Improvement of the Blankenbecler-Sugar Approximation*

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We present an approximate solution to the Bethe-Salpeter equation. It retains the attractive features of the Blankenbecler-Sugar approximation, while improving upon their results. For one-particle-exchange interactions, there is no increase in calculational difficulty with this technique.

SEVERAL years ago, Zemach and Schwartz[1] (ZS) presented an exact solution to the Bethe-Salpeter[2] (BS) equation for two scalar particles interacting via the exchange of a third scalar particle. These exact solutions, as a comparable standard, gave impetus to several investigators to find approximation techniques for solving the BS equation. The hope was to obtain a useful method of solution to physical problems. Several of these techniques have been referenced and compared by Cohen and Pagnamenta.[3]

One such approximation was presented by Blankenbecler and Sugar[4] (BBS). With the BBS approximation, the integral equation for the scattering of two particles is reduced from a four-dimensional to a three-dimensional integral equation. Projection of the angular momentum amplitudes further reduces this to an integral equation over a single variable, the magnitude of the 3-momentum carried by the internal particles (see Fig. 1).

In this paper, we present an approximation which is an improvement over the BBS method but is no more difficult to use. It is essentially a logical extension to an approximate technique we had presented earlier,[5] hereafter called CPT.

For simplicity, we consider the BS equation for the scattering of two scalar particles, both of mass μ, interacting via the exchange of a third particle of mass m. The partial-wave BS amplitude for this process satisfies

$$M_l(q_0,|\mathbf{q}|)=B_l(q_0,|\mathbf{q}|)+\frac{ig^2}{(2\pi)^4}\int d^4k \times\frac{P_l(\theta_{qk})M_l(k_0,|\mathbf{k}|)}{[(q-k)^2-m^2][(p+k)^2-\mu^2][(p-k)^2-\mu^2]}, \quad (1)$$

which is diagrammed in Fig. 1. The phase shifts are related to the on-shell amplitude by

$$M_l(s)=-4\pi[s(s-4\mu^2)]^{1/2}e^{i\delta}\sin\delta.$$

The mass-shell values of q_0 and $|\mathbf{q}|$ are found by setting $(p\pm q)^2=\mu^2$ to obtain $\hat{q}_0=0$, $\hat{q}^2=\frac{1}{4}(s-4\mu^2)$. We have set $M_l(s)=M_l(\hat{q}_0,|\hat{q}|)$.

The BBS approximation replaces the Green's function

$$G=[(p+k)^2-\mu^2]^{-1}[(p-k)^2-\mu^2]^{-1}$$

by a function which retains the two particle cut of G, i.e.,

$$E_2=2\pi\int_{4\mu^2}^{\infty}\frac{ds'}{s'-s}\delta[(p'+k)^2-\mu^2]\delta[(p-k)^2-\mu^2], \quad (2)$$

where $4p'^2=s'$. Then $G=iE_2+R$, where R contains no two-particle structure. R is ignored and G is replaced by

$$iE_2=i\pi\delta(k_0)/2\omega(k^2-\hat{q}^2),$$

where $\omega^2=k^2+\mu^2$. The $\delta(k_0)$ allows us to do trivially the dk_0 integral in Eq. (1), leaving a one-dimensional integral equation to solve.

The CPT approximation involves removing $M_l(k_0,|\mathbf{k}|)$ from the d^4k integral, assuming that $M_l(k_0,|\mathbf{k}|)$ does not vary much from its mass-shell value. An algebraic approximation results; i.e., we take

$$M_l(q_0,|\mathbf{q}|)\simeq B_l(q_0,|\mathbf{q}|)+\frac{ig^2}{(2\pi)^4}M_l(\hat{q}_0,|\hat{q}|)\int d^4k \times\frac{P_l(\theta_{qk})}{[(q-k)^2-m^2][(p+k)^2-\mu^2][(p-k)^2-\mu^2]}. \quad (3)$$

Neither the BBS nor the CPT approximations gives absurd results although the BBS method is markedly better.

