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Minimal Factorized Veneziano Amplitudes for the Reactions $A_1\pi \to A_1\pi$, $A_1\pi \to \pi\pi$, $\pi\pi \to \pi\pi$*

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Explicit minimal amplitudes are constructed for the reactions $A_1\pi \to A_1\pi$, $A_1\pi \to \pi\pi$, and $\pi\pi \to \pi\pi$. All amplitudes have correct Regge asymptotic behavior corresponding to the (degenerate) ρ-f trajectory dominant in all channels. Signature is systematically enforced to leading order so that the poles on the leading trajectory have the correct spin and parity. The parity-conserving helicity amplitudes for $A_1\pi \to A_1\pi$ factorize, and multichannel factorization for the three coupled amplitudes is enforced for the leading trajectory. Multichannel factorization requires the extension of the $A_1\pi \to \pi\pi$ amplitude, but the parameters in the new amplitude are all determined, except for two which can be identified with the (*independent*) $A_1\rho\pi$ couplings G_S and G_D. Two acceptable $A_1\pi \to A_1\pi$ amplitudes are found, each involving two arbitrary constants in addition to $f_\rho = f_{\rho\pi\pi}$, G_S, G_D, and the three $A_1A_1\rho$ couplings g_1, g_2, and g_3. These couplings are not constrained by the model except for one relation connecting G_D and g_3. These results are compared with the hard-pion analysis by Schnitzer and Weinberg.

I. INTRODUCTION

In the present paper we derive an *explicit* and complete "minimal" set of Veneziano amplitudes for the related reactions $A_1\pi \to A_1\pi$, $A_1\pi \to \pi\pi$, and $\pi\pi \to \pi\pi$. We assume these amplitudes to be dominated by the (degenerate) (ρ,f) trajectory. The resulting amplitudes have manifest crossing symmetry, signature, correct pole structure, and asymptotic behavior corresponding to the leading Regge trajectory. The factorization relations are satisfied by the leading trajectory for the parity-conserving helicity amplitudes in $A_1\pi \to A_1\pi$ and for the coupled-channel amplitudes $A_1\pi \to A_1\pi$, $A_1\pi \to \pi\pi$, and $\pi\pi \to \pi\pi$. The $A_1\rho\pi$ couplings G_S and G_D and the $A_1A_1\rho$ couplings g_1, g_2, and g_3 are aribtrary except for one relation connecting g_3 and G_D. [These constants are defined in Eq. (2.8).] Two additional arbitrary constants appear in the $A_1\pi \to A_1\pi$ amplitude, for which two acceptable solutions are found.

The complex relations required among the coefficients of the beta-function expansion of the amplitudes for $A_1\pi \to A_1\pi$ given in a previous paper[1] have been simplified and solved to eliminate all dependent quantities. In addition, the overly stringent signature requirements of I have been relaxed so that signature is now *systematically* enforced to leading order. In Sec. II we derive two explicit families of amplitudes for $A_1\pi \to A_1\pi$ in terms of the couplings $f_\rho = f_{\rho\pi\pi}$, G_S, G_D, g_1, g_2, g_3, and two other arbitrary constants, and note the constraint $G_D^2 = 4f_\rho g_3$. Next we ask whether the minimal $A_1\pi \to \pi\pi$ amplitude of I is compatible with multichannel factorization if we describe $\pi\pi$ scattering by the simple amplitude given by Lovelace[2] and by Shapiro.[3] As suggested by Whippman,[4] the multichannel factorization provides further strong constraints on the problem. Although the minimal $A_1\pi \to \pi\pi$ amplitude of I satisfies factorization, it requires the relation $G_S = \frac{1}{2}m_\rho^2 G_D$, so it is reasonable to make this amplitude more flexible by adding further terms (Sec. III). Assuming[5] that the ρ trajectory satisfied $\alpha(m_A^2) = \frac{3}{2}$, one finds that adding terms of the form $(a+bs+ct)\, B_{12}$, etc., does not lead to a new amplitude once all conditions are taken into account. However, adding terms like B_{22} times a quadratic polynomial leads to the basic amplitudes (3.8), which contain five arbitrary constants in addition to the (unconstrained) coupling constants f_ρ, G_S, and G_D. In Sec. IV it is found that multichannel factorization allows us to eliminate all five arbitrary constants from the $A_1\pi \to \pi\pi$ amplitude. Therefore, the full set of amplitudes involves two arbitrary constants in addition to the relevant couplings among A_1, ρ, and π, which are constrained only by the aforementioned relation connecting G_D and g_3.

Since the axial-vector and vector currents are intimately related to the A_1, π, and ρ mesons, it is of interest to compare our results with those of "hard-pion" calculations of Schnitzer and Weinberg (SW)[6] and others. Our couplings have only one constraint (which can no doubt be removed by adding more terms) so there can be no contradiction. However, since the "smoothness" assumption eliminates the coupling g_3, our amplitude would have to have $G_D = 0$ ($\delta = 0$ in the notation of Ref. 6) which does not lead to acceptable experimental results. This mild conflict can

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[1] P. Carruthers and F. Cooper, Phys. Rev. D 1, 1223 (1970), hereafter referred to as I.

[2] C. Lovelace, Phys. Letters 28B, 264 (1968).

[3] J. Shapiro, Phys. Rev. 179, 1345 (1969).

[4] M. Whippman, Phys. Rev. D 1, 701 (1970).

[5] This is an assumption which cannot be derived from the Adler condition unless *ad hoc* assumptions are made concerning the Veneziano amplitude. Similarly, the Lovelace-Shapiro amplitude (Refs. 2 and 3) for $\pi\pi$ scattering amounts to assuming that $\alpha(m_\pi^2) = \frac{1}{2}$. See Ref. 1 for a critical discussion. In the present work we analyze our equations using the relations $m_A^2 = 2m_\rho^2$, $m_\pi^2 = 0$.

