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Unitary and Crossing-Symmetric S-, P-, and D-Wave Low-Energy Pion-Pion Scattering

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Crossing-symmetric solutions to the pion-pion partial-wave dispersion relations are obtained, using the inverse-amplitude method. The presence of the ρ meson with its physical mass and width is assumed, the amplitudes are constrained to contain Adler zeros, and general sum rules and inequalities are satisfied. Favored solutions contain a very broad isoscalar S-wave resonance, and have a small decreasing isospin-2 S-wave phase shift. The D waves are very small. The over-all agreement with phenomenology and with other S-matrix calculations is very good.

I. INTRODUCTION

THIS paper presents predictions of S-, P- and D-wave π-π scattering up to about 1 GeV c.m. energy, obtained by solving coupled partial-wave dispersion relations using the inverse-amplitude method.[1] The mass and width of the ρ resonance are assumed given. Crossing-symmetric polynomial expressions deduced previously[2] under simple and general assumptions are used to provide S- and D-wave subtraction constants and information about zeros below threshold. Consequently, the solutions automatically satisfy sum rules,[3] and obey to a high degree of accuracy rigorous inequalities.[4]

The over-all features of our results agree well with experiment[5] and with other S-matrix calculations.[6–9] We find a large positive isoscalar S-wave phase shift, and some of the solutions contain a σ resonance. The isospin-2 S-wave phase shift is small and negative, while both D waves are very small.

The paper is organized as follows. Section II summarizes the relevent results of I, Sec. III discusses the partial-wave dispersion relations and explains the methods used to solve them, and Secs. IV and V contain numerical results and discussion.

II. POLYNOMIAL AMPLITUDES

In I the π-π amplitudes within the Mandelstam triangle (i.e., for $0<s,\ t,\ u<4\mu^2$)[10] are constructed from a single invariant function F symmetric in its two arguments[11]:

$$F(s,t)=a+b(s+t)+cst+d(s^2+t^2)\,,\tag{1}$$

where a, b, c, and d are real constants. One of these is eliminated if the amplitudes have an Adler zero[12]—that is, if[13]

$$F(\mu^2,\mu^2)=0\,,\tag{2}$$

and two other constants may be expressed in terms of the ρ-meson mass (m) and width (Γ) by matching the P-wave threshold to a resonance-dominated partial wave[14] of the form[15]

$$A_1{}^1=q^2a_1[(1-q^2/k^2)-iq^2a_1\rho]^{-1}\,,\tag{3}$$

where $4q^2=s-4\mu^2$, $4k^2=m^2-4\mu^2$, and

$$\rho=[(s-4\mu^2)/s]^{1/2},\tag{4}$$

with

$$a_1=m^2\Gamma/8k^5.\tag{5}$$

The fourth and last parameter,

$$X=\tfrac{4}{3}k^2(4d-c)/a_1\,,\tag{6}$$

is fixed in sign ($X>0$) if the π^0-π^0 S-wave amplitude is to obey simultaneously all of a set of eight constraints which follow from crossing together with weak analyticity and unitarity assumptions.[4]

In I the magnitude of X is constrained only by the relatively weak self-consistency requirements that the unitarity branch points, neglected in (1), should be unimportant singularities. This is taken to be equivalent to asking that the S-wave scattering lengths should be small, and this condition is weak because terms proportional to X provide only corrections of order μ^2/k^2 ($\sim 15\%$) to these quantities.

Subsequent work with dispersion relations showed[16] that under certain (fairly strong) assumptions about

* S. R. C. Research Student.
[1] J. W. Moffat, Phys. Rev. **121**, 926 (1961).
[2] R. H. Graham and R. C. Johnson, Phys. Rev. **188**, 2362 (1969), hereafter referred to as I.
[3] R. Roskies, Nuovo Cimento **66A**, 494 (1970); J. L. Basdevant, G. Cohen-Tannoudji, and A. Morel, *ibid.* **64A**, 585 (1969).
[4] A. Martin, Nuovo Cimento **47**, 265 (1967); A. K. Common, *ibid.* **53A**, 946 (1968); G. Auberson, O. Piguet, and G. Wanders, Phys. Letters **28B**, 41 (1968).
[5] See, for a review, Proceedings of the Conference on $\pi\pi$ and $K\pi$ Interactions, edited by F. Loeffler and E. Malamud, Argonne National Laboratory Report, 1969 (unpublished).
[6] E. P. Tryon, in Ref. 5, p. 665.
[7] D. Morgan and G. Shaw, Nucl. Phys. **B10**, 261 (1969) (summarized in Ref. 5, p. 726); Phys. Rev. D **1**, 520 (1970).
[8] R. C. Johnson and P. D. B. Collins, Phys. Rev. **185**, 2020 (1969).
[9] R. C. Johnson, Phys. Letters **32B**, 199 (1970).
[10] The usual s, t, and u variables, connected by $s+t+u=4\mu^2$ (where μ is the pion mass), are used throughout.
[11] A. Yahil, Phys. Rev. **185**, 1786 (1969).
[12] S. L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).
[13] Equation (2) applies strictly off shell where $s+t+u=3\mu^2$. We are assuming its validity on shell, where $s+t+u=4\mu^2$.
[14] M. G. Olsson, Phys. Rev. **162**, 1338 (1967).
[15] Notation for partial-wave amplitudes is $A_l{}^I$, where $I=0,1,2$ is the isospin and $l=0,1,2,\ldots$ is the angular momentum.
[16] R. C. Johnson, Phys. Rev. D (to be published), hereafter referred to as II.

