PHYSICAL REVIEW D VOLUME 2, NUMBER 9 1 NOVEMBER 1970

Covariant Phase-Space Calculations of n-Body Decay and Production Processes. II

RAJENDRA KUMAR
Department of Physics and Astrophysics, University of Delhi, Delhi-7, India
(Received 21 May 1970)

Lorentz-invariant phase-space integrals in the multicluster form are transformed into definite integrals over a set of "symmetric" Mandelstam variables. The momentum configuration of the particle clusters (in the center-of-mass system and in the lab system) as well as of the individual particles (in the cm. system, in the lab system, and in the rest frame of the decaying cluster) is explicitly given in terms of a complete set of kinematical variables, i.e., $3N-10$ independent Mandelstam variables and signs of $N-4$ quadrilinear invariants. This enables one to express all possible scalar products formed from the 4-momenta of particles in terms of the independent variables. Expressions for the angular correlation between two of the final-state particles or clusters and the simultaneous distribution in energies of two of the final-state particles are given explicitly, and distributions in invariant momentum transfers are discussed.

I. INTRODUCTION

THE kinematics of N-particle systems has been studied by several authors[1–4] in different ways. In a previous paper,[3] we described some simple transformations of Lorentz-invariant phase-space integrals for n-body decay and production processes, into definite integrals over the independent Mandelstam-like variables. In the present paper we study some aspects of the problem not dealt with in I. One of these is associated with the fact that the total number of scalar products ($p_i \cdot p_j$, $i \neq j$) in a reaction involving N particles is ${}^NC_2$, which increases quadratically with increasing N, whereas the number of independent variables ($3N-10$) increases only linearly. On using the 4-momentum conservation equation and contracting with each of the N 4-momenta, one gets N linear relations among the scalar products, thus reducing the number of linearly independent scalar products to ${}^NC_2-N$. For $N=4$ or 5, this number equals the number of independent variables and, therefore, there is apparently no problem. However, for $N>5$, the number of linearly independent scalar products exceeds $3N-10$:

$$ {}^NC_2-N=3N-10+{}^{N-4}C_2. $$

Evidently, the additional ${}^{N-4}C_2$ number of linearly independent scalar products depend on the $3N-10$ Mandelstam variables in a complicated manner; once these are determined, all other scalar products ($p_i \cdot p_j$) can be expressed as linear superpositions of these and the $3N-10$ independent Mandelstam variables. A closely related problem is that of determining the 4-momenta of N particles in a certain fixed frame of reference—which, in practical cases, is either the center-of-mass (c.m.) system or the laboratory system—given the $3N-10$ independent Mandelstam variables. However, the fact is that the $3N-10$ independent scalar products do not suffice to completely define the momentum configuration of an N-particle system. This is because the dependence on the azimuthal angle $(\phi_i-\phi_j)$ of the scalar product $p_i \cdot p_j$ comes only through $\cos(\phi_i-\phi_j)$, so that the scalar product has the same value for two different values of $\phi_i-\phi_j$ in the range $(0,2\pi)$. Thus, a description in terms of $3N-10$ independent Mandelstam variables gives a 2^{N-4}-fold degeneracy. This degeneracy is lifted if the signs of $N-4$ pseudoscalars—each formed from four linearly independent 4-momenta—are specified.[5]

Given a complete set of kinematical variables, namely, a set of $3N-10$ independent Mandelstam variables and $N-4$ independent "kinematical signatures," the simplest way to obtain the above-mentioned ${}^{N-4}C_2$ number of scalar products is to get expressions for the 4-momenta of all particles in a certain fixed frame of reference in terms of the independent variables, provided that the independent Mandelstam variables are such that the construction of momentum vectors does not require a knowledge of these scalar products. The sets of variables given by Eqs. (3) and (29) of I have precisely this property and are therefore best suited for this purpose, whereas the invariant variables of the multi-Regge model[4] involving $n-1$ subenergies, $n-1$ invariant momentum transfers, and $n-2$ invariant masses (which may be chosen in various different ways) are not convenient for this purpose. In general, however, the set of independent kinematical variables should be such as to correspond to the picture where the final-state particles are grouped together into various clusters,[6] with each cluster containing an arbitrary number of particles. In Sec. II, we discuss a set of "symmetric" variables of this type and write down the phase-space integral in terms of these variables with a view to making the subsequent discussions of a

[1] N. Byers and C. N. Yang, Rev. Mod. Phys. **36**, 595 (1964); P. Nyborg, H. S. Song, W. Kernan, and R. H. Good, Jr., Phys. Rev. **140**, B914 (1965); P. Nyborg, *ibid.* **140**, B921 (1965).

[2] R. A. Morrow, J. Math. Phys. **7**, 844 (1966); see also R. J. Eden, P. V. Landshoff, D. I. Oliver, and J. C. Polkinghorne, *The Analytic-S-Matrix* (Cambrige U. P., Cambridge, England, 1966), Chap. 4.

[3] R. Kumar, Phys. Rev. **185**, 1865 (1969), hereafter referred to as I.

[4] E. Byckling and K. Kajantie, Phys. Rev. **187**, 2018 (1969).

[5] M. Goldberg, F. Rohrlich, and J. Leitner, Phys. Rev. **140**, B1592 (1965).

[6] F. Zachariasen and G. Zweig, Phys. Rev. **160**, 1322 (1967); N. F. Bali, G. F. Chew, and A. Pignotti, *ibid.* **163**, 1572 (1967); S. J. Chang and R. Rajaraman, *ibid.* **183**, 1517 (1969); R. Rajaraman, Phys. Rev. D **1**, 118 (1970).

more general nature. In Sec. III, the momentum configuration of a system of N particles is discussed; this enables us to obtain all scalar products. The momentum configuration of an N-particle system may also be useful in the Monte Carlo[7] generation of events. Section IV deals with the angular correlation between two of the final-state particles or clusters, simultaneous distribution in the energies of two of the final-state particles, and some distributions in the invariant momentum transfers. A compact notation for the limits of integrations and some other kinematical quantities used in this paper is explained in the Appendix.

II. MULTICLUSTER FORM OF PHASE-SPACE INTEGRAL

Let the final-state particles in the production process

$$p_a+p_b\to\sum_{k=1}^{n}p_k\,,\quad n\geqslant 2 \tag{2.1}$$

be grouped into m clusters $(1\leqslant m\leqslant n)$ with momenta P_i $(1\leqslant i\leqslant m)$ and let the ith cluster contain n_i number of particles with momenta ${}^ip_j$ $(1\leqslant j\leqslant n_i)$. Kinematically, the collision process (2.1) may then be viewed as a two-step process: the scattering of initial-state particles resulting in m clusters $(p_a+p_b\to\sum_{i=1}^{m}P_i)$ followed by the decay of these clusters into individual particles $(P_i\to\sum_{j=1}^{n_i}{}^ip_j)$. Thus, the phase-space integral gets "factorized":

$$\mathcal{P}_n(p_a,p_b;p_k)=\mathcal{P}_m(p_a,p_b;P_i)\times\prod_{i=1}^{m}\bar{\mathcal{P}}_{n_i}(P_i,Q_{i-1},Q_i;{}^ip_j)\,, \tag{2.2}$$

where

$$Q_i=p_a-\sum_{l=1}^{i}P_l\,. \tag{2.3}$$

The production of m clusters is described by $4(m-1)$ Mandelstam variables (and, of course, $m-2$ kinematical signatures), the m extra variables appearing here being

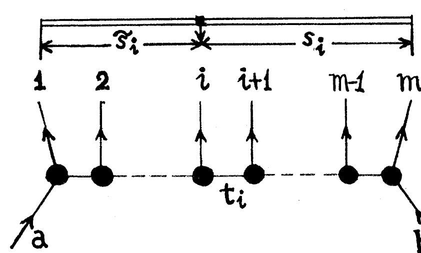

FIG. 1. The "symmetric" Mandelstam variables in a multiparticle production process. Each outgoing line represents a group of particles, in general. The diagram also depicts the Mandelstam variables for decay of particle a into $m+1$ clusters if we make the replacement $p_b\to -P_{m+1}$.

[7] Program FOWL, CERN Library W-505 (unpublished); E. Byckling and K. Kajantie, Nucl. Phys. **B9**, 568 (1969); E. Byckling *et al.*, University of Helsinki Report No. 3-69 (unpublished).

the squared invariant masses of the clusters. The c.m. energy (s) and the $m-1$ invariant momentum transfers are a natural choice for other m variables. The remaining $2m-4$ variables can be chosen in various ways. It is unlikely that a certain set of variables is a priviliged one, though, depending on the nature of problem, a particular set of variables may have some advantages over others. Here, we wish to draw attention to a set of symmetric variables s_i and $\tilde{s}_i$ depicted in Fig. 1. Explicitly, the independent variables are defined as follows:

$$\begin{aligned}
&t_i=-Q_i^2\,,\quad 1\leqslant i\leqslant m-1\,;\\
&s_i=-(p_a+p_b-\sum_{l=1}^{i-1}P_l)^2\,,\quad \tilde{s}_i=-(\sum_{l=1}^{i}P_l)^2\,,\\
&\qquad\qquad 2\leqslant i\leqslant m-1\,;\\
&s\equiv s_1=\tilde{s}_m=-(p_a+p_b)^2=-(\sum_{i=1}^{m}P_i)^2\,,\\
&M_i^2=-P_i^2\,,\quad 1\leqslant i\leqslant m\,;\\
&\epsilon_i=\epsilon[p_ap_b(\sum_{l=1}^{i-1}P_l)P_i]\,,\quad 2\leqslant i\leqslant m-1\,.
\end{aligned} \tag{2.4}$$

The Mandelstam variables in Eq. (2.4) are linearly related to the independent variables defined in I and are identical with the invariant variables of the multi-Regge model for $m=2$ and 3.

The decay of cluster P_i may be visualized as the production process

$$Q_{i-1}+(-Q_i)\to\sum_{j=1}^{n_i}{}^ip_j\,. \tag{2.5}$$

However, note that because of the existence of vectors $\mathbf{p}_a$, $\mathbf{p}_b$, and $\mathbf{P}_l$ $(1\leqslant l\leqslant m)$, the azimuthal symmetry about the collision axis (along $\mathbf{Q}_{i-1}$ or $\mathbf{Q}_i$ in the rest system of the cluster P_i) need not exist any more. Consequently, apart from the mass M_i of the cluster [already defined in Eq. (2.4)], the number of independent Mandelstam variables is $3n_i-4$ and the number of kinematical signatures is n_i-1. These variables may be defined as follows:

$$\begin{aligned}
&{}^is_j=-(P_i-\sum_{k=1}^{j-1}{}^ip_k)^2\,,\qquad 2\leqslant j\leqslant n_i-1\,;\\
&{}^it_j=-[p_b-(P_i-\sum_{k=1}^{j}{}^ip_k)]^2\,,\\
&{}^i\tilde{s}_j=-[p_a+p_b-(P_i-\sum_{k=1}^{j}{}^ip_k)]^2\,,\\
&{}^i\epsilon_j=\epsilon[p_ap_b(P_i-\sum_{k=1}^{j-1}{}^ip_k)\,{}^ip_j]\,,\quad 1\leqslant j\leqslant n_i-1\,.
\end{aligned} \tag{2.6}$$

The total number of Mandelstam variables in (2.4) and (2.6) sums up to $3n-4$ and the total number of kinematical signatures is $n-2$, as it should be.