[6] H. J. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967), hereafter referred to as SW.

be equally well regarded as due to the *ad hoc* character of the smoothness approximation or the *ad hoc* omission of "higher" terms in our Veneziano amplitudes.

A number of papers[7–10] have purported to show that Veneziano amplitudes predict or give results identical to various schemes such as ρ-meson universality, chiral symmetry, hard-pion results, etc. These claims have been based on amplitudes too simple to satisfy the general criteria given in the first paragraph. (Similar remarks[1] apply to the "mass quantization rules" of Ref. 11.) Our considered opinion is that the Veneziano amplitude first of all has to satisfy these basic criteria and that secondly it should be flexible enough to accommodate such desirable features as universal ρ couplings, etc. Indeed, the problem in developing Veneziano amplitudes for spinning particles has mainly been to reduce the number of arbitrary constants to a tractable number. In the present work we have been able to reduce the number of constants to six, having begun with about 50 constants. In order to perform this reduction, it has been essential to study in systematic detail the far-reaching algebraic relations arising from the basic constraints. We also emphasize that analysis of only part of the problem almost inevitably leads to overconstrained or inconsistent amplitudes.

The end result is a set of model amplitudes parametrized in terms of several important couplings, the trajectory slope, and two extra constants. We do not have a "theory" of these reactions, nor do we satisfy unitarity. The Pomeranchuk trajectory has no place in this model. In view of the large number of intricate physical conditions met by our "minimal" amplitudes, the latter may be regarded as fairly economical in the number of required constants.

In addition to the papers already mentioned, or referred to in I, we call attention to the study of $\rho\pi\to\rho\pi$ by Abers and Teplitz[12] and the reaction $\omega\pi\to\omega\pi$ analyzed by the authors.[13] Other works concerned with the scattering of vector and scalar particles are given in Refs. 14–17. References 18–20 are concerned with possible relations between chiral symmetry, partial conservation of axial-vector current (PCAC), and the Veneziano model.

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II. VENEZIANO AMPLITUDE FOR $A_1\pi\to A_1\pi$

In a previous paper[1] we constructed a Veneziano amplitude for $A_1\pi$ scattering satisfying a large set of physical requirements. Here, we complete the analysis of that amplitude, simplifying and revising the previous results. (For details of the notation and conventions employed, see I.)

We review the construction of the amplitude. Crossing symmetry enables us to write s- and t-channel isospin amplitudes for $A_1\pi$ scattering on the mass shell in terms of eight independent functions. We take these to be the s- and t-channel $I=2$ amplitudes (four for each channel). This choice is convenient since the $I=2$ Veneziano amplitudes have no direct-channel poles in the (assumed) absence of exotic resonances. Invariant amplitudes having definite isospin in the s channel may then be written

$$\begin{aligned} T_i^0(s,t,u)&=\tfrac{1}{2}[3\epsilon_i f_i^s(t,s)+3f_i^t(s,u)-f_i^s(t,u)],\\ T_i^1(s,t,u)&=\epsilon_i f_i^s(t,s)-f_i^t(s,u),\\ T_i^2(s,t,u)&\equiv f_i^s(t,u). \end{aligned}\tag{2.1}$$

The t-channel isospin amplitudes are

$$\begin{aligned} T_i^0(t,s,u)&=\tfrac{1}{2}[3f_i^s(t,u)+3\epsilon_i f_i^s(t,s)-f_i^t(s,u)],\\ T_i^1(t,s,u)&=\epsilon_i f_i^s(t,s)-f_i^s(t,u),\\ T_i^2(t,s,u)&\equiv f_i^t(s,u). \end{aligned}\tag{2.2}$$

$\epsilon_i=+1$ $(i=1,2,4)$; $\epsilon_i=-1$ $(i=3)$. The only consequence of crossing symmetry not explicit in (2.1) and (2.2) is the Bose symmetry requirement $f_i^t(s,u)=\epsilon_i f_i^t(u,s)$.

Examination of the parity-conserving helicity amplitudes[21] reveals restrictions on the asymptotic behavior of certain linear combinations of the invariant amplitudes necessary if they are to be dominated by the positive normality ρ-f trajectories. These restrictions simultaneously ensure that the leading trajectory has the correct spin-parity content and eliminate its wrong-parity partner. The expected asymptotic behavior of the invariant amplitudes is

$$\begin{aligned} s\text{ channel:}\quad & T_1,\ T_2-T_3,\ T_3-T_4\sim t^{\alpha(s)-1},\\ & 2T_3-T_2-T_4\sim t^{\alpha(s)-2}; \end{aligned}\tag{2.3}$$

$$\begin{aligned} t\text{ channel:}\quad & T_1,\ T_4\sim s^{\alpha(t)},\\ & T_2\sim s^{\alpha(t)-2},\\ & T_3\sim s^{\alpha(t)-1}. \end{aligned}\tag{2.4}$$

These restrictions do not fully determine the allowed behavior of the individual T_i which must, however,

[21] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).

obey the Regge bounds

$$\begin{aligned} T_1 &\sim t^{\alpha(s)-1}, & s^{\alpha(t)} \\ T_2 &\sim t^{\alpha(s)}, & s^{\alpha(t)-2} \\ T_3 &\sim t^{\alpha(s)}, & s^{\alpha(t)-1} \\ T_4 &\sim t^{\alpha(s)}, & s^{\alpha(t)}. \end{aligned} \tag{2.5}$$

The additional restrictions on linear combinations of the T_i contained in (2.3) relative to (2.5) are what we termed "conspiracies" in I and bear no relation to the usual conspiracies between Regge trajectories. Since (2.3)–(2.5) must be satisfied for all isospins, $\epsilon_i f_i^s(t,s)$ and $f_i^t(s,u)$ must separately have the listed asymptotic behavior.

