gamma^\nu],$$

and where φ stands for the boson fields. Note that the tensor coupling constant in H_V has been normalized by the nucleon mass m. The next step is the calculation of $\mathcal{K}^{(2)}$ corresponding to the above interactions. This calculation has been performed by several authors.[6,30] However, since the notation is not standard, and for completeness, we shall reproduce some of the intermediate results. Using the appropriate propagators for the scalar and vector particles, we arrive at the following expressions for $\mathcal{K}^{(2)}$:

$$\mathcal{K}_S{}^{(2)}=-\left(\frac{G_s{}^2}{2\pi^2}\right)\frac{I^{(1)}I^{(2)}}{\omega^2},$$

$$\mathcal{K}_P{}^{(2)}=\left(\frac{G_P{}^2}{2\pi^2}\right)\frac{\gamma^{5(1)}\gamma^{5(2)}}{\omega^2},$$

$$\begin{aligned} \mathcal{K}_V{}^{(2)}=\frac{1}{2\pi^2}\Big[&G_V{}^2\gamma^{\lambda(1)}\gamma_\lambda{}^{(2)}\\ &+\frac{G_VF_V}{m}\gamma^{\nu(1)}\gamma_\nu{}^{(2)}k_\mu(\gamma^{\mu(1)}-\gamma^{\mu(2)})\\ &+\frac{F_V{}^2}{m^2}(k_\mu\gamma^{\mu(1)}\gamma^{\nu(1)}\gamma_\nu{}^{(2)}k_\lambda\gamma^{\lambda(2)}-k^\mu k_\mu)\Big]_{\omega^2}^{1}, \quad (5.2) \end{aligned}$$

where k^μ is the momentum transfer $p'^\mu-p^\mu$, $\omega^2=\boldsymbol{\Delta}^2+m_B{}^2$, and $\boldsymbol{\Delta}$ is the spatial part of k^μ as before. The symbol m_B is used for the mass of a boson.

In converting the above amplitudes into potentials, we follow the procedures of Sec. IV, with one important exception: Our sole approximation here will consist in expanding the amplitudes in powers of $\mathbf{Q}/m$ and retaining terms to first order. The lack of energy dependence in OBEP enables us to avoid the additional approximations. Moreover, as we shall see later, the relatively large masses of the resonances to be considered make it imperative to avoid an expansion in $\boldsymbol{\Delta}^2/m^2$, which quantity is eventually transformed into $-m_B{}^2/m^2$. Anticipating later results, we note that avoiding an expansion in $\boldsymbol{\Delta}^2/m^2$ causes a negligible change in OPEP on account of the small pion-to-nucleon mass ratio. Thus we are consistent in keeping these terms for OPEP (having dropped them for TPEP) and treating all OBEP contributions uniformly. The terms higher than $\mathbf{Q}/m$ are dropped for the same reasons as in Sec. IV A. Indeed, the larger boson masses here make the transition region sharper than that for TPEP. At any rate, keeping terms higher than $\mathbf{Q}/m$ would not be consistent with the procedure of Sec. IV.

The prescription for evaluating the γ matrices and passing to the Pauli-Schrödinger representation has been explained in Sec. IV. Here, only the approximation stated above is applied in the passage to the Pauli-Schrödinger representation. The resulting potentials in momentum space are

$$V_S(1)=-\frac{G_S{}^2}{2\pi^2}\frac{D}{m}\frac{1}{\omega^2},$$

$$V_S(2)=\frac{G_S{}^2}{2\pi^2}\frac{4(D-m)}{m\boldsymbol{\Delta}^2}\frac{1}{\omega^2},$$

$$V_P(3)=\frac{G_P{}^2}{2\pi^2}\frac{1}{12mD}\frac{1}{\omega^2},$$

$$V_P(4)=-V_P(3),$$

$$V_V(1)=\frac{1}{2\pi^2}\left(G_V{}^2\frac{m}{D}-F_VG_V\frac{\boldsymbol{\Delta}^2}{mD}+F_V{}^2\frac{\boldsymbol{\Delta}^4}{4m^3D}\right)\frac{1}{\omega^2},$$

[30] N. Hoshizaki, I. Lin, and S. Machida, Progr. Theoret. Phys. (Kyoto) 26, 680 (1961).

$$V_V(2)=\frac{1}{2\pi^2}\Big[G_V{}^2\Big(\frac{-4m}{\Delta^2 D}+\frac{1}{mD}+\frac{4}{\Delta^2}\Big)+G_VF_V\Big(\frac{8}{mD}-\frac{4}{m^2}\Big)-F_V{}^2\Big(\frac{-4}{mD}+\frac{\Delta^2}{m^2D}+\frac{4}{m^2}\Big)\Big]\frac{1}{\omega^2},$$

$$V_V(3)=\frac{-1}{2\pi^2}(G_V+2F_V)^2\frac{1}{12mD}\frac{1}{\omega^2},$$

$$V_V(4)=\frac{-1}{2\pi^2}(G_V+2F_V)^2\frac{1}{6mD}\frac{1}{\omega^2},$$

$$D=(m^2+\tfrac{1}{4}\Delta^2)^{1/2}. \tag{5.3}$$

The configuration-space potentials are obtained from the above by Fourier transformation as before. To illustrate the method, we shall carry out the transformation for $V_P(3)$. Dropping inessential factors, we have

$$V_P(3)=\int\frac{d\Delta\exp(i\Delta\cdot\mathbf{r})}{(m^2+\frac{1}{4}\Delta^2)^{1/2}}\frac{1}{m_B{}^2+\Delta^2}$$
$$=\frac{4\pi}{2ir}\int_{-\infty}^{+\infty}\frac{d\Delta\,\Delta\exp(i\Delta r)}{(m^2+\frac{1}{4}\Delta^2)^{1/2}}\frac{1}{m_B{}^2+\Delta^2}$$
$$=-\frac{4\pi}{r}\int_{2m}^{\infty}\frac{ydy\exp(-yr)}{(\frac{1}{4}y^2-m^2)^{1/2}(y^2-m_B{}^2)}+\frac{2\pi^2}{r}\frac{\exp(-m_Br)}{(m^2-\frac{1}{4}m_B{}^2)^{1/2}}$$
$$=\frac{2\pi^2}{r(m^2-\frac{1}{4}m_B{}^2)^{1/2}}\Big\{\exp(-m_Br)-\frac{2}{\pi}\int_{2m}^{\infty}\frac{ydy\exp(-yr)}{y^2-m_B{}^2}\Big[\frac{m^2-\frac{1}{4}m_B{}^2}{\frac{1}{4}y^2-m^2}\Big]^{1/2}\Big\}.$$

The last step in the above calculation involves a contour integration around the branch cut caused by the square-root factor. The usual derivations of OBEP, such as that of Bryan and Scott,[11] employ an expansion of the square-root factor about $\Delta^2/4m^2=0$. As a result, the branch cut is eliminated, and only the first term in the above result is retained with the square-root term expanded in $m_B{}^2/4m^2$. In the case of small m_B/m, as for the pion, the integral along the branch cut very slightly alters the spin-spin potential, while the expansion in $m_B{}^2/4m^2$ has an inappreciable effect. For $m_B\approx m$, as for the ρ and ω mesons, it is the expansion in $m_B{}^2/4m^2$ that modifies the result somewhat. We shall give a quantitative discussion of these observations when we present numerical results (Sec. VI).

In the following, we will record the four potentials corresponding to the π, η, ρ, and ω mesons. (See the notation in Sec. IV B.) The masses of these particles are denoted by m_{π^+} and m_{π^0} for the charged and neutral pions, and m_η, m_ρ, and m_ω for the next three, respectively. The corresponding coupling constants are denoted by G_π, G_η, (G_ρ,F_ρ), and (G_ω,F_ω), respectively. Before recording the potentials, we make the following auxiliary definitions:

$$G(r|m_B,n)=(2m)^n\int_1^\infty d\xi\frac{1-\alpha^2}{\xi^2-\alpha^2}\frac{\xi^n}{(\xi^2-1)^{1/2}}\exp(-2mr\xi),$$

$$\alpha=m_B/2m,$$

$$H(r|m_B,n)=\frac{(2\pi)^2(m_B)^n\exp(-m_Br)}{2mr},$$

$$J(r|m_B,n)=\frac{(2\pi)^2(m_B)^n}{(4m^2-m_B{}^2)^{1/2}}\frac{e^{-m_Br}}{r}-\frac{8\pi}{4m^2-m_B{}^2}\frac{G(r|m_B,n+1)}{r}, \tag{5.4}$$

$$J_C(r|m_B,n)=(-)^{n/2}J(r|m_B,n),$$

$$J_{SO}(r|m_B,n)=(-)^{n/2+1}\Big[\frac{1}{r}J(r|m_B,n+1)+\frac{1}{r^2}J(r|m_B,n)\Big],$$

$$H_{SO}(r|m_B,n)=(-)^{n/2+1}\Big[\frac{1}{r}H(r|m_B,n+1)+\frac{1}{r^2}H(r|m_B,n)\Big], \tag{5.5}$$

$$J_{TN}(r|m_B,n)=(-)^{n/2}\Big[J(r|m_B,n+2)+\frac{3}{r}J(r|m_B,n+1)+\frac{3}{r^2}J(r|m_B,n)\Big],$$

$$H_{TN}(r|m_B,n)=(-)^{n/2}\Big[H(r|m_B,n+2)+\frac{3}{r}H(r|m_B,n+1)+\frac{3}{r^2}H(r|m_B,n)\Big].$$

The definition of the potentials follows.

(*i*) *Pion:* We first define the quantities C by

$$C(1(3))=\frac{G_\pi^2}{2\pi^2}\frac{1}{12m}J_{TN}(r|m_{\pi^+},0),$$

$$C(0(3))=\frac{G_\pi^2}{2\pi^2}\frac{1}{12m}J_{TN}(r|m_{\pi^0},0),$$

$$C(1(4))=-\frac{G_\pi^2}{2\pi^2}\frac{1}{12m}J_C(r|m_{\pi^+},2),$$

$$C(0(4))=-\frac{G_\pi^2}{2\pi^2}\frac{1}{12m}J_C(r|m_{\pi^0},2).$$

In terms of these quantities and using the rules of Sec. IV C, we have

$$V(\mathrm{PI1}+(\alpha))=C(0(\alpha)),$$
$$V(\mathrm{PI10}(\alpha))=-C(0(\alpha))+2C(1(\alpha)),$$
$$V(\mathrm{PI0}(\alpha))=-C(0(\alpha))-2C(1(\alpha)).$$

