Energy Estimation from the Angular Distribution of 20-GeV/c Pion Interactions in Photographic Emulsion*

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Interactions of 20-GeV/c negative pions in photographic emulsion have been analyzed in order to compare and test the methods of energy estimation commonly used in high-energy cosmic-ray investigations. The energy of each pion interaction has been determined from the angular distribution of the secondary particles using the median-angle, Castagnoli, E_{ch}, and $E(\theta)$ methods. The interactions have been divided into various groups according to the number N_h of evaporation prongs and the number n_s of secondary particles. The median-angle, Castagnoli, and $E(\theta)$ methods all overestimate the energy of the group with $N_h \leq 5$ ($n_s \geq 4$) by factors of 1.5, 1.2, and 1.1, respectively. These same methods underestimate the energy of the group with $N_h > 5$ ($n_s \geq 4$) by factors of 1.3, 2.0, and 2.3, respectively. The E_{ch} method underestimates the energies of the groups with $N_h \leq 5$ ($n_s \geq 4$) and $N_h > 5$ ($n_s \geq 4$) by factors of 1.3 and 1.5, respectively. These underestimates by the E_{ch} method become 0.9 and 1.0 if the general practice of including the effect of neutral secondaries to the E_{ch} method is adopted. For all the various groupings of N_h and n_s considered, the E_{ch} method yields the most consistent and uniform results with the smallest standard deviations than any of the other three methods.

I. INTRODUCTION

STUDIES of the interactions of high-energy primary particles with nuclei in photographic emulsion have frequently used three methods to estimate the energy of the primary particle. These methods are based on the observed angular distribution in the laboratory system (LS) of the charged secondary particles (secondaries) produced by the interaction of a primary particle with emulsion nuclei. Bradt *et al.*[1] suggested in 1950 that the median angle of emission of the secondaries might be used to obtain an estimate E_m of the energy of the primary particle. Castagnoli *et al.*[2] in 1953 developed a method of using the angular distribution of the secondaries to obtain an estimate E_{Cast} of the primary particle's energy. The International Cooperative Emulsion Flight[3] (ICEF) in 1964 suggested a method of using the constancy of the mean transverse momentum of the secondaries and the constancy of the inelasticity to obtain an estimate E_{ch} of the primary particle energy. One of the present authors[4] suggested a fourth method using a kinematic parameter $\eta(\theta)$. This fourth method of obtaining an estimate $E(\theta)$ of the primary particle energy is also based on the observations of the angular distribution of the secondaries in the LS.

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[1] H. Bradt, M. Kaplon, and B. Peters, Helv. Phys. Acta **23**, 24 (1950).

[2] C. Castagnoli, G. Cortini, C. Franzinetti, A. Manfredini, and A. Moreno, Nuovo Cimento **10**, 1539 (1953).

[3] International Cooperative Emulsion Flight Collaboration, Nuovo Cimento Suppl. **1**, 1039 (1963); Suppl. 1, 1091 (1963).

[4] C. O. Kim, Phys. Rev. **158**, 1261 (1967).

The energies of secondary particles from the interactions of primary cosmic rays have also been estimated by Gierula *et al.*[5] using these methods. Since many of these secondaries are pions, a comparison of these methods of energy estimation and their applicability has been made using the interactions of accelerator-produced pions. The angular distribution of the secondaries of interactions, produced by 20-GeV/c pions in photographic emulsion, has been analyzed. The dependence of the averages of $\eta(\theta)$, $\langle\eta(\theta)\rangle$, on N_h and n_s has also been investigated and compared with the analysis in Ref. 4.

II. EXPERIMENTAL PROCEDURE

Exposure of Emulsion, Scanning, and Measurements

A stack of Kodak NTB-4 photographic emulsion was exposed to a momentum-analyzed beam of negative pions from the Brookhaven Alternating Gradient Synchrotron. The momentum of these pions was 20 GeV/c.

All of the pion interactions were found by on-track scanning in the emulsion. The interactions were classified according to the number N_h of secondary tracks having $g^* > 1.4$ and the number n_s of secondaries with $g^* \leq 1.4$. The ratio of the blob density of a secondary track to that of the incident pion track is defined to be g^*.

The measurements were made in a manner analogous to the ICEF measurements[3] using a Koristka MS-2 microscope. However, here the angles of the secondaries were determined using measurements based on the extrapolation of the incident pion direction. This technique made it possible to measure very accurately small emission angles of secondaries. The emulsion stack had

[5] J. Gierula and S. Krzywdzinski, Nukleonika **13**, 77 (1968).

been carefully aligned with the beam direction so that typical dip angles of the incident pions were within ±1 mrad of the emulsion plane. Wide-angle secondaries were measured with a standard goniometer attachment to the Koristka microscope.

Multiplicity Distribution

A summary of the secondary particle multiplicities for the different groups of N_h is given in Table I. The over-all charged-particle multiplicity for all events is

$$\langle n_s \rangle = 5.8 \pm 0.3. \qquad (1)$$

Out of a total of 427 interactions found in the scanning, the angles of the secondaries of 318 interactions were measured and analyzed for this paper. A correlation plot for N_h and n_s for the 318 interactions used is shown in Fig. 1. All interactions with $N_h \leq 7$ were measured. Only an unbiased sample of all the interactions found with $N_h > 7$ was measured and included in the energy-estimation analysis.

III. ENERGY ESTIMATION FROM ANGULAR DISTRIBUTION

Methods of Energy Estimation

A brief discussion of the median angle and Castagnoli methods appears in Aly *et al.*[6] A brief summary of all four methods used in this analysis to estimate the energy of the incident pions follows.