We construct a Veneziano-model amplitude for $A_1\pi$ scattering which is free of ancestors and double poles by writing f_i^s and f_i^t as sums of (polynomials)×(beta functions). Our initial amplitudes are the most general Bose symmetric amplitudes having the asymptotic behavior of (2.5) and constructed from B_{11}, B_{12}, and B_{21},[22] multiplied by polynomials linear in s, t, and u. Such a set of amplitudes was given in I, but 16 of the 46 constants occurring in the previous amplitudes may be eliminated,[23] with *no* loss of generality, by exploiting the identity $B_{11}=B_{12}+B_{21}$. We defer writing the explicit form of these amplitudes until a later stage of the analysis.

Requiring the leading trajectory to have proper signature, we find that the following linear combinations of invariant amplitudes must have odd signature to the indicated orders:

$$\begin{aligned} &T_2,\ T_3,\ T_4 \sim t^{\alpha(s)}, \\ &T_1,\ T_2-T_3,\ T_3-T_4 \sim t^{\alpha(s)-1}, \\ &2T_3-T_2-T_4 \sim t^{\alpha(s)-2}. \end{aligned} \tag{2.6}$$

Although we have not investigated this question in general, proper signature for the present t-channel amplitudes is guaranteed by Bose symmetry in conjunction with the asymptotic conditions (2.3) and (2.4). In I, we further required T_i ($i=2$, 3, 4) to have odd signature to order $t^{\alpha(s)-1}$. This was done to simplify the algebra. Subsequent analysis has shown that this assumption, combined with the factorization conditions given in I, leads to a vanishing d-wave $A\rho\pi$ coupling constant G_D when we normalize our amplitudes to the ρ pole. For the sake of greater generality, we drop this (unnecessary) restriction.

Working out the asymptotic expansions of our amplitudes (in the mild approximation $a=bm_\rho^2=\frac{1}{2}$) and imposing the conspiracy and signature conditions (2.3), (2.4), and (2.6) yields 15 independent relations among the remaining 30 parameters:

$$\begin{aligned} &a_{11}^t=-a_{12}^s+(1/4b)(b_{13}^s-b_{12}^s), \\ &b_{12}^t=-\tfrac{1}{2}(b_{12}^s+b_{13}^s), \\ &b_{21}^s=b_{43}^s=-b_{32}^s=b_{22}^t=c_{32}^t=b_{42}^t, \\ &c_{22}^t=-c_{42}^t, \\ &c_{32}^s=-2c_{22}^t-b_{33}^s, \\ &c_{42}^s=4b_{21}^s, \\ &c_{43}^s=4c_{22}^t-b_{42}^s, \\ &a_{43}^s=-2a_{32}^s-a_{21}^s+(1/b)(b_{33}^s+\tfrac{1}{2}b_{23}^s+\tfrac{1}{2}b_{42}^s), \\ &a_{32}^t=a_{21}^t+a_{21}^s+a_{32}^s-(1/2b)(b_{32}^t+b_{23}^s+c_{22}^t+b_{33}^s), \\ &a_{41}^t=a_{21}^t+2a_{21}^s+2a_{32}^s-(1/b)(b_{23}^s+c_{22}^t+b_{33}^s), \\ &a_{33}^s=2(a_{21}^t+a_{21}^s+a_{32}^s)-\tfrac{1}{2}a_{42}^s \\ &\qquad -(1/b)(\tfrac{3}{2}b_{33}^s+b_{23}^s+\tfrac{1}{4}b_{42}^s+c_{22}^t). \end{aligned} \tag{2.7}$$

Here a_{12}^s, a_{13}^s, a_{21}^t, a_{21}^s, a_{32}^s, a_{42}^s, b_{21}^s, b_{12}^s, b_{13}^s, b_{23}^s, b_{32}^t, b_{33}^s, b_{42}^s, c_{13}^s, and c_{22}^t are to be regarded as the independent variables.

Define $A_1\rho\pi$, $A_1A_1\rho$, and $\rho\pi\pi$ couplings by means of the effective Lagrangian densities:

$$\begin{aligned} &\mathcal{L}_{\rho\pi\pi}=f_\rho\epsilon_{abc}\rho_\mu{}^a\pi^b\partial^\mu\pi^c, \\ &\mathcal{L}_{A\rho\pi}=G_S\epsilon_{abc}\pi^aA_\mu{}^b\rho^{c\mu}+G_D\epsilon_{abc}\pi^a\partial_\mu A_\nu{}^b\partial^\nu\rho^{c\mu}, \\ &\mathcal{L}_{AA\rho}=\epsilon_{abc}(g_1A_\mu{}^a\partial_\nu A^{b\mu}\rho^{c\nu}+g_2A_\mu{}^a\partial^\mu A_\nu{}^b\rho^{c\nu} \\ &\qquad +g_3\partial^\lambda A_\mu{}^a\partial^\mu A_\nu{}^b\partial^\nu\rho_\lambda{}^c). \end{aligned} \tag{2.8}$$

These can be used to calculate Born terms for $A_1\pi\to A_1\pi$ and $A_1A_1\to\pi\pi$ (given in I). Matching these Born terms with the spin-1 portions of the Veneziano amplitudes at the ρ pole, we can relate the conventional coupling constants to the Veneziano parameters with the results

$$\begin{aligned} &G_S^2=-[a_{12}^s+(1/2b)b_{12}^s]/b, \quad G_D^2=2b_{21}^s/b, \\ &2G_SG_D=[a_{21}^s+a_{32}^s-(1/2b)(b_{23}^s+b_{33}^s+2c_{22}^t)]/b, \\ &f_\rho g_1=c_{13}^s/2b, \quad f_\rho g_3=b_{21}^s/2b, \\ &f_\rho g_2=-[2(a_{21}^t+a_{21}^s+a_{32}^s)-\tfrac{1}{2}a_{42}^s \\ &\qquad -(1/b)(b_{33}^s+b_{23}^s+\tfrac{1}{4}b_{42}^s+c_{22}^t)]/4b. \end{aligned} \tag{2.9}$$

Here the relations (2.7) have been used to cast these results in terms of independent Veneziano parameters. Note that we have the new relation

$$G_D^2=4f_\rho g_3. \tag{2.10}$$

In I it was shown that the Adler consistency condition[24] leads to the constraint

$$a_{12}^s=a_{13}^s+(c_{13}^s-b_{12}^s)/b, \tag{2.11}$$

provided that we drop terms of order m_π^2/m_ρ^2 and make the approximation $m_A^2=2m_\rho^2$ (in addition to the usual ones $a=bm_\rho^2=\frac{1}{2}$). The same result continues to hold here. In I, we (inadvertently) imposed odd signature on

[22] $B_{mn}(s,t)=\Gamma(m-\alpha(s))\Gamma(n-\alpha(t))/\Gamma(m+n-\alpha(s)-\alpha(t))$.