(*ii*) η *meson:*

$$V(\mathrm{ET1}(3))=\frac{G_\eta^2}{2\pi^2}\frac{1}{12m}J_{TN}(r|m_\eta,0),$$

$$V(\mathrm{ET1}(4))=-\frac{G_\eta^2}{2\pi^2}\frac{1}{12m}J_C(r|m_\eta,2),$$

$$V(\mathrm{ET0}(\alpha))=V(\mathrm{ET1}(\alpha)).$$

(*iii*) ρ *meson:*

$$V(\mathrm{RH1}(1))=\frac{1}{2\pi^2}\left\{G_\rho^2 mJ_C(r|m_\rho,0)-\frac{G_\rho F_\rho}{m}J_C(r|m_\rho,2)+\frac{F_\rho^2}{4m^3}J_C(r|m_\rho,4)\right\},$$

$$V(\mathrm{RH1}(2))=\frac{1}{2\pi^2}\left\{G_\rho^2\left[-4mJ_{\mathrm{SO}}(r|m_\rho,-2)+\frac{1}{m}J_{\mathrm{SO}}(r|m_\rho,0)+4mH_{\mathrm{SO}}(r|m_\rho,-2)\right]\right.$$
$$\left.+G_\rho F_\rho\left[\frac{8}{m}J_{\mathrm{SO}}(r|m_\rho,0)-\frac{4}{m}H_{\mathrm{SO}}(r|m_\rho,0)\right]-F_\rho^2\left[\frac{-4}{m}J_{\mathrm{SO}}(r|m_\rho,0)+\frac{1}{m^3}J_{\mathrm{SO}}(r|m_\rho,2)+\frac{4}{m}H_{\mathrm{SO}}(r|m_\rho,0)\right]\right\},$$

$$V(\mathrm{RH1}(3))=\frac{1}{2\pi^2}\left[\frac{-(G_\rho+2F_\rho)^2}{12m}J_{TN}(r|m_\rho,0)\right],$$

$$V(\mathrm{RH1}(4))=\frac{1}{2\pi^2}\left[-\frac{(G_\rho+2F_\rho)^2}{6m}J_C(r|m_\rho,2)\right],$$

$$V(\mathrm{RH0}(\alpha))=-3V(\mathrm{RH1}(\alpha)).$$

(*iv*) ω *meson:* The ω meson potentials $V(\mathrm{OM}(\alpha))$ are obtained from $V(\mathrm{RH1}(\alpha))$ by the replacements $m_\rho\rightarrow m_\omega$, $G_\rho\rightarrow G_\omega$, $F_\rho\rightarrow F_\omega$ and the equation

$$V(\mathrm{OM0}(\alpha))=V(\mathrm{OM1}(\alpha)).$$

VI. NUCLEON-NUCLEON POTENTIAL

The nucleon-nucleon potential is defined as the sum of OBEP and TPEP. The parts that comprise OBEP are the pion, η, ρ, and ω meson exchanges. The integrals appearing in the definition of these potentials have been calculated numerically. (See Appendix B for details of these calculations.) The parameters used in the potential are shown in Table I. The masses have been taken from the Rosenfeld tables.[31] The nucleon mass represents an average of proton and neutron masses. The η coupling constant is not well determined experimentally, so one usually appeals to $SU(3)$ for its estimation.[32] Such estimations place the η coupling constant (squared) somewhere between 0.5 and 2, depending on the F/D ratio used. Our choice of the η coupling constant represents a compromise between the two bounds. At any rate, the η contribution is confined to the tensor and spin-spin parts, and it is very small. The ρ coupling constants are relatively well determined by experiments. Our choice of the vector coupling constant is based on the recent data on the leptonic decay and

TABLE I. Coupling constants and masses used in the potentials.

Particle	Mass (MeV)	Coupling constant
Nucleon	$m=938.9$	
Pion (neutral)	$m_{\pi^0}=135.0$	$G_\pi^2=14.4$
Pion (charged)	$m_{\pi^+}=139.6$	$G_\pi^2=14.4$
η meson	$m_\eta=549.0$	$G_\eta^2=1.0$
ρ meson	$m_\rho=765.0$	$G_\rho^2=0.53$, $F_\rho/G_\rho=1.83$
ω meson	$m_\omega=782.8$	$G_\omega^2=6.36$, $F_\omega/G_\omega=-0.06$

[31] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).

[32] E. L. Lomon and H. Feshbach, Ann. Phys. (N. Y.) **48**, 94 (1968).

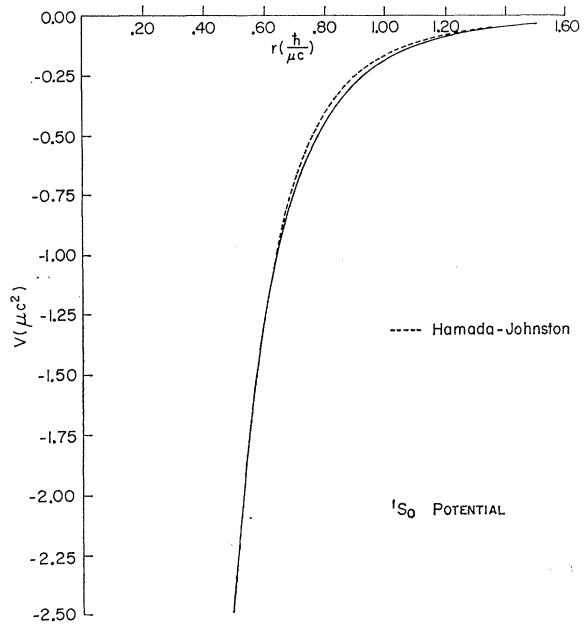

FIG. 2. Complete singlet-S potential.

photoproduction of ρ.[33,34] The coupling constant obtained in this way is in good agreement with other determinations.[35,36] The ω and ϕ coupling constants are not well known. The $SU(3)$ prediction (with ϕ-ω mixing) for the relative (squared) coupling constants of the vector mesons is 1:4.5:9, corresponding to ρ, ϕ, and ω,

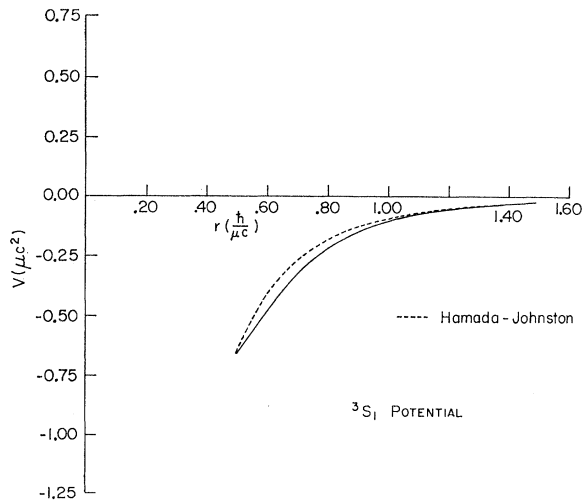

FIG. 3. Complete triplet-S potential.

[33] Note that our $G_\rho^2=0.53$ corresponds to Sakurai's 2.12 (Ref. 36).

[34] S. C. C. Ting, in *Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968* (CERN, Geneva, 1968).

[35] S. C. C. Ting, DESY Internal Report No. F31-68/1, 1968 (unpublished).

[36] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).

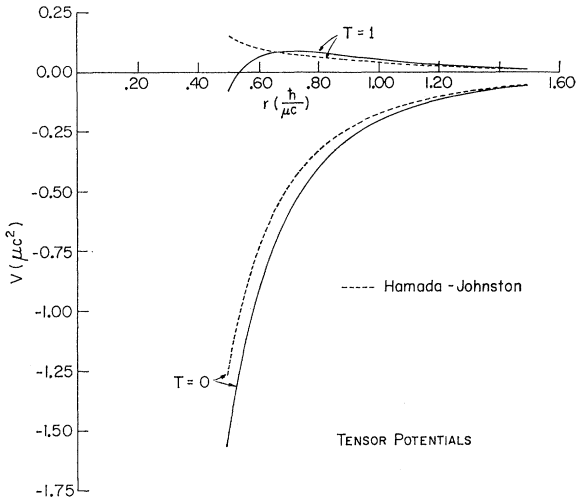

FIG. 4. Complete tensor potential.

respectively. On the other hand, the calculation of Oakes and Sakurai[37] predicts roughly 1:7:14 for these ratios. The results of Ref. 34 based on the leptonic decays of these mesons are in rough accord with the $SU(3)$ prediction. However, the latter prediction definitely disagrees with the suppressed photoproduction of ϕ mesons, while the Oakes-Sakurai prediction reduces the disagreement. Because of the above uncertainties and the higher mass of the ϕ meson ($m_\phi/m_\rho=1.3$), we have not included the ϕ in our potential. Instead, our choice of the ω-ρ coupling-constant ratio (namely, 12) is a com-

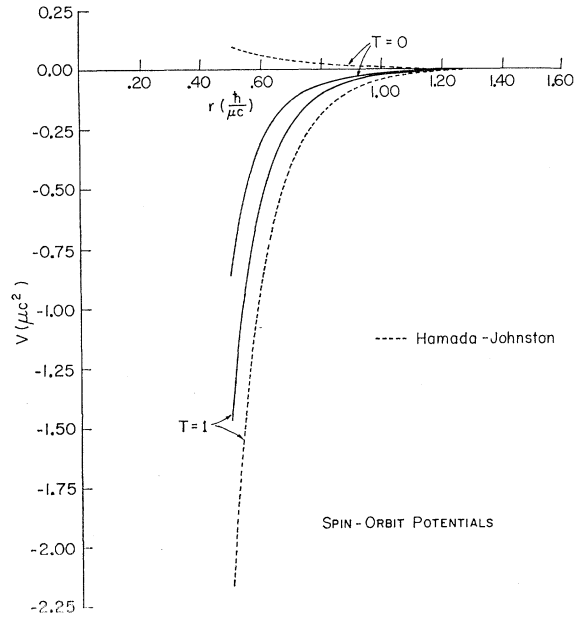

FIG. 5. Complete spin-orbit potential.

[37] R. J. Oakes and J. J. Sakurai, Phys. Rev. Letters **19**, 1266 (1967).

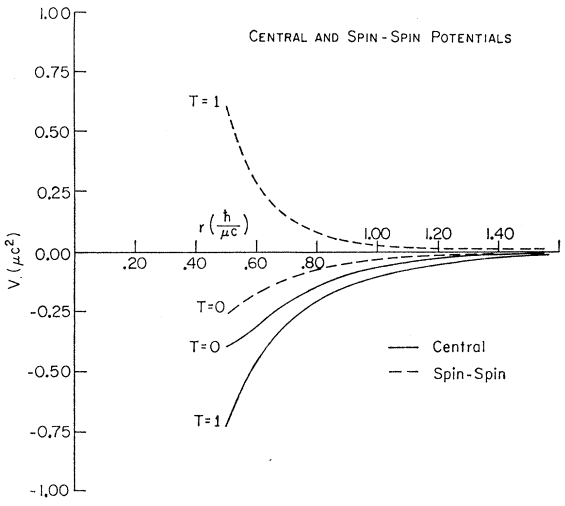

FIG. 6. Complete central and spin-spin potentials.

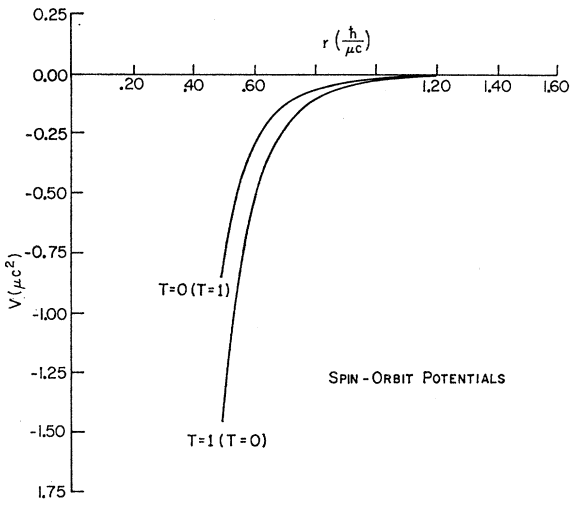

FIG. 8. Complete spin-orbit potentials versus TPEP.

promise between the experimental indications and theoretical predictions, taken about 1 standard deviation on the large side of the experimental value of Ref. 34 to account for the combined contributions of the ϕ and ω mesons. At any rate, the inclusion of both ϕ and ω with coupling constants predicted by $SU(3)$ would decrease the small difference between our potential and a phenomenological potential. Our OBEP also does not include scalar mesons. The customary inclusion of fictitious scalar mesons in OBE models is meant to simulate the effect of TPEP. Recently, however, there has been some experimental evidence for a two-pion isoscalar, S-wave resonance.[38] Its mass seems to be about 730 MeV, with a rather broad width, much higher than the 500 MeV required in fitting data with a pure OBE potential. The contribution of such a resonance to the potential is probably similar to the high-mass part of the two-pion continuum on account of its broad width. At any rate, should such a resonance definitely exist, care must be exercised in accounting for it so as to avoid double counting with TPEP.