average behavior of the π-π amplitudes above 1 GeV c.m. energy, the neglect of thresholds and the use of quadratic polynomial approximation are completely consistent with current phenomenological knowledge of the physical scattering phase shifts,[5] provided they are not too large at low energies (where the experimental possibilities[5] are still open). Under these conditions, the sum rules allow the order-of-magnitude estimate $X\sim 1$.

In terms of X we have from I

$$a=-\mu^2 a_1[3-\tfrac{1}{4}(13+5X)\mu^2/k^2], \quad (7)$$

$$b=a_1[\tfrac{3}{2}-(2+X)\mu^2/k^2], \quad (8)$$

$$c=\tfrac{1}{4}(X+2)a_1/k^2, \quad (9)$$

$$d=\tfrac{1}{4}(X+\tfrac{1}{2})a_1/k^2. \quad (10)$$

For $0<s<4\mu^2$, the partial-wave amplitudes are

$$A_0{}^0=\tfrac{1}{4}[5a+4b(s+2\mu^2)-\tfrac{1}{6}c(19s-4\mu^2)(s-4\mu^2)+\tfrac{2}{3}d(11s^2-16\mu^2 s+32\mu^4)], \quad (11)$$

$$A_2{}^0=(c-4d)(s-4\mu^2)^2/120, \quad (12)$$

$$A_1{}^1=\tfrac{1}{6}(s-4\mu^2)[b+cs+d(4\mu^2-s)], \quad (13)$$

$$A_0{}^2=\tfrac{1}{2}[a-b(s-4\mu^2)+\tfrac{1}{6}(c+2d)(s-4\mu^2)^2], \quad (14)$$

$$A_2{}^2=(2d-c)(s-4\mu^2)^2/60. \quad (15)$$

The two S waves [(11) and (14)] have Adler zeros near $s=\frac{1}{2}\mu^2$ and $s=2\mu^2$, respectively, while the P and D waves have the usual threshold angular momentum zeros. As we shall describe in Sec. III, the solutions to the partial-wave dispersion relations are constructed to have all these zeros, exactly as (11)–(15) prescribe, and in addition all but the P wave are constrained to match these polynomial expressions at the symmetry point through a subtraction. A subtraction above threshold in the P wave is used to insert the ρ resonance at the experimental mass[17] ($\sim$765 MeV).

III. PARTIAL-WAVE DISPERSION RELATIONS

A pion-pion partial-wave amplitude[18] $A(s)$ has well-known analytic properties[19] in s, which are summarized in the dispersion relation[20]

$$\pi A(s)=\int_{-\infty}^{0}\frac{\mathrm{Im}A(s')ds'}{s'-s}+\int_{4\mu^2}^{\infty}\frac{\mathrm{Im}A(s')ds'}{s'-s}. \quad (16)$$

Elastic unitarity, expressed by

$$\mathrm{Im}A(s)=\rho|A|^2, \quad (17)$$

where ρ is given by (4), is strictly valid only for $4\mu^2<s<16\mu^2$, but is known to be a good approximation for $s\lesssim 50\mu^2$ ($\sqrt{s}\lesssim 1$ GeV). Here we assume (17) is valid for all $s>4\mu^2$.

For $s<0$, $\mathrm{Im}A(s)$ is related to physical scattering in all angular momentum states in the crossed channels. However, because of the presence of the third double spectral function the t- and u-channel partial-wave series converge only for $s>-32\mu^2$, so that in principle the power of crossing to relate the different amplitudes is limited.

Nevertheless, we follow the usual practice of ignoring the divergence, and we calculate the left-hand cut discontinuity for $s\gtrsim -50\mu^2$ through crossing as if the third double spectral function were absent. (The phase-shift results are insensitive to the precise behavior of $\mathrm{Im}A$ for $s\lesssim -32\mu^2$.) That is, we use on the left-hand cut the formula[19,21]

$$\mathrm{Im}A_l{}^I(s)=\frac{2}{4\mu^2-s}\int_{4\mu^2}^{4\mu^2-s}P_l\left(1+\frac{2s'}{s-4\mu^2}\right)\sum_{I'=0}^{2}\beta^{II'}\times\sum_{l'=0}^{2}(2l'+1)\,\mathrm{Im}A_{l'}{}^{I'}(s')P_{l'}\left(1+\frac{2s}{s'-4\mu^2}\right)ds', \quad (18)$$

where

$$\beta^{II'}=\begin{pmatrix}\frac{1}{3} & 1 & 5/3\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix}, \quad (19)$$

and where, as indicated, the crossed-channel partial-wave series are truncated at D waves.