The phase-space integrals $\mathcal{P}_m$ and $\bar{\mathcal{P}}_{n_i}$ are of the form

$$\mathcal{P}_m=\prod_{i=1}^{m}\left[\int dM_i^2\int d^4P_i\,\delta(P_i^2+M_i^2)\theta(E_i)\right]\delta^4(p_a+p_b-\sum_{i=1}^{m}P_i)F(p_a,p_b;P_i),$$
$$\bar{\mathcal{P}}_{n_i}=\prod_{j=1}^{n_i}\left[\int d^4\,{}^ip_j\,\delta({}^ip_j^2+{}^im_j^2)\theta({}^iE_j)\right]\delta^4(Q_{i-1}-Q_i-\sum_{j=1}^{n_i}{}^ip_j)G_i(P_l,Q_{i-1},Q_i;{}^ip_j).\qquad(2.7)$$

The transformation of $\mathcal{P}_m$ into definite integrals over the Mandelstam variables defined in Eq. (2.4) is done in a straightforward way[3]; we have

$$\prod_{i=1}^{m}\left[\int d^4P_i\,\delta(P_i^2+M_i^2)\theta(E_i)\right]\delta^4(p_a+p_b-\sum_{i=1}^{m}P_i)$$
$$=\prod_{i=2}^{m-1}\left[\int\int\int ds_idt_id\tilde{s}_i\right]\int dt_1\int d^4P_1\,\delta(P_1^2+M_1^2)\delta((p_a+p_b-P_1)^2+s_2)\delta((p_a-P_1)^2+t_1)\theta(E_1)\theta(E_a+E_b-E_1)$$
$$\times\prod_{i=2}^{m-1}\left[\int d^4P_i\,\delta(P_i^2+M_i^2)\delta((p_a+p_b-\sum_{l=1}^{i}P_l)^2+s_{i+1})\delta((\sum_{l=1}^{i}P_l)^2+\tilde{s}_i)\right.$$
$$\left.\times\delta((p_a-\sum_{l=1}^{i}P_l)^2+t_i)\theta(E_i)\theta(E_a+E_b-\sum_{l=1}^{i}E_l)\right],\qquad(2.8)$$

so that, suppressing the step function $\theta(\sqrt{s}-\sum_{k=1}^{n}m_k)$, which implies a condition that is always satisfied in all physical processes, we finally get

$$\mathcal{P}_m=2^{7-4m}\pi[\lambda(s,m_a^2,m_b^2)]^{-1/2}\left[\prod_{i=1}^{m}\int dM_i^2\right]\int ds_2\int dt_1\prod_{i=2}^{m-2}\left[\int ds_{i+1}\int dt_i\int d\tilde{s}_i|\varepsilon_i|^{-1}\right]$$
$$\times\int dt_{m-1}\int d\tilde{s}_{m-1}|\varepsilon_{m-1}|^{-1}\sum_{\epsilon_i}F(s_i;t_i;\tilde{s}_i;\epsilon_i),\qquad(2.9)$$

where the summation over the ϵ_i's implies the sum of 2^{m-2} values of F (cf. Sec. III) corresponding to two values of each of the $(m-2)$ ϵ_i's, and

$$|\varepsilon_i|=|\varepsilon(s_i;M_i^2,s_{i+1};\tilde{s}_{i-1},s;t_{i-1},m_b^2;m_a^2,t_i,\tilde{s}_i)|.\qquad(2.10)$$

The limits of integrations are

$$M_{i-}=\sum_{j=1}^{n_i}{}^im_j,\qquad M_{i+}=\sqrt{s}-\sum_{k=1}^{i-1}M_k-\sum_{k=i+1}^{m}M_{k-},$$
$$s_{i-}=(\sum_{k=i}^{m}M_k)^2,\qquad s_{i+}=(\sqrt{s_{i-1}}-M_{i-1})^2,\qquad(2.11)$$
$$t_{i\pm}=L_\pm(s_i;M_i^2,s_{i+1};t_{i-1},m_b^2),$$
$$\tilde{s}_{i\pm}=L_\pm[s_i;M_i^2,s_{i+1},(t_i);\tilde{s}_{i-1},s,(m_a^2);t_{i-1},m_b^2].$$

The result is conveniently obtained by carrying out the d^4P_i integration in (2.8) in the frame of reference characterized by $\mathbf{p}_a+\mathbf{p}_b-\sum_{l=1}^{i-1}\mathbf{P}_l=0$, with $\mathbf{p}_a-\sum_{l=1}^{i-1}\mathbf{P}_l$ along the z axis and $\sum_{l=1}^{i-1}\mathbf{P}_l$ lying in the zx plane and having a positive x component. The condition $|\mathbf{P}_i|\geqslant 0$ together with

$$\theta(E_i)\theta(E_a+E_b-\sum_{l=1}^{i}E_l)$$

is expressed by the inequality $\sqrt{s_i}\geqslant\sqrt{s_{i+1}}+M_i$, which gives the upper bound on s_{i+1}. These inequalities lead to the essential requirement

$$\sqrt{s}\geqslant\sum_{i=1}^{m}M_i,$$

which, in turn, places upper bounds on the M_i's. The limits on t_i and $\tilde{s}_i$ integrations arise from bounds on the angular integrations (cf. Appendix). The d^4P_i integration in (2.8) gives $(16|\varepsilon_i|)^{-1}$, $(2\leqslant i\leqslant m-1)$, when the integration over the azimuthal angle ϕ_i is done in the range $(0,\pi)$. The same value is obtained when ϕ_i varies in the range $(\pi,2\pi)$ but ϵ_i changes sign. If F is independent of some of the variables, $\mathcal{P}_m$ may be written as a lower-rank integral; integration over $\tilde{s}_i$ $(2\leqslant i\leqslant m-1)$ changes the integrand to $(\pi/4)[\lambda(s_i,t_{i-1},m_b^2)]^{-1/2}$ in place of $(16|\varepsilon_i|)^{-1}$, and further integration over t_i changes it to $(\pi/4s_i)[\lambda(s_i,s_{i+1},M_i^2)]^{1/2}$. Similarly, if F does not depend on some of the ϵ_i's, the summation over each of these ϵ_i's may be replaced by a factor of 2.

The phase-space integrals $\bar{\mathcal{P}}_{n_i}$ are similarly transformed into definite integrals over the set of variables defined in Eq. (2.6). We have

$$\bar{\mathcal{P}}_{n_i}=2^{-4(n_i-1)}\prod_{j=1}^{n_i-2}\left[\int d\,{}^is_{j+1}\int d\,{}^it_j\int d\,{}^i\tilde{s}_j|{}^i\varepsilon_j|^{-1}\right]$$
$$\times\int d\,{}^it_{n_i-1}\int d\,{}^i\tilde{s}_{n_i-1}|{}^i\varepsilon_{n_i-1}|^{-1}$$
$$\times\sum_{{}^i\epsilon_j} G_i(s_l,t_l,\tilde{s}_l,\epsilon_l;\,{}^is_j,{}^it_j,{}^i\tilde{s}_j,{}^i\epsilon_j)\,, \quad (2.12)$$

where

$$|{}^i\varepsilon_j|=|\varepsilon({}^is_j;\,{}^im_j{}^2,{}^is_{j+1};\,{}^i\tilde{s}_{j-1},s;\,{}^it_{j-1},m_b{}^2;\,m_a{}^2,{}^it_j,{}^i\tilde{s}_j)| \quad (2.13)$$

and the limits of integrations are as follows:

$${}^is_{j-}=\Big(\sum_{k=j}^{n_i}{}^im_k\Big)^2,\quad {}^is_{j+}=(\sqrt{{}^is_{j-1}}-{}^im_{j-1})^2,$$
$${}^it_{j\pm}=L_\pm({}^is_j;\,{}^im_j{}^2,\,{}^is_{j+1};\,{}^it_{j-1},\,m_b{}^2)\,,$$
$${}^i\tilde{s}_{j\pm}=L_\pm[{}^is_j;\,{}^im_j{}^2,{}^is_{j+1},({}^it_j);\,{}^i\tilde{s}_{j-1},s,(m_a{}^2);\,{}^it_{j-1},m_b{}^2]\,. \quad (2.14)$$

Successive integrations over ${}^i\tilde{s}_j$ and ${}^it_j$ change the integrand to $(\pi/4)[\lambda({}^is_j,{}^it_{j-1},m_b{}^2)]^{-1/2}$ and $(\pi/4\,{}^is_j)\times[\lambda({}^is_j,{}^im_j{}^2,{}^is_{j+1})]^{1/2}$, respectively, in place of $(16|{}^i\varepsilon_j|)^{-1}$, if G_i has no dependence on these variables.

The "factorization" of the phase-space integral enables one to define the independent variables in various ways corresponding to the different ways in which m and n_i's can be chosen for a given n. It should be emphasized that it is not necessary that each cluster consist of at least two particles; in general, some or all of the "clusters" could be just single particles. If P_i is a single-particle cluster, then $\bar{\mathcal{P}}_{n_i}$ is simply unity and the mass (M_i) of the cluster is now a constant (denoted by m_i) so that there is no integration over $M_i{}^2$. Needless to say, the number of independent variables remains the same.[8] Similarly, for the special case of $m=1$, we have $M_1{}^2=s$, $\mathcal{P}_m=1$, and ${}^1\tilde{s}_1={}^1m_1{}^2$, so that there is no integration over ${}^1\tilde{s}_1$,

$$[16|{}^1\varepsilon_1|]^{-1}\rightarrow\tfrac{1}{2}\pi[\lambda({}^1s_1,m_a{}^2,m_b{}^2)]^{-1/2}\,,$$

and there is no summation over ${}^1\epsilon_1$.

The same set of independent variables can also be used to describe a decay process with $m+1$ particle clusters in the final state, if we make the replacement $p_b\rightarrow -P_{m+1}$ and integrate over the momenta of all particles which constitute the $(m+1)$th cluster. Explicitly, we have in this case

$$\mathfrak{D}_n(p_a;\,p_k)=\mathfrak{D}_{m+1}(p_a;\,P_i)\prod_{i=1}^{m+1}\bar{\mathcal{P}}_{n_i}(p_a,P_l;\,{}^ip_j)\,, \quad (2.15)$$

where

$$\mathfrak{D}_{m+1}=\frac{\pi}{2m_a{}^2}\int dM_{m+1}{}^2\int ds_1[\lambda(m_a{}^2,M_{m+1}{}^2,s_1)]^{1/2}\times\mathcal{P}_m(m_b{}^2\rightarrow M_{m+1}{}^2)\,, \quad (2.16)$$

with the upper limit on M_i in Eq. (2.11) modified as follows[9]:

$$M_{i+}=m_a-\sum_{k=1}^{i-1}M_k-\sum_{k=i+1}^{m+1}M_{k-}\,,\quad 1\leqslant i\leqslant m+1\,. \quad (2.17)$$

The lower bound on s_1 is given by Eq. (2.11) and the upper limit is given by

$$s_{1+}=(m_a-M_{m+1})^2\,. \quad (2.18)$$

III. MOMENTUM CONFIGURATION AND SCALAR PRODUCTS

Given the values of independent kinematical variables in the physical region, the momentum configuration of the N-particle system is obtained as follows.