Method Based on Constancy of Transverse Momentum and Inelasticity

E_{ch}. The transverse momentum of secondaries has been studied extensively in cosmic-ray and accelerator experiments.[7–9] The average value of the transverse momentum has been found to be rather small (about

TABLE I. Multiplicity distribution for 20-GeV/c pion interactions.

N_h	Number of interactions	Average multiplicity $\langle n_s \rangle$
0	100	3.7±0.4
1	44	3.9±0.6
0, 1	144	3.7±0.3
2, 3, 4	113	5.5±0.5
5, 6, 7	57	6.7±0.9
≥8	113	8.3±0.8

[6] H. H. Aly, C. M. Fisher, and A. Mason, Nuovo Cimento **28**, 1117 (1963).
[7] D. H. Perkins, in *Progress in Elementary Particle and Cosmic Ray Physics*, edited by J. G. Wilson and S. A. Wouthuysen (North-Holland, Amsterdam, 1960), Vol. 5, Chap. 4, p. 259.
[8] D. H. Perkins, in *Proceedings of the International Conference on Theoretical Aspects of Very-High-Energy Phenomena* (CERN 61-22, Geneva, 1961), p. 99.
[9] M. Koshiba, in "Experimental Program Requirements for a 300–1000 GeV Accelerator," Brookhaven National Laboratory, 1961 (unpublished).

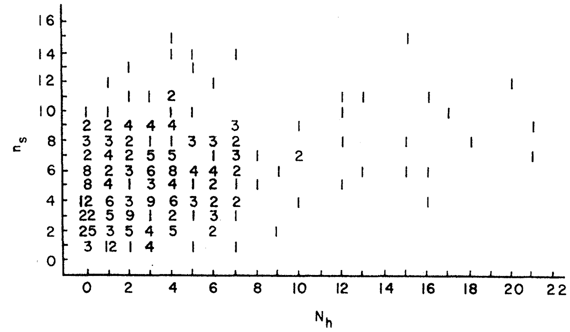

FIG. 1. Correlation plot for the number n_s of secondary particles and the number N_h of evaporation prongs. This plot applies to a sample of 20-GeV/c pion interactions in photographic emulsion.

0.4 GeV/c) and appears to be approximately independent of the emission angle of the secondaries and of the energy of the primary particle. Use has been made of this observation in order to estimate the energy E_{ch} carried away from an interaction in the form of charged particles. This energy is calculated for an interaction using the formula[3]

$$E_{ch} = \sum_i \frac{0.4 \text{ GeV}}{\sin\theta_i}, \qquad (2)$$

where θ_i is the emission angle of the ith secondary particle emitted from the interaction, and the summation is carried out over all the charged secondaries. All of the calculations and estimates of the energies, as determined by this method, have been obtained and tabulated by assuming that all the available energy from an interaction was carried away by the charged secondaries. The inelasticity K_{ch} has been assumed to be a constant for all interactions and equal to one. With these assumptions, E_{ch} as calculated using Eq. (2) becomes equal to the energy of the primary particle. [Often the energy carried away by the neutral secondaries is taken into account on a statistical basis by multiplying E_{ch} in Eq. (2) by a factor of 1.5.] The energy estimate obtained by this method should yield an estimate which is independent of the mass of the effective target.[3] Furthermore, the energy estimate should be insensitive to any secondary nuclear cascading inside the target nucleus.

Methods Based on Symmetry in Center-of-Mass System

The other three methods of energy estimation are commonly based on the assumption that in the LS, the velocity of the c.m. system (CMS), $\beta_{c.m.}$, is equivalent to that of a "symmetric" system (SS), β_S. In this SS (where the secondaries possess forward-backward symmetry) it is assumed that

(a) there is no correlation between the angles of emission of the secondaries, and
(b) there is no correlation between the energies of the secondaries.

Since the CMS of a nucleon-nucleon or nucleon–quasifree-nucleon collision system possesses forward-backward symmetry before the collision, the same symmetry can be expected to hold statistically after an interaction with multiple production of secondaries. This same symmetry after an interaction is assumed even for a pion-nucleon collision system. (Before the collision, a pion-nucleon system obviously does not possess forward-backward symmetry in the CMS.) With these assumptions, the energy estimate E_p of the incident primary particle is given by

$$E_p=M_t\{(\gamma_{\mathrm{c.m.}}^2-1)+\gamma_{\mathrm{c.m.}}[\gamma_{\mathrm{c.m.}}^2-1+(M_i/M_t)^2]^{1/2}\}\,, \quad (3a)$$

where M_i and M_t are the masses of the incident and target particles, respectively, and $\gamma_{\mathrm{c.m.}}=1/(1-\beta_{\mathrm{c.m.}}^2)^{1/2}$. In this experiment M_i is taken to be the pion mass and M_t is assumed to be the nucleon mass. A commonly used approximation of Eq. (3a) for $\gamma_{\mathrm{c.m.}}\gg1$ is

$$E_p\simeq 2M_t\gamma_{\mathrm{c.m.}}^2\,. \quad (3b)$$

Median-angle method (E_m). To find the velocity of a SS, β_S, let us assume that a secondary is emitted at an angle of $\frac{1}{2}\pi$ in the SS with the velocity β^*, and that its angle of emission in the LS becomes θ_m. The relation[1]

$$\tan\theta_m=\beta^*/\gamma_S\beta_S\,, \quad (4)$$

where $\gamma_S=1/(1-\beta_S^2)^{1/2}$, can be readily obtained. In the relativistic limit of $\beta_S/\beta^*\simeq1$,

$$\tan\theta_m\simeq 1/\gamma_S\,. \quad (5)$$

If the particles are emitted with equal numbers forward and backward in the SS, the median angle of the angular distribution in the LS may be taken as a measure of θ_m. Using γ_S obtained from Eq. (5), the estimate E_m can be obtained from Eq. (3a) by assuming $\beta_S=\beta_{\mathrm{c.m.}}$.