To solve (16), we introduce[1]

$$B=A^{-1}. \quad (20)$$

With our approximations on the left- and right-hand cuts, there are two advantages of considering the inverse amplitude.

The first is the fact that, independent of isospin and angular momentum, from (17) we have

$$\mathrm{Im}B=-\rho \quad (21)$$

for $s>4\mu^2$, and the second is that

$$\mathrm{Im}B\sim 1/s \quad (22)$$

as $s\to -\infty$ [while from (18), $\mathrm{Im}A\sim s$].

As a result of (21) and (22), a once-subtracted dispersion relation for B converges adequately, and the right-hand cut integral in each partial wave can be evaluated once and for all in closed form.

The disadvantage of the inverse-amplitude is that a solution to the dispersion relation for B may not satisfy (16)—either because B so constructed has zeros on the physical sheet, or because A should have similar zeros which are not taken into account, or both.

However, in practice it turns out that the presence of important physical-sheet zeros of B can be recognized

[17] Particle Data Group, Rev. Mod. Phys. **41**, 1 (1969).

[18] Isospin and angular momentum labels are dropped when inessential.

[19] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[20] The possibility of bound-state poles is ignored, the $+i\epsilon$ prescription for s approaching the cuts is to be understood, and we are assuming for the moment that (16) coverges.

[21] Isospin and angular momentum labels are essential here.

and dealt with, while the fact that our results largely agree with experiment[5] and with conventional dispersion-relation calculations[6–9] leads us to believe that the only important zeros of A are those which are explicitly inserted.

The dispersion relation for B may be written as[1]

$$B=L+H-H_0+B_0+P\,, \qquad (23)$$

where the terms on the right-hand side have the following significance.

The left-hand-cut integral is

$$L(s,s_0)=\frac{s-s_0}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}B(s')\,ds'}{(s'-s)(s'-s_0)}\,, \qquad (24)$$

where s_0 is a subtraction point. The right-hand cut integral is the universal phase-space function[1,19,22]

$$H(s)=\frac{1}{\pi}\rho\ln\left(\frac{\rho+1}{\rho-1}\right), \qquad (25)$$

and $H_0\equiv H(s_0)$. $B_0\equiv B(s_0)$, a subtraction constant, and P is a term arising from physical-sheet zeros of A, i.e., poles of B.

For one simple pole at $s=s_p$, we have

$$P=\frac{s-s_0}{(s-s_p)(s_p-s_0)}\left(\frac{d}{ds}A(s)\Big|_{s=s_p}\right)^{-1}, \qquad (26)$$

and for a double pole at threshold (such as occurs in a D-wave amplitude) we have

$$P=\frac{s-s_0}{(4\mu^2-s)(4\mu^2-s_0)}\left[\tilde{a}\left(\frac{1}{4\mu^2-s}+\frac{1}{4\mu^2-s_0}\right)-\tilde{b}\right], \qquad (27)$$

where $\tilde{a}$ and $\tilde{b}$ are essentially scattering-length and effective-range parameters:

$$(s-4\mu^2)^2\,\mathrm{Re}B=\tilde{a}+\tilde{b}(s-4\mu^2) \qquad (28)$$

for $s\approx4\mu^2$. [$\tilde{b}=0$ in both A_2^0 and A_2^2 of I; see (12) and (15).]

For a choice of m, Γ, X, and s_0, the model summarized in Sec. II supplies B_0 and P for each partial wave, and only L is unknown. It is convenient to solve (23) iteratively,[23] as follows:

(i) Guess L;
(ii) calculate ReB;
(iii) construct $A=B^{-1}$;
(iv) use crossing to find ImA for $s<0$;
(v) calculate ImB for $s<0$;
(vi) reestimate L;
(vii) go to (ii), and cycle to unchangedness.

Step (iii) needs knowledge of ImB for $s<0$, and step (v) needs similar knowledge of ReA. Both are supplied by the results of the previous iteration. The output amplitude vanishes logarithmically at large s, and (16) exists.

This procedure is the same as that used in the original application of the inverse-amplitude method,[23] and its convergence has been proved.[24]

IV. RESULTS

To start with, we set $m=765$ MeV, $\Gamma=120$ MeV as before,[2] and choose as subtraction point $s_0=\frac{4}{3}\mu^2$ in all but the P wave. There, we insert the ρ resonance by putting $s_0=m^2$, $B_0=0$. The polynomial expressions (11)–(15) provide all the pole terms P, plus B_0 in the S and D waves.