A. Momentum Configuration in Cluster Production Process and Scalar Products $P_i\cdot P_j$

In the c.m. system, we have

$$\begin{aligned}
&\mathbf{p}_b=-\mathbf{p}_a\quad\text{along the } z \text{ axis}\,,\\
&|\mathbf{p}_a|=|\mathbf{p}_b|=[\lambda(s,m_a{}^2,m_b{}^2)]^{1/2}/2\sqrt{s}\,,\\
&E_i=(s_i+\tilde{s}_i-s_{i+1}-\tilde{s}_{i-1})/2\sqrt{s}\,,\\
&|\mathbf{P}_i|=[\lambda(s,M_i{}^2,s_i')]^{1/2}/2\sqrt{s}\,,\\
&\cos\theta_i=\langle\mathbf{p}_b\cdot\mathbf{P}_i\rangle=\eta_i=-C(s;\,m_a{}^2,m_b{}^2;\,M_i{}^2,s_i';\,t_i')\,,\quad 1\leqslant i\leqslant m-1
\end{aligned} \quad (3.1)$$

where

$$\begin{aligned}
s_i'&=-(p_a+p_b-P_i)^2=s+M_i{}^2+s_{i+1}+\tilde{s}_{i-1}-s_i-\tilde{s}_i\,,\\
t_i'&=-(p_a-P_i)^2=m_a{}^2+M_i{}^2+\tilde{s}_{i-1}-\tilde{s}_i+t_i-t_{i-1}\,.
\end{aligned} \quad (3.2)$$

In order to determine the azimuthal angles[10] of the momentum vectors, we now completely specify the orientation of the coordinate reference system by choosing the x axis in such a direction that $\mathbf{P}_1$ lies in the zx plane with a positive x component, i.e., $\phi_1=0$. The expression for the azimuthal angle (ϕ_i) of the vector $\mathbf{P}_i$ ($3\leqslant i\leqslant m-1$) is a bit complicated because $(P_1+P_i)^2$ cannot be expressed as a linear superposition of the independent Mandelstam variables. The angle ϕ_i is obtained as a sum of $i-1$ azimuthal angles:

$$\phi_i=\sum_{k=2}^{i-1}\Phi_k+\phi_i'\,, \quad (3.3)$$

[8] The decrease in the number of independent variables in the cluster production process due to constancy of mass M_i is compensated by the increase in the number of independent variables in the cluster decay process: For $n_i=1$ the number of independent variables is zero, which is one more than that given by $3n_i-4$.

[9] It is implied that the $M_{m+1}{}^2$ and s_1 integrations are placed to the left of the s_2 integration in (2.9) and $\theta(\sqrt{s}-\sum_{k=1}^n m_k)$ in (2.9) is replaced by $\theta(m_a-\sum_{k=1}^n m_k)$.

[10] In high-energy hadron collision processes, where one is interested in generating only those Monte Carlo events which correspond to small transverse momenta of particles, one may not proceed any further (to determine the azimuthal angles of momentum vectors) if the transverse momenta of particles exceed the desired limit. This results in better efficiency.

where Φ_k is the azimuthal angle of the vector $\sum_{l=1}^{k} \mathbf{P}_l$ in a coordinate reference system whose x axis is oriented such that $\sum_{l=1}^{k-1} \mathbf{P}_l$ lies in the zx plane and has a positive component along the x axis, and ϕ_i' is the azimuthal angle of the vector $\mathbf{P}_i$ in a coordinate reference system whose x axis is oriented such that the azimuthal angle of $\sum_{l=1}^{i-1} \mathbf{P}_l$ in this reference system is zero. The cosines of angles ϕ_i' and Φ_k are given by

$$\begin{aligned}\cos\phi_i' &= \langle(\mathbf{p}_b\times\sum_{l=1}^{i-1}\mathbf{P}_l)\cdot(\mathbf{p}_b\times\mathbf{P}_i)\rangle=\omega_i\,,\\ \cos\Phi_k &= \langle(\mathbf{p}_b\times\sum_{l=1}^{k-1}\mathbf{P}_l)\cdot(\mathbf{p}_b\times\sum_{l=1}^{k}\mathbf{P}_l)\rangle=\Omega_k\,,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\omega_k &= C(\zeta_k;\,\xi_k,\eta_k)\,,\\ \Omega_k &= C(Z_k;\,\xi_k,\xi_{k+1})\\ &= \{[\lambda(s,s_k,\tilde{s}_{k-1})(1-\xi_k^2)]^{1/2}+\omega_k[\lambda(s,M_k^2,s_k')\\ &\quad\times(1-\eta_k^2)]^{1/2}\}[\lambda(s,s_{k+1},\tilde{s}_k)(1-\xi_{k+1}^2)]^{-1/2}\,,\\ \xi_k &= \langle\mathbf{p}_b\cdot\sum_{l=1}^{k-1}\mathbf{P}_l\rangle=-C(s;\,m_a^2,m_b^2;\,\tilde{s}_{k-1},s_k;\,t_{k-1})\,,\\ \zeta_k &= \langle\mathbf{P}_k\cdot\sum_{l=1}^{k-1}\mathbf{P}_l\rangle=-C(s;\,M_k^2,s_k';\,\tilde{s}_{k-1},s_k;\,\tilde{s}_k)\\ &= -C(s;\,M_k^2,s_k';\,\tilde{s}_{k-1},s_k;\,\tilde{s}_k,s_{k+1})\,,\\ Z_k &= \langle(\sum_{l=1}^{k-1}\mathbf{P}_l)\cdot(\sum_{l=1}^{k}\mathbf{P}_l)\rangle\\ &= C(s;\,\tilde{s}_{k-1},s_k;\,\tilde{s}_k,s_{k+1};\,m_k^2)\,.\end{aligned} \tag{3.5}$$

It is clear that the above equations do not determine the azimuthal angles uniquely, since, for a given value of $\cos\phi$, there are two possible values of the angle ϕ in the range $(0,2\pi)$. However, as mentioned earlier, the degeneracy is lifted if the signs of quadrilinear invariants defined by Eq. (2.4) are known. It is easy to see that not only the sign of $\sin\phi_i'$ is given by ϵ_i, but the sign of $\sin\Phi_k$ is also given by ϵ_k, so that finally we get the following unambigous expression for the azimuthal angles:

$$\phi_i=\sum_{k=2}^{i-1}\epsilon_k\cos^{-1}\Omega_k+\epsilon_i\cos^{-1}\omega_i\,,\quad 2\leqslant i\leqslant m-1 \tag{3.6}$$

where the inverse of a cosine implies its principal value, i.e., the angle lying in the range $(0,\pi)$. In this way, all 4-momenta, except P_m, are determined in the c.m. system; the latter is, of course, obtained from the energy-momentum conservation:

$$\begin{aligned}E_m &= \sqrt{s}-\sum_{i=1}^{m-1}E_i\,,\\ \cos\theta_m &= \langle\mathbf{p}_b\cdot\mathbf{P}_m\rangle=\eta_m\equiv-|\mathbf{P}_m|^{-1}\sum_{i=1}^{m-1}|\mathbf{P}_i|\eta_i\,,\\ \phi_m &= \epsilon_m\cos^{-1}\{-[|\mathbf{P}_m|(1-\eta_m^2)^{1/2}]^{-1}\\ &\quad\times\sum_{i=1}^{m-1}|\mathbf{P}_i|(1-\eta_i^2)^{1/2}\cos\phi_i\}\,,\\ \epsilon_m &\equiv \operatorname{sgn}[-\sum_{i=1}^{m-1}|\mathbf{P}_i|(1-\eta_i^2)^{1/2}\sin\phi_i]\,.\end{aligned} \tag{3.7}$$

The momentum configuration in the frames characterized by $\mathbf{p}_a=0$ or $\mathbf{p}_b=0$ can also be determined in a straightforward manner, using the same set of variables. In the lab system $(\mathbf{p}_a=0)$, using a coordinate system of reference which is oriented such that $\mathbf{p}_b$ is along the polar axis and $\phi_1=0$, the momentum configuration in the cluster production process is given by the following equations:

$$\begin{aligned}|\mathbf{p}_b| &= [\lambda(s,m_a^2,m_b^2)]^{1/2}(2m_a)^{-1}\,,\\ E_i &= (\tilde{s}_i-\tilde{s}_{i-1}+t_{i-1}-t_i)(2m_a)^{-1}\,,\\ |\mathbf{P}_i| &= [\lambda(m_a^2,M_i^2,t_i')]^{1/2}(2m_a)^{-1}\,,\\ \cos\theta_i &= \langle\mathbf{p}_b\cdot\mathbf{P}_i\rangle=\bar{\eta}_i\,,\quad 1\leq i\leq m-1\\ \phi_i &= \sum_{k=2}^{i-1}\epsilon_k\cos^{-1}\bar{\Omega}_k+\epsilon_i\cos^{-1}\bar{\omega}_i\,,\quad 2\leq i\leq m-1\\ E_m &= (s+m_a^2-m_b^2)(2m_a)^{-1}-\sum_{i=1}^{m-1}E_i\,,\\ \cos\theta_m &= \bar{\eta}_m\equiv[|\mathbf{p}_b|-\sum_{i=1}^{m-1}|\mathbf{P}_i|\bar{\eta}_i]/|\mathbf{P}_m|\,,\\ \phi_m &= \epsilon_m\cos^{-1}\{-[\sum_{i=1}^{m-1}|\mathbf{P}_i|(1-\bar{\eta}_i^2)^{1/2}\cos\phi_i]\\ &\quad\times[|\mathbf{P}_m|(1-\bar{\eta}_m^2)^{1/2}]^{-1}\}\,,\\ \epsilon_m &\equiv \operatorname{sgn}[-\sum_{i=1}^{m-1}|\mathbf{P}_i|(1-\bar{\eta}_i^2)^{1/2}\sin\phi_i]\,,\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}\bar{\omega}_k &= C(\bar{\zeta}_k;\,\bar{\xi}_k,\bar{\eta}_k)\,,\\ \bar{\Omega}_k &= C(\bar{Z}_k;\,\bar{\xi}_k,\bar{\xi}_{k+1})\,,\\ \bar{\xi}_k &= \langle\mathbf{p}_b\cdot\sum_{l=1}^{k-1}\mathbf{P}_l\rangle=C(m_a^2;\,s,m_b^2;\,\tilde{s}_{k-1},t_{k-1};\,s_k)\,,\\ \bar{\eta}_k &= \langle\mathbf{p}_b\cdot\mathbf{P}_k\rangle=C(m_a^2;\,s,m_b^2;\,M_k^2,t_k';\,s_k')\,,\\ \bar{\zeta}_k &= \langle\mathbf{P}_k\cdot\sum_{l=1}^{k-1}\mathbf{P}_l\rangle=-C(m_a^2;\,M_k^2,t_k';\,\tilde{s}_{k-1},t_{k-1};\,\tilde{s}_k)\\ &= -C(m_a^2;\,M_k^2,t_k';\,\tilde{s}_{k-1},t_{k-1};\,\tilde{s}_k,t_k)\,,\\ \bar{Z}_k &= \langle\sum_{l=1}^{k-1}\mathbf{P}_l\cdot\sum_{l=1}^{k}\mathbf{P}_l\rangle=C(m_a^2;\tilde{s}_{k-1},t_{k-1};\tilde{s}_k,t_k;\,m_k^2)\,.\end{aligned} \tag{3.9}$$

If, instead of the invariants $\tilde{s}_i$ [cf. Eq. (2.4)], we define invariant subenergies $-(P_{i-1}+P_i)^2$ as the independent variables, we have the variables of the multi-Regge model. The $m-2$ kinematical signatures in this case are

$$\epsilon[p_bQ_{i-1}P_{i-1}P_i]\,,\quad 2\leqslant i\leqslant m-1\,. \tag{3.10}$$

Using this set of variables, the magnitudes of all momenta $(|\mathbf{P}_i|)$ and $\langle\mathbf{P}_{i-1}\cdot\mathbf{P}_i\rangle$ can be simply obtained in the reference frame $\mathbf{p}_b=0$. However, the polar angles of all momentum vectors with respect to a fixed axis cannot be easily calculated for $m>3$, because that would require the knowledge of those scalar products which are not linear superpositions of the independent Mandelstam variables.