The numerical results[39] are displayed in Figs. 2–23. All quantities are expressed in natural units with the mass of the charged pion as the unit of mass. When an isotriplet component of the potential is illustrated in a graph, it is actually the $T_3=1$ component corresponding to the proton-proton (or neutron-neutron) state. Recall that because of pion mass splitting, OPEP and TPEP are only charge-symmetric. The remaining potentials are charge-independent.

Figures 2–6 display the nucleon-nucleon potential and compare it with the phenomenological potential of

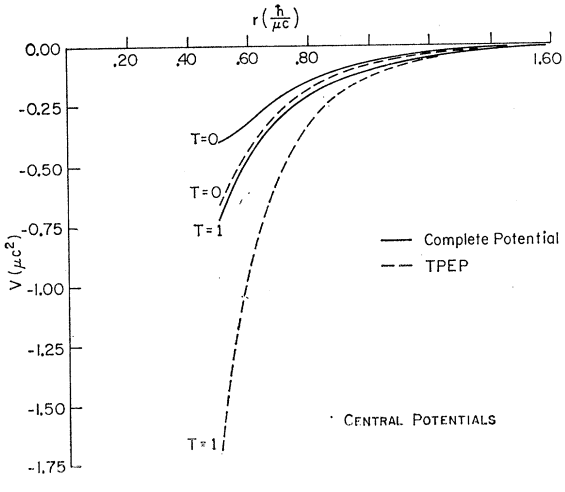

FIG. 7. Complete central potentials versus TPEP.

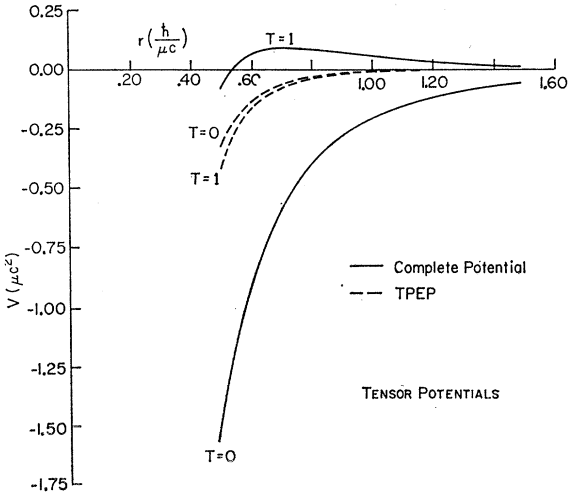

FIG. 9. Complete tensor potentials versus TPEP.

[38] M. Feldman *et al.*, Phys. Rev. Letters **22**, 316 (1969); G. A. Smith and R. J. Manning, Phys. Rev. **171**, 1399 (1968); E. Malamud and P. E. Schlein, Phys. Rev. Letters **19**, 1056 (1967).

[39] We can supply numerical tables of the potential on request.

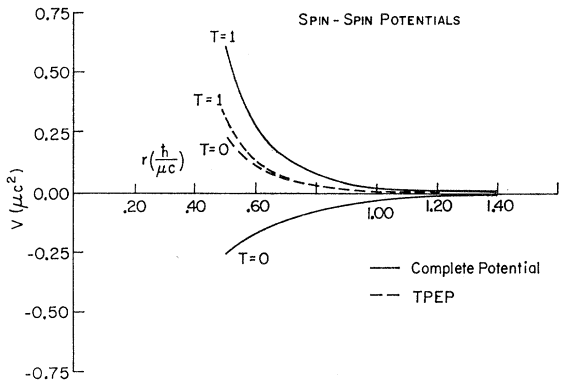

FIG. 10. Complete spin-spin potentials versus TPEP.

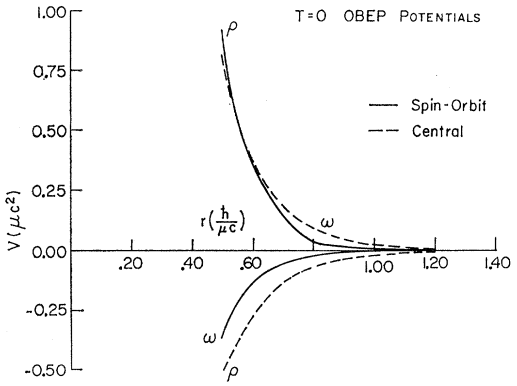

FIG. 11. Central and spin-orbit potentials of ρ and ω mesons.

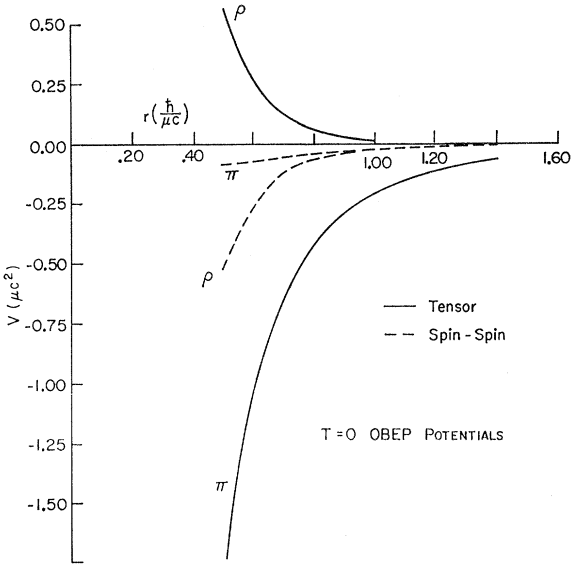

FIG. 12. Tensor and spin-spin potentials of pion and ρ meson.

Hamada and Johnston.[40] Figures 2 and 3 show the appropriate combinations of the central and spin-spin potentials for the two S states. These potentials are

[40] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).

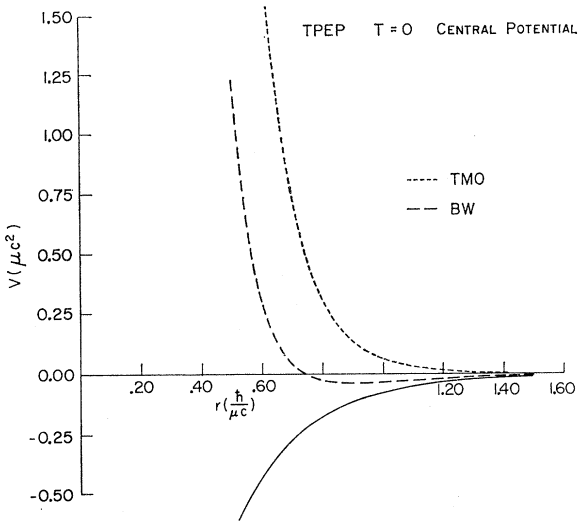

FIG. 13. Isosinglet central TPEP versus (modified) TMO and BW.

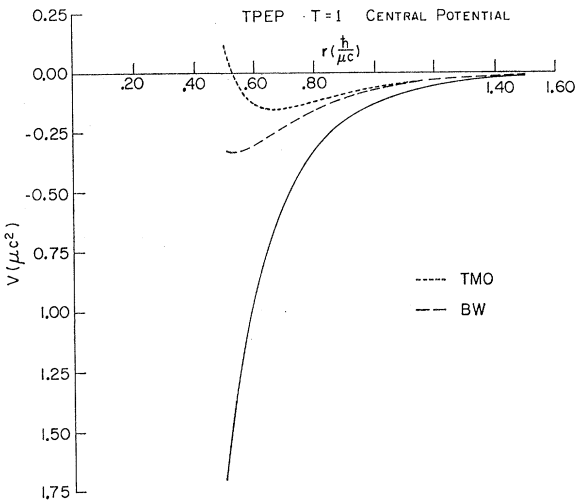

FIG. 14. Isotriplet central TPEP versus (modified) TMO and BW.

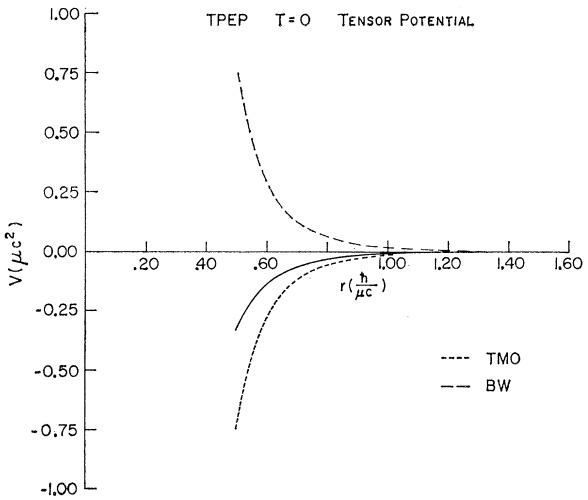

FIG. 15. Isosinglet tensor TPEP versus TMO and BW.

determined rather uniquely from the experimental data for $r \gtrsim 0.5$, as a comparison with the Yale potential[41] and others would show. Figure 4 compares the tensor potentials. The agreement with phenomenological potentials (moderately well determined by the data) is again very good. In Fig. 5 the isotriplet spin-orbit potential is shown to be in moderate agreement with that of Hamada and Johnston, while the isosinglet spin-orbit potential is in disagreement. However, the spin-orbit potentials are not so well determined phenomenologically, especially in the isosinglet case, where it first appears in D waves. As shown in Ref. 32, the region $r \lesssim 0.5$ can completely determine the phenomenological spin-orbit effects. Figure 6 shows the central and spin-spin potentials separately.

The above comparison is not intended to establish the validity of our potential on the basis of agreement with

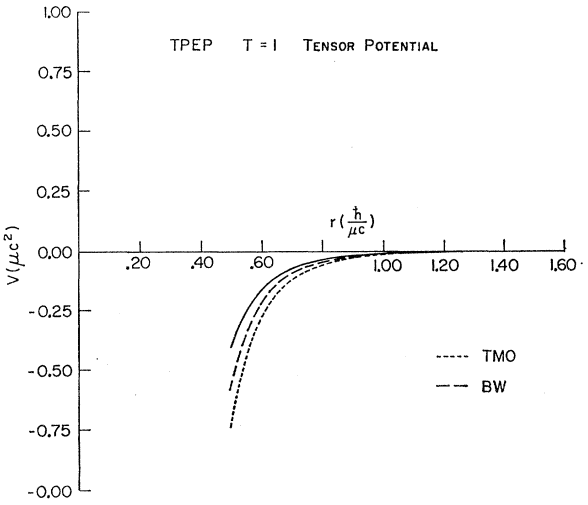

FIG. 16. Isotriplet tensor TPEP versus TMO and BW.

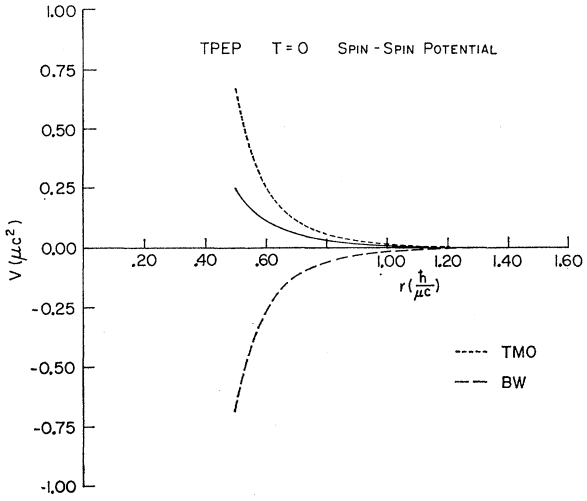

FIG. 17. Isosinglet spin-spin TPEP versus TMO and BW.