The initial approximation $L=0$ is a satisfactory starting point for the iterative solution of (23). Figure 1 shows the corresponding first approximation to the phase shifts, with $X=0.5$. Already, these show features in fair agreement with experiment.[5]

In following the iterative procedure (i)–(vii) above, experience shows that sufficiently rapid convergence

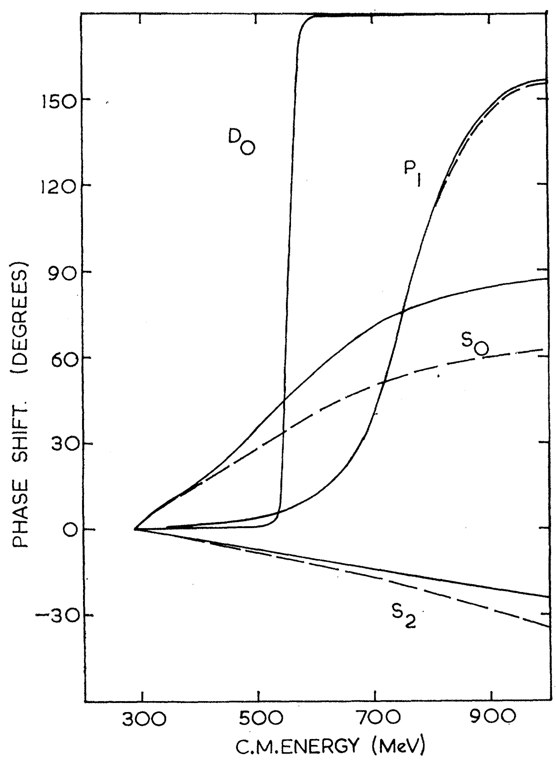

FIG. 1. Input (dashed line) and output (solid line) phase shifts, as described in the text. Notation is as in Ref. 26. The input D_0 and input and output D_2 phase shifts are indistinguishable from each other and from zero on the scale of the graph.

[22] For $s<4\mu^2$, H is to be evaluated by circling the branch points in ρ in a counterclockwise sense. See. Ref. 1 for the explicit analytic continuations.

[23] B. H. Bransden and J. W. Moffat, Nuovo Cimento 21, 505 (1961); Phys. Rev. Letters 8, 145 (1962).

[24] B. H. Bransden and J. W. Moffat, Nuovo Cimento 32, 159 (1964). The treatment given in this paper is easily extended to the case considered here.

and accurate results are obtained by calculating on a sequence of points in s separated by $\frac{1}{2}\mu^2$, and performing the integrals by the simple trapezium rule [with appropriate careful treatment[25] of the principal-value singularity in (24) for $s<0$].

For convenience, the integral for L is cut off at $s=\Lambda=-50\mu^2$, and for $s<\Lambda$ we assume

$$\mathrm{Im}B(s)=\mathrm{Im}B(\Lambda)\times(s/\Lambda)^\alpha. \tag{29}$$

The results for the phase shifts for $s\lesssim 50\mu^2$ (where they are most likely to be reliable) are completely insensitive to the precise values of Λ and α, provided $\alpha\leqslant 0$ and $|\Lambda|$ is not small ($\Lambda<-32\mu^2$, say). For all the results quoted here, $\alpha=0$.

After three cycles of iteration the phase shifts begin to settle down, and after five cycles they remain constant to within a few (<5) percent.

Figure 1 includes the results after five iterations, showing that, except in the D_0 channel,[26] the presence of a left-hand-cut contribution makes no major change in the behavior of the phase shift.

In S_0, it rises quickly from threshold, reaching 40° at 500 MeV, 75° under the ρ, and rising to just less than 90° at 1 GeV. In S_2 it falls smoothly through $-8°$, $-15°$, to $-20°$ at these energies.

To a very good approximation, the P_1 wave is unaltered by the iteration. From threshold to 1 GeV it remains dominated by the pole term and the subtraction. As Fig. 2 shows, the effective-range plot is almost exactly linear, in agreement with (3) and with Olsson's results.[14] We find that this is true in all our solutions, whatever choices we make for X, Γ, etc., and so hereafter we omit explicit references to this partial wave.

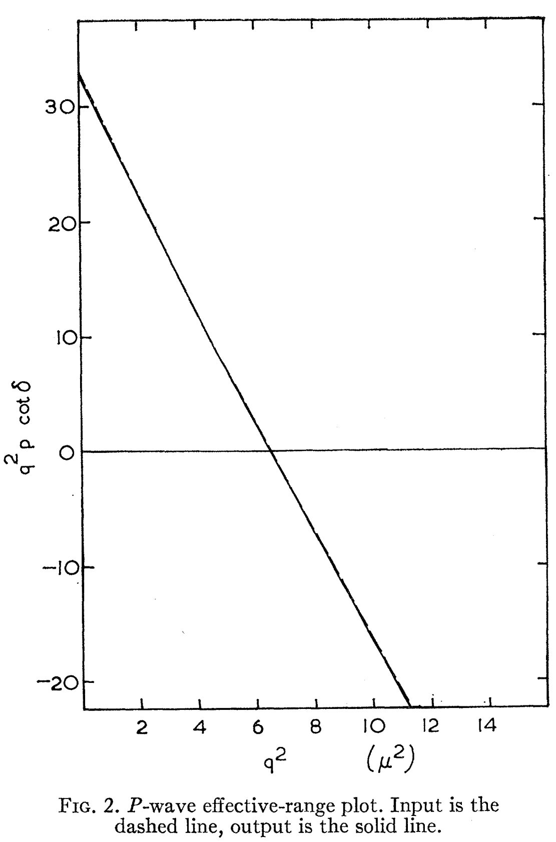

FIG. 2. P-wave effective-range plot. Input is the dashed line, output is the solid line.