Castagnoli method (E_{Cast}). Let β^* be the velocity and θ^* the direction of a secondary in the SS, relative to the direction of the primary particle. The corresponding quantities in the LS are β and θ. The angular relationship between the direction of a secondary in the SS to that in the LS is given by

$$\tan\theta=\frac{\sin\theta^*}{\gamma_S(\cos\theta^*+\beta_S/\beta^*)}\,. \quad (6)$$

In the relativistic limit of $\beta_S/\beta^*\simeq1$, one obtains[2]

$$\ln\gamma_S\simeq-\langle\ln\tan\theta\rangle\,, \quad (7)$$

where the averaging procedure is carried out over the n_s charged secondaries in the LS. Thus the Lorentz factor γ_S of the SS can be obtained from Eq. (7), and the energy estimate E_{Cast} can be obtained from Eq. (3a) by assuming $\beta_S=\beta_{\mathrm{c.m.}}$.

$E(\theta)$ method. A kinematic parameter

$$\eta=\mathrm{arctanh}(\beta\cos\theta)$$

transforms to $\eta^*=\mathrm{arctanh}(\beta^*\cos\theta^*)$ by a Lorentz transformation as

$$\eta=\mathrm{arctanh}\beta_S+\eta^*\,, \quad (8)$$

where β or β^* is the velocity and θ or θ^* is the angle of emission in the LS or SS, respectively. From the definition of the SS, we can assume that $\langle\eta^*\rangle=0$. One then obtains[4]

$$\mathrm{arctanh}\beta_S=\langle\eta\rangle\,. \quad (9)$$

Equation (9) usually reduces to

$$\ln\gamma_S=-\langle\ln\tan\theta\rangle-\langle\ln[(1+x^2)^{1/2}/x]\rangle\,. \quad (10)$$

It is convenient to introduce the parameter $x=p_t/m$,[4] where p_t is the transverse momentum and m is the mass of the secondary particle. Equation (10) becomes a correction to Eq. (7) in the usual situation in which $x>1$. In place of the knowledge of values of β to calculate η of the secondaries, the rather well-known distribution of the transverse momentum is used. A parameter $\eta(\theta)$ ($\simeq0.46-\ln\tan\theta$), which depends only on θ, but which is consistent with the definition of η, is introduced. Then Eq. (9) is approximated by

$$\mathrm{arctanh}\beta_S\simeq\langle\eta(\theta)\rangle\,. \quad (11)$$

The $E(\theta)$ estimate can be obtained from Eq. (3a) by assuming that $\beta_S=\beta_{\mathrm{c.m.}}$ in Eq. (11).

Experimental Results

Estimates of the energy of the 318 interactions produced by the 20-GeV/c incident pions, as determined by the four methods of energy estimation described above, have been made. The dependence of these energy estimation methods on N_h and n_s has also been investigated. In particular, a comparison of these energy estimates has been made between different groupings of N_h and n_s. The groupings of N_h are 0–1, 2–4, 5–7, ≤5, >5, and ≥8. Two groupings of n_s were used, $n_s\geq4$ and $n_s\geq7$.

Shown in Fig. 2 is a comparison between the distributions of the logarithm of the energy of events with $n_s\geq4$ as estimated by E_{Cast}, E_{ch}, and $E(\theta)$ for different groupings of N_h. Properties of the logarithmic distributions[10] are used here because the logarithmic distributions are generally more nearly symmetrical and Gaussian in appearance than the linear distributions. Furthermore, the antilogarithm of the mean 10^m divided by the incident pion energy E_0 is the average factor by which the energy estimates are in error. The antilogarithm of the standard deviation σ, 10^σ, is the factor which defines the approximate 68% confidence interval for statistical fluctuations of individual estimates of the primary energy about the average factor. The solid vertical line in Fig. 2 is the mean of each distribution, and the dashed line corresponds to the known energy

[10] R. D. Settles and R. W. Huggett, Phys. Rev. **133**, B1305 (1964).