[41] K. E. Lassila *et al.*, Phys. Rev. **126**, 881 (1962).

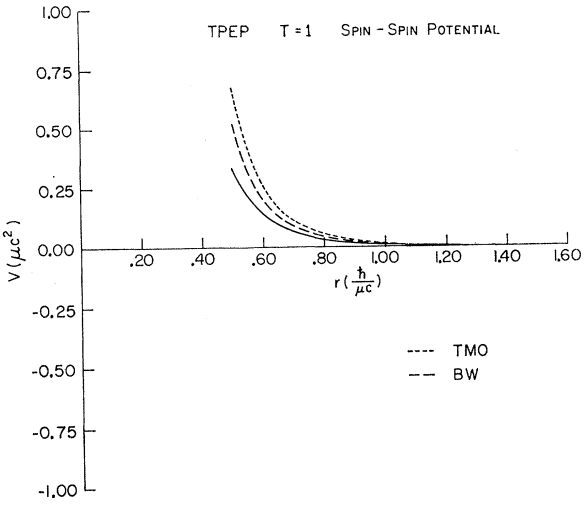

FIG. 18. Isotriplet spin-spin TPEP versus TMO and BW.

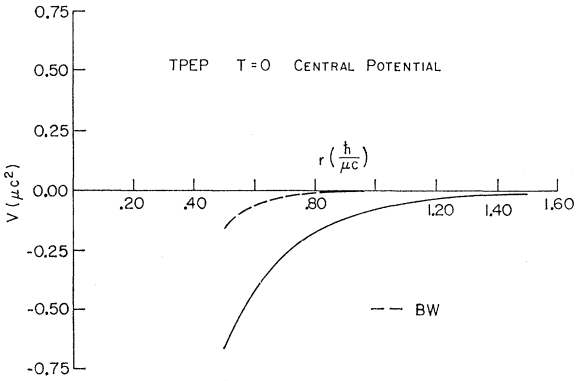

FIG. 19. Isosinglet central TPEP versus BW.

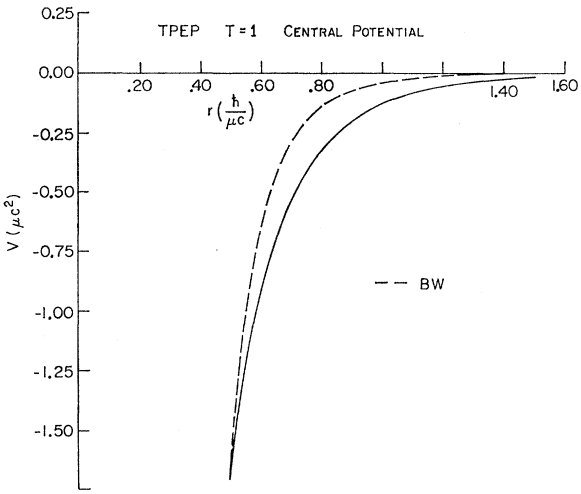

FIG. 20. Isotriplet central TPEP versus BW.

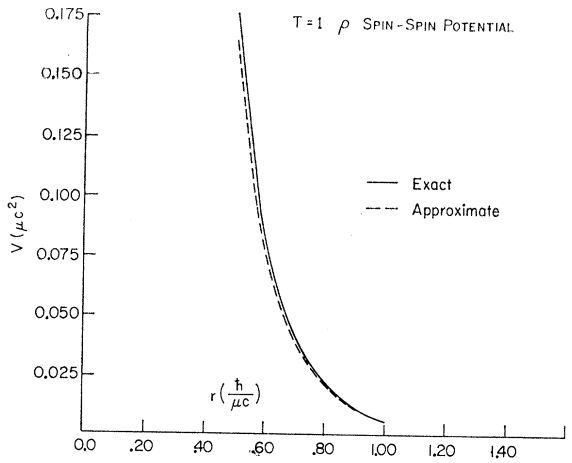

FIG. 21. Exact versus approximate ρ spin-spin potential.

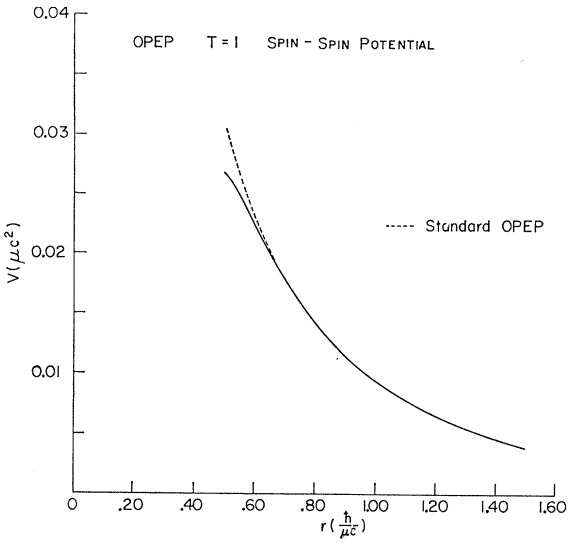

FIG. 22. Exact versus approximate (standard) pion spin-spin potential.

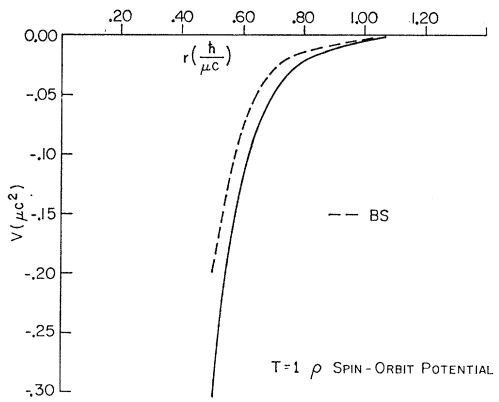

FIG. 23. ρ spin-orbit potential versus the modified Bryan-Scott III.

a phenomenological one. Indeed, the Hamada-Johnston potential includes quadratic spin-orbit terms (designed principally for the purpose of fitting D waves[42])which have no counterpart here. Furthermore, by exploiting the uncertainties in the coupling parameters used in the potential, one can no doubt achieve a closer agreement. Our intention is rather to show that the potential presented here is reasonably close to one that is determined by experiments with some degree of uniqueness.[43] The rather close agreement achieved here is no doubt partially accidental in view of the uncertainties in the ϕ and ω coupling constants. Moreover, the corrections due to nucleon resonances and pair-suppression effects to be discussed in Sec. VII (which have not been included here) will certainly alter the quantitative aspects of the potential somewhat.

Figures 7–10 compare TPEP with the complete potential. Note that the difference is made up of OPEP and resonance exchanges. This comparison is intended to show the extent to which the two-pion continuum contributes to the potential. It is seen that TPEP is a major contribution except for the tensor potentials that are largely determined by OPEP. An important fact to notice here is that TPEP definitely does not resemble an isoscalar exchange (as it is sometimes asserted) in central and spin-orbit potentials, although it does so for tensor and spin-spin parts. This fact explains the necessity of employing an isovector (in addition to an isoscalar) scalar meson in OBE models. The size of TPEP contributions, on the other hand, explains the unreasonably large ρ and ω meson coupling constants demanded by these models.[11]

Figures 11 and 12 show the relative contributions of OBEP. Figure 11 shows the isosinglet contributions of ρ and ω. Note that ω contributes importantly to the central potentials. As mentioned before, the inclusion of both ϕ and ω with coupling constants predicted by $SU(3)$ would cut down the attraction in the S-wave potentials, bringing the latter even closer to the phenomenological ones. It is interesting to observe that the TPEP contribution to the spin-orbit potential is larger in magnitude than that of ρ in both isospin states. Figure 12 shows the major contributions of OBEP to the isosinglet tensor and spin-spin parts. The tensor part is, of course, dominated by OPEP, and the spin-spin part is largely dominated by ρ. The η and ω contributions are small in these parts and are not shown.

Figures 13–20 compare our TPEP with those of TMO[8] and BW.[9] Although the latter potentials (as originally given by their authors) do not include pair contributions, in Figs. 13 and 14 we have added these contributions[32] for a more direct comparison. (Note

[42] The fact that these terms fail completely for higher-angular-momentum waves (see Refs. 40 and 41) is an indication of the poor convergence of the series, as noted in Sec. III.

[43] It must be recognized in this connection that the phenomenological potentials are somewhat dependent on their assumed form, particularly in the core region.

that the pair terms, as calculated by these authors, only contribute to the central potentials.) Figures 19 and 20 show the (original) BW central potentials without the pair terms. Generally speaking, our TPEP is quite different from the other two. Somewhat surprisingly, pair suppression in the BW central potentials improves the agreement. However, we believe that the static approximation, particularly for the one-pair terms, is sufficiently disastrous to produce unexpected results. Incidentally, the sign of the BW alteration to TMO is always substantiated by our TPEP, as expected theoretically.[44]

Figures 21–23 show the effect of the expansion in $\mathbf{\Delta}^2/4m^2$ for OBEP. (Recall that we do not make this expansion for OBEP.) In terms of the functions J and H of Sec. V, this expansion is usually carried out for J, the lowest-order approximation of which is H. Figure 21 compares the ρ spin-spin potential with the lowest approximation thereof. Since the two are proportional to J and H, respectively, this plot also shows the effect of setting $\mathbf{\Delta}^2/4m^2=0$ in J. As explained in Sec. V, in the case of the resonances, the bulk of the difference comes from the expansion in $m_B{}^2/4m^2$ in the configuration representation. The next plot shows the effect of the expansion of OPEP. Only the spin-spin part of OPEP is modified to an appreciable extent. This modification is seen to be quite small, however.

As a means of comparing our OBEP with the usual ones, we considered the Bryan-Scott III[11] potential for ρ exchange, dropped their $\mathbf{P}^2/m^2$ terms, and used our values of the coupling constants. Except for the spin-orbit potential, the differences are small and characterized by Fig. 21, which shows the effect of the $\mathbf{\Delta}^2/4m^2$ expansion. The spin-orbit terms, however, differ substantially, as seen in Fig. 23. This difference can be traced to a term that is proportional to the square of the tensor coupling constant, and which is absent in the Bryan-Scott potential.

VII. FURTHER DEVELOPMENTS

The nucleon-nucleon potential defined in Sec. VI includes single-boson exchanges and the two-pion continuum as derived from our formalism. In this section, we shall discuss further extensions of the program to include other potentially significant effects. These effects arise from higher-order irreducible Feynman graphs in which only two mesons are exchanged between nucleons, i.e., each extra meson is emitted and absorbed by the same nucleon line. Such extra mesons added to the basic OPEK graph only give rise to renormalization effects. Their effect is completely absorbed by the use of physical masses and coupling constants, and our OPEK completely includes such meson radiative corrections.

Many radiative corrections to TPEK graphs are similarly absorbed into renormalized constants; but not those (such as the simplest cases illustrated in Fig. 24) which link the vertices of both exchanged mesons. These have the same range as TPEP and, because of the large value of G_π, cannot all be expected to produce negligible amplitudes. However, we may hope that the largest of such effects are also apparent in pion-nucleon scattering, which contains the same radiative corrections. When such pion-nucleon scattering effects are noted, one may then inquire into their proper continuation to the class of diagrams represented by Fig. 24.