With the choice of variables defined by Eq. (2.4), the ${}^{m-2}C_2$ linearly independent scalar products which cannot be expressed as linear superpositions of the $4m-4$ independent Mandelstam variables are $P_i\cdot P_j$ ($i<j$; $2\leqslant i$, $j\leqslant m-1$). Since the 4-momenta P_i are known (in the c.m. system, for example) in terms of the invariants, these scalar products are given by

$$P_i\cdot P_j=-E_iE_j+|\mathbf{P}_i||\mathbf{P}_j| \times\{\eta_i\eta_j+[(1-\eta_i^2)(1-\eta_j^2)]^{1/2}\cos(\phi_j-\phi_i)\}\,, \quad (3.11)$$

with

$$\phi_j-\phi_i=\sum_{k=i}^{j-1}\epsilon_k\cos^{-1}\Omega_k+\epsilon_j\cos^{-1}\omega_j-\epsilon_i\cos^{-1}\omega_i\,. \quad (3.12)$$

Since $\cos(\sum_i\alpha_i)$, when expressed in terms of the products of sines and cosines of α_i's, contains an even number of $\sin\alpha_i$ factors, it follows from (3.11) and (3.12) that the above-mentioned ${}^{m-2}C_2$ number of scalar products depend on products of even number of ϵ_i's. Thus, even if the squared T-matrix element does not depend linearly on the ε_i's (which are noninvariant under space inversion and time reversal), it may, in general, depend on all scalar products and hence on products of an even number of ϵ_i's.

B. Momentum Configuration in Cluster Decay Process

The kinematical variables defined in Eq. (2.6) for the decay of a cluster P_i are such that the momentum configuration of the decay products can be easily computed in any of the inertial frames $\mathbf{p}_a=0$, $\mathbf{p}_b=0$, or $\mathbf{p}_a+\mathbf{p}_b=0$, in exactly the same way as in the case of a cluster production process. For example, in the c.m. system ($\mathbf{p}_a+\mathbf{p}_b=0$), with the axes of the reference coordinate system oriented in a manner described in Sec. III A, the momenta of the particles are given by the following equations:

$$\begin{aligned}
&{}^iE_j=({}^is_j+{}^i\tilde{s}_j-{}^i\tilde{s}_{j-1}-{}^is_{j+1})/2\sqrt{s}\,,\\
&|{}^i\mathbf{p}_j|=[\lambda(s,{}^im_j{}^2,{}^is_j{}')]^{1/2}/2\sqrt{s}\,,\\
&\cos{}^i\theta_j=\langle\mathbf{p}_b\cdot{}^i\mathbf{p}_j\rangle={}^i\eta_j\,,\\
&{}^i\phi_j=\phi_i-\sum_{k=1}^{j-1}{}^i\epsilon_k\cos^{-1}{}^i\Omega_k+{}^i\epsilon_j\cos^{-1}{}^i\omega_j\\
&\qquad(1\leqslant i\leqslant m,\quad 1\leqslant j\leqslant n_i-1)\,,
\end{aligned} \quad (3.13)$$

where ϕ_i is given by Eq. (3.6) [or Eq. (37)] and

$$\begin{aligned}
&{}^i\omega_k=C({}^i\zeta_k;\,{}^i\xi_k,{}^i\eta_k)\,,\quad {}^i\Omega_k=C({}^iZ_k;\,{}^i\xi_k,{}^i\xi_{k+1})\,,\\
&{}^i\xi_k=\langle\mathbf{p}_b\cdot(\mathbf{P}_i-\sum_{l=1}^{k-1}{}^i\mathbf{p}_l)\rangle=C(s;\,m_b{}^2,m_a{}^2;\,{}^is_k,{}^i\tilde{s}_{k-1};\,{}^it_{k-1})\,,\\
&{}^i\eta_k=\langle\mathbf{p}_b\cdot{}^i\mathbf{p}_k\rangle=C(s;\,m_b{}^2,m_a{}^2;\,{}^im_k{}^2,{}^is_k{}';\,{}^it_k{}')\,,\\
&{}^i\zeta_k=\langle{}^i\mathbf{p}_k\cdot(\mathbf{P}_i-\sum_{l=1}^{k-1}{}^i\mathbf{p}_l)\rangle=C(s;\,{}^is_k,{}^i\tilde{s}_{k-1};\,{}^im_k{}^2,{}^is_k{}';\,{}^is_{k+1})\,,\\
&{}^iZ_k=\langle(\mathbf{P}_i-\sum_{l=1}^{k-1}{}^i\mathbf{p}_l)\cdot(\mathbf{P}_i-\sum_{l=1}^{k}{}^i\mathbf{p}_l)\rangle=C(s;\,{}^is_k,{}^i\tilde{s}_{k-1};\,{}^is_{k+1},\,{}^i\tilde{s}_k;\,{}^im_k{}^2)\,,\\
&{}^is_k{}'=-(p_a+p_b-{}^ip_k)^2=s+{}^im_k{}^2+{}^is_{k+1}+{}^i\tilde{s}_{k-1}-{}^is_k-{}^i\tilde{s}_k\,,\\
&{}^it_k{}'=-(p_b-{}^ip_k)^2=m_b{}^2+{}^im_k{}^2+{}^is_{k+1}+{}^it_{k-1}-{}^is_k-{}^it_k\,.
\end{aligned} \quad (3.14)$$

The 4-momentum ${}^ip_{n_i}$ is obtained from the energy-momentum conservation as before. Having determined the momentum configuration of the N-particle system in this way, any scalar product of the type ${}^ip_j\cdot{}^kp_l$ can be computed. However, the variables defined by Eq. (2.6) are not suitable for a determination of particle momenta in the rest frame of the decaying cluster. For this purpose, a set of appropriate variables is

$$\begin{aligned}
&{}^is_j=-(P_i-\sum_{k=1}^{j-1}{}^ip_k)^2\,,\\
&{}^i\tilde{s}_j=-(\sum_{k=1}^{j}{}^ip_k)^2\,,\qquad 2\leqslant j\leqslant n_i-1\\
&{}^it_j=-(Q_{i-1}-\sum_{k=1}^{j}{}^ip_k)^2\,,\qquad 1\leqslant j\leqslant n_i-1\\
&{}^i\epsilon_j=\epsilon[P_iQ_{i-1}(\sum_{k=1}^{j-1}{}^ip_k){}^ip_j]\,,\quad 2\leqslant j\leqslant n_i-1\\
&{}^i\tilde{s}_1=-[p_a+p_b-(P_i-{}^ip_1)]^2\,,\\
&{}^i\epsilon_1=\epsilon[P_iQ_{i-1}(p_a+p_b){}^ip_1]\,.
\end{aligned} \quad (3.15)$$

The momentum configuration of the decay products in the rest frame of the decaying cluster ($\mathbf{P}_i=0$), with coordinate axes of the reference system oriented such that the polar axis is along $\mathbf{Q}_{i-1}$ and ${}^i\mathbf{p}_1$ lies in the zx plane with a positive x component, is given by equations analogous to Eqs. (3.1)–(3.7). All that remains to be done is to find the orientation of $\mathbf{Q}_{i-1}$ with respect to the vectors $\mathbf{p}_a$ and $\mathbf{p}_b$ and the azimuthal angle of ${}^i\mathbf{p}_1$ with $\mathbf{Q}_{i-1}$ as the polar axis, in the rest frame of the decaying cluster. The orientations of $\mathbf{Q}_{i-1}$ and ${}^i\mathbf{p}_1$ are given by the following equations:

$$
\begin{aligned}
\langle \mathbf{p}_a\cdot\mathbf{Q}_{i-1}\rangle &= C(M_i^2;\, m_a^2,t_i';\, t_{i-1},t_i;\, \tilde{s}_{i-1})\,,\\
\langle \mathbf{p}_b\cdot\mathbf{Q}_{i-1}\rangle &= -C(M_i^2;\, m_b^2,\tilde{t}_i;\, t_{i-1},t_i;\, s_i)\,,\\
\langle \mathbf{p}_a\cdot\mathbf{p}_b\rangle &= -C(M_i^2;\, m_a^2,t_i';\, m_b^2,\tilde{t}_i;\, s)\,,\\
\epsilon[p_a p_b Q_{i-1} P_i] &= -\epsilon_i\,,\\
\tilde{t}_i &= -(p_b-P_i)^2 = m_b^2+M_i^2+s_{i+1}-s_i+t_{i-1}-t_i\,,
\end{aligned}
\tag{3.16}
$$

and

$$
\begin{aligned}
\langle \mathbf{Q}_{i-1}\cdot {}^i\mathbf{p}_1\rangle &= C(M_i^2;\, t_{i-1},t_i;\, {}^im_1^2,{}^is_2;\, {}^it_1)\,,\\
\langle (\mathbf{p}_a+\mathbf{p}_b)\cdot {}^i\mathbf{p}_1\rangle &= -C(M_i^2;\, s,s_i';\, {}^is_2,{}^im_1^2;\, {}^i\tilde{s}_1)\,,\\
\langle (\mathbf{p}_a+\mathbf{p}_b)\cdot \mathbf{Q}_{i-1}\rangle &= C(M_i^2;\, s,s_i';\, t_{i-1},t_i;\, 2(m_b^2+\tilde{s}_{i-1})-\tilde{t}_{i-1})\,,\\
\epsilon[P_i Q_{i-1}(p_a+p_b)\, {}^ip_1] &= {}^i\epsilon_1\,,\\
t_i &= -(p_b-\sum_{k=1}^{i} P_k)^2 = s_{i+1}+\tilde{s}_i+m_a^2+m_b^2-s-t_i.
\end{aligned}
\tag{3.17}
$$

IV. DISTRIBUTIONS IN ANGLES, ENERGIES, AND INVARIANT MOMENTUM TRANSFERS

The sequence of integrations over the various variables in the phase-space integral $\mathcal{P}_m$ [cf. Eq. (2.9)] is such that the limits of integrations are simple. The following changes in the order of integrations can be easily made.

(a) Integrations over the various M_i^2's may be done in any order. The changes in the limits of integrations are obvious [cf. Eq. (11) of I]. Similarly, the order of integrations over the s_i's can be permuted in all possible ways. Moreover, the integration over any s_l may be brought to the front, i.e.,

$$
\prod_{i=1}^{m}\int dM_i^2 \prod_{i=2}^{m-1}\int ds_i \to \prod_{i=1}^{m}\int dM_i^2 \int ds_l \prod_{\alpha=2}^{l-1}\int ds_\alpha \times \prod_{\beta=l+1}^{m-1}\int ds_\beta \to \int ds_l \prod_{i=1}^{m}\int dM_i^2 \prod_{\alpha=2}^{l-1}\int ds_\alpha \times \prod_{\beta=l+1}^{m-1}\int ds_\beta\,. \tag{4.1}
$$

The limits of integrations are modified thus:

$$
\begin{aligned}
s_{l-} &= (\sum_{i=l}^{m} M_{i-})^2\,, \qquad s_{l+} = (\sqrt{s}-\sum_{i=1}^{l-1} M_{i-})^2\,,\\
M_{i+} &= (\sqrt{s}-\sum_{k=1}^{i-1} M_k - \sum_{k=i+1}^{l-1} M_{k-} - \sqrt{s_l})\,, \qquad 1\leqslant i\leqslant l-1\\
&= (\sqrt{s}-\sum_{k=1}^{i-1} M_k - \sum_{k=i+1}^{m} M_{k-})\,, \qquad l\leqslant i\leqslant m\\
s_{\alpha-} &= (\sum_{k=\alpha}^{l-1} M_k+\sqrt{s_l})^2.
\end{aligned}
\tag{4.2}
$$

M_{i-}, $s_{\alpha+}$, and $s_{\beta\pm}$ remain unchanged.