In the approximation presented here, we use the spirit of the CPT approximation, keeping the BBS technique in mind. We allow $M_l(k_0,|\mathbf{k}|)$ to be approximated by $M_l(\hat{q}_0,|\mathbf{k}|)$; that is, we assume that the off-shell amplitude is well approximated by putting k_0 on shell (as in the BBS approximation) but we retain the

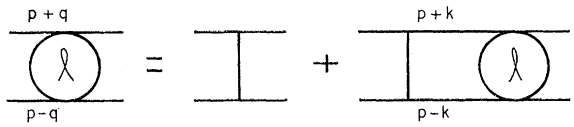

FIG. 1. Diagramatic representation of the partial-wave BS equation.

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[1] C. Zemach and C. Schwartz, Phys. Rev. **141**, 1454 (1966). See also C. Schwartz, *ibid.* **137**, B717 (1965); M. Levine, J. Tjon, and J. Wright, *ibid.* **154**, 1433 (1967), called LTW in the text.

[2] H. Bethe and E. Salpeter, Phys. Rev. **84**, 1231 (1951).

[3] H. Cohen and A. Pagnamenta, Phys. Rev. **181**, 2098 (1969).

[4] R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966).

[5] H. Cohen, A. Pagnamenta, and J. G. Taylor, Nuovo Cimento **50**, 796 (1967).

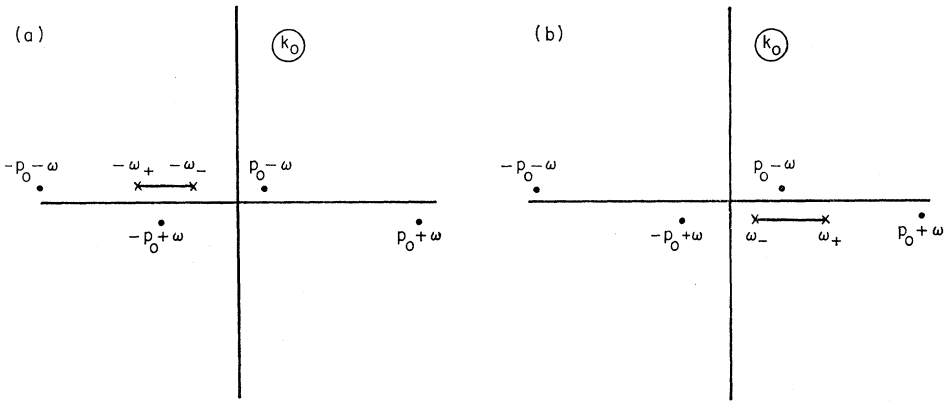

FIG. 2(a). Singularities of GL_+. (b) Singularities of GL_-.

full off-shell behavior of the known functions (as in the CPT method). As with the BBS approximation, we are left with a one-dimensional integral equation

$$M_l(q)\simeq B_l(q)+\frac{ig^2}{(2\pi)^4}\int_0^\infty k^2dk\,M_l(k)\times\int_{-\infty}^{\infty}dk_0\int d\Omega\frac{P_l(\theta)G(p+k,p-k)}{[(q-k)^2-m^2]},\quad(4)$$

but unlike BBS, we retain the full Green's function, rather than an approximate form.

For s waves, the integral in the kernel of Eq. (4) becomes (setting $q_0=0$)

$$I\equiv\frac{\pi}{qk}\int_{-\infty}^{\infty}dk_0\frac{\ln[(k_0^2-\omega_-^2+i\epsilon)/(k_0^2-\omega_+^2+i\epsilon)]}{[(p+k)^2-\mu^2][(p-k)^2-\mu^2]},\quad(5)$$

where $\omega_\pm^2=(q\pm k)^2+m^2$.

We point out that we are able to evaluate this integral explicitly. Since the singular nature of one-particle-exchange kernels is the same as will be described below, explicit evaluation of the approximate BS kernel will be possible for a wide class of physically interesting problems. If a more complicated potential is considered, the k_0 integral may have to be done numerically, but this added inconvenience may well be worthwhile.