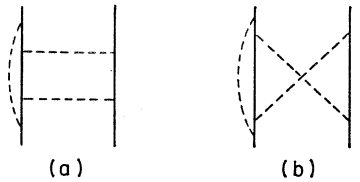

FIG. 24. Simplest meson radiative corrections not absorbed by renormalizations.

There are two features of pion-nucleon scattering which imply important pion radiative corrections. One is the existence of nucleon resonances (isobars). The low-mass ($J=\frac{3}{2}$, $I=\frac{3}{2}$) Δ resonance, in an intermediate state as shown in Fig. 25, is particularly close to the physical region of low-energy nucleon-nucleon scattering. A method for including this correction is discussed in Sec. VII A.

The other observed feature of pion-nucleon scattering demanding important radiative corrections is the value of the scattering lengths; they are much smaller than the second-order amplitude and have a different isotopic dependence. The possible implications of this for the nucleon-nucleon interaction are discussed in Sec. VII B.

Although neither calculation has yet been made, it is indicated that these are small, though probably significant, corrections. If so, it will indicate that remaining radiative corrections, which do not appear as

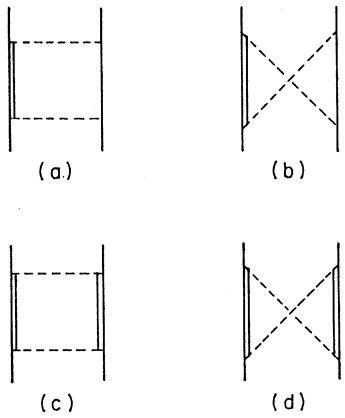

FIG. 25. Nucleon isobars in intermediate states.

[44] N. Fukuda, K. Sawada, and M. Taketani, Progr. Theoret. Phys. (Kyoto) 12, 156 (1954).

important effects in pion-nucleon reactions, are negligible for nucleon-nucleon forces.

A. Isobars in Intermediate States

Since the discovery of the N^* resonances in pion-nucleon scattering, the necessity of accounting for them (and especially of the low-mass Δ resonance) in the nucleon-nucleon interaction has been recognized. Several attempts have been made in this direction,[19,45–47] some relying on the analytic continuation of the pion-nucleon amplitude to the unphysical region appropriate for nucleon-nucleon interactions, and some treating the resonances as Rarita-Schwinger particles as in the diagrams of Fig. 24. Actually, the one-dimensional dispersion relations of Cini and Fubini[48] show that the particle treatment is a well-defined approximation to the dispersion-relations method. Often the P-wave partial amplitude is projected out and adjusted so as to reflect the effect of the resonance. In the following, we shall discuss a preliminary treatment of the nucleon resonances in the spirit of our formalism. Explicitly, the resonances will be considered as particles coupled to the nucleons via the pion field as in the graphs of Fig. 26.

The iteration by the Schrödinger equation of the potential determined by these one-pion-exchange graphs gives rise to the contributions of graphs (a) and (c) of Fig. 25, in the approximation of a nonrelativistic propagator. The corrections due to the relativistic propagator and the graphs (b) and (d) of Fig. 25 can then be obtained from the analog of $\mathbf{U}^{(4)}$ in Eq. (3.18). However, the on-shell contribution of (3.18) is given by the off-diagonal potential.

To obtain the dominant effect of N^* intermediate states the interaction will thus be represented in a coupled-channel formalism. The outstanding advantage of such a procedure is the correct threshold behavior for resonance production, which contributes significant energy dependence below threshold. On the other hand, if one goes to the trouble of directly calculating the contributions of graphs (a) and (c) of Fig. 25, one avoids a coupled-channel calculation at the expense of losing the threshold effect in the usual adiabatic approximation.

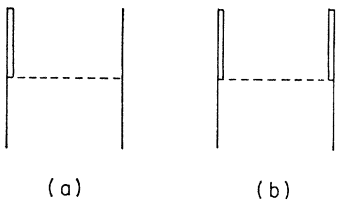

FIG. 26. One-pion-exchange isobar production diagrams.

[45] M. Konuma, A. Miyazawa, and S. Otsuki, Progr. Theoret. Phys. (Kyoto) **19**, 17 (1958).
[46] A. Klein and B. H. McCormick, Progr. Theoret. Phys. (Kyoto) **20**, 870 (1958).
[47] H. Sugawara and F. von Hippel, Phys. Rev. **172**, 1764 (1968).
[48] M. Cini and S. Fubini, Ann. Phys. (N. Y.) **10**, 352 (1960).

For convenience, we consider the case of two neutral scalar fields N and N^* representing the nucleon (mass m) and the resonance (mass m^*), coupled via a third neutral scalar field π representing the pion (mass μ). The generalization to the realistic case should be straightforward. The Bethe-Salpeter wave function for the two-baryon states may be represented by a four-component object as follows:

$$\begin{bmatrix}(m,m)\\(m,m^*)\\(m^*,m)\\(m^*,m^*)\end{bmatrix}.$$

All particles have been treated as distinguishable, and every component is specified by an ordered pair of particles identified with their masses. Note that the inclusion of (m,m^*) and (m^*,m) as two distinct states is merely an algebraic device at this stage. We now define the Bethe-Salpeter wave function $\Psi(x_1,x_2)$ in the usual way:

$$\Psi(x_1,x_2)=\langle \mathrm{vac}|\, T\begin{bmatrix}N(x_1)N(x_2)\\N(x_1)N^*(x_2)\\N^*(x_1)N(x_2)\\N^*(x_1)N^*(x_2)\end{bmatrix}|\mathrm{in}\rangle, \quad (7.1)$$

where $|\mathrm{in}\rangle$ and $|\mathrm{vac}\rangle$ are the incoming and vacuum states, T is Wick's chronological operator, and all quantities are expressed in the Heisenberg representation. The resulting Bethe-Salpeter equation in the ladder approximation is

$$\mathfrak{D}(x_1,x_2)\Psi(x_1,x_2)=\lambda\Delta_F(\mu|\,x_1-x_2)\mathfrak{F}\Psi(x_1,x_2), \quad (7.2)$$

where λ is a numerical constant, Δ_F is the Feynman propagator for the meson, and

$$\mathfrak{D}(x_1,x_2)=\begin{bmatrix}(\Box_1+m^2)(\Box_2+m^2) & & & \\ & (\Box_1+m^2)(\Box_2+m^{*2}) & & \\ & & (\Box_1+m^{*2})(\Box_2+m^2) & \\ & & & (\Box_1+m^{*2})(\Box_2+m^{*2})\end{bmatrix},$$

$$\mathfrak{F}=\begin{bmatrix}f_1^2 & f_1f_{12} & f_1f_{12} & f_{12}^2\\ f_1f_{12} & f_1f_2 & f_{12}^2 & f_{12}f_2\\ f_1f_{12} & f_{12}^2 & f_1f_2 & f_{12}f_2\\ f_{12}^2 & f_{12}f_2 & f_{12}f_2 & f_2^2\end{bmatrix}.$$

The coupling constants for the $N\pi N$, $N\pi N^*$, and $N^*\pi N^*$ vertices have been denoted by f_1, f_{12}, and f_2, respectively. The incoming state $|\mathrm{in}\rangle$ is specified by a total four-momentum $P=(2W,0,0,0)$ in the center-of-mass system. For definiteness, and with no loss of generality, we let the incoming state be one of two nucleons. The center-of-mass motion can be eliminated in (7.1) by using Lorentz invariance:

$$\Psi(x_1,x_2)=\exp[-iP\cdot\tfrac{1}{2}(x_1+x_2)]\chi(x_1-x_2), \quad (7.3)$$

whereupon (6.2) may be written in the integral form

$$\chi(x)=\chi_0(x)+\lambda'\int d^4k d^4y\,\exp[ik\cdot(x-y)]\times\mathfrak{D}^{-1}(k,W)\mathfrak{F}\Delta_F(\mu|\,y)\chi(y), \quad (7.4)$$

where the baryon propagator is given by

$$\mathfrak{D}(k|W)=\begin{bmatrix}[(W+k)^2-m^2+i\epsilon][(W-k)^2-m^2+i\epsilon] & & & \\ & [(W+k)^2-m^2+i\epsilon][(W-k)^2-m^{*2}+i\epsilon] & & \\ & & [(W+k)^2-m^{*2}+i\epsilon][(W-k)^2-m^2+i\epsilon] & \\ & & & [(W+k)^2-m^{*2}+i\epsilon][(W-k)^2-m^{*2}+i\epsilon]\end{bmatrix}.$$

We now observe that with χ as defined in (7.3), the corresponding *free* incoming wave function χ_0 has a (relative) time dependence in the second and third components, that is, in the components involving two unequal masses. On the other hand, the corresponding Schrödinger representation must be free of (relative) time dependence, and, moreover, the two representations must agree as to the expression of *free* incoming and outgoing waves. Thus the appropriate wave function to consider is (the off-diagonal components of the matrix vanish)

$$\psi(x)=\begin{bmatrix}1 & & & \\ & \exp[ix^0(m^2-m^{*2})/4W] & & \\ & & \exp[ix^0(m^{*2}-m^2)/4W] & \\ & & & 1\end{bmatrix}\chi(x)$$
$$\equiv\Theta(x^0)\chi(x), \quad (7.5)$$

the corresponding free waves for which are time-independent. The equation satisfied by ψ is

$$\psi(x)=\psi_0(x)+\lambda''\int d^4k d^4y\,\exp[ik\cdot(x-y)]\mathcal{G}(k|W)\Theta(y^0)\mathfrak{F}\Theta^{-1}(y^0)\Delta_F(\mu|y)\psi(y), \quad (7.6)$$

where

$$\mathcal{G}^{-1}(k|W)=\begin{bmatrix}[(W+k)^2-m^2+i\epsilon][(W-k)^2-m^2+i\epsilon] & & & \\ & \left[\left(W+\frac{m^2-m^{*2}}{4W}+k\right)^2-m^2+i\epsilon\right]\left[\left(W-\frac{m^2-m^{*2}}{4W}-k\right)^2-m^{*2}+i\epsilon\right] & & \\ & & \left[\left(W+\frac{m^{*2}-m^2}{4W}+k\right)^2-m^{*2}+i\epsilon\right]\left[\left(W-\frac{m^{*2}-m^2}{4W}-k\right)^2-m^2+i\epsilon\right] & \\ & & & [(W+k)^2-m^{*2}+i\epsilon][(W-k)^2-m^{*2}+i\epsilon]\end{bmatrix}. \quad (7.7)$$

Note that the quantities W and $(m^2-m^{*2})/4W$, formally written as four-momenta, have only a time component. The Green's function $\mathcal{G}$ is now in a proper form for reduction by means of the modified Blankenbecler-Sugar[23] prescription of Sec. II. According to that prescription,

$$g_{ii}(k|W)=\frac{2\pi}{i}\int\frac{d\mathbf{q}_i'^2}{\mathbf{q}_i^2-\mathbf{q}_i'^2+i\epsilon}\delta([\ \ ]_i)\delta([\ \ ]_i), \quad (7.8)$$

where the square brackets indexed by i are those appearing in the ith row of $\mathcal{G}^{-1}$. The quantities $\mathbf{q}^2$ and $\mathbf{q}'^2$ are defined differently for each channel as the notation implies. For the channel i involving the masses $m_i^{(1)}$ and $m_i^{(2)}$, they are defined by

$$(m_i^{(1)2}+\mathbf{q}^2)^{1/2}+(m_i^{(2)2}+\mathbf{q}^2)^{1/2}=2W, \quad (7.9)$$

and similarly for $\mathbf{q}'^2$. The meaning of (7.9) is obvious. The expression for g is

$$g_{ii}(k|W)=\frac{1}{(m_i^{(1)2}+\mathbf{k}^2)^{1/2}+(m_i^{(2)2}+\mathbf{k}^2)^{1/2}}\times\frac{\delta(k^0)}{2(\mathbf{q}_i^2-\mathbf{k}^2+i\epsilon)}. \quad (7.10)$$

Note that the appearance of the nonrelativistic denominator in (7.10) is owing to our modification of the Blankenbecler-Sugar prescription in writing the dispersion integral in terms of $\mathbf{q}'^2$ rather than W'^2. This modification does not alter the result in the equal-mass case.