While the D_2 phase shift is entirely as expected[5,9] (namely, very small and negative), the D_0 phase shift, after starting out small and positive, shows a sudden rise of nearly 180° corresponding to a very narrow resonance ($\Gamma_{D_0}<5$ MeV) at about 550 MeV.

It is tempting to identify this object with the narrow $\pi^+\pi^-$ enhancement at 530 MeV discussed by Kalogeropoulos[27] (or perhaps with the similar effect discussed by Dubal and Roos[28]).

However, its position and width prove to be unaffected by wide variations of the input quantities Γ, X, etc.—that is, its properties appear unconnected to essential parameters of the dynamics. Thus one is led to suspect that it is not a real resonance at all but rather is a CDD-type effect[29] peculiar to this calculation.[30]

In fact, the "resonance" is so narrow, and the D_0 phase shift a few half-widths away so small ($<1\frac{1}{2}°$, modulo π—which is what counts), that relative to other contributions, its effects (plus those of the D_2 wave) in L are entirely negligible for all I and l.

If in (18) the D waves are taken to be identically zero, the output S- and P-wave phase shifts are altered so little that the corresponding changes in Fig. 1 are undetectable on the scale of the graph.

This is true in all our solutions, and so henceforth the D waves generally will be ignored as being unreliable in detail, and we will concentrate on the S waves.

The parameter X controls satisfaction of rigorous conditions on the polynomial model,[2] and because all the important features of the model are explicitly built in to our amplitudes, the sign of X continues to determine to a good approximation their validity as solutions to (16). [In fact it is found that Eqs. (20)–(27) of I are all satisfied for $X>0.1$; the two integral constraints (26) and (27) are the first to be violated because $A_0{}^0$ is slightly larger than the polynomial expression.]

[25] The principal-value singularity in Eq. (24) involves the derivative of the tabulated integrand, which is evaluated by quadratic interpolation.

[26] The "spectroscopic" notation, (angular momentum)$_{\text{isospin}}$, is convenient.

[27] T. E. Kalogeropoulos, Phys. Rev. **185**, 2030 (1969).

[28] L. Dubal and M. Roos, in Ref. 5, p. 285.

[29] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

[30] It is found that the "resonance" can easily be made to disappear in a crossing-symmetric way by including in the D_0-wave threshold pole term at the start of the iteration a nonzero effective-range parameter [see (27) and (28)]. For example, with $\tilde{b}=m_f{}^3/\Gamma_f$, corresponding to the tail of an (elastic) f^0 meson at $s=m_f{}^2$ with width Γ_f [$m_f=1260$ MeV, $\Gamma_f=110$ MeV, (Ref. 17)], the D_0 phase shift rises smoothly and slowly from threshold about $2\frac{1}{2}°$ at 1 GeV.

Figure 3 shows the S waves corresponding to values of X ranging from -0.5 to 1.5. For $X \leqslant 0$ (when the polynomial expression of Sec. II violate the rigorous constraints[4]) the S_2 phase shift falls rapidly downward, passing through $-90°$ within (or very close to) the energy region of validity of our solutions. For $X=-0.5$, it reaches $-90°$ at $\sqrt{s}=850$ MeV.[31]

This behavior indicates the presence of an important physical-sheet zero of B, which means there is a pole of A in violation of analyticity. Thus these solutions are to be rejected.

It must be stressed that it is when and only when we try to construct amplitudes that violate rigorous constraints that such physical-sheet zeros appear in the solutions of (23). Otherwise the results satisfy the partial-wave dispersion relation (16).

At this point it is worth emphasizing that whatever the value of X, the iterative solutions we find to (23) are crossing symmetric in the low-energy region, and so automatically obey the sum rules of Ref. 3 as well as the symmetry-point conditions given by Chew and Mandelstam.[19,32]

It is evident from Fig. 3 that although the S_0 phase shift is large and positive for all $X>0$, the detailed behavior of the amplitude depends strongly on X. For $X \gtrsim 0.5$ the phase shift passes through 90°, and the solutions contain a "super-broad σ" resonance, as Morgan and Shaw favor.[7] For $X \approx 0.6$ the σ is close to the ρ in mass. For $X \gtrsim 0.8$ the resonance is rather lighter than phenomenology indicates,[5] and for $X>1$ the D_2 phase shift (whose sign is controlled by the sign of the scattering length[2]) is positive, in contradiction to general arguments.[9]

As II shows,[16] a large S_0 phase shift near threshold can affect the adequacy of the polynomial model, and lead to significant errors in an extrapolation to the S-wave thresholds.[33] This is the reason to choose the subtraction point well away from threshold, i.e., at $s_0=\frac{4}{3}\mu^2$. Then this allows consistent solutions containing a large isoscalar interaction (as well as the possibility of others).