[23] Our initial amplitudes may be obtained from those in Eqs. (4.6) of I by putting $a_{11}^s=b_{11}^s=c_{11}^s=b_{11}^t=a_{22}^s=b_{22}^s=c_{22}^s=b_{21}^t=a_{31}^s=b_{31}^s=c_{31}^s=a_{31}^t=a_{41}^s=b_{41}^s=c_{41}^s=b_{41}^t=0$.

[24] S. L. Adler, Phys. Rev. **137**, B1022 (1965).

the amplitude T_1 to order $t^{\alpha(s)-2}$. This, seemingly minor, specialization gives two more relations between the parameters entering T_1 [in addition to (2.11) and the first two of Eqs. (2.7)], and leads to the relation $G_S{}^2=\frac{4}{3}m_\rho{}^2 f_\rho g_1$. This result is discussed (and rejected) in Sec. V.

Next we investigate the consequences of requiring that the helicity amplitudes for $A_1\pi \to A_1\pi$ factorize for the leading trajectory. The factorization conditions here take the simple form

$$\begin{aligned} C^2&=A_2{}^s(8bA_1{}^s+J)\,,\\ 2CD&=A_2{}^s(8bB_1{}^s+H-J)\,,\\ D^2&=-A_2{}^sH\,,\end{aligned}\tag{2.12}$$

(see I for the more general form and a description of the notation) since $E=F=G=0$ follows from Eqs. (2.7). Adding these three equations yields $(C+D)^2=8bA_2{}^s(A_1{}^s+B_1{}^s)$, which simply enforces the consistency of the first three of Eqs. (2.9). Using $H=4b_{21}{}^s/b$, the third of Eqs. (2.12) becomes $(b_{21}{}^s)^2=(c_{22}{}^t)^2$ or $b_{21}{}^s=\pm c_{22}{}^t$. This sign ambiguity results in two families of Veneziano amplitudes, each satisfying all our requirements, and each reducing the solution of I when more stringent signature requirements are imposed. We no longer have $D=\frac{3}{2}A_2{}^s$ as in I, but the preceding result implies $D=\pm 2A_2{}^s$, and the final factorization condition becomes $4C=\pm(8bB_1{}^s+H-J)$. The implications of factorization may then be summarized by case 1:

$$\begin{aligned} c_{22}{}^t=b_{21}{}^s &\Rightarrow 2(a_{21}{}^s+a_{32}{}^s)+a_{21}{}^t-(b_{23}{}^s+b_{33}{}^s)/b\\ &\quad=2(b_{12}{}^s+b_{13}{}^s)+b_{21}{}^s/b\,;\end{aligned}$$

case 2: (2.13)

$$c_{22}{}^t=-b_{21}{}^s \Rightarrow a_{21}{}^t=2(b_{11}{}^s+b_{13}{}^s)\,.$$

The eight relations contained in (2.9), (2.11), and (2.13) may now be used to eliminate further of the original 30 parameters. Making extensive use of identities such as $(m_\rho{}^2-s)B_{12}(s,t)=(m_\rho{}^2-u)B_{21}(s,t)$, we find that the resulting amplitudes depend on two independent combinations of parameters rather than the anticipated seven $(30-15-8=7)$. This is not surprising since the expansion (polynomials)×(beta functions), although useful for calculational purposes, tends to mask the true number of independent parameters due to the presence of such identities. (Similar remarks apply to the usual Veneziano expansion in terms of generalized beta functions.) The resulting amplitudes may be written

$$\begin{aligned}
f_1{}^s(t,s)&=[-2b^2G_S{}^2s]B_{12}(s,t)+\{-(2bG_S{}^2+2f_\rho g_1)+[\tfrac{1}{8}G_D{}^2+2b^2G_S{}^2\pm bG_SG_D-bf_\rho g_2\\
&\qquad+\tfrac{1}{8}(a_{42}{}^s+m_\rho{}^2b_{42}{}^s)]t+2bf_\rho g_1 s\}B_{21}(s,t)\,,\\
f_1{}^t(s,u)&=\{bG_S{}^2+\tfrac{1}{2}[\tfrac{1}{8}G_D{}^2\pm bG_SG_D-bf_\rho g_2+\tfrac{1}{8}(a_{42}{}^s+m_\rho{}^2b_{42}{}^s)]\times(m_\rho{}^2-s)\}B_{12}(s,u)+(s\leftrightarrow u)\,,\\
f_2{}^s(t,s)&=\{[2bG_SG_D-(a_{32}{}^s-m_\rho{}^2b_{33}{}^s)\pm\tfrac{1}{2}G_D{}^2]+\tfrac{1}{2}bG_D{}^2t\}B_{12}(s,t)\,,\\
f_2{}^t(s,u)&=\{[-2bf_\rho g_2-2bG_SG_D+\tfrac{1}{4}(a_{42}{}^s+m_\rho{}^2b_{42}{}^s)+\tfrac{1}{2}bG_D{}^2u\pm\tfrac{1}{2}bG_D{}^2](s-m_\rho{}^2)\}B_{12}(s,u)+(s\leftrightarrow u)\,,\\
f_3{}^s(t,s)&=[(a_{32}{}^s-m_\rho{}^2b_{33}{}^s)-\tfrac{1}{2}bG_D{}^2t\mp bG_D{}^2s]B_{12}(s,t)-4bf_\rho g_2B_{21}(s,t)\,,\\
f_3{}^t(s,u)&=[-2bf_\rho g_2+\tfrac{1}{4}(a_{42}{}^s+m_\rho{}^2b_{42}{}^s)+\tfrac{1}{2}bG_D{}^2u]B_{12}(s,u)-(s\leftrightarrow u)\,,\\
f_4{}^s(t,s)&=\{[-2bG_SG_D-(a_{32}{}^s-m_\rho{}^2b_{33}{}^s)\mp\tfrac{1}{2}G_D{}^2]+\tfrac{1}{2}bG_D{}^2t\pm 2bG_D{}^2s\}B_{12}(s,t)+[(a_{42}{}^s+m_\rho{}^2b_{42}{}^s)+2bG_D{}^2s]B_{21}(s,t)\,,\\
f_4{}^t(s,u)&=\{[2bG_SG_D-2bf_\rho g_2+\tfrac{1}{4}(a_{42}{}^s+m_\rho{}^2b_{42}{}^s)]+\tfrac{1}{2}bG_D{}^2u\pm\tfrac{1}{2}bG_D{}^2(m_\rho{}^2-s)\}B_{12}(s,u)+(s\leftrightarrow u)\,.
\end{aligned}\tag{2.14}$$