If now g is inserted in (7.6) in place of $\mathcal{G}$, the resulting equation gives the lowest-order potential. Reverting to the differential form, we have

$$(\nabla^2+\mathbf{q}_i^2)\psi_i(x)=\lambda'''\int\frac{d\mathbf{k}d^4y\,\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]}{(m_i^{(1)2}+\mathbf{k}^2)^{1/2}+(m_i^{(2)2}+\mathbf{k}^2)^{1/2}}\Delta_F(\mu|y)\times[\Theta(y^0)\mathfrak{F}\Theta^{-1}(y^0)]_{ij}\psi_j(y), \quad (7.11)$$

where a summation over j is implied. The square-root factors in the denominator are connected with relativistic phase-space factors as explained in Sec. II. For the purpose of this discussion, it suffices to neglect these factors and consider ψ as a Schrödinger wave function; hence

$$(\nabla^2+\mathbf{q}_i^2)\psi_i(\mathbf{x})=\lambda''''\int\frac{d\mathbf{k}\exp(i\mathbf{x}\cdot\mathbf{k})}{(\theta_i-\theta_j)^2-(\mathbf{k}^2+\mu^2)+i\epsilon}\mathfrak{F}_{ij}\psi_j(\mathbf{x}), \quad (7.12)$$

where

$$\theta_i=(0,(m^2-m^{*2})/4W,\ (m^{*2}-m^2)/4W,0).$$

Note that some of the off-diagonal elements of the potential

$$V_{ij}(\mathbf{x})=\mathfrak{F}_{ij}\int\frac{d\mathbf{k}\exp(i\mathbf{x}\cdot\mathbf{k})}{(\theta_i-\theta_j)^2-(\mathbf{k}^2+\mu^2)+i\epsilon} \tag{7.13}$$

may be complex (for all scattering energies) if the $i\epsilon$ in the Feynman propagator is formally retained. Furthermore, when the second and third channels are appropriately combined to get the physical NN^* channel, a complex diagonal element appears in the potential. The source of the difficulty lies in the continuation of the potential off the energy shell. As far as the on-energy-shell consequences (e.g., scattering amplitude) are concerned, the real part of the potential will do in this lowest-order approximation. Indeed, the treatment of similar denominators in Ref. 30 on the basis of the Fukuda-Sawada-Taketani[44] (FST) method is equivalent to taking the real part of (7.13). Be that as it may, we feel that a definite choice of off-shell continuation could perhaps be found on firmer grounds. At any rate, a complex potential is not out of the question.[24] Note, incidentally, that the complex components have an oscillatory behavior decreasing as r^{-1}. This circumstance is not troublesome as the wave functions multiplying these potentials are strongly damped for large r. The damping below the production threshold increases as the elastic nucleon-nucleon threshold is approached. The effect of isobar intermediate states is therefore expected to be comparatively small at lower energies ($\lesssim$100 MeV c.m. kinetic energy). The phenomenological potentials are largely determined by the lower-energy range, and an effective alteration of them at higher energies may improve their fit to the data.

B. Pion-Nucleon S States and Nucleon-Antinucleon Pair Suppression

The main contribution to S-state pion-nucleon scattering of the second-order Feynman graph arises from the time-ordered component of Fig. 27(a), which has an antinucleon in the intermediate state. For this reason the small experimental scattering length is often attributed to a pair-suppression or pair-damping effect.

We now consider the possible effect of such pair suppression on nucleon-nucleon graphs containing one or two nucleon-antinucleon pairs as illustrated in Figs. 27(b) and 27(c). Within the field-theoretical framework, the arguments for pair damping in these cases are based on selective treatments of radiative corrections.[49] In addition, there is the argument based on the anomalously small pion-nucleon S-wave scattering lengths.

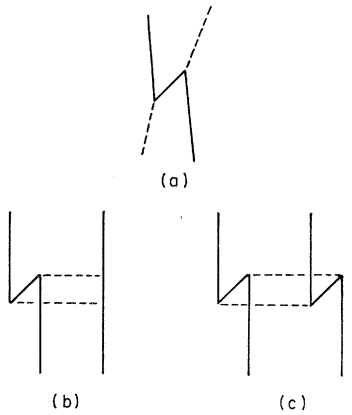

FIG. 27. Nucleon-antinucleon pair diagrams for pion-nucleon and nucleon-nucleon scattering.

Finally, the phenomenological pion-nucleon Lagrangians based on soft-pion physics predict a cancellation at pion-nucleon scattering threshold of the pair terms by counterterms arising from the exchange of an isoscalar, scalar boson.

It is clear that the above arguments based upon the pion-nucleon interaction involve continuation to the physical region of nucleon-nucleon scattering or hard-pion momenta. Also, the treatment relying on *selected* radiative corrections to field-theoretical diagrams can at best be considered as an indication for pair damping.

To include any such effect, one can resort to either the old-fashioned method of suppressing the pair terms or the inclusion of a scalar boson as a first step. The latter method is the easier one and does not require any new formalism.

The only physical candidate for such a scalar boson[38] has a high mass (730 MeV) and a large width ($\sim$200 MeV).A high-mass-scalar exchange gives a comparatively small contribution to OBEP. The coupling constant would be adjusted to cancel the second-order pion-nucleon scattering lengths, and is not expected to be anomalously large. In addition, when the large width is taken into account, the contribution is expected to resemble that high-mass part of the continuum contribution which it replaces.

The pair-suppression method can also be incorporated into our program by the following simple precsription. Consider the expression for the Feynman propagator S_F:

$$S_F(x)=\langle\text{vac}|\,T(\psi(x)\bar{\psi}(0))\,|\text{vac}\rangle. \tag{7.14}$$

One can consider S_F as a superposition of "particle" and "antiparticle" propagators by retaining, respectively, the positive- or negative-frequency parts of the field operators in (7.14). In symbols,

$$S_F=S_F^{(+)}+S_F^{(-)},$$

[49] K. Brueckner, M. Gell-Mann, and M. Goldberger, Phys. Rev. 90, 476 (1953); A. Klein, *ibid.* 95, 1061 (1954).

and, explicitly,

$$\frac{\gamma\cdot p+m}{p^2-m^2+i\epsilon}=\frac{1}{2E(\mathbf{p})}\left[\frac{\gamma^0E(\mathbf{p})-\boldsymbol{\gamma}\cdot\mathbf{p}+m}{p^0-E(\mathbf{p})+i\epsilon}-\frac{-\gamma^0E(\mathbf{p})-\boldsymbol{\gamma}\cdot\mathbf{p}+m}{p^0+E(\mathbf{p})-i\epsilon}\right]. \quad (7.15)$$

Using $S_F^{(-)}$ ($S_F^{(+)}$) in place of S_F for a nucleon line corresponds to projecting out the pair (no-pair) contribution of that line. Alternatively, one may introduce pair damping by replacing S_F with $S_F^{(+)}+\lambda S_F^{(-)}$, where λ is a damping parameter.

It has been noted[32] that there is a large cancellation between full-strength one-pair and two-pair terms within the static approximation. This greatly reduces the effect obtained on varying λ between 1 and 0.

VIII. OMITTED CONTRIBUTIONS AND EXPANSION LIMITATIONS

In this section we shall discuss various points pertaining to the validity of results obtained with our formalism. As stated in Sec. II, limitations arise from two different aspects: (i) the difficulty of calculating high-order irreducible Feynman amplitudes and (ii) approximations made in expanding the potential into a convenient form for many-body calculations.

The basis of our calculation is covariant perturbation theory in the form of selected irreducible Feynman graphs. These graphs form the kernel of the Bethe-Salpeter equation whose amplitude represents the sum of ladder iterations of those graphs. We have demonstrated that a potential can be produced in detail which reproduces the Bethe-Salpeter amplitudes to a good approximation. The validity of the result then depends on the adequacy of the selection of Feynman graphs.

The main justifications for the use of covariant perturbation theory as input in the nucleon-nucleon problem are the mass-range relationship and the possibility of accounting for some of the principal corrections such as rescattering and pion-pion interactions by appealing to other experimental and theoretical information. Thus, for example, one-dimensional dispersion relations indicate that to a good approximation, pion-nucleon resonances appearing in intermediate states may be treated as particles. Similarly, phenomenological Lagrangians based on current algebra and soft-pion physics may provide information on higher-order vertex corrections.

Our inclusion of the meson resonances in the nucleon-nucleon potential is in the spirit of the OBE hypothesis. It is designed to account for the pion-pion interaction by including the correlated multimeson exchanges. With respect to the problem of double counting, of the resonances included only the ρ meson is a two-pion exchange; however, the high mass of this resonance safely separates it from the two-pion continuum. As mentioned before, the (possible) isoscalar, scalar two-pion resonance (σ or ϵ^0) may have some overlap with the two-pion continuum. The addition of this resonance may therefore necessitate some subtraction from TPEP.

Thus OBEP and TPEP corrected for the above higher-order processes (perhaps along the lines indicated in Sec. VII) may be expected to dominate for internucleon distance where three-meson continuum effects are unimportant. The remaining question is whether three-meson continuum effects are significant for $r>0.5\ \mu^{-1}$.

Calculations of the meson-theoretic potentials have rarely gone beyond the fourth order because of the great computational work involved. The very few works that exist[50,51] are quite limited in their objectives and results. A very rough estimate of the three-pion continuum may be obtained by assuming a scaling of the strength of the two-pion contribution. Using this together with the characteristic range of a three-pion exchange, one arrives at the result that the three-pion continuum is unimportant outside half a pion Compton wavelength ($r>0.5$). The results of Ref. 50, on the other hand, indicate a non-negligible three-pion contribution near $r=0.5$ in some parts. However, this treatment is a selective one, and it relies on doubtful approximations that render its results unreliable. A careful examination of the three-pion continuum is in order.

Assuming an adequate selection of graphs for use in constructing the Bethe-Salpeter kernel, we now turn to the potential representation itself.

Throughout this paper inherent restrictions on the potential representation have been noted. The complexity of the Schrödinger potential required to simulate the Bethe-Salpeter equation in some regions may be summarized, with great simplification, in the impossibility of approximating $(\mathbf{p}^2+m^2)^{1/2}$ by $m+\mathbf{p}^2/2m$ near the branch points. This is related to the impossibility of localizing a (relativistic) particle more narrowly than within its Compton wavelength. In this case of the nucleon-nucleon system, the effective Compton wavelength of the nucleons is increased by virtue of the coupling to the pion field. Thus it is found that the resulting potentials become extremely nonlocal near and inside $r=0.5$. This fact, together with the importance of higher-mass exchanges (which fortunately arise at the same radius), sets a boundary for the potential representation somewhere near $r=0.5$. At shorter distances it seems necessary to resort to a phenomenological form. The nature of our results indicates that the form should be very nonlocal, such as a separable-potential or a boundary-condition representation.[12]

The approximation of expanding the configuration-

[50] S. Machida and K. Senba, Progr. Theoret. Phys. (Kyoto) **13**, 389 (1955).