(b) The order of integrations over any pair of variables $\tilde{s}_i$ and t_i may be interchanged. The limits of integrations over $\tilde{s}_i$ and t_i are modified as follows (cf. Appendix):

$$
\begin{aligned}
\tilde{s}_{i\pm} &= L_\pm(s_i;\, M_i^2,s_{i+1};\, \tilde{s}_{i-1},s)\,,\\
t_{i\pm} &= L_\pm[s_i;\, M_i^2,s_{i+1},(\tilde{s}_i);\, t_{i-1},m_b^2,(m_a^2);\, \tilde{s}_{i-1},s].
\end{aligned}
\tag{4.3}
$$

(c) The order of integrations over s_{i+1} and $\tilde{s}_i$ ($2\leqslant i \leqslant m-2$) may be interchanged. The range of s_{i+1} and t_i in the physical region is obtained from $\tilde{s}_{i\pm}$ and $t_{i\pm}$ in Eq. (2.11) by making the interchanges $\tilde{s}_i\leftrightarrow s_{i+1}$, $\tilde{s}_{i-1}\leftrightarrow s_i$, $m_a^2\leftrightarrow m_b^2$, and the range of integration over $\tilde{s}_i$ is given by

$$
\tilde{s}_{i-} = (\sqrt{\tilde{s}_{i-1}}+M_i)^2\,,\quad \tilde{s}_{i+} = (\sqrt{s}-\sum_{k=i+1}^{m} M_k)^2. \tag{4.4}
$$

(d) The order of integrations in Eq. (2.9) may be completely reversed, with the modification of the limits of integrations which is obvious. This is possible because we have used a set of symmetric variables (cf. Fig. 1).

Change of variables or change in the order of integrations is often required in order to obtain the various distributions[3,11] of interest which can be compared with the experimental measurements. Here, we discuss a few such distributions.

A. Angular Correlation between Two Final-State Clusters

In order to derive an expression for the angular correlation between two of the final-state clusters (or particles) P_1 and $P_2 (m\geqslant 3)$ in the c.m. system we delete identity integral

$$
\int d\tilde{s}_2\, \delta(\tilde{s}_2+(P_1+P_2)^2)
$$

in Eq. (2.8) and integrate over d^4P_2 in the c.m. system with $\mathbf{P}_1$ along the z axis. Integrations over $|\mathbf{P}_2|$ and ϕ_2 may be done with the help of $\delta(P_2^2+M_2^2)$ and $\delta((p_a-P_1-P_2)^2+t_2)$, respectively; the relevant integral for our purpose here is of the form

$$
\int_{-1}^{+1} d\zeta_2 \int_{-\infty}^{+\infty} dE_2\, \delta(s_3-s_2-M_2^2+2(\sqrt{s}-E_1)E_2 +2|\mathbf{P}_1|(E_2^2-M_2^2)^{1/2}\zeta_2)\,, \tag{4.5}
$$

where $\zeta_2=\langle \mathbf{P}_1\cdot\mathbf{P}_2\rangle$ in the c.m. system [cf. Eq. (3.5)]. Integration over E_2 is done using the δ function. The condition that the argument of the δ function vanishes for some real value (or values) of E_2 is

$$
\zeta_2^2 \geqslant 1-(s_1/M_2^2)[\lambda(M_2^2,s_2,s_3)/\lambda(M_1^2,s_1,s_2)]. \tag{4.6}
$$

Evidently, the right-hand side of this inequality is $\leqslant 1$ in the physical region where both the λ's are positive.

[11] K. Kajantie and P. Lindblom, Phys. Rev. **175**, 2203 (1968); R. A. Morrow, *ibid.* **176**, 2147 (1968); P. Nyborg *et al.*, Ref. 1.

Hence $\zeta_2=\pm 1$ satisfy the inequality for all values of s_i's and M_i's lying in the physical region [cf. Eq. (2.11)].

Note that by setting the argument of the δ function equal to zero, one gets a unique solution of E_2. An easy way to choose between the two real roots of the quadratic equation in E_2 obtained by setting the argument of the δ function equal to zero is to use the fact that for $\zeta_2=+1$ we must have $\langle \mathbf{P}_1\cdot\mathbf{P}_2\rangle=+1$ in the frame of reference characterized by

$$\sum_{i=2}^{m}\mathbf{P}_i=0.$$

Using this value of E_2 (in the c.m. system), the invariant $\tilde{s}_2$ in terms of the independent variables (including ζ_2) is given by the following expression:

$$\tilde{s}_2=M_1^2-s_2+s_3+\tfrac{1}{2}\{(s_1+s_2-M_1^2)(s_2+M_2^2-s_3) -\zeta_2[\lambda(s_1,s_2,M_1^2)]^{1/2}[\lambda(s_2,M_2^2,s_3)-(M_2^2/s_1)(1-\zeta_2^2) \times\lambda(s_1,s_2,M_1^2)]^{1/2}\}[s_2+\lambda(s_1,M_1^2,s_2) \times(1-\zeta_2^2)/4s_1]^{-1}. \quad (4.7)$$

The condition implied by the inequality (4.6), when written in a form so as to restrict the range of variation of s_2 rather than that of ζ_2, reads

$$[s_2-s_{2-}(\zeta_2,s_3)][s_2-s_{2+}(\zeta_2,s_3)]\geqslant 0, \quad (4.8)$$

where

$$s_{2\pm}(\zeta_2,s_3)=\{\alpha\pm[\alpha^2-\beta(1-(M_2^2/s_1)(1-\zeta_2^2))]^{1/2}\} \times[1-(M_2^2/s_1)(1-\zeta_2^2)]^{-1},$$
$$\alpha=[s_3+M_2^2-(M_2^2/s_1)(1-\zeta_2^2)(s_1+M_1^2)],$$
$$\beta=(s_3-M_2^2)^2-(M_2^2/s_1)(1-\zeta_2^2)(s_1-M_1^2)^2. \quad (4.9)$$

Hence, the phase-space integral takes the form[12]

$$\mathcal{P}_m\sim\int_{-1}^{+1}d\zeta_2\left(\prod_{i=1}^{m}\int dM_i^2\right)\int ds_3\int ds_2 \times\theta[(s_2-s_{2-}(\zeta_2,s_3))(s_2-s_{2+}(\zeta_2,s_3))], \quad (4.10)$$

where the integrand is the same as in Eq. (2.9) except that

$$(16|\varepsilon_2|)^{-1}\rightarrow\tfrac{1}{2}s[\lambda(s_1,M_2^2,s_2')]^{1/2}[\lambda(s,m_a^2,m_b^2)]^{-1/2} \times(1-\xi_2^2-\eta_2^2-\zeta_2^2+2\xi_2\eta_2\zeta_2)^{-1/2}\{[\lambda(s_1,M_2^2,s_2')]^{1/2} \times(s_1+s_2-M_1^2)+\zeta_2[\lambda(s_1,M_1^2,s_2)]^{1/2} \times(s_1+M_2^2-s_2')\}^{-1}. \quad (4.11)$$

The range of varation of s_3 and s_2 is given by

$$s_{3-}=(\sum_{i=3}^{m}M_i)^2,\qquad s_{3+}=(\sqrt{s_1}-M_1-M_2)^2,$$
$$s_{2-}=(\sqrt{s_3}+M_2)^2,\quad s_{2+}=(\sqrt{s_1}-M_1)^2. \quad (4.12)$$

The next thing to be done is to express the implications of the step function explicitly so as to get the actual range of the variables s_2 for a given value of ζ_2 and s_3. To do this, we observe that since the inequalities (4.6) and (4.8) are two different ways of expressing the same condition, substituting $s_2=(\sqrt{s_3}+M_2)^2$ and $s_2=(\sqrt{s_1}-M_1)^2$ successively in (4.6) and (4.8) yields

$$[(\sqrt{s_3}+M_2)^2-s_{2-}(\zeta_2,s_3)] \times[(\sqrt{s_3}+M_2)^2-s_{2+}(\zeta_2,s_3)]\leqslant 0 \quad \text{if} \quad \zeta_2^2\leqslant 1, \quad (4.13)$$

and

$$[(\sqrt{s_1}-M_1)^2-s_{2-}(\zeta_2,s_3)] \times[(\sqrt{s_1}-M_1)^2-s_{2+}(\zeta_2,s_3)]\geqslant 0 \quad \text{if} \quad \zeta_2^2\geqslant-\infty. \quad (4.14)$$

Since $0\leqslant\zeta_2^2\leqslant 1$ and $(\sqrt{s_1}-M_1)^2\geqslant(\sqrt{s_3}+M_2)^2$, the following inequality holds:

$$s_{2-}(\zeta_2,s_3)\leqslant(\sqrt{s_3}+M_2)^2 \leqslant s_{2+}(\zeta_2,s_3)<(\sqrt{s_1}-M_1)^2. \quad (4.15)$$

Hence, for a given s_3 and ζ_2, the lower bound on s_2 is given by $s_{2+}(\zeta_2,s_3)$ and the step function in (4.10) can now be dropped. For the special case of $m=3$, we have $s_3=M_3^2$ so that the integration over s_3 should be dropped.

The angular correlation in the lab system ($\mathbf{p}_a=0$) is obtained by integrating over E_2 using

$$\delta(t_2+(p_a-P_1-P_2)^2).$$

The counterparts of Eqs. (4.6), (4.7), and (4.11) are obtained by the replacements

$$\sqrt{s}\leftrightarrow m_a,\quad s_2\rightarrow t_1,\quad s_3\rightarrow t_2,\quad (\xi_2,\eta_2,\zeta_2)\rightarrow(\bar{\xi}_2,\bar{\eta}_2,\bar{\zeta}_2).$$

However, it is not easy to give the upper and lower limits of variables t_1 and t_2 explicitly for given values of $\bar{\zeta}_2$, so that the counterpart of the condition (4.6) has to be expressed by a step function in the t_2 integration.

B. Distributions in Invariant Momentum Transfers

Let us first obtain the differential cross section $(\partial\sigma/\partial t)$ in a multiple-particle production process, where t denotes the invariant momentum transfer from an initial-state particle to a particle in the final state. Without any loss of generality one may choose $t=-(p_a-P_1)^2=t_1$, where P_1 is the 4-momentum of a single particle (i.e., $n_1=1$, $M_1=m_1$). The phase-space integral can be written in the form[12] [cf. Eq. (4.1)]

$$\mathcal{P}_m=\int_{s_{2-}}^{s_{2+}}ds_2\int_{t_{1-}(s_2)}^{t_{1+}(s_2)}dt_1\cdots, \quad (4.16)$$

with the following limits of integrations:

$$s_{2-}=(\sum_{i=2}^{m}M_{i-})^2,\quad s_{2+}=(\sqrt{s}-m_1)^2,$$
$$t_{1\pm}(s_2)=L\pm(s;\quad m_1^2,s_2;\quad m_a^2,m_b^2). \quad (4.17)$$

In order to reverse the order of integrations over s_2 and t_1, the condition which restricts the range of variation

[12] All other integrations except those which are relevant for our purpose are dropped. Similarly, the integrand is also suppressed.