Figures 4 and 5 illustrate this point. Figure 4 compares the solutions of (23) to expressions (11)–(15) for $0 \leqslant s \leqslant 4\mu^2$, with $X=0.5$. There is evidently a $\sim 25\%$ mismatch between the two versions of $A_0{}^0$ at threshold. The dispersion-relation amplitude has the larger scattering length, as one would expect, of 0.17 compared to 0.14 of I. This is perhaps a slight improvement,[6–9] however.

[31] It is interesting to note the similarity of our solutions with $X \leqslant 0$ to those obtained by L. S. Brown and R. L. Goble [Phys. Rev. Letters 20, 346 (1968)] by unitarizing a simpler amplitude which violates the rigorous constraints of Ref. 4 (see also Ref. 2). There the presence of an unwanted physical-sheet zero at about 800 MeV is evident.

[32] G. F. Chew and S. Mandelstam, Nuovo Cimento 19, 752 (1961).

[33] This is the extrapolation that leads to the sum rule $2a_0-5a_2 \approx 18a_1\mu^2$.

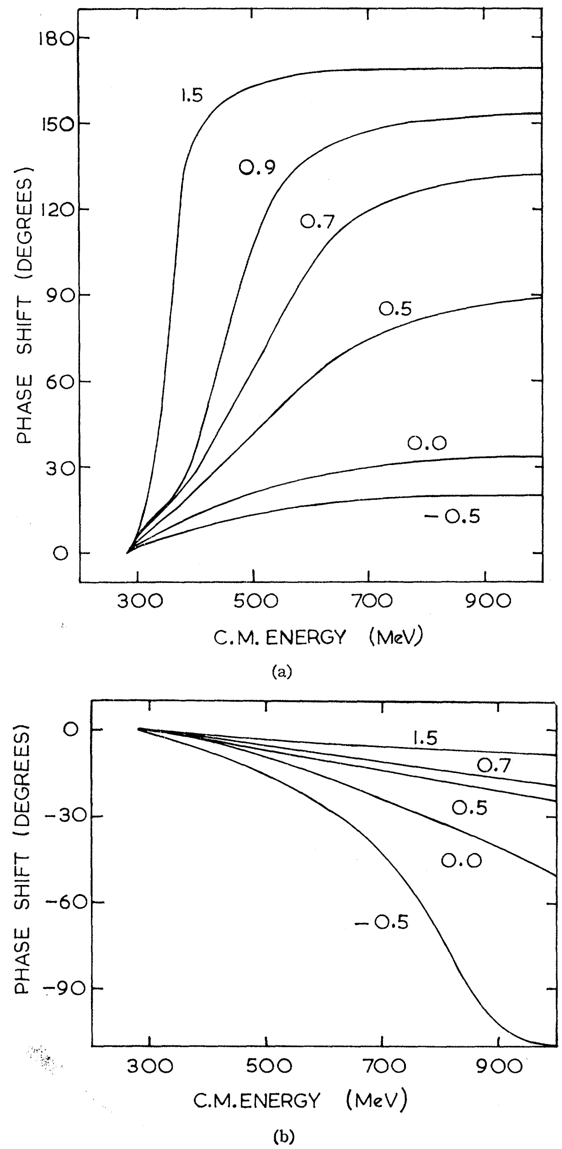

FIG. 3. S-wave phase shifts for various values of X, as indicated. (a) isospin-0; (b) isospin-2.

Figure 5 shows the S-wave phase shifts before and after iteration, with $m=765$ MeV, $\Gamma=120$ MeV, $X=0.5$, and $s_0=3\mu^2$ (instead of $\frac{4}{3}\mu^2$). Forcing the solution for $A_0{}^0$ to agree with (11) closer to threshold permits (for all X) only rather small S_0 phase shifts, which disagree with phenomenology.[5]

Our favored solutions of those in Fig. 3 are for X in the range 0.5–0.7. This choice is based on experimental evidence.[5] For $X=0.6\pm0.1$, we have scattering lengths