Upper and lower signs refer to cases 1 and 2 of (2.13), respectively. The two arbitrary parameters are $(a_{32}{}^s-m_\rho{}^2b_{33}{}^s)$ and $(a_{42}{}^s+m_\rho{}^2b_{42}{}^s)$. The coupling constants G_S, G_D, f_ρ, g_1, and g_2 are unrestricted, while g_3 is given by (2.10).

III. VENEZIANO AMPLITUDE FOR $A_1\pi\to\pi\pi$

A simple Veneziano amplitude for the reaction $A_1\pi\to\pi\pi$ was constructed in I. Here we present a more general amplitude which allows factorization between a simple $\pi\pi\to\pi\pi$ amplitude and the $A_1\pi\to A_1\pi$ amplitude of Sec. II, but does not require the special relation $G_S\cong\frac{1}{2}m_\rho{}^2G_D$ found previously.

Crossing symmetry may be used to express the six invariant amplitudes for $A_1\pi\to\pi\pi$ (two for each isospin) in terms of two independent functions $g(t,u)$ and $g'(t,u)$. The s-channel isospin amplitudes are

$$\begin{aligned}
M_2(s,t,u)&=g(t,u)\,,\\
M_1(s,t,u)&=\tfrac{1}{2}[g(t,s)-g'(t,s)+2g(s,u)]\,,\\
M_0(s,t,u)&=\tfrac{1}{2}\{\tfrac{3}{2}[g(t,s)-g'(t,s)]-3g(s,u)-g(t,u)\}\,,\\
M_2{}'(s,t,u)&=g'(t,u)\,,\\
M_1{}'(s,t,u)&=-\tfrac{1}{2}[3g(t,s)+g'(t,s)+2g'(s,u)]\,,\\
M_0{}'(s,t,u)&=-\tfrac{1}{2}\{\tfrac{3}{2}[3g(t,s)+g'(t,s)]-3g'(s,u)+g'(t,u)\}\,.
\end{aligned}\tag{3.1}$$

We found previously that a satisfactory solution, satisfying crossing symmetry, proper asymptotic behavior, and the Adler self-consistency condition, could be obtained by allowing only the single beta function $B_{11}(t,u)$ to occur in $g(t,u)$ and $g'(t,u)$. At that time, we suggested that a more general amplitude constructed from B_{12} and B_{21} could probably satisfy these same restrictions without producing $G_S/G_D \cong \frac{1}{2}m_\rho^2$. This turns out not to be the case. Such an amplitude certainly contains the previous one as a special case and, before the various restrictions are imposed, contains twice as many parameters as the previous solution (12 versus 6). However, the beta functions $B_{11}(s,u)$ and $B_{11}(s,t)$ automatically vanish at the Adler point $(s,t,u)=(m_A^2,0,0)$ owing to the (approximate) relations $\alpha(m_A^2)=\frac{3}{2}$, $\alpha(0)=\frac{1}{2}$, with the consequence that requiring the helicity amplitudes to vanish at this point gives just one restriction on the model. For amplitudes constructed from B_{12} and B_{21}, the Adler condition instead yields three relations [cf. with (3.6)] which eventually lead us to recover precisely the result of I with no gain in generality.

Consider instead the model arising from

$$g(t,u)=(a_1+b_1t+c_1u)B_{12}(t,u)+(a_2+b_2t+c_2u)B_{21}(t,u)+(a_3+b_3t+c_3u+d_3tu+e_3t^2+f_3u^2)B_{22}(t,u) \quad (3.2)$$

and a corresponding definition for $g'(t,u)$ with $a_1,\ldots,f_3 \to a_1',\ldots,f_3'$. The Bose symmetry requirements are most conveniently expressed in terms of the isospin amplitudes $3M_I+M_I'$ and M_I+M_I'. They take the form

$$\begin{aligned}(3M_I+M_I')(s,t,u)&=(-1)^I(3M_I+M_I')(s,u,t)\,,\\(M_I-M_I')(s,t,u)&=(-1)^{I+1}(M_I-M_I')(s,u,t)\,.\end{aligned} \quad (3.3)$$