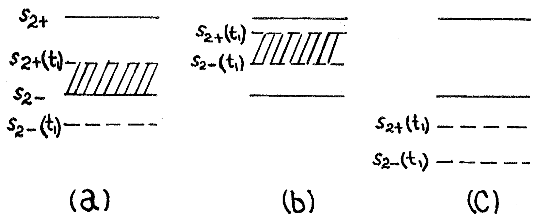

FIG. 2. The inequalities among $s_{2\pm}(t_1)$ and $s_{2\pm}$. (a) $t_{1-}\leq t_1\leq t_{1+}$; (b) $t_{1+}\leq t_1\leq (m_a-m_1)^2$ and $\nu_1>0$; (c) $t_1<t_{1-}$ or $t_1>t_{1+}$ but $\nu_1<0$. The physical region is indicated by the shaded area.

of t_1 should be expressed in the form of a restriction on the range of variation of s_2 so that t_1 is free to have all values. This constraint is given by

$$s_{2\pm}(t_1)=L_\pm(m_a^2;\,s,m_b^2;\,m_1^2,t_1) \tag{4.18}$$

(cf. Appendix). To find the overlap of the conditions $s_{2-}\leqslant s_2\leqslant s_{2+}$ and $s_{2-}(t_1)\leqslant s_2\leqslant s_{2+}(t_1)$, note that the latter inequality is alternatively expressed by the inequality $t_{1-}(s_2)\leqslant t_1\leqslant t_{1+}(s_2)$, so that if one of them is satisfied, the other too is satisfied and vice-versa. It follows that s_{2+} lies outside the range $(s_{2-}(t_1),s_{2+}(t_1))$, for all values of t_1, and $s_{2-}(t_1)\leqslant s_{2-}\leqslant s_{2+}(t_1)$, if $t_{1-}\leqslant t_1\leqslant t_{1+}$, where $t_{1\pm}\equiv t_{1\pm}(s_{2-})$. Also taking into account the fact that $s_{2-}\leqslant(\sqrt{s}-m_1)^2$, we have the inequality [cf. Fig. 2(a)]

$$s_{2-}(t_1)\leqslant s_{2-}\leqslant s_{2+}(t_1)\leqslant s_{2+}. \tag{4.19}$$

If, however, t_1 does not lie in this range, the two possible inequalities among $s_{2\pm}(t_1)$ and $s_{2\pm}$ are expressed by Figs. 2(b) and 2(c). Only the former corresponds to the physical region, the condition for which is $s_{2-}(t_1)>s_{2-}$. Since $s_{2-}(t_1=(m_a-m_1)^2)=s_{2+}(t_1=(m_a-m_1)^2)$, we have, for any $t_1\leqslant t_{1\,\max}$, $s_{2-}(t_1)\leqslant s_{2-}(t_{1\,\max})$. Hence the inequalities implied by Fig. 2(b) are satisfied if $s_{2-}(t_{1\max})>s_{2-}$ and $t_1>t_{1+}$, so that we finally have

$$\mathcal{P}_m\sim\int_{t_{1-}}^{t_{1+}}dt_1\int_{s_{2-}}^{s_{2+}(t_1)}ds_2 +\theta(\nu_1)\int_{t_{1+}}^{(m_a-m_1)^2}dt_1\int_{s_{2-}(t_1)}^{s_{2+}(t_1)}ds_2\,, \tag{4.20}$$

where $\nu_1=s+m_1^2-(m_1/m_a)(s+m_a^2-m_b^2)-s_{2-}$.

We can now relax the restriction $n_1=1$ and get the distribution $\partial\sigma/\partial t_1$ in the invariant momentum transfer from an initial-state particle p_a to a group of n_1 particles in the final state. $\mathcal{P}_m$ may be written in the form

$$\mathcal{P}_m\sim\int dM_1^2\int ds_2\int dt_1\cdots\,, \tag{4.21}$$

with the limits of the M_1^2 integration given by Eq. (2.11) and

$$s_{2-}=\Big(\sum_{i=2}^{m}M_{i-}\Big)^2,\quad s_{2+}=(\sqrt{s}-M_1)^2,$$
$$t_{1\pm}(M_1,s_2)=L_\pm(s;\,M_1^2,s_2;\,m_a^2,m_b^2)\,. \tag{4.22}$$

Changing the order of integrations we have

$$\mathcal{P}_m\sim\Bigg[\int_{t_{1-}}^{t_{1+}}dt_1\int_{M_{1-}^2}^{M_{1+}^2(t_1)}dM_1^2 +\theta(\alpha)\int_{t_{1+}}^{(m_b-\sqrt{s_{2-}})^2}dt_1\int_{M_{1-}^2(t_1)}^{M_{1+}^2(t_1)}dM_1^2\Bigg]\int_{s_{2-}}^{s_{2+}(M_1,t_1)}ds_2$$
$$+\theta(\beta)\int_{M_{1-}^2}^{M_{1+}^2}dM_1^2\int_{t_{1+}(M_1)}^{(m_a-M_1)^2}dt_1\int_{s_{2-}(M_1,t_1)}^{s_{2+}(M_1,t_1)}ds_2\,, \tag{4.23}$$

where

$$t_{1\pm}\equiv t_{1\pm}(M_{1-},s_{2-})\,,$$
$$t_{1\pm}(M_1)\equiv t_{1\pm}(M_1,s_{2-})\,,$$
$$M_{1\pm}^2(t_1)=L_\pm(m_b^2;\,t_1,s_{2-};\,m_a^2,s)\,,$$
$$s_{2\pm}(M_1,t_1)=L_\pm(m_a^2;\,M_1^2,t_1;\,s,m_b^2)\,,$$
$$\alpha=(\sqrt{s}-\sqrt{s_{2-}})^2-M_{1-}^2-(\sqrt{s_{2-}})\times[(\sqrt{s}-m_b)^2-m_a^2]m_b^{-1}\,,$$
$$\beta=(\sqrt{s}-M_{1-})^2-s_{2-}-(M_{1-})\times[(\sqrt{s}-m_a)^2-m_b^2]m_a^{-1}. \tag{4.24}$$

The order of the t_1 and M_1^2 integrations in the last term of (4.23) cannot be easily interchanged so that it has to be done graphically.

Next, we consider the simultaneous distribution in two invariant momentum transfers. Of particular interest is the case with three particles in the final state ($n=m=3$). Writing the t_2 integration in (4.20) explicitly, we have

$$\mathcal{P}_3\sim\int_{t_{1-}}^{t_{1+}}dt_1\int_{s_{2-}}^{s_{2+}(t_1)}ds_2\int_{t_{2-}(t_1,s_2)}^{t_{2+}(t_1,s_2)}dt_2 +\theta(\nu_1)\int_{t_{1+}}^{(m_a-m_1)^2}dt_1\int_{s_{2-}(t_1)}^{s_{2+}(t_1)}ds_2\int_{t_{2-}(t_1,s_2)}^{t_{2+}(t_1,s_2)}dt_2\,, \tag{4.25}$$

where

$$t_{2\pm}(t_1,s_2)=L_\pm(s_2;\,m_2^2,m_3^2;\,t_1,m_b^2)\,. \tag{4.26}$$

In order to reverse the order of the s_2 and t_2 integrations, we express this condition in the form

$$s_{2\pm}(t_1,t_2)=L_\pm(t_2;\,m_2^2,t_1;\,m_3^2,m_b^2)\,. \tag{4.27}$$

The inequalities, similar to those expressed by Fig. 2, among $s_{2\pm}(t_1,t_2)$, $s_{2+}(t_1)$, and s_{2-} [or $s_{2-}(t_1)$, depending on the range of t_1], for negative and for positive values of t_2 lead to the following form of $\mathcal{P}_3$:

$$\mathcal{P}_3\sim\int_{t_{1-}}^{t_{1+}}dt_1\int_{t_{2-}(t_1)}^{t_{2+}(t_1)}dt_2\int_{s_{2-}(t_1,t_2)}^{s_{2+}(t_1)}ds_2+\theta(\nu_1)\int_{t_{1+}}^{(m_a-m_1)^2}dt_1\int_{t_{2-}(t_1)}^{t_{2+}(t_1)}dt_2\int_{S_{2-}(t_1,t_2)}^{s_{2+}(t_1)}ds_2$$
$$+\theta(T_{2+})\int_{t_{1-}}^{T_{1+}}dt_1\int_0^{T_{2+}}dt_2\theta[-(s_{2+}(t_1,t_2)-s_{2-}(t_1))(s_{2+}(t_1,t_2)-s_{2+}(t_1))]\int_{S_{2-}(t_1,t_2)}^{s_{2+}(t_1,t_2)}ds_2\,, \tag{4.28}$$

where

$$t_{2\pm}(t_1)\equiv t_{2\pm}[t_1,s_{2+}(t_1)],$$
$$S_{2-}(t_1,t_2)=\max[s_{2-}(t_1),s_{2-}(t_1,t_2)],$$

and $T_{1+}=\theta(-\nu_1)t_{1+}+\theta(\nu_1)(m_a-m_1)^2$ gives the maximum value of t_1 in the physical region. Similarly, T_{2+} and t_{2-} are the maximum and the minimum values of t_2 in the physical region and are obtained from T_{1+} and t_{1-}, respectively, by the interchange $m_a\leftrightarrow m_b$, and $m_1\leftrightarrow m_3$. The limits $t_{2\pm}(t_1)$ are real numbers since $\lambda(s_{2+}(t_1),m_2^2,m_3^2)\geqslant 0$ [since $s_{2+}(t_1)\geqslant(m_2+m_3)^2$, if $t_{1-}\leqslant t_1\leqslant T_{1+}$], and

$$\lambda(s_{2\pm}(t_1),t_1,m_b^2)=(4m_a^2)^{-1}\{(s-m_a^2-m_b^2)\times[\lambda(m_a^2,t_1,m_1^2)]^{1/2}\pm(m_a^2+t_1-m_1^2)\times[\lambda(s,m_a^2,m_b^2)]^{1/2}\}^2>0. \quad (4.29)$$

The physical region in the plane of the two invariant momentum transfers is a curve enclosed in a rectangle formed by the straight lines $t_1=t_{1-}$, $t_2=t_{2-}$, $t_1=T_{1+}$, and $t_2=T_{2+}$. Apart from a possible contribution from the last term in (4.28), the physical region in the t_1-t_2 plane is given by

$$t_{1-}\leqslant t_1\leqslant T_{1+},\quad t_{2-}(t_1)\leqslant t_2\leqslant t_{2+}(t_1). \quad (4.30)$$

Since

$$\lambda(s_{2+}(t_{1-}),m_2^2,m_3^2)=\lambda((m_2+m_3)^2,m_2^2.m_3^2)=0,$$

we have, in the t_1-t_2 plot, $t_{2+}(t_{1-})=t_{2-}(t_{1-})$, i.e., the boundary curve in the t_1-t_2 plane is tangent to the line $t_1=t_{1-}$. Similarly, the boundary curve is tangent to the line $t_2=t_{2-}$. Now, depending on whether $\nu_1<0$ or $\nu_1>0$, we have $s_{2\pm}(t_{1+})=(m_2+m_3)^2$, so that if $\nu_1<0$, the curve is also tangent to the line $t_1=t_{1+}$. On the other hand, if $\nu_1>0$, the line $t_1=(m_a-m_1)^2$ is itself a part of the boundary curve. Similarly, if $T_{2+}=t_{2+}$, the curve is tangent to the line $t_2=t_{2+}$, and if $T_{2+}=(m_b-m_3)^2$, the line $t_2=(m_b-m_3)^2$ is a part of the boundary curve. The last term in (4.28) can possibly contribute for positive values of t_2 but it does not appear advantageous to write it more explicitly[13] by writing the restriction implied by the step function on the masses of the particles, the c.m. energy, and the momentum transfers.