$$a_0 \approx 0.18\pm0.01\,, \tag{30}$$

$$a_2 \approx -0.04\,, \tag{31}$$

or

$$2a_0-5a_2 \approx 0.56\pm0.02\,. \tag{32}$$

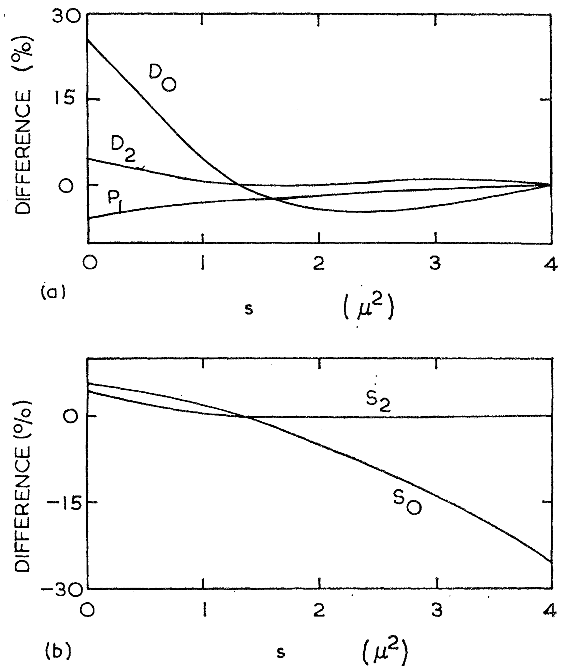

FIG. 4. Comparison of computed amplitudes and polynomial amplitudes of Eqs. (11)–(15). The quantity plotted on the ordinate is $[(A-A_{\text{poly}})/A_{\text{poly}}]\times 100\%$. Notation is that of Ref. 26.

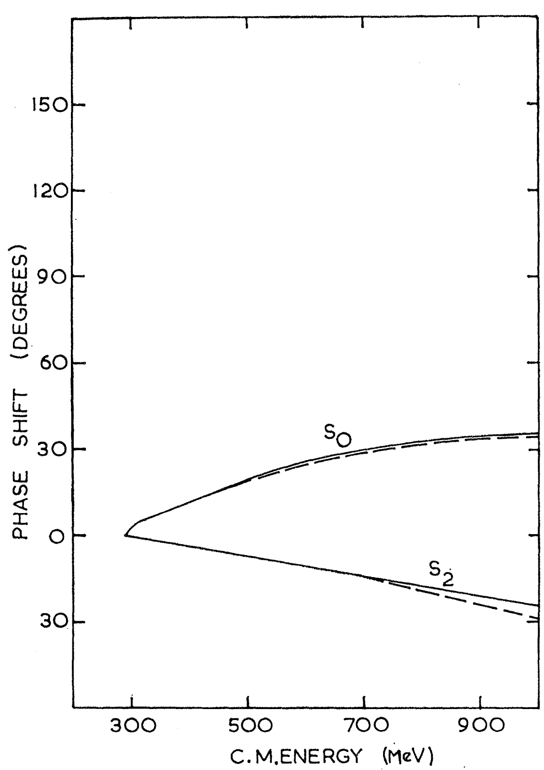

FIG. 5. S-wave phase shifts with subtraction-point $s_0=3\mu^2$. Notation is that of Ref. 26.

Equation (32) agrees well with, e.g., Ref. 7. The ratio

$$a_0/a_2 \sim -4.5 \tag{33}$$

is not in serious disagreement with expectation.[34]

It is not absolutely certain that these calculations find the unique class of solutions consistent with crossing, analytically, Adler zeros, etc., although the conclusions reached by Dilley[35] would suggest that given the presence of the ρ meson and of Adler zeros in the S waves, there can be no radically different types of phase shift for $\sqrt{s}\lesssim 400$ MeV. This view is reinforced by the results of Arbab and Donohue.[36] We have been unable to find a different set of solutions to (23), and would suspect that no others exist. (See, for discussion of uniqueness, Ref. 24.)

In this connection it is significant that the general features of Fig. 1 persist under rather wide variations of Γ, which is the parameter controlling the absolute size of the amplitudes near threshold.

Figure 6 shows the variation of solutions for Γ ranging from 40 to 200 MeV. The rise of the S_0 phase shift to about 90° at 1 GeV is always present. In fact this is true even if the ρ is not inserted at all, but the

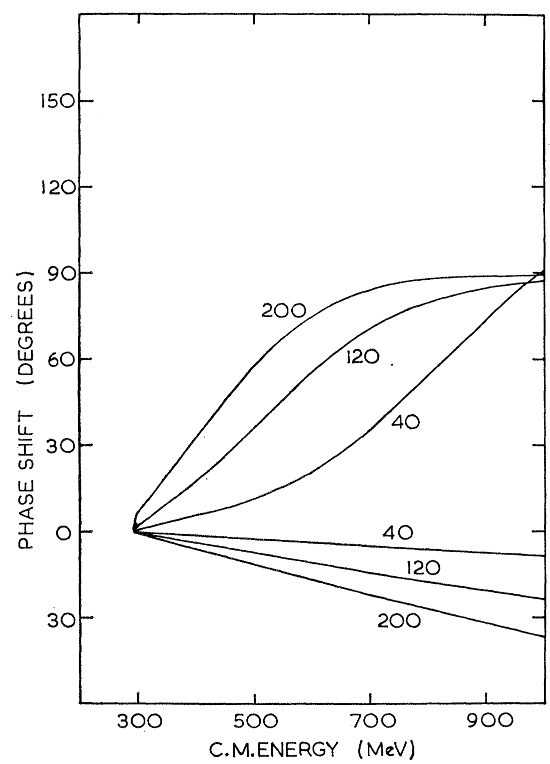

FIG. 6. S-wave phase shifts for three values of Γ (in MeV) as indicated, with $X=0.5$.