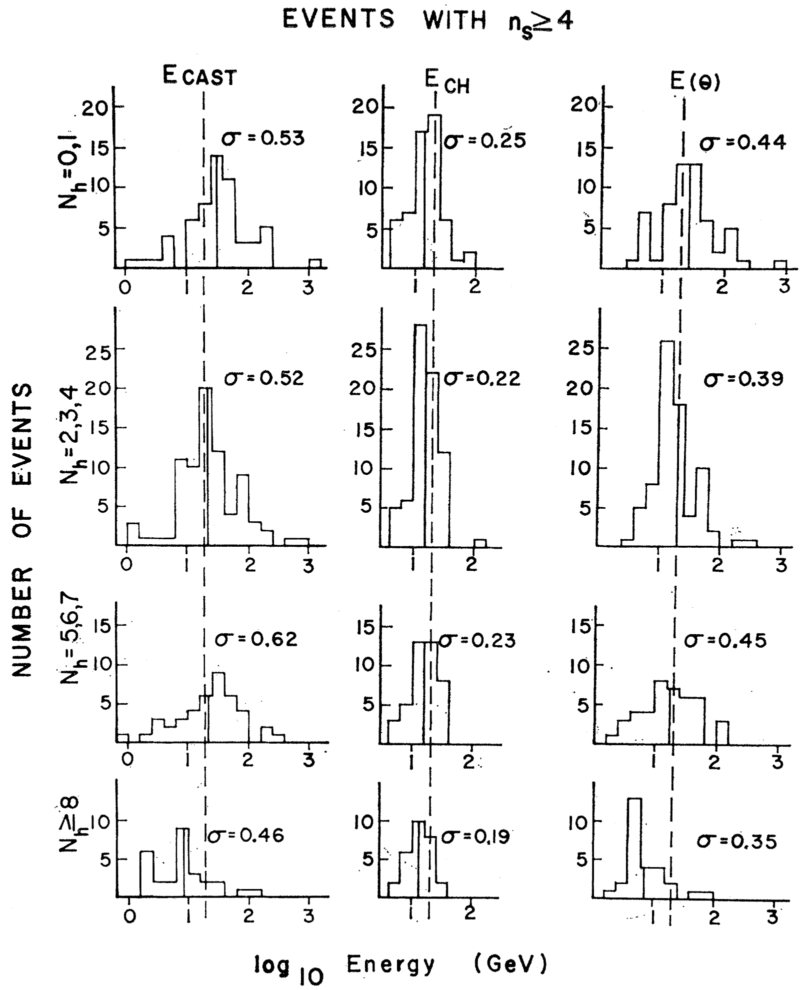

FIG. 2. Distribution of the logarithm of the energy as estimated by the methods of E_{Cast}, E_{ch}, and $E(\theta)$ for pion interactions with $n_s \geq 4$ at 20 GeV/c in photographic emulsion. The groupings of N_h are 0, 1; 2, 3, 4; 5, 6, 7; and ≥ 8. [The solid vertical line is the mean of each distribution and the dashed line corresponds to the known energy E_0 (20 GeV) of the incident pions.]

E_0 (20 GeV) of the incident pions. The standard deviation of each distribution is shown in Fig. 2 for each group of N_h and for each method of energy estimation. Table II gives a tabular comparison of the data shown in Fig. 2. For each grouping of N_h is indicated the number of interactions, the numerical value of the mean of the logarithm of the estimated energy (divided by the known primary energy E_0), the standard error of the mean, and the standard deviation.

A comparison between the four different methods of energy estimation is given in Table III for various groups of N_h with $n_s \geq 7$. The energy estimates for

TABLE II. Comparison of various methods of energy estimation[a,b] for $n_s \geq 4$.

N_h	Number of inter-actions	$\langle\log_{10}(E_{Cast}/E_0)\rangle$	$\langle\log_{10}(E_{ch}/E_0)\rangle$	$\langle\log_{10}[E(\theta)/E_0]\rangle$
0	36	0.21± 0.10	−0.10± 0.04	0.14± 0.08
		(0.56)	(0.26)	(0.47)
1	22	0.15± 0.10	−0.17± 0.05	0.04± 0.08
		(0.44)	(0.22)	(0.38)
0, 1	58	0.19± 0.07	−0.13± 0.03	0.10± 0.06
		(0.53)	(0.25)	(0.44)
2, 3, 4	79	0.05± 0.06	−0.10± 0.03	−0.02± 0.04
		(0.52)	(0.22)	(0.39)
5, 6, 7	42	0.03± 0.10	−0.12± 0.04	−0.06± 0.07
		(0.62)	(0.23)	(0.45)
≥ 8	28	−0.37± 0.09	−0.19± 0.04	−0.44± 0.07
		(0.46)	(0.19)	(0.35)

[a] The incident pion energy is E_0 (20 GeV).
[b] Each number in parentheses is the standard deviation of the distribution of individual quantities whose mean is given directly above.

TABLE III. Comparison of various methods of energy estimation[a] for $n_s \geq 7$.

N_h	Number of inter-actions	E_{Cast}	E_{ch}	$E(\theta)$	E_m(comp)	E_{Cast}(comp)
0	8	24_{-4}^{+5}	19_{-3}^{+3}	19_{-4}^{+4}	22±8	···
1	10	35_{-9}^{+11}	20_{-2}^{+2}	24_{-4}^{+9}	36±11	···
0, 1	18	30_{-5}^{+6}	20_{-2}^{+2}	21_{-4}^{+5}	29±7	31_{-5}^{+6}
2, 3, 4	36	14_{-2}^{+2}	18_{-1}^{+1}	13_{-1}^{+1}	18±3	15_{-2}^{+2}
5, 6, 7	21	22_{-4}^{+5}	19_{-2}^{+2}	16_{-3}^{+3}	32±7	25_{-4}^{+4}
≥8	20	7_{-1}^{+1}	14_{-1}^{+1}	6_{-1}^{+1}	9±2	8_{-1}^{+2}

[a] All energies are in GeV.