In the particular case of $n=3$, it may be of some interest to get the distribution in a different pair of invariant momentum transfers, viz., t_1' and t_2' [cf. Eq. (3.2)]. In terms of the new variables, we have

$$\mathcal{P}_3=\tfrac{1}{4}\pi^2[\lambda(s,m_a^2,m_b^2)]^{-1/2}\times\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2}ds_2\int_{t_{1-}'(s_2)}^{t_{1+}'(s_2)}dt_1'\int_{t_{2-}'(s_2,t_1')}^{t_{2+}'(s_2,t_1')}dt_2'\times[\lambda(s_2,m_a^2,\bar{t}_1)]^{-1/2}, \quad (4.31)$$

where t_1' and $t_{1\pm}'(s_2)$ are the same as t_1 and $t_{1\pm}(s_2)$, respectively, and

$$t_{2\pm}'(s_2,t_1')=L_\pm(s_2;\,m_a^2,\bar{t}_1;\,m_2^2,m_3^2). \quad (4.32)$$

The order of the s_2 and t_1' integrations is reversed as before, and in order to interchange the sequence of s_2 and t_2' integrations, the constraint on t_2' is now shifted to s_2, giving

$$s_{2\pm}(t_1',t_2')=L_\pm(m_a^2;\,m_2^2,t_2';\,m_3^2,\bar{t}_2;+), \quad (4.33)$$

where

$$\bar{t}_2=-(p_b-P_1-P_2)^2=2m_a^2+m_b^2+m_1^2+m_2^2+m_3^2-s-t_1'-t_2'. \quad (4.34)$$

From $s_{2\pm}(t_1',t_2')$, it is evident that $t_2'{}_+(s_2,t_1')$ has a maximum at $s_2=s_{2\pm}(t_1',(m_a-m_2)^2)$ and $t_{2-}'(s_2,t_1')$ has a minimum at $s_2=s_{2\pm}(t_1',\tau_2)$, where τ_2 is the value of t_2' corresponding to $\bar{t}_2=(m_a-m_3)^2$, viz.,

$$\tau_2=m_a^2+m_b^2+m_1^2+m_2^2+2m_am_3-s-t_1'. \quad (4.35)$$

The overlap of region $(m_2+m_3)^2\leqslant s_2\leqslant s_{2+}(t_1')$ [or $s_{2-}(t_1')\leqslant s_2\leqslant s_{2+}(t_1')$] with $s_{2-}(t_1',t_2')\leqslant s_2\leqslant s_{2+}(t_1',t_2')$ is obtained as before from the inequalities among the four limits. Thus,

$$\mathcal{P}_3\sim\int_{t_{1-}'}^{t_{1+}'}dt_1'\int_{t_{2-}'(t_1')}^{t_{2+}'(t_1')}dt_2'\int_{s_{2-}(t_1',t_2')}^{s_{2+}(t_1')}ds_2+\theta(\nu_1)\int_{t_{1+}'}^{(m_a-m_1)^2}dt_1'\int_{t_{2-}'(t_1')}^{t_{2+}'(t_1')}dt_2'\int_{S_{2-}(t_1',t_2')}^{s_{2+}(t_1')}ds_2$$
$$+\int_{t_{1-}'}^{T_{1+}'}dt_1'\,\theta(\alpha_+(t_1'))\int_{t_{2+}'(t_1')}^{T_{2+}'(t_1')}dt_2'\int_{S_{2-}(t_1',t_2')}^{s_{2+}(t_1',t_2')}ds_2+\int_{t_{1-}'}^{T_{1+}'}dt_1'\,\theta(\beta_+(t_1'))\int_{T_{2-}'(t_1')}^{t_{2-}'(t_1')}dt_2'\int_{S_{2-}(t_1',t_2')}^{s_{2+}(t_1',t_2')}ds_2, \quad (4.36)$$

where

$$\begin{aligned}
&T_{1+}'\equiv T_{1+},\quad s_{2\pm}(t_1')\equiv s_{2\pm}(t_1),\\
&t_{2\pm}'(t_1')=t_{2\pm}'(s_{2\pm}(t_1'),t_1'),\\
&\alpha_\pm(t_1')=s_{2\pm}(t_1')-s_{2+}(t_1',(m_a-m_2)^2),\\
&\beta_\pm(t_1')=s_{2\pm}(t_1')-s_{2+}(t_1',\tau_2),\\
&T_{2+}'(t_1')=\theta[-\alpha_-(t_1')](m_a-m_2)^2+\theta[\alpha_-(t_1')]t_{2+}'(s_{2-}(t_1')),\\
&T_{2-}'(t_1')=\theta[-\beta_-(t_1')]\tau_2+\theta[\beta_-(t_1')]t_{2-}'(s_{2-}(t_1')),\\
&S_{2-}(t_1',t_2')\equiv\max[s_{2-}(t_1',t_2'),s_{2-}(t_1')].
\end{aligned} \quad (4.37)$$

[13] According to last term of (4.28), the curves $t_2=(m_2+\sqrt{t_1})^2$ or $t_2=(m_2-\sqrt{t_1})^2$ could also be parts of the boundary curves for $t_1>0$ if $m_a<m_1$, $m_b>(m_2+m_3)$ or $m_a>m_1$, respectively. These conditions are necessary but not sufficient. In any case, the region defined by (4.30) as well as that obtained from it by the interchanges $m_a\leftrightarrow m_b$, $m_1\leftrightarrow m_3$ is physical. The entire physical region is obtained by joining the intersection of these two curves (if at all they intersect) by one of the curves $t_1=(m_a-m_1)^2$, $t_2=(m_b-m_3)^2$, or $t_2=(m_2\pm\sqrt{t_1})^2$, as the case may be. A detailed discussion of the physical region in a somewhat different way has been given by Kajantie and Lindblom, Ref. 11.

The constraints implied by the step functions in the t_1' integration in (4.36) on the masses of particles and the Mandelstam variables can be written explicitly so that the last two terms in (4.36) may be rewritten as

$$\theta(\nu_2)\int_{t_{1-}'(2)}^{t_{1+}'(2)}dt_1'\int_{t_{2+}'(t_1')}^{(m_a-m_2)^2}dt_2'\int_{S_{2-}(t_1',t_2')}^{s_{2+}(t_1',t_2')}ds_2+\theta(\nu_3)\int_{t_{1-}'(3)}^{t_{1+}'(3)}dt_1'\int_{\tau}^{t_{2-}'(t_1')}dt_2'\int_{S_{2-}(t_1',t_2')}^{s_{2+}(t_1',t_2')}ds_2$$
$$+\theta(\nu_2-M^2)\int_{t_{1+}'(2)}^{(m_a-m_1)^2}dt_1'\int_{t_{2+}'(t_1')}^{t_{2+}'(t_1)}dt_2'\int_{s_{2-}(t_1')}^{s_{2+}(t_1',t_2')}ds_2+\theta(\nu_3-M^2)\int_{t_{1+}'(3)}^{(m_a-m_1)^2}dt_1'\int_{t_{2-}'(t_1)}^{t_2'-(t_1')}dt_2'\int_{s_{2-}(t_1')}^{s_{2+}(t_1',t_2')}ds_2, \quad (4.38)$$

where ν_2 and ν_3 are obtained from ν_1 by interchaning $m_1\leftrightarrow m_2$ and $m_1\leftrightarrow m_3$, respectively, and

$$\mu_2^2=\nu_2+(m_1+m_3)^2,$$
$$\mu_3^2=\nu_3+(m_1+m_2)^2,$$
$$M^2=(m_1/m_a)[s+m_a^2-m_b^2-2m_a(m_1+m_2+m_3)],$$
$$t_{2\pm}'(t_1)\equiv t_{2\pm}'(s_{2-}(t_1'),t_1'),$$
$$t_{1\pm}'(2)=m_a^2+m_1^2-(s+m_a^2-m_b^2-2m_am_2)(\mu_2^2+m_1^2-m_3^2)/2\mu_2^2\pm[\lambda(s,m_a^2,m_b^2)\lambda(\mu_2^2,m_1^2,m_3^2)]^{1/2}/2\mu_2^2, \quad (4.39)$$

and $t_{1\pm}'(3)$ is obtained from $t_{1\pm}'(2)$ by the interchange $m_2\leftrightarrow m_3$, $\mu_2\leftrightarrow\mu_3$. In the range $t_{1-}'(2)\leqslant t_1'\leqslant t_{1+}'(2)$, we have $s_{2-}(t_1')\leqslant s_{2\pm}(t_1',(m_a-m_2)^2)\leqslant s_{2+}(t_1')$, and for $t_1'>t_{1+}'(2)$, we have $s_{2\pm}(t_1',(m_a-m_2)^2)\leqslant s_2-(t_1')$. Similarly, $s_{2-}(t_1')\leqslant s_{2\pm}(t_1',\tau_2)\leqslant s_{2+}(t_1')$, if $t_{1-}'(3)\leqslant t_1'\leqslant t_{1+}'(3)$, and $s_{2\pm}(t_1',\tau_2)\leqslant s_{2-}(t_1')$, if $t_1'\geqslant t_{1+}'(3)$. Hence the limits of t_1' and s_2 integrations as given in (4.38) follow from (4.36) and (4.37).

Finally, the simultaneous distribution in the invariant momentum transfers t_1 and t_2 for a production process with n particles in the final state is only a slight modification of (4.28). We have

$$\mathcal{P}_n\sim\left[\int_{t_{1-}}^{t_{1+}}dt_1\int_{t_{2-}(t_1)}^{t_{2+}(t_1)}dt_2\int_{s_{2-}(t_1,t_2)}^{s_{2+}(t_1)}ds_2+\theta(\nu_1)\int_{t_{1+}}^{(m_a-m_1)^2}dt_1\int_{t_{2-}(t_1)}^{t_{2+}(t_1)}dt_2\int_{S_{2-}(t_1,t_2)}^{s_{2+}(t_1)}ds_2\right]\int_{s_{3-}}^{s_{3+}(t_1,t_2,s_2)}ds_3$$
$$+\theta(T_{1+})\int_{t_{1-}}^{T_{1+}}dt_1\int_0^{T_{2+}}dt_2\theta(-[s_{2+}(t_1,t_2)-s_{2-}(t_1)][s_{2+}(t_1,t_2)-s_{2+}(t_1)])\int_{S_{2-}(t_1,t_2)}^{s_{2+}(t_1,t_2)}ds_2\int_{s_{3-}}^{s_{3+}(t_1,t_2,s_2)}ds_3$$
$$+\theta(T_{1+})\theta(T_{2+})\int_{T_{1-}}^{T_{1+}}dt_1\int_{T_{2-}(t_1)}^{T_{2+}(t_1)}dt_2\int_{S_{2-}(t_1)}^{s_{2+}(t_1)}ds_2\,\theta[s_{3-}(t_1,t_2,s_2)-s_{3-}]\int_{s_{3-}(t_1,t_2,s_2)}^{s_{3+}(t_1,t_2,s_2)}ds_3, \quad (4.40)$$

where

$$T_{1-}=\max(0,t_{1-}),\quad S_{2-}(t_1)=\max[s_{2-},s_{2-}(t_1)],$$
$$s_{3\pm}(t_1,t_2,s_2)=L_\pm(t_1;\,s_2,m_b^2;\,m_2^2,t_2),$$
$$t_{2\pm}(t_1)=L_\pm(s_{2+}(t_1);\,m_2^2,s_{3-};\,t_1,m_b^2),$$
$$s_{2\pm}(t_1,t_2)=L_\pm(t_2;\,m_2^2,t_1;\,s_{3-},m_b^2), \quad (4.41)$$
$$T_{2-}(t_1)=0,\quad T_{2+}(t_1)=(m_2-\sqrt{t_1})^2\quad\text{if}\quad m_a>m_1;$$
$$T_{2-}(t_1)=(m_2+\sqrt{t_1})^2,\quad T_{2+}(t_1)=\min\{\max[(m_a-m_1-m_2)^2,(m_a-(\sqrt{s}-\sum_{i=3}^n m_i))^2],$$
$$\max[(m_b-\sum_{i=3}^n m_i)^2,(m_b-(\sqrt{s}-m_1-m_2))^2]\}\quad\text{if}\quad m_a<m_1.$$