Inspection of (3.1) reveals that these equations are equivalent to $3g(t,u)+g'(t,u)=3g(u,t)+g'(u,t)$ and $g(t,u)-g'(t,u)=-g(u,t)+g'(u,t)$ or

$$\begin{aligned}a_1'&=2a_2-a_1\,, & a_2'&=2a_1-a_2\,,\\ b_1'&=2c_2-b_1\,, & c_2'&=2b_1-c_2\,,\\ c_1'&=2b_2-c_1\,, & b_2'&=2c_1-b_2\,,\\ b_3'&=2c_3-b_3\,, & c_3'&=2b_3-c_3\,,\\ e_3'&=2f_3-e_3\,, & f_3'&=2e_3-f_3\,,\\ a_3'&=a_3\,, & d_3'&=d_3\,.\end{aligned} \quad (3.4)$$

The required Regge asymptotic behavior, $(3M_I+M_I')(s,t,u)\sim t^{\alpha(s)}$ and $(M_I-M_I')(s,t,u)\sim t^{\alpha(s)-1}$, is equivalent to $g(t,s)\sim t^{\alpha(s)-1}$ and $g(s,u)+g'(s,u)\sim t^{\alpha(s)-1}$. The former requires

$$e_3=b_2=0\,, \quad (3.5)$$

while the latter leads to $f_3+f_3'=c_1+c_1'=0$, which is guaranteed by (3.4) and (3.5). Proper signature for the leading trajectory is automatically imposed by the Bose symmetry conditions together with (3.5).

As usual, we assume that our (on-shell) amplitude is a satisfactory vehicle for analytic continuation to the Adler point. More precisely, we work in the approximation $m_\pi=0$, in which case the Adler point coincides with the physical threshold. At the point $(s,t,u)=(m_A^2,0,0)$, we have

$$(M_2-M_2')=(3M_1+M_1')=(M_0-M_0')\equiv 0\,,$$

since these combinations are antisymmetric under $t\leftrightarrow u$. Requiring $(3M_2+M_2')$, (M_1-M_1'), and $(3M_0+M_0')$ to vanish at this point, we find

$$\begin{aligned}a_3&=-4(a_1+a_2)\,,\\ m_A^4f_3&=2a_1+6a_2+m_A^2(2c_2-c_1-c_3)\,,\\ m_A^2b_3&=6a_1+2a_2+2m_A^2b_1\,,\end{aligned} \quad (3.6)$$

respectively. Consideration of the Adler point $(s,t,u)=(0,m_A^2,0)$ yields no further relations.

Normalizing our Veneziano amplitude to the ρ pole, we obtain

$$\begin{aligned}f_\rho G_S&=-(a_2+c_2m_\rho^2)/b\,,\\ f_\rho G_D&=-2c_1/b\,.\end{aligned} \quad (3.7)$$

Of course, $s=m_\rho^2$ is not a physically accessible point, so this procedure also assumes an analytic continuation. The product $g_{\epsilon\pi\pi}g_{A\epsilon\pi}$ of coupling constants[25] of the $J^P=0^+$ ρ daughter ϵ may be easily computed by going to $t=m_\rho^2$ in the $I_s=2$ amplitudes (see I). We no longer have $g_{A\epsilon\pi}=0$.

Equations (3.4)–(3.7) serve to eliminate 19 of the original 24 parameters, allowing the amplitudes to be written as

$$\begin{aligned}g(t,u)={}&[a_1+b_1t-\tfrac{1}{2}bf_\rho G_Ds]B_{12}(t,u)+[a_2-(2b^2f_\rho G_S\\&+2ba_2)u]B_{12}(t,u)+\{-4(a_1+a_2)\\&+(6ba_1+2ba_2+2b_1)t+c_3u+d_3tu\\&+[2b^2(a_1+a_2)+b^2f_\rho G_D-4b^3f_\rho G_S\\&-bc_3]u^2\}B_{22}(t,u)\,,\\ g'(t,u)={}&[(2a_2-a_1)+(-4b^2f_\rho G_S-4ba_2-b_1)t\\&+\tfrac{1}{2}bf_\rho G_Du]B_{12}(t,u)+[(2a_1-a_2)-bf_\rho G_Dt\\&+(2b_1+2b^2f_\rho G_S+2ba_2)u]B_{21}(t,u)\\&+\{-4(a_1+a_2)+(2c_3-6ba_1-2ba_2-2b_1)t\\&+(12ba_1+4ba_2+4b_1-c_3)u+d_3tu\\&+[2b^2(a_1+a_2)+b^2f_\rho G_D-4b^3f_\rho G_S\\&-bc_3](2t^2-u^2)\}B_{22}(t,u)\,.\end{aligned} \quad (3.8)$$

IV. FACTORIZATION

The Veneziano representation violates unitarity since it incorporates a narrow-resonance approximation in the form of trajectory functions having vanishing imaginary part.[26] We have seen that it is, nevertheless, possible to impose factorization of the helicity amplitudes in the reaction $A_1\pi \to A_1\pi$ for the leading trajectory. Here we study the implications of factorization of the ρ-trajectory contribution to the coupled reactions $\pi\pi\to\pi\pi$, $A_1\pi\to\pi\pi$, and $A_1\pi\to A_1\pi$. We use the simple $\pi\pi\to\pi\pi$ model of Lovelace,[2] Shapiro,[3] and others. The

[25] These coupling constants are defined by the effective Lagrangian densities $\mathfrak{L}_{\epsilon\pi\pi}=g_{\epsilon\pi\pi}\epsilon\boldsymbol{\pi}\cdot\boldsymbol{\pi}$ and $\mathfrak{L}_{A\epsilon\pi}=g_{A\epsilon\pi}\mathbf{A}_\mu\cdot\partial^\mu\boldsymbol{\pi}\epsilon$.