Referring to Eq. (5), we note that the integral contains the elastic threshold $s=4\mu^2$. When $s\geqq 4^2$, the poles $k_0=-p_0+\omega-i\epsilon$ and $k_0=p_0-\omega+i\epsilon$ from the Green's function can coincide, pinching the k_0 contour. We also point out that the contour can be pinched by a Green's-function pole (e.g., $k_0=p_0-\omega+i\epsilon$) and a branch point from the logarithm (e.g., $k_0=\omega_\pm-i\epsilon$). This can occur for $s\geqq(2m+2\mu)^2$. So our approximation contains the two-particle production threshold as well.

To evaluate the integral of Eq. (5), we divide the logarithm term into two parts:

$$\ln\left(\frac{k_0^2-\omega_-^2+i\epsilon}{k_0^2-\omega_+^2+i\epsilon}\right)=\ln\left(\frac{k_0+\omega_--i\epsilon}{k_0+\omega_+-i\epsilon}\right)+\ln\left(\frac{k_0-\omega_-+i\epsilon}{k_0-\omega_++i\epsilon}\right)\equiv L_++L_-.$$

The singularities of GL_+ and GL_- in the complex k_0 plane are shown in Figs. 2(a) and 2(b), respectively. We see that if we close the contour in the lower half plane for GL_+ and in the upper half plane for GL_-, we avoid the cuts arising from the exchange terms. We can thus evaluate the integrals as sums of residues:

$$I=\frac{\pi}{qk}\int_{-\infty}^{\infty}dk_0(GL_+-GL_-)=\frac{\pi}{qk}(-2\pi i\sum R_-+2\pi i\sum R_+),$$

where R_+ and R_- are the residues of the Green's-function poles in the upper and lower half planes, respectively. We find

$$I=\frac{-i\pi^2}{2p_0\omega qk}\left[(p_0+\omega)\ln\left(\frac{\omega+\omega_--p_0}{\omega+\omega_+-p_0}\right)+(p_0-\omega)\ln\left(\frac{\omega+\omega_-+p_0}{\omega+\omega_++p_0}\right)\right]\Big/(p_0^2-\omega^2)\quad(6a)$$

and

$$I\to\frac{-i\pi^2}{qk\omega^2}\left[\frac{1}{\omega+\omega_-}-\frac{1}{\omega+\omega_+}-\frac{1}{\omega}\ln\left(\frac{\omega+\omega_-}{\omega+\omega_+}\right)\right]\quad\text{as } p_0\to 0.\quad(6b)$$

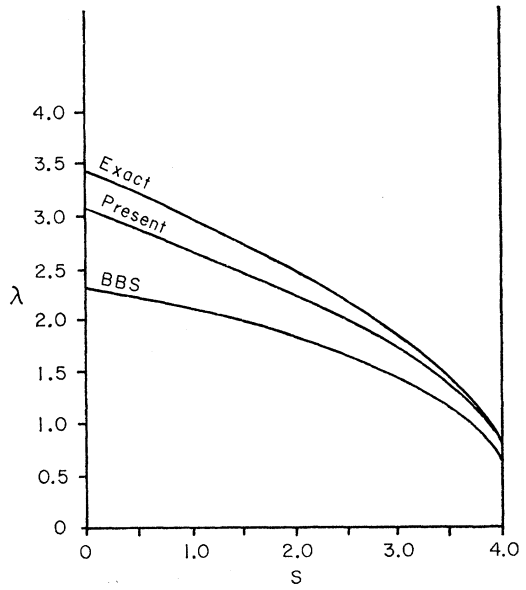

FIG. 3. λ-versus-s curves for the exact (BS), present, and BBS kernels.