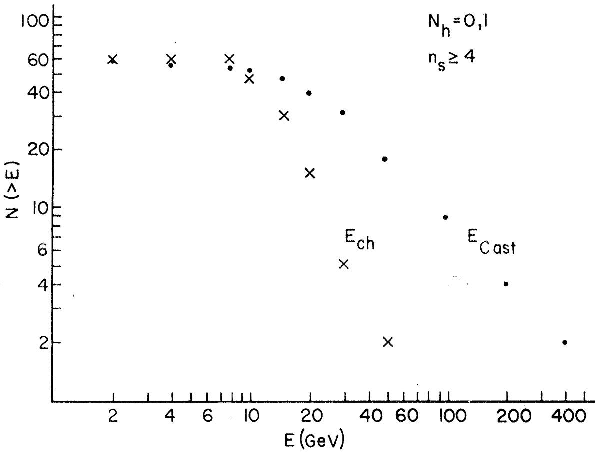

FIG. 3. Integral distribution of 20-GeV/c pion interactions in photographic emulsion with $N_h=0$, 1 and $n_s\geq4$, as estimated using the methods of E_{Cast} and E_{ch}.

E_{Cast}, E_{ch}, and $E(\theta)$ were obtained in a manner analogous to the analysis of the interactions with $n_s\geq4$, as shown in Fig. 2. The estimate E_m(comp) was obtained as a result of a composite distribution of all secondaries contained within each group of N_h. (The errors in E_m are based upon the number of interactions in each group of N_h.) Also given in Table III is the result of composite distributions for each group of N_h as estimated by E_{Cast}. In this estimate [as with E_m(comp)], E_{Cast}(comp) is obtained by taking all interactions within each group of N_h and treating the result as a single interaction. It is worthwhile to mention that the deletion of the smallest-angle track in the individual interactions with $N_h=0$, 1 in the composite distribution of E_{Cast}(comp) yields an energy estimate which, on the average, is in very good agreement with the incident energy.

A further comparison between E_m(comp), E_{Cast}, E_{ch}, and $E(\theta)$ is given in Table IV. The overestimates and underestimates of energy are shown for the two groups of N_h, $N_h\leq5$ and $N_h>5$, for all interactions with $n_s\geq4$.

TABLE IV. Comparison of overestimates and underestimates in energy estimation for $n_s\geq4$.

	E_{Cast}		E_{ch}		$E(\theta)$		E_m(composite)	
N_h	Over	Under	Over	Under	Over	Under	Over	Under
≤5	1.2	···	···	1.3	1.1	···	1.5	···
>5	···	2.0	···	1.5	···	2.3	···	1.3

An integral energy distribution is shown in Fig. 3 for all interactions with $N_h=0$, 1 and $n_s\geq4$. The wide range of energies obtained by individual estimates of the primary energy using the angular distribution of secondaries in the LS is readily seen in this figure. The comparison between the E_{ch} and E_{Cast} integral distributions is very similar to the 30-GeV/c proton results which were included in the ICEF data.[3]

IV. DEPENDENCE OF $\langle\langle\eta(\theta)\rangle\rangle$ ON N_h AND n_s

The averages of $\langle\eta(\theta)\rangle$, $\langle\langle\eta(\theta)\rangle\rangle$, as a function of N_h and n_s are shown in Table V for the 318 interactions produced by 20-GeV/c pions. (These interactions were analyzed as were the 30-GeV/c proton interactions in Ref. 4.) For comparison, $\langle\langle\eta\rangle\rangle$ should be 1.89 if the symmetric system has the velocity of the CMS, $\beta_{\text{c.m.}}$, for a collision of a 20-GeV/c pion with a target proton which is at rest in the LS. Essentially this kind of analysis shows why the large fluctuations of energy estimates by E_m, E_{Cast}, or $E(\theta)$, exist as a function of N_h and n_s.

The trends of $\langle\langle\eta(\theta)\rangle\rangle$ shown in Table V are

(a) $\langle\langle\eta(\theta)\rangle\rangle$ becomes larger as n_s decreases, and
(b) $\langle\langle\eta(\theta)\rangle\rangle$ becomes smaller as N_h increases.

These trends have been observed and discussed in Ref. 4 for 30.9-GeV proton interactions in photographic emulsion. The present study was an attempt to parametrize the above two trends for the interactions with even n_s and $n_s\leq8$. The study of the trends for the interactions with odd n_s and $n_s\leq7$ is given in the Appendix. By limiting the study to interactions with a small number

TABLE V. Dependence of $\langle\langle\eta(\theta)\rangle\rangle$ on n_s and N_h.

n_s \ N_h	0	1	2, 3	4, 5	6–8	9–15	≥16
1	1.7±0.1	3.9±0.1	3.5±0.3	2.5	2.2	···	···
2	2.8±0.2	2.9±0.5	3.0±0.4	2.4±0.3	2.8±0.9	−0.4	···
3	2.8±0.2	2.2±0.2	2.1±0.3	2.1±0.6	2.2±0.3	···	···
4	2.2±0.2	2.0±0.2	2.2±0.1	2.4±0.2	2.2±0.5	1.9	2.8
5	2.0±0.2	1.5±0.2	2.2±0.1	1.8±0.1	1.5±0.2	1.7±0.1	···
6	1.9±0.2	2.2±0.1	1.8±0.1	1.8±0.1	1.5±0.2	1.2±0.1	1.1
7	1.9±0.2	1.8±0.3	1.6±0.1	1.8±0.2	1.7±0.2	1.7±0.3	1.6
8	1.9±0.1	2.3±0.4	1.9±0.1	1.7±0.1	1.7±0.2	1.2±0.3	1.1
9	1.8±0.5	2.1±0.7	1.6±0.0	1.4±0.2	1.5±0.3	1.4	2.0±0.4
10	1.7	1.6	···	2.1±0.1	···	1.1	1.4
11	···	···	1.8±0.2	1.5±0.1	2.3	1.1	1.3
12	···	···	···	···	2.2	1.2	0.7
13	···	···	1.6	1.8	···	···	···
14	···	···	···	1.6±0.1	1.3±0.1	···	···
15	···	···	···	1.5	···	1.2	···

of produced particles, the analysis has been confined mainly to single pion-nucleon collisions. Successive collisions, if any occur in the same nucleus, would be expected to increase the number of particles produced. For the group of interactions with $n_s \geq 10$, one would expect to have effects of complex collisions. The four following plausible assumptions were made for the analysis of the 20-GeV/c interaction with even n_s:

(a) The target is a proton.