[26] R. Z. Roskies [Phys. Rev. Letters 21, 1851 (1968)] shows that $\alpha(s)$ must have an imaginary part which increases with s.

s-channel amplitude having $I=1$ may be written

$$M^1(s,t,u)=-2f_\rho^2\{[1-\alpha(s)-\alpha(t)]B_{11}(s,t) -[1-\alpha(s)-\alpha(u)]B_{11}(s,u)\}. \quad (4.1)$$

We may express the requirements of factorization either in terms of the residues of our amplitudes at $\alpha(s)=J$ for arbitrary integral J or, equivalently, in terms of the asymptotic forms of these amplitudes at large t (or z_s). We adopt the latter method. Factorization for the leading trajectory requires that the amplitudes for the three reactions obey

$$(M_0^{I+})^2\sim M^I M_{00}^{I+}, \quad z_s(M_1^{I+})^2\sim 2M^I M_{11}^{I+}, \quad (4.2)$$

as $z_s\to\infty$. Here, I can be either 0 or 1, but, in this limit, the $I=0$ amplitudes are just $\frac{3}{2}$ times the corresponding $I=1$ amplitudes (except, of course, for a change in the signature factor between $I=0$ and $I=1$) for all three reactions, and we lose nothing by restricting our attention to $I=1$.

In order to write Eqs. (4.2) in a compact form, we make a series of definitions. For $I_s=1$, we let

$$\begin{aligned} &M\to k(bt)^{\alpha(s)}+\cdots && (\pi\pi\to\pi\pi),\\ &(3M_1+M_1')\to C_1(bt)^{\alpha(s)}+\cdots && (A\pi\to\pi\pi),\\ &(M_1-M_1')\to C_2(bt)^{\alpha(s)-1}+\cdots,\\ &T_1\to t_1(bt)^{\alpha(s)-1}+\cdots, && (4.3)\\ &T_2\to t_2(bt)^{\alpha(s)}+\cdots,\\ &T_3-T_4\to v(bt)^{\alpha(s)-1}+\cdots,\\ &2T_3-T_2-T_4\to u(bt)^{\alpha(s)-2}+\cdots && (A\pi\to A\pi), \end{aligned}$$

as $t\to\infty$. Here we have suppressed a common factor $(1-e^{-i\pi\alpha(s)})\Gamma(1-\alpha(s))$ which should be understood to multiply each term on the right. Employing Eqs. (2.16) and (9.16) of I, the factorization conditions (4.1) are easily found to take the form

$$C_2^2+2kbt_1+\tfrac{1}{4}ku=0, \quad (4.4)$$

$$p^2(\omega+E)^2(C_1^2-4kt_2) -2[E(\omega+E)/b](C_1C_2+kv)=0. \quad (4.5)$$

These equations are quite general, and depend only on the requirements of leading order signature and proper asymptotic behavior. In a specific model, k, C_1, C_2, ... will be particular polynomials in $\alpha(s)$.

For the models of Eq. (4.1) and Secs. II and III, we find

$$\begin{aligned} k&=2f_\rho^2,\\ C_1&=4[-x\alpha(s)+(x+\tfrac{1}{2}f_\rho G_D)],\\ C_2&=(2d_3/b^2)\alpha^2(s)+(y-4bf_\rho G_S-3d_3/b^2)\alpha(s) +(-y+2bf_\rho G_S+d_3/b^2),\\ t_1&=-bG_S^2+(\alpha(s)-1)[G_D^2/8b\pm G_SG_D-f_\rho g_2 +(1/8b)(a_{42}^s+m_\rho^2b_{42}^s)],\\ t_2&=\tfrac{1}{2}G_D^2, \quad (4.6)\\ v&=2bG_SG_D\pm G_D^2[1-\alpha(s)],\\ u&=[1-\alpha(s)]\{-8bf_\rho g_2+(a_{42}^s+m_\rho^2b_{42}^s) +G_D^2[2\alpha(s)-1]\}, \end{aligned}$$

where $x\equiv 2(a_1+a_2)-c_3/b-4bf_\rho G_S+f_\rho G_D$ and $y\equiv 12a_1+6b_1/b$. Since kt_1 and ku are at most quadratic in $\alpha(s)$, Eq. (4.4) requires the quadratic terms in C_2 to vanish:

$$d_3=0. \quad (4.7)$$

Given $d_3=0$, the first factorization condition then reduces to a set of three equations in the two unknowns y and $(a_{42}^s+m_\rho^2b_{42}^s)$. These equations are uniquely solved if we put

$$y=4bf_\rho G_S\pm f_\rho G_D. \quad (4.8)$$

In treating (4.5), it is helpful to note that (in our approximations) the kinematical factors may be written $p^2(\omega+E)^2=[2\alpha(s)-3]^2/16b^2$ and $E(\omega+E)=[2\alpha(s)+1]/4b$. Using (4.7) and (4.8), $C_2=(\pm f_\rho G_D)\alpha(s)-(2bf_\rho G_S\pm f_\rho G_D)$, so C_1C_2+kv is only quadratic in $\alpha(s)$. Therefore, if (4.5) is to hold for all s, the coefficient of $\alpha^2(s)$ in $C_1^2-4kt_2$ must vanish, i.e.,

$$x=0. \quad (4.9)$$

It is easy to verify that (4.7)–(4.9) guarantee that $C_1C_2+kv=C_1^2-4kt_2=0$, so that factorization contains no further restrictions.

It is perhaps surprising that our model allows a solution to the factorization problem, since (4.6) contains only three parameters which must satisfy a rather complicated set of equations. [Counting powers of $\alpha(s)$ in Eqs. (4.4) and (4.5) for the model polynomials of (4.6), we would expect factorization to yield nine independent constraints on our model amplitudes.] This suggests that we may have, in some sense, a set of minimal amplitudes for the three coupled reactions. Equations (4.7)–(4.9) may be written

$$\begin{aligned} d_3&=0,\\ 12a_1+6b_1/b&=4bf_\rho G_S\pm f_\rho G_D, \quad (4.10)\\ 2(a_1+a_2)-c_3/b&=4bf_\rho G_S-f_\rho G_D. \end{aligned}$$