Equation (4) now becomes

$$N(q)=C(q)+\frac{\lambda}{2p_0}\int_0^\infty dk\frac{N(k)}{\omega}\Big[(p_0+\omega)\ln\Big(\frac{\omega+\omega_--p_0}{\omega+\omega_+-p_0}\Big)+(p_0-\omega)\ln\Big(\frac{\omega+\omega_-+p_0}{\omega+\omega_+-p_0}\Big)\Big]\Big/(p_0^2-\omega^2+i\epsilon),\quad(7)$$

where $N(q)=qM_l(q)$, $C(q)=qB_l(q)$, and $\lambda=g^2/16\pi^2$.

Like BBS, our approximation makes a Wick rotation unnecessary since the k_0 integral can be done for all s.

To compare our approximation to the BBS approximation and the exact results, we present two sets of data. The first is a computation of the smallest eigenvalue of the kernal of Eq. (7) for several values of s, in the bound-state region, with the exchanged mass $m=\mu$. We have computed λ versus s for the BBS kernel as well. We approximated the integral in the homogeneous form of Eq. (7) by a sum and used a standard iteration technique[6] to find the largest λ^{-1}. We reproduce the BBS curve of Fig. 11 of LTW except at threshold. LTW seem to indicate that the BBS eigenvalue and the exact eigenvalue at threshold are essentially equal. We suspect that this is not correct. It would be unusual if the part of the Green's function neglected by the BBS approximation had no effect on the threshold value of λ. At threshold, we find $\lambda_{\rm BBS}=0.570$ while that obtained with the new approximation is $\lambda_{\rm new}=0.732$. This is to be compared with the exact threshold value of 0.762 reported by ZS.

We have also computed the s-wave phase shifts (again with $m=\mu$) for both the BBS and present approximate kernels. Referring to Eq. (7), with $p_0^2-\omega^2+i\epsilon=\hat{q}^2-k^2+i\epsilon$, we set

$$(\hat{q}^2-k^2+i\epsilon)^{-1}=P(\hat{q}^2-k^2)^{-1}-i\pi\delta(\hat{q}^2-k^2).\quad(8)$$

The principal-value integral was computed by dividing the integral into two parts:

$$\int_0^\infty=\int_0^{\hat{q}}+\int_{\hat{q}}^\infty.$$

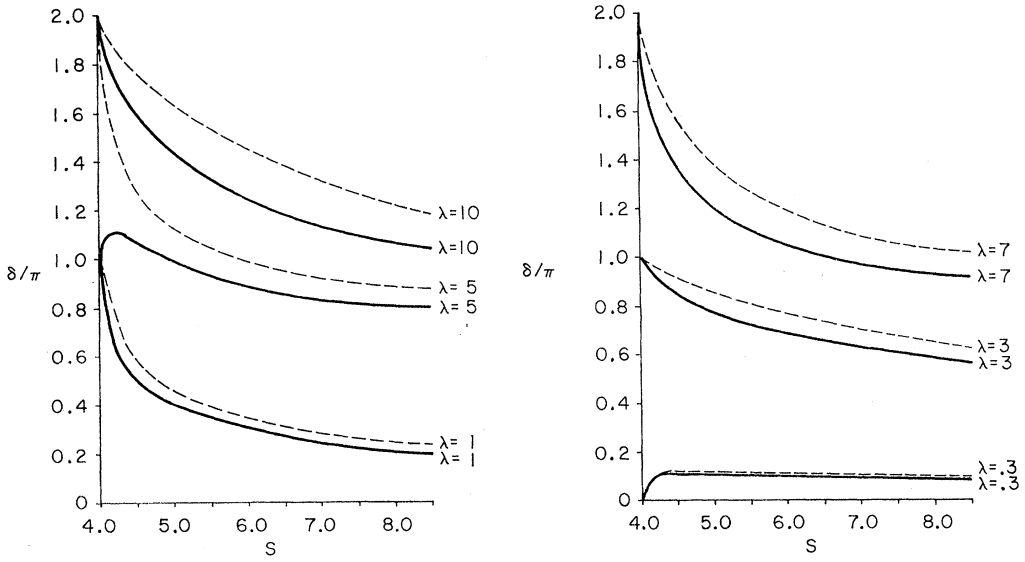

FIG. 4. s-wave phase shifts for various values of the coupling constant. The solid curves were obtained with the present approximation; the dashed curves are from the BBS approximation.