(b) Strong interactions do not depend on the charges involved in the interaction.

(c) (n_s-2) of the charged particles are produced pions. (The incident pion and the target proton are assumed to be included among the charged secondaries.)

(d) As in Ref. 4, a formula for $\langle\langle\eta(\theta)\rangle\rangle$ as a function of n_s in each grouping of N_h has the form

$$\langle\langle\eta(\theta)\rangle\rangle = A + B/n. \tag{12}$$

In Eq. (12), A and B are constants, and n is the total number of produced secondaries which may be deduced from the number of charged secondaries.

Now, with these assumptions,

$$n = \tfrac{3}{2}(n_s-2)+2 = \tfrac{3}{2}(n_s-\tfrac{2}{3}). \tag{13}$$

The constants, A and B, for the interactions with $n_s = 2$, 4, 6, and 8 and with $n_s = 4$, 6, and 8 were obtained by least-squares fits of the values of $\langle\langle\eta(\theta)\rangle\rangle$ to Eq. (12). Table VI lists the values of A and B for even n_s, one group for $n_s = 2$, 4, 6, and 8, and, inside the parentheses, for the group with $n_s = 4$, 6, and 8. The straight-line plots of Eq. (12) with the values of A and B obtained are shown in Fig. 4. The solid lines are for the interactions with $n_s = 4$, 6, and 8. The values of $[n_s/(n_s+1)]\langle\langle\eta(\theta)\rangle\rangle$ for the interactions with odd n_s are plotted as the unfilled circles shown in Fig. 4, in order to facilitate a comparison with the cases of even n_s which are plotted as the filled circles. The explanation for the values shown in Fig. 4 for odd n_s is given in the Appendix. The dependence of $\langle\langle\eta(\theta)\rangle\rangle$ on n_s and N_h is well expressed by Eq. (12). The values of A and B given in Table VI are to be compared with the corresponding values in Ref. 4. The obtained values of A, 1.7±0.1 and 1.9±0.2, for interactions of $N_h = 0$ and 1, respectively, agree well with the expected value of 1.89. As N_h increases, the value of A decreases, which was explained previously by the increase of the target mass.[4] The weighted average of the difference of the values of A between 20-GeV/c pion and 30-GeV/c proton interactions is 0.3±0.08. This value is very close to the difference of the arctanh$\beta_{\mathrm{c.m.}}$ between the pion and proton interactions, 0.21, where $\beta_{\mathrm{c.m.}}$ is the velocity of the CMS with respect to the LS. On the other hand, the averages of B, 1.5±0.5 and 2.8±0.8 for the proton and pion interactions, respectively, seem to disagree. If the same kind of homogeneity of emission angles between the incident pion and the target proton holds in the CMS, the value of B for the pion interactions (based on the 30.9-GeV proton interactions in Ref. 4) should have been 0.75±0.25. It appears that there is observed intrinsic breakdown of forward-backward symmetry in the CMS of the pion interactions.

The averages of the standard deviations σ of the $\eta(\theta)$ distributions, $\langle\sigma\rangle$, as a function of n_s and N_h are given in Table VII. As shown in Table VII, the values of $\langle\sigma\rangle$ are uniform and independent of n_s and N_h.

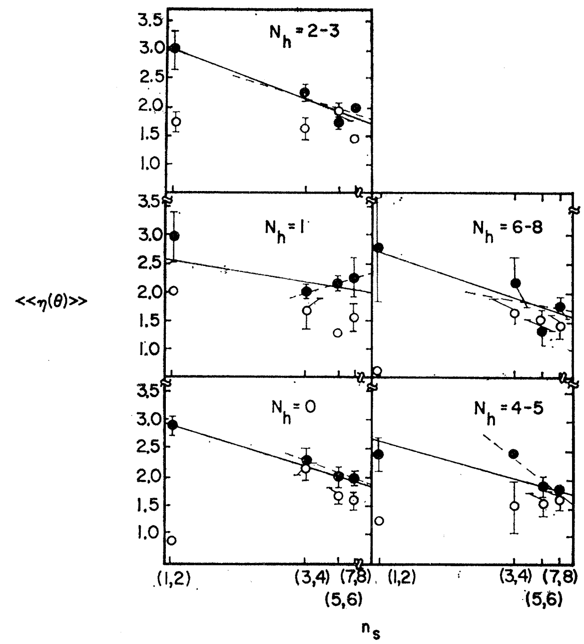

FIG. 4. Dependence of $\langle\langle\eta(\theta)\rangle\rangle$ of 20-GeV/c pion interactions on even n_s, according to the groupings of N_h of 0, 1, 2–3, 4–5, and 6–8. The filled circles are for even n_s; the unfilled circles are for the values of $[n_s/(n_s+1)]\langle\langle\eta(\theta)\rangle\rangle$ for odd n_s. The solid straight lines represent the least-squares fits for $n_s = 2$, 4, 6, and 8; correspondingly, the dashed lines are for the interactions with $n_s = 4$, 6, and 8.