[51] S. Furuichi and M. Yonezawa, Progr. Theoret. Phys. (Kyoto) **38**, 1200 (1967).

space potential in powers of the momentum operator is founded on the above considerations. The nonlocalities have a short range of about 0.5 and are quite strong inside this distance. Although it is possible to retain second-order terms in this expansion and obtain an energy-dependent potential (some quadratic spin-orbit terms were computed and found to be small outside $r=0.5$), the merits of such a procedure are doubtful in view of the asymptotic nature of this expansion. The expansion has the effect of smearing the nonlocalities beyond their actual range. This is rather similar to the effect of L^2 (and similar) terms that are used in phenomenological potentials (mainly) for the purpose of fitting D waves.[40,41] It is found that these terms result in too large an interaction for higher partial waves.[52] We feel that the inclusion of second-order momentum terms, although perhaps a useful phenomenological tool, is not theoretically justified.

IX. CONCLUSIONS

In this paper we have developed the formalism and presented the calculation of a nucleon-nucleon potential which accurately represents dominant classes of Feynman amplitudes. We have improved substantially on the accuracy of previous theoretical potential representations. This required that we resolve the ambiguity arising from the difference between relativistic and nonrelativistic iterations of the irreducible diagrams. This ambiguity has in the past led to very large errors[19] and to important differences between calculations based on similar approximation schemes.[8,9]

The method allows us to obtain the momentum-space potentials [Appendix A, Eqs. (4.16), (4.17), and (4.19), and the discussion following (4.18)] approximated only in that Eq. (3.7) is iterated to a finite order in the coupling constant, as given by Eq. (3.18). But the unitarity condition on the Bethe-Salpeter amplitude is carried over to the Schrödinger equation in an exact manner [Eqs. (3.13) and (3.17)]. The iteration of (3.7) is carried out to the same order in the coupling as the covariant perturbation expansion which provides the Bethe-Salpeter kernel. In addition, all the higher-order ladder iterations are produced nonrelativistically on solving the Schrödinger equation.

The calculation of the energy-dependent nonlocal configuration-space potential $V(\mathbf{r},\mathbf{r}'|W)$ can be done without further approximation. Although many-dimensional numerical integration is required, the calculation is feasible. Supplementing such a potential with a short-range phenomenological part (or the above momentum-space version with high-momentum parts), one may solve the Schrödinger equation and compare the results to experiment. However, the flexibility of the phenomenological addition would probably mask the difference between the exact form and the simpler local expansion. It would be more interesting to compare the Schrödinger amplitude produced by the exact form of potential with the calculated Bethe-Salpeter amplitude. The latter calculation is not yet feasible for the nucleon-nucleon case (even with OPEK alone) but may eventually succumb to future generations of computers.

We have shown that the expansion to first order in momentum (or, equivalently, to an energy-independent local plus spin-orbit potential in configuration space) converges well for $r>0.5$. The three-pion continuum and other corrections are expected to become significant at $r\lesssim 0.5$. The resulting potential was computed and is exhibited in detail. The result is in good agreement with phenomenological potentials. One expects discrepancies both because the phenomenological potentials are constrained by their assumed forms (and not completely determined by data) and because the theoretical corrections discussed in Sec. VII have not yet been added. The discrepancies are in fact greatest for those potentials least well determined by the data. This indicates that the corrections to the present theoretical potential may be comparatively small.

Comparing the various components of our potential (Figs. 7–10), it is clear that TPEP is more important than OBEP, although OBEP contributions are very significant. The necessity in OBEP models of employing two hypothetical low-mass scalar mesons and unreasonably large coupling constants for the vector mesons is explained by their neglect of TPEP. In addition, previously published OBEP models had serious quantitative errors arising from the use of expansions in momentum transfer (Fig. 23).

The probable major remaining corrections to the potential can be calculated according to the prescriptions of Sec. VII with no special difficulty. We are proceeding with those calculations. Furthermore, the method is applicable to the calculation of accurate potentials for many hyperon-hyperon, meson-hyperon, and meson-meson systems. For the ΛN system the simple expanded potential should be nearly as accurate as for the NN case. However, for the $\pi N \leftrightarrow \rho N$ off-diagonal potential, for example, there are very unequal initial- and final-state mass ratios. This circumstance will bring the relativistic kinematic singularities into the physical region, and require drastic approximations to obtain a simple form. We expect the method to be very useful in treating most hyperon-hyperon interactions, and to be useful, if not accurate, in the description of some meson-hyperon and meson-meson interactions.

Finally, we summarize our formalism for the derivation of a Schrödinger-equation potential from a field-theoretical model. This formalism is based on a careful reduction of the Bethe-Salpeter equation to a Lippmann-Schwinger form. The reduction is effected by means of a nonrelativistic two-body propagator that possesses the (elastic) unitarity properties of the original rela-

[52] R. Reid, Ann. Phys. (N. Y.) **50**, 411 (1968).

tivistic one and embodies all the appropriate kinematic properties. Although based on the technique of Blankenbecler and Sugar, it was extended to unequal-mass particles with spin. Furthermore, we clarified the identification of the nonrelativistic potential and amplitude by using unitarity conditions. The extension to arbitrary masses permits the extension to a variety of particle interactions.

An important consideration in our work has been the avoidance of the static approximation. The definition of the potential in Secs. IV and V, unlike previous derivations, is free of such approximations. As mentioned before, these approximations are particularly harmful when applied to integrals involving intermediate particles. As a result, in our calculation the sensitive cancellation in the iteration on the physical cut is valid to all orders in momenta and coupling strength. The usual statement about the inapplicability of μ/m expansions refers to the static approximation where recoil effects are ignored and all meson energies (including rest mass) are neglected relative to the nucleon mass. An expansion of the final result in terms of μ/m is of course another matter and of a mathematical nature. An examination of the potentials as analytic functions of μ/m reveals a branch point at $\mu/m=0$ (this is verified by the dispersion-theoretic amplitudes of Ref. 48), resulting in an asymptotic part in the expansion. On the whole, the convergence is moderately rapid, though perhaps not sufficiently so. At any rate, we have not found these expansions useful or necessary in our work.

The nature of the momentum dependence obtained indicates that strong nonlocality sets in at $r\approx 0.5$ over a narrow transition region. This confirms the arguments of Ref. 12 and suggests that a boundary-condition model (or some other extremely nonlocal form, such as a separable potential) is required for $r\lesssim 0.5$.

On the whole, the formalism presented in this paper appears to be a useful one for the construction of a theoretical nucleon-nucleon potential. It achieves a consistency largely lacking in previous works, and it appears to yield meaningful results.

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APPENDIX A

The task of this appendix is to obtain suitable integral representations for the quantities $\mathcal{I}$ and $\mathcal{J}$ defined by Eqs. (4.15). First we consider $\mathcal{J}$. Combining denominators à la Feynman, and translating the momentum variable, we get

$$\mathcal{J}=\frac{3!i}{2\pi}\int d^4k\, d(\alpha\beta\nu\lambda)\delta(1-\alpha-\beta-\nu-\lambda)\{\gamma\cdot[W+p+\beta(p'-p)+\nu(W-p)-\lambda(W+p)+k]-m\}^{(1)}$$
$$\times\frac{\{\gamma\cdot[W-p-\beta(p'-p)-\nu(W-p)+\lambda(W+p)-k]-m\}^{(2)}}{\{k^2-[\beta(p'-p)+\nu(W-p)-\lambda(W+p)]^2+\beta(p'-p)^2+\nu(W-p)^2+\lambda(W+p)^2-m^2(\nu+\lambda)-\mu^2(\alpha+\beta)+i\epsilon\}^4}. \quad \text{(A1)}$$

Next, we rotate the contour, carry out the four-dimensional Euclidian integration, and eliminate α using the δ function. The result is

$$\mathcal{J}=-\tfrac{1}{4}\pi\int_0^1 d\beta\int_0^{1-\beta}d\nu\int_0^{1-\beta-\nu}d\lambda\left\{\frac{\gamma^{(1)\mu}\gamma_\mu{}^{(2)}}{D}+2\{\gamma\cdot[W+p+\beta(p'-p)+\nu(W-p)-\lambda(W+p)]-m\}^{(1)}\right.$$
$$\left.\times\frac{\{\gamma\cdot[W-p-\beta(p'-p)-\nu(W-p)+\lambda(W+p)]-m\}^{(2)}}{D^2}\right\},$$

where

$$D=[\beta(p'-p)+\nu(W-p)-\lambda(W+p)]^2-\beta(p'-p)^2-\nu(W-p)^2-\lambda(W+p)^2+m^2(\nu+\lambda)+\mu^2(1-\nu-\lambda)-i\epsilon.$$

Changing variables according to

$$\zeta=\nu+\lambda,\quad \xi=\nu-\lambda,$$

and integrating the first term by parts over ξ, we get

$$\mathcal{J}=-\tfrac{1}{4}\pi\int_0^1\int_0^{1-\beta}d\beta d\zeta\left\{\zeta\frac{\gamma^{\mu(1)}\gamma_\mu{}^{(2)}}{D'}-2\int_0^\zeta d\xi\frac{W^2\xi^2\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}}{D^2}\right.$$
$$\left.+2\int_0^\zeta d\xi\frac{\{\gamma^0W-\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(1)}\{\gamma^0W+\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(2)}}{D^2}\right\}, \quad \text{(A2)}$$

where

$$D' = D|_{\xi=\zeta}.$$

Note that in (A2) we have explicitly set $p^0 = p'^0 = 0$.

Turning to $\mathcal{G}$, we write it as a sum of three terms:

$$\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3,$$

where

$$\mathcal{G}_1 = -\tfrac{1}{4}\int d\mathbf{k} \frac{\gamma^{0(1)}(\boldsymbol{\gamma}\cdot\mathbf{k}-m)^{(2)} + \gamma^{0(2)}(-\boldsymbol{\gamma}\cdot\mathbf{k}-m)^{(1)}}{[(\mathbf{k}-\mathbf{p}')^2+\mu^2][E^2(\mathbf{k})-W^2-i\epsilon][(\mathbf{k}-\mathbf{p})^2+\mu^2]},$$

$$\mathcal{G}_2 = -\tfrac{1}{4}\int d\mathbf{k} \frac{\gamma^{0(1)}\gamma^{0(2)}}{E(\mathbf{k})[(\mathbf{k}-\mathbf{p}')^2+\mu^2][(\mathbf{k}-\mathbf{p})^2+\mu^2]},$$

$$\mathcal{G}_3 = -\tfrac{1}{4}\int d\mathbf{k} \frac{(\boldsymbol{\gamma}\cdot\mathbf{k}+m)^{(1)}(-\boldsymbol{\gamma}\cdot\mathbf{k}+m)^{(2)} + W^2\gamma^{0(1)}\gamma^{0(2)}}{E(\mathbf{k})[(\mathbf{k}-\mathbf{p}')^2+\mu^2][E^2(\mathbf{k})-W^2-i\epsilon][(\mathbf{k}-\mathbf{p})^2+\mu^2]},$$

We shall demonstrate the calculation for $\mathcal{G}_3$, and record the result for the other two, since the procedure is the same in all cases. To apply the Feynman technique, we must eliminate the square-root dependence in $E(\mathbf{k})$. This is achieved by means of the integral representation

$$\frac{1}{E(\mathbf{k})} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dZ}{Z^2+E^2(\mathbf{k})}.$$