[6] See, e.g., R. Hamming, *Numerical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1962), p. 367.

Both regions of integration were then mapped into the region (0,1) by taking $k=\hat{q}x$ for $0\leqq k\leqq\hat{q}$, and $k=\hat{q}/x$ for $\hat{q}\leqq k\leqq\infty$. This is a slightly less complicated mapping than that used by LTW[7] and is quite satisfactory. By using Gaussian quadrature points for each piece of the integral, we never encounter endpoint values. We were able to reproduce the BBS phase shifts given in Fig. 8 of LTW. With this technique, we find that 10 integration points are sufficient for each piece of the integral.

The results of our new approximation seem to be indistinguishable from the exact results of Fig. 1 of LTW (or Fig. 3 of ZS). Comparing with the BBS results, we point out, in particular, the significant difference we obtain for the phase-shift structure for $\lambda=5$ in the region slightly above threshold.

The two aforementioned sets of data are presented in Figs. 3 and 4.

As a final note, we point out that Thompson[8] has recently proposed an alternative to the BBS approximation in the case of nucleon-nucleon scattering. He suggests that the amplitude and the exchange potential be approximated by their values at $k_0=\hat{q}_0=0$, but the Green's function be retained under the k_0 integral.

Applying this to our problem, we achieve the Thompson approximation by removing the logarithm from the integral in Eq. (5) with $k_0=0$. Thus, we obtain

$$I=\frac{2\pi}{qk}\ln\left(\frac{\omega_-}{\omega_+}\right)\int_{-\infty}^{\infty}\frac{dk_0}{[(p+k)^2-\mu^2][(p-k)^2-\mu^2]},\tag{9}$$

which is easily evaluated to give

$$I=\frac{-i\pi^2\ln(\omega_+/\omega_-)}{\omega qk(\omega^2-p_0^2)}.\tag{10}$$

With this approximation, the BS equation then becomes

$$N(q)=C(q)+\lambda\int_0^{\infty}dk\frac{N(k)\ln(\omega_+/\omega_-)}{\omega(\omega^2-p_0^2-i\epsilon)}.\tag{11}$$

Comparing this to the s-wave form of Eq. (3.1) of LTW, we see that this is exactly the BBS approximation.

[7] J. Wright (private communication).
[8] R. Thompson, Phys. Rev. D 1, 110 (1970), in particular, "Method II" on p. 113.

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Pion Production and the Algebraic Realization of Chiral Symmetry

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We treat single-pion production within the framework of the algebraic realization of chiral symmetry introduced by Weinberg. Our objective is to test the consistency of Weinberg's hypotheses and procedure and to see if further restrictions on the axial-vector helicity coupling and the mass matrices result. We conclude that the procedure is consistent, at least in the case of production of a single pion in pion-target collision processes.

I. INTRODUCTION

STRONG interactions are known to obey an approximate dynamic symmetry, chiral $SU(2)\otimes SU(2)$, which is realized by inhomogeneous transformations on the pion field, with the chiral-transformation properties of other general fields being determined by their isospin.[1] In an elegant paper, Weinberg[2] demonstrated how the dynamic chirality group $SU(2)\otimes SU(2)$ can yield algebraic consequences. The fundamental hypothesis of his procedure is that the scattering amplitude of a zero-mass pion on a target (stable) calculated in the tree approximation by means of a chiral-invariant Lagrangian should not behave at asymptotic energies worse than the true amplitude; i.e., since one knows that the individual behavior of the trees is, in fact, bad at high energies, the basic hypothesis demands that cancellations occur among those terms that have a bad behavior at high energies. The cancellation should occur among the terms coming from the various trees alone and should not involve the continuum, which does not contribute to the tree approximation. The high-energy behavior of the actual scattering amplitude is taken to be given by Regge-pole theory. Specifically, Weinberg

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