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[34] L. J. Gutay, F. T. Meiere, and J. H. Scharenguivel, Phys. Rev. Letters **23**, 431 (1969); D. Cline, K. J. Braun, and V. R. Scherer, Nucl. Phys. **B18**, 77 (1970).

[35] J. P. Dilley, University of Ohio report, 1970 (unpublished).

[36] F. Arbab and J. T. Donohue, Phys. Rev. D **1**, 217 (1970).

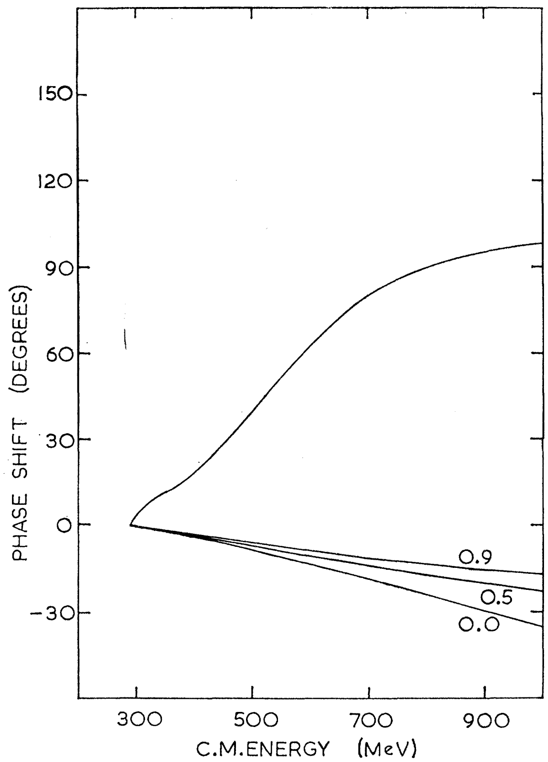

FIG. 7. S-wave phase shifts for three values of X as indicated, with the σ inserted through a subtraction at $\sqrt{s}=800$ MeV. The variation of the isoscalar phase shift with X is negligible on the scale of the graph.

P-wave subtraction is made at the symmetry point.[37]

This is in contrast to the variation of solution with X (Fig. 3). Thus it seems that the presence of the σ is linked more to the shape of the amplitude and the fulfillment of general requirements rather than to its absolute magnitude and the presence of the ρ. We would expect a ρ-σ bootstrap to be unsuccessful.

It should be noted that the presence of an S-wave isoscalar resonance near the ρ is certainly not *inconsistent* with a range of values of X. Figure 7 shows perfectly acceptable solutions which have a σ inserted through a subtraction at $s=s_0=(800\text{ MeV})^2$.

The dependence on X is rather small, provided $X>0$ [although the best over-all agreement with expressions (11)–(15) below threshold is for $X\approx 0.5$–0.7]. If one takes the view that the existence of a σ is experimentally established beyond doubt, then Fig. 7 is a prediction of detailed S-wave phase shifts.

V. CONCLUSIONS

In I a model of subthreshold π-π scattering was presented which was suggested as the simplest possible consistent with established theoretical principles. The present calculations find amplitudes valid over a wider range of energy which contain the essential ingredients of I. The agreement with phenomenology is very good,[5] and provides confirmation of the results of others.[6–9,35,36]

There are several points which deserve further investigation. The first concerns the need for an independent estimate of the size of the parameter X. The technique used in II depends very strongly on the extra ingredient of an assumption of effective average Regge asymptotic behavior for $\sqrt{s}\gtrsim 1.1$ GeV, and is probably inaccurate in the presence of a large low-energy isoscalar interaction (which causes an overestimate of asymptotic contributions). The order-of-magnitude result $X\sim 1$ is useful, nevertheless.

A second point concerns the assumption of elastic unitarity for all $s>4\mu^2$. Preliminary estimates show that the effects of reasonable inelasticities above $\sqrt{s}=1$ GeV are not very big below $\sqrt{s}\approx 800$ MeV, but definite conclusions cannot be quoted.

Interesting dynamical questions concerning bootstrap solutions are also waiting to be answered, but on present information a ρ-σ analog[38] of the N-N^* reciprocal bootstrap[39] seems unlikely.

To conclude, it is rapidly becoming evident that nearly all reasonable theories of the low-energy dipion interaction agree with each other and with the quickly growing mass of experimental information—a rather remarkable state of affairs in hadron physics.

[37] In this case the P_1 phase shift rises smoothly to a moderate positive value, e.g., to 45° at 1 GeV for $X=0.5$.

[38] This possibility is raised in Ref. 7 (see also Ref. 6).

[39] G. F. Chew, Phys. Rev. Letters 9, 233 (1962).