C. Simultaneous Distribution in Energies of Two Particles

In order to get $\partial^2\sigma/\partial E_1\partial E_2$ in the c.m. system for $n>3$, the phase-space integral must be written in the form

$$\mathcal{P}_n\sim\int ds_2\int ds_2'\cdots.$$

Starting from the phase-space integral written in the form[3]

$$\mathcal{P}_n \sim \int_{s_{2-}}^{(\sqrt{s}-m_1)^2} ds_2 \int_{s_{3-}}^{(\sqrt{s_2}-m_2)^2} ds_3 \int_{s_{2-}'(s_2,s_3)}^{s_{2+}'(s_2,s_3)} ds_2' \tag{4.42}$$

and interchanging the order of s_3 and s_2' integrations, we have

$$\mathcal{P}_n \sim \int_{s_{2-}}^{(\sqrt{s}-m_1)^2} ds_2 \int_{s_{2-}'(s_2)}^{s_{2+}'(s_2)} ds_2' \int_{s_{3-}}^{s_{3+}(s_2,s_2')} ds_3 + \int_{s_{2-}(s_1)}^{(\sqrt{s}-m_1)^2} ds_2 \int_{s_{2+}'(s_2)}^{(\sqrt{s_2}-m_2)^2} ds_2' \int_{s_{3-}(s_2,s_2')}^{s_{3+}(s_2,s_2')} ds_3 , \tag{4.43}$$

where

$$s_{i-} = (\sum_{k=i}^{n} m_k)^2 , \qquad s_{2-}(s_1) = [(s_{3-}-m_2^2)\sqrt{s}+m_2(s-m_1^2)](\sqrt{s}-m_2)^{-1} ,$$
$$s_{2\pm}'(s_2,s_3) = L_\pm(s_2;\, m_2^2,s_3;\; s,m_1^2) ,$$
$$s_{3\pm}(s_2,s_2') = L_\pm(s;\, s_2,m_1^2;\, m_2^2,s_2') , \quad s_{2\pm}'(s_2) \equiv s_2{}'_\pm(s_2,s_{3-}) . \tag{4.44}$$

$\partial^2\sigma/\partial E_1\partial E_2$ is then simply related to $\partial^2\sigma/\partial s_2\partial s_2'$.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor S. N. Biswas and Dr. R. Rajaraman for their interest, and Dr. D. Bondyopadhyaya and Dr. D. Basu for reading the manuscript.

APPENDIX: NOTATION

$$\langle \mathbf{A}\cdot\mathbf{B}\rangle \equiv (\mathbf{A}\cdot\mathbf{B})/|\mathbf{A}|\,|\mathbf{B}| , \tag{A1}$$

$$[ABCD] \equiv i\epsilon_{\mu\nu\lambda\sigma}A_\mu B_\nu C_\lambda D_\sigma , \tag{A2}$$

$$\epsilon[ABCD] \equiv \mathrm{sgn}[ABCD] , \tag{A3}$$

$$C(s;\, a,b;\, c,d;\, t) \equiv [(s+a-b)(s+c-d)-2s(a+c-t)][\lambda(s,a,b)\lambda(s,c,d)]^{-1/2} , \tag{A4}$$

$$C(s;\, a,b;\, c,d;\, t,\bar{t}) \equiv \tfrac{1}{2}[\lambda(s,a,b)+\lambda(s,c,d)-\lambda(s,t,\bar{t})][\lambda(s,a,b)\lambda(s,c,d)]^{-1/2} , \tag{A5}$$

$$C(\zeta;\, \xi,\eta) \equiv (\zeta-\xi\eta)[(1-\xi^2)(1-\eta^2)]^{-1/2} , \tag{A6}$$

$$L_\pm(s;\, a,b;\, c,d) \equiv a+c-(s+a-b)(s+c-d)/2s \pm [\lambda(s,a,b)\lambda(s,c,d)]^{1/2}/2s , \tag{A7}$$

$$L_\pm(s;\, a,b;\, c,d;\, +) \equiv a+c+(s+a-b)(s+c-d)/2s \pm [\lambda(s,a,b)\lambda(s,c,d)]^{1/2}/2s , \tag{A8}$$

$$L_\pm[s;\, a_1,b_1,(c_2);\, a_2,b_2,(c_1);\, a_3,b_3] \equiv a_1+a_2-(s+a_1-b_1)(s+a_2-b_2)(2s)^{-1}+[\lambda(s,a_1,b_1)\lambda(s,a_2,b_2)]^{1/2}(2s)^{-1}\{\xi\eta \pm [(1-\xi^2)(1-\eta^2)]^{1/2}\} , \tag{A9}$$

$$|\varepsilon(s;\, a_1,b_1;\, a_2,b_2;\, a_3,b_3;\, c_1,c_2,c_3)| \equiv (8s)^{-1}[\lambda(s,a_1,b_1)\lambda(s,a_2,b_2)\lambda(s,a_3,b_3)(1-\xi^2-\eta^2-\zeta^2+2\xi\eta\zeta)]^{1/2} , \tag{A10}$$

where ξ, η, and ζ in (A9) and (A10) denote the following:

$$\xi \equiv C(s;\, a_3,b_3;\, a_2,b_2;\, c_1) , \quad \eta \equiv C(s;\, a_3,b_3;\, a_1,b_1;\, c_2) , \quad \zeta \equiv C(s;\, a_1b_1;\, a_2,b_2;\, c_3) . \tag{A11}$$

Note that (A9) does not depend on ζ, and is symmetric in ξ and η, and (A10) is symmetric in ξ, η, and ζ.

If $s>0$, $C(s;\, a,b;\, c,d;\, t)$ gives $\langle \mathbf{p}_a\cdot\mathbf{p}_c\rangle$ in the frame of reference characterized by $\mathbf{p}_s=0$, where the 4-momenta are such that

$$-p_\alpha^2 = \alpha ,$$
$$p_s = \pm(p_a \pm p_b) = \pm(p_c \pm p_d) , \tag{A12}$$
$$t = -(p_a-p_c)^2 .$$

Evidently, $C(s;\, a,b;\, c,d;\, t)$ is invariant under the interchange $(a,b) \leftrightarrow (c,d)$ or $a \leftrightarrow b$ and $c \leftrightarrow d$ simultaneously. With the invariant $\bar{t}$ such that $p_{\bar{t}} = \pm(p_s \pm p_t)$, $C(s;\, a,b;\, c,d;\, t,\bar{t})$ also gives $\langle \mathbf{p}_a\cdot\mathbf{p}_c\rangle$ in the form

$$\cos\theta = \tfrac{1}{2}(\lambda_1+\lambda_2-\lambda_3)(\lambda_1\lambda_2)^{-1/2} .$$

When written in this form, we have

$$\sin\theta = \tfrac{1}{2}[-\lambda(\lambda_1,\lambda_2,\lambda_3)]^{1/2}(\lambda_1\lambda_2)^{-1/2} .$$

$C(\zeta;\xi,\eta)$ obviously gives the cosine of the azimuthal angle between two vectors which are such that their polar angles are $\cos^{-1}\xi$ and $\cos^{-1}\eta$ and the cosine of the angle between the two vectors is ζ.

Integration over the solid angle using two Dirac δ functions which define suitable Mandelstam variables gives constraints of the type $-1\leqslant\eta\leqslant+1$ and $-1\leqslant C(\zeta;\xi,\eta)\leqslant+1$. The former constraint restricts the range of the invariant c_2 in the physical region, viz.,

$$c_{2-}\leqslant c_2\leqslant c_{2+} \quad \text{if} \quad s>0,$$
$$(c_2-c_{2-})(c_2-c_{2+})\geqslant 0 \quad \text{if} \quad s<0,$$

where

$$c_{2\pm}=L_\pm(s;a_3,b_3;a_1,b_1). \tag{A13}$$

Note that $c_{2-}>c_{2+}$ if $s<0$. The condition $\eta^2\leqslant 1$ may alternatively be expressed so as to restrict the range of any one of the invariants a_1, b_1, a_3, b_3 or s. We have

$$s_\pm=L_\pm(c_2;a_1,a_3;b_1,b_3),$$
$$a_{1\pm}=L_\pm(b_3;s,a_3;b_1,c_2), \tag{A14}$$

and the restrictions on any of the remaining three invariants follow from (A14) by use of the symmetry properties of (A4) or (A7). The condition $-1\leqslant C(\zeta;\xi,\eta)\leqslant+1$ similarly restricts the range of variation of the invariant c_3:

$$c_{3\pm}=L_\pm[s;a_1,b_1,(c_2);a_2,b_2,(c_1);a_3,b_3],$$

i.e.,

$$c_{3-}\leqslant c_3\leqslant c_{3+} \quad \text{if} \quad s>0,$$
$$(c_3-c_{3-})(c_3-c_{3+})\geqslant 0 \quad \text{if} \quad s<0. \tag{A15}$$

(A9) is clearly invariant under the following interchanges:

(i) $(a_1,a_2,a_3)\leftrightarrow(b_1,b_2,b_3)$

and (A16)

(ii) $(a_1,b_1,c_2)\leftrightarrow(a_2,b_2,c_1)$.

Other forms of the constraint $[C(\zeta;\xi,\eta)]^2\leqslant 1$ are

$$\begin{aligned}a_{1\pm}&=L_\pm[b_2;c_3,b_1,(c_2);a_2,s,(a_3);c_1,b_3]\\&=L_\pm[b_3;c_2,b_1,(c_3);a_3,s,(a_2);c_1,b_2],\\s_\pm&=L_\pm[c_3;a_1,a_2,(b_3);b_1,b_2,(a_3);c_1,c_2]\\&=L_\pm[c_1;a_2,a_3,(b_1);b_2,b_3,(a_1);c_2,c_3]\\&=L_\pm[c_2;a_3,a_1,(b_2);b_3,b_1,(a_2);c_3,c_1].\end{aligned} \tag{A17}$$

Constraints on the range of variation of any of the remaining seven invariants may be similarly written using the fact that the condition $[C(\zeta;\xi,\eta)]^2\leqslant 1$ is symmetric in ξ, η, and ζ and by employing the symmetries of the latter quantities under the interchange of the various indices. Also, note that the simultaneous conditions $-1\leqslant\eta\leqslant+1$ and $-1\leqslant C(\zeta;\xi,\eta)\leqslant+1$ are equivalent to the conditions $-1\leqslant\zeta\leqslant+1$ and $-1\leqslant C(\eta;\xi,\zeta)\leqslant+1$.

Finally, (A10) gives the magnitude of the pseudoscalar $[p_s p_{a_1} p_{a_2} p_{a_3}]$.