We then use (4.10) to eliminate d_3, b_1, and c_3 from (3.8). These amplitudes then depend on two parameters a_1 and a_2. Using the identity $(m_\rho^2-t)B_{12}(t,u)=(m_\rho^2-u)B_{21}(t,u)$ and $g'(t,u)$ depend only on the combination $B_{21}(t,u)$, we find that $g(t,u)$ and $g'(t,u)$ depend only on the combination (a_1+a_2). Finally, employing $(m_\rho^2-u)B_{21}(t,u)=(4m_\rho^2-t-u)B_{21}(t,u)$, even this combination cancels out of both $g(t,u)$ and $g'(t,u)$, leaving the amplitudes completely determined in terms of the conventional coupling constants

$$g(t,u)=[(\tfrac{2}{3}b^2f_\rho G_S\pm\tfrac{1}{6}bf_\rho G_D)t-\tfrac{1}{2}bf_\rho G_D u]B_{12}(t,u) -2b^2f_\rho G_S uB_{21}(t,u)+[(\tfrac{4}{3}b^2f_\rho G_S\pm\tfrac{1}{3}bf_\rho G_D)t +(-4b^2f_\rho G_S+bf_\rho G_D)u]B_{22}(t,u),$$

$$g'(t,u)=\{[-(14/3)b^2f_\rho G_S\mp\tfrac{1}{6}bf_\rho G_D]t+\tfrac{1}{2}bf_\rho G_D u\}\times B_{12}(t,u)+\{-bf_\rho G_D t+[(10/3)b^2f_\rho G_S\pm\tfrac{1}{3}bf_\rho G_D]u\}B_{21}(t,u)+\{[-(28/3)b^2f_\rho G_S+2bf_\rho G_D\mp\tfrac{1}{3}bf_\rho G_D]t+[(20/3)b^2f_\rho G_S\pm\tfrac{2}{3}bf_\rho G_D-bf_\rho G_D]u\}B_{22}(t,u).$$

Here (as elsewhere) the upper and lower signs refer, respectively, to cases 1 and 2 for the $A_1\pi\rightarrow A_1\pi$ amplitude. Note that if $G_S/G_D=\frac{1}{2}m_\rho^2$ and we take the lower sign in (4.1), the coefficients of $B_{22}(t,u)$ vanish and we recover the minimal solution of I

$$[\text{i.e., } g(t,u)=-2b^2f_\rho G_S uB_{11}(t,u)$$

and

$$g'(t,u)=-2b^2f_\rho G_S(2t-u)B_{11}(t,u)].$$

Thus the solution of I permits factorization with $A_1\pi\rightarrow A_1\pi$ for case 2 of Sec. II but not case 1.

V. COUPLING-CONSTANT RELATIONS AND MINIMAL VENEZIANO AMPLITUDES

The most attractive feature of the Veneziano model is that it enables one, for the first time, to write analytic expressions for strong-interaction scattering amplitudes which simultaneously display crossing symmetry and proper Regge asymptotic behavior in all channels, in addition to the pole structure suggested by Feynman diagrams. However, for reactions involving particles with nonzero spin or isospin, the first two features are not automatic and must be ensured by imposing stringent conditions on the model parameters. Because of the strong interrelationships arising from these conditions, one cannot properly construct a Veneziano model for a scattering process without treating all possible isospin states and all invariant (helicity) amplitudes. Conversely, if one writes a Veneziano representation for just a few isospin states or invariant amplitudes, one risks violating restrictions due to crossing symmetry and amplitude "conspiracies" which relate the amplitudes being treated to those ignored. We find it more reasonable, if not demonstrably more correct, to treat each invariant amplitude and all isospin states on the same footing— forming each out of the same beta functions—and then to apply the various physical restrictions to the whole system of amplitudes.

Adopting the viewpoint of the preceding paragraph, we believe that many papers employing the Veneziano model are open to criticism. We consider one particular example in detail. Reference 7 considers only the single charge state $\pi^-A^+\rightarrow\pi^-A^+$, and writes Veneziano representations for only two of the four invariant amplitudes. In our notation[27] these amplitudes are

$$T_1^{\pi^-A^+}(s,t,u)=-2f_\rho g_1[2-\alpha(s)-\alpha(t)]\times B_{21}(s,t)-bG_S^2B_{11}(s,t),\quad(5.1)$$

$$T_4^{\pi^-A^+}(s,t,u)=\tfrac{1}{2}G_D^2[1-\alpha(s)-\alpha(t)]B_{11}(s,t).$$

In order to compare with our results, note that

$$T_i^{\pi^-A^+}=\tfrac{1}{6}T_i^{I_s=2}+\tfrac{1}{2}T_i^{I_s=1}+\tfrac{1}{3}T_i^{I_s=0}=\epsilon_i f_i^s(t,s).\quad(5.2)$$

It is easy to show that (5.1) can correspond to a special case of our amplitudes (2.14) only if $G_S=G_D=f_\rho g_1=0$, in which case $T_1^{\pi^-A^+}=T_4^{\pi^-A^+}\equiv0$ in (5.1). This corresponds to no $A_1\rho\pi$ coupling and to a ρ which does not couple to the isospin current of the A_1. Of course, this comparison does not prove that the amplitudes of Ref. 7 are "wrong," since it implicitly assumes that the amplitudes T_2 and T_4 (not treated in Ref. 7) contain only B_{11}, B_{12}, and B_{21} multiplied by polynomials linear in s, t, and u. (These amplitudes affect T_1 and T_3 via the conspiracy and factorization conditions.) We have not excluded the possibility that more complicated models, in which further terms B_{mn} are admitted to T_2 and T_4, might satisfy all our physical requirements and yet reduce to the model of Ref. 7 for the amplitudes $T_1^{\pi^-A^+}$ and $T_3^{\pi^-A^+}$.

Nevertheless, we believe that the conclusions of Ref. 7, in particular the statement that "the Veneziano representation appears to require universal ρ coupling in order to satisfy the Adler-Weisberger (AW) low-energy theorem," are unjustified. In this paper, we have presented a Veneziano model for the reaction $A_1\pi\rightarrow A_1\pi$, treating all charge states and all invariant amplitudes. Besides satisfying