TABLE VI. Results of least-squares fits of $\langle\langle\eta(\theta)\rangle\rangle$ of even n_s to a formula[a] $A+B/n$.

N_h	A	B	χ^2
0	1.70±0.10	2.28±1.80	0.20
	(1.68±0.23)	(2.67±2.80)	(0.28)
1	1.91±0.23	1.24±1.23	2.65
	(2.41±0.38)	(−2.12±2.41)	(0.02)
2, 3	1.53±0.13	2.86±0.82	4.20
	(1.57±0.21)	(2.53±1.62)	(4.13)
4, 5	1.58±0.11	2.45±0.68	6.63
	(1.16±0.18)	(5.86±1.74)	(1.35)
6–8	1.34±0.28	2.64±2.10	2.00
	(1.42±0.50)	(1.85±4.22)	(1.94)
Average		2.47±0.45	
		(2.84±0.83)	

[a] Numbers inside parentheses are those corresponding to fits with $n_s = 4$, 6, and 8; the other numbers correspond to fits with $n_s = 2$, 4, 6, and 8.

TABLE VII. Dependence of $\langle\sigma\rangle$ on n_s and N_h.

n_s \ N_h	0	1	2, 3	4, 5	6–8	9–15	≥16
1							
2	0.78±0.08	0.79±0.27	0.63±0.23	0.75±0.16	0.99±0.95	2.12	···
3	0.60±0.07	0.91±0.26	0.84±0.10	0.53±0.08	0.46±0.07	···	···
4	0.87±0.12	0.61±0.12	0.77±0.13	0.81±0.17	0.96±0.11	0.77	1.10
5	0.78±0.12	0.64±0.15	0.62±0.11	0.85±0.19	0.97±0.12	1.42	···
6	0.94±0.14	0.67±0.25	0.92±0.13	0.81±0.09	0.78±0.05	0.66±0.21	0.53
7	0.75±0.01	0.93±0.11	0.84±0.12	0.82±0.21	0.79±0.21	0.81±0.24	1.09
8	0.88±0.08	0.79±0.16	0.88±0.29	0.74±0.09	0.57±0.09	1.26±0.12	0.75
9	0.94±0.44	0.92±0.11	0.78±0.08	1.04±0.16	0.89±0.11	1.05	0.79±0.22
10	0.52	0.62	···	1.05±0.06	···	1.12	0.46
11	···	···	0.90±0.17	0.75±0.04	0.53	1.13±0.07	0.88±0.15
12	···	···	···	···	0.77	···	···
13	···	···	0.75	···	···	···	···
14	···	···	···	0.93±0.06	0.81	···	···
15	···	···	···	1.12	···	1.13	···

Discussion

E_{ch}. For all of the various groupings of N_h and n_s considered, E_{ch} yields more consistent and uniform results with smaller standard deviations than do any of the other methods. In Fig. 2 and Table II, one can see that there is only a small variation in the mean values of E_{ch} with N_h. The standard deviations of the E_{ch} distributions shown in Fig. 2 and those obtained in Table III are typically one-half that of the other methods. There is better agreement between E_{ch} and the incident pion energy E_0 for all the groupings of N_h with $n_s \geq 7$ than for $n_s \geq 4$. This is probably due to the fact that the charged-particle multiplicity for $n_s \geq 7$ is more closely related to the total particle multiplicity (including neutral particles). However, for all the groupings of N_h with $n_s \geq 4$ (as shown in Tables II and IV) the energy estimates of E_{ch} are approximately consistent with an underestimate of the incident pion energy by a factor of 1.5. This factor is in good agreement with the generally accepted practice of multiplying Eq. (2) by 1.5 to include the effect of neutral secondaries.

E_m. This method generally overestimates the energy E_0 as is shown in Table III. There is also considerable variation in E_m with N_h and n_s, the latter not shown. A composite distribution of all interactions with $n_s \geq 4$ and $N_h \leq 4$ yields a value for θ_m of 14.3±0.5 deg. This value agrees closely with the empirical relation, $\tan\theta_m = 0.67/\gamma_S$, suggested by Aly *et al.*[4] as a correction to Eq. (5). Composite distributions for the groups with $N_h \leq 5$, $n_s \geq 4$ and $N_h > 5$, $n_s \leq 4$ (as shown in Table IV) overestimate and underestimate the energy by factors of 1.5 and 1.3, respectively.

E_{Cast}. Generally, E_{Cast} consistently overestimates the energies of the incident pions. The values of E_{Cast} also show considerable variation with n_s and N_h, as can be seen by comparing Tables II–IV and by examining the groupings of N_h within these tables. The best agreement of E_{Cast} with E_0 is obtained in the group of events with $N_h \leq 5$ and $n_s \geq 4$, as is shown in Table IV. The overestimate by a factor of 1.2 for the group with $N_h \leq 5$, $n_s \geq 4$ and the underestimate by a factor of 2.0 for the group with $N_h > 5$, $n_s \geq 4$ are in good agreement with the results obtained by Lohrmann *et al.*[11]

$E(\theta)$. The energy estimates using $E(\theta)$ are generally an improvement over the estimates made by E_{Cast} for the groups with $N_h \leq 7$. The values of $E(\theta)$ do evidence the same general types of variations in the estimated energy with N_h and n_s as does E_{Cast}; however, these variations (for $N_h \leq 7$) are not as large. The groups with $N_h \leq 5$, $n_s \geq 4$ and $N_h > 5$, $n_s \geq 4$ (as shown in Table IV) are overestimates and underestimates of the energy by factors of 1.1 and 2.3, respectively.