Using this representation, we combine denominators and translate the variable, whereupon we get

$$\mathcal{G}_3 = -\frac{3!}{4\pi}\int dZ d\mathbf{k}\, d(\alpha\beta\nu\lambda)\delta(1-\alpha-\beta-\nu-\lambda) \times \frac{[\boldsymbol{\gamma}\cdot(\mathbf{k}+\alpha\mathbf{p}+\beta\mathbf{p}')+m]^{(1)}[-\boldsymbol{\gamma}\cdot(\mathbf{k}+\alpha\mathbf{p}+\beta\mathbf{p}')+m]^{(2)} + W^2\gamma^{0(1)}\gamma^{0(2)}}{[\mathbf{k}^2-(\alpha\mathbf{p}+\beta\mathbf{p}')^2+\alpha\mathbf{p}^2+\beta\mathbf{p}'^2+(\alpha+\beta)\mu^2+\nu(m^2-W^2)+\lambda(Z^2+m^2)-i\epsilon]^4}.$$

Carrying out the angular integration, integrating by parts over $|\mathbf{k}|$, and eliminating λ leads to

$$\mathcal{G}_3 = -\int_{-\infty}^{+\infty} dZ \int_0^\infty d|\mathbf{k}| \int_0^1 d\beta \int_0^{1-\beta} d\alpha \int_0^{1-\alpha-\beta} d\nu \times \frac{-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}\mathbf{k}^2 + [\boldsymbol{\gamma}\cdot(\alpha\mathbf{p}+\beta\mathbf{p}')+m]^{(1)}[-\boldsymbol{\gamma}\cdot(\alpha\mathbf{p}+\beta\mathbf{p}')+m]^{(2)} + W^2\gamma^{0(1)}\gamma^{0(2)}}{[\mathbf{k}^2-(\alpha\mathbf{p}+\beta\mathbf{p}')^2+\alpha\mathbf{p}^2+\beta\mathbf{p}'^2+(\alpha+\beta)\mu^2+\nu(m^2-W^2)+(1-\alpha-\beta-\nu)(Z^2+m^2)-i\epsilon]^3}.$$

The integration over ν is elementary and can be performed immediately. The result is

$$\mathcal{G}_3 = -\tfrac{1}{2}\int_{-\infty}^{+\infty} dZ \int_0^\infty d|\mathbf{k}| \int_0^1 d\beta \int_0^{1-\beta} d\alpha \{-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}\mathbf{k}^2 + [\boldsymbol{\gamma}\cdot(\alpha\mathbf{p}+\beta\mathbf{p}')+m]^{(1)}[-\boldsymbol{\gamma}\cdot(\alpha\mathbf{p}+\beta\mathbf{p}')+m]^{(2)} + W^2\gamma^{0(1)}\gamma^{0(2)}\} \times \frac{1}{W^2+Z^2}\left\{\frac{1}{[\mathbf{k}^2-(\alpha\mathbf{p}+\beta\mathbf{p}')^2+\alpha\mathbf{p}^2+\beta\mathbf{p}'^2+(\alpha+\beta)\mu^2+(1-\alpha-\beta)(m^2-W^2)-i\epsilon]^2} - \frac{1}{[\mathbf{k}^2-(\alpha\mathbf{p}+\beta\mathbf{p}')^2+\alpha\mathbf{p}^2+\beta\mathbf{p}'^2+(\alpha+\beta)\mu^2+(1-\alpha-\beta)(Z^2+m^2)-i\epsilon]^2}\right\}.$$

The first denominator is now independent of Z, and we perform the Z integration for that part. Next, we change

variables according to $\alpha=1-\beta-\xi$, $|\mathbf{k}|=\xi W$, and obtain

$$\mathcal{I}_3=-\tfrac{1}{2}\pi\int_0^\infty d\xi\int_0^1 d\beta\int_0^{1-\beta} d\zeta\,\frac{N}{\{\xi^2W^2-[\beta(\mathbf{p}'-\mathbf{p})-\zeta\mathbf{p}]^2+\beta(\mathbf{p}-\mathbf{p}')^2+\zeta\mathbf{p}^2+(1-\zeta)\mu^2+\zeta(m^2-W^2)-i\epsilon\}^2}$$
$$+\tfrac{1}{2}W\int_{-\infty}^{+\infty} dZ\int_0^\infty d\xi\int_0^1 d\beta\int_0^{1-\beta} d\zeta\,\frac{N}{(Z^2+W^2)\{\xi^2W^2-[\beta(\mathbf{p}'-\mathbf{p})-\zeta\mathbf{p}]^2+\beta(\mathbf{p}-\mathbf{p}')^2+\zeta\mathbf{p}^2+(1-\zeta)\mu^2+\zeta(Z^2+m^2)\}^2},\tag{A3}$$

where

$$N=\{\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(1)}\{-\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(2)}-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}W^2\xi^2+W^2\gamma^{0(1)}\gamma^{0(2)}.$$

A similar procedure for $\mathcal{I}_1$ and $\mathcal{I}_2$ leads to the expressions

$$\mathcal{I}_1=\tfrac{1}{2}W\pi\int_0^\infty d\xi\int_0^1 d\beta\int_0^{1-\beta} d\zeta\,\frac{\gamma^{0(1)}\{-\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(2)}+\gamma^{0(2)}\{\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(1)}}{\{\xi^2W^2-[\beta(\mathbf{p}'-\mathbf{p})-\zeta\mathbf{p}]^2+\beta(\mathbf{p}-\mathbf{p}')^2+\zeta\mathbf{p}^2+(1-\zeta)\mu^2+\zeta(m^2-\mu^2)-i\epsilon\}^2},$$
$$\mathcal{I}_2=-\tfrac{1}{2}W\int_{-\infty}^{+\infty} dZ\int_0^\infty d\xi\int_0^1 d\beta\int_0^{1-\beta} d\zeta\,\frac{\gamma^{0(1)}\gamma^{0(2)}}{\{W^2\xi^2-[\beta(\mathbf{p}'-\mathbf{p})-\zeta\mathbf{p}]^2+\beta(\mathbf{p}-\mathbf{p}')^2+\zeta\mathbf{p}^2+(1-\zeta)\mu^2+\zeta(Z^2+m^2)\}^2}.\tag{A4}$$

The final result of this appendix is obtained by combining $\mathcal{I}_1$ with the first integral contributing to $\mathcal{I}_3$ in Eq. (A3), and recording the resulting expression for $\mathcal{I}$;

$$\mathcal{I}=-\tfrac{1}{2}\pi\int_0^\infty d\xi\int_0^1 d\beta\int_0^{1-\beta} d\zeta$$
$$\times\frac{-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}W^2\xi^2+\{\gamma^0W-\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(1)}\{\gamma^0W+\boldsymbol{\gamma}\cdot[\beta(\mathbf{p}'-\mathbf{p})+\mathbf{p}(1-\zeta)]-m\}^{(2)}}{[W^2\xi^2+R+(1-\zeta)\mu^2+\zeta(m^2-W^2)-i\epsilon]^2}$$
$$-\tfrac{1}{2}W\int_{-\infty}^{+\infty} dZ\int_0^\infty d\xi\int_0^1 d\beta\int_0^{1-\beta} d\zeta\,\frac{\gamma^{0(1)}\gamma^{0(2)}}{[W^2\xi^2+R+(1-\zeta)\mu^2+\zeta(Z^2+m^2)]^2}+\tfrac{1}{2}W\int_{-\infty}^{+\infty} dZ\int_0^\infty d\xi\int_0^1 d\beta\int_0^{1-\beta} d\zeta$$
$$\times\frac{-\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}W^2\xi^2+\{\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(1)}\{-\boldsymbol{\gamma}\cdot[(1-\zeta)\mathbf{p}+\beta(\mathbf{p}'-\mathbf{p})]+m\}^{(2)}+W^2\gamma^{0(1)}\gamma^{0(2)}}{(Z^2+W^2)[W^2\xi^2+R+(1-\zeta)\mu^2+\zeta(Z^2+m^2)]^2},\tag{A5}$$

where

$$R=\beta(\mathbf{p}-\mathbf{p}')^2+\zeta\mathbf{p}^2-[\beta(\mathbf{p}'-\mathbf{p})-\zeta\mathbf{p}]^2.$$

APPENDIX B

In this appendix details of the numerical calculations will be presented. All numerical integrations were performed on an IBM 360/65 computer using Simpson's rule. The integrals treated were those of TPEP and OBEP, given, respectively, by Eqs. (4.31) and (5.4). During the course of computations, it was recognized that the form of the integrands given in (4.31) was not suitable for the integration method used. The difficulty stemmed from the fact that for moderately large r, the exponential factor in the integrand peaked in some regions of integration and was therefore not adequately probed by a uniform mesh of reasonable size. This difficulty was easily overcome by changing the variables of integration so as to spread the integrand over the volume of integration. The following changes of variables were found adequate: for integrals involving X, $\zeta=u^3$; for integrals involving Y, $\zeta=u^6$; for integrals involving W, $\zeta=u^3$ and $\xi=\omega^{-3}$; and for integrals involving Z, $\zeta=u^6$ and $\xi=\omega^{-3}$.

The variable β was not changed. The number of points used was 21 in each dimension. As a check on the accuracy of the integration procedure, a run was made with the number of points doubled in each dimension. The resulting change occurred in the third significant figure for the worst cases, and the over-all change in the potential amounted to a few tenths of a percent for a small number of points, and less than one-tenth of a percent for the rest.

The functions G defined by (5.4) can all be computed

using the auxiliary integrals f defined by

$$f(r|\alpha,n)=\int_1^\infty d\xi \frac{\xi^n(\xi^2-1)^{1/2}}{\xi^2-\alpha^2}\exp(-2mr\xi). \quad \text{(B1)}$$

The relations between these two sets of functions are easily ascertained by simple manipulation. Defining the dimensionless quantities g by

$$g(r|\alpha,n)=(2m)^{-n}G(r|m_\beta,n), \quad \alpha=m_\beta/2m$$

we get the following set of expressions:

$$g(r|\alpha,5)=(1-\alpha^2)[K_1(z)+(1/z)K_2(z)]+\alpha^2 g(r|\alpha,3),$$
$$g(r|\alpha,4)=(1-\alpha^2)\tfrac{1}{2}[K_0(z)+K_2(z)]+\alpha^2 g(r|\alpha,2),$$
$$g(r|\alpha,3)=(1-\alpha^2)K_1(z)+\alpha^2 g(r|\alpha,1),$$
$$g(r|\alpha,2)=(1-\alpha^2)K_0(z)+\alpha^2 g(r|\alpha,0),$$
$$g(r|\alpha,n)=K_n(z)-f(r|\alpha,n), \quad n=0,1$$
$$g(r|\alpha,-1)=g(r|\alpha,1)-(1-\alpha^2)f(r|\alpha,-1),$$

where $z=2mr$. The functions K are the modified Bessel functions defined by

$$K_n(z)=\frac{(\sqrt{\pi})(\frac{1}{2}z)^n}{\Gamma(n+\frac{1}{2})}\int_1^\infty (t^2-1)^{n-1/2}\exp(-zt)\,dt.$$

The integrals f are needed for the values of $n=-1, 0, 1$. These were numerically integrated, truncating the range of integration at $\xi=4$. The contribution thus neglected is easily estimated to be no greater than 10^{-6}, the total magnitude of the integral in all cases. As in the case of the integrals comprising TPEP, the integrand in (B1) peaked sharply near the lower end of the range of integration. This difficulty was circumvented by using 41 points in the interval [1, 1.1] and an equal number in the interval [1.1, 4]. Doubling the number of points in each segment resulted in no change to within one-tenth of a percent. Comparison with asymptotic expansions for large r and the tests mentioned above indicate that the numerical integrations are accurate to within a few parts in a thousand.