V. CONCLUSIONS

The primary-particle energy values as estimated by E_m, E_{Cast}, and $E(\theta)$ are all subject to considerable variation with respect to N_h and n_s, as well as generally overestimating the energy. The principal areas of disagreement, however, are in those areas where the experimental data probably do not warrent being treated by these methods, namely, for low n_s and $N_h \geq 5$. For these cases it would appear that the data are not consistent with the assumptions of forward-backward symmetry in the CMS for pion-nucleon collisions. It does appear that $E(\theta)$ is an improvement over E_{Cast}, since in almost all cases there is better agreement with the incident pion energy.

The values of E_{ch}, however, show much less variation with n_s and N_h than do the other three methods. It is also significant that the standard deviations in the energy distributions for E_{ch} for the various groupings of N_h with $n_s \geq 4$ and $n_s \geq 7$ are considerably less than that of the other methods. The standard deviations of the distributions of E_{ch} are generally about one-half that of the other methods. Of course, the values of E_{ch} are subject to the assumption of the value of the transverse momentum as well as the constancy of that value and the constancy of the inelasticity. However, there is only slight variation in E_{ch} with N_h and n_s. The E_{ch} data do seem to show that the energy estimate is independent of

[11] E. Lohrmann, M. W. Teucher, and M. Schein, Phys. Rev. 122, 672 (1961).

the target mass and of secondary nuclear cascading inside the target nucleus. Since the transverse momentum and the inelasticity are approximately constant over a wide energy range, it would appear that E_{ch} may be a fairly reliable parameter for estimating the energies of primary particle interactions in photographic emulsion.

It should be mentioned that individual estimates of the incident pion energy by any of the methods are subject to large variations, as can be seen by examining Fig. 3. Estimates of the primary energy using E_{ch} vary from about 5 to 50 GeV for interactions with $N_h=0$, 1 and $n_s\geq 4$. Corresponding values for E_{Cast} can vary from about 7 to as high as 400 GeV.

From the analysis of $\langle\langle\eta(\theta)\rangle\rangle$ as a function of N_h and n_s, one can conclude that:

(a) The values of $\langle\langle\eta(\theta)\rangle\rangle$ for the same n_s and N_h groupings have the same trends as those of the proton-nucleon interactions of Ref. 4. This relationship can be expressed by the formula $\langle\langle\eta(\theta)\rangle\rangle=A+B/n$.

(b) The averages $\langle\sigma\rangle$ of the standard deviations of the $\eta(\theta)$ distributions are independent of the number of produced secondary particles.

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APPENDIX: DEPENDENCE OF $\langle\langle\eta(\theta)\rangle\rangle$ ON N_h AND ODD n_s FOR 20-GeV/c PION INTERACTIONS

In order to analyze the interactions with odd n_s, the following assumptions have been made:

(a) the target is a neutron,

(b) there is no charge exchange between the incident pion and the target neutron, and

(c) of the charged secondary particles, n_s-1 are created pions. Using the above assumptions, we have

$$n=\tfrac{3}{2}(n_s-1)+2=\tfrac{3}{2}(n_s+\tfrac{1}{3}).$$

It is necessary to include the effect of a surviving target neutron having an emission angle θ_n. This is because of the dominant role played by the surviving baryons and their possible correlation with the number of produced particles. Therefore

$$\frac{n_s\langle\langle\eta(\theta)\rangle\rangle+\eta(\tilde{\theta}_n)}{n_s+1}=A+\frac{2}{3(n_s+\frac{1}{3})}B, \qquad \text{(A1)}$$

where $\eta(\tilde{\theta}_n)$ is the most probable value of $\eta(\theta)$ for the emission angle of the target neutron, θ_n. Equation (A1) is coupled with Eqs. (12) and (13) to yield

$$\frac{\eta(\tilde{\theta}_n)}{n_s+1}=\langle\langle\eta(\theta)\rangle\rangle_{2\lambda}-\frac{n_s}{n_s+1}\langle\langle\eta(\theta)\rangle\rangle_{2\lambda-1}, \qquad \text{(A2)}$$

where $\langle\langle\eta(\theta)\rangle\rangle_{2\lambda-1}$ is the value of the average of $\langle\eta(\theta)\rangle$ for events with odd n_s. The corresponding average for the events with even n_s ($\lambda=1$, 2, 3, and 4) is $\langle\langle\eta(\theta)\rangle\rangle_{2\lambda}$. Table VIII lists the values of $\eta(\tilde{\theta}_n)/(n_s+1)$ for the cases of $\lambda=1$, 2, 3, and 4.

TABLE VIII. $\eta(\tilde{\theta}_n)/2\lambda$ deduced from "even-odd" differences of neighboring $\langle\langle\eta(\theta)\rangle\rangle$.

Combinations of n_s $(2\lambda)-(2\lambda-1)$ \ N_h	0	1	2, 3	4, 5	6, 7, 8
2–1	1.97±0.24	0.97±0.49	1.26±0.44	1.16±1.05	1.68±1.01
4–3	0.12±0.25	0.33±0.30	0.16±0.28	0.87±0.47	0.52±0.51
6–5	0.25±0.19	0.88±0.18	−0.11±0.16	0.26±0.24	0.23±0.35
8–7	0.28±0.15	0.70±0.48	0.49±0.13	0.19±0.17	0.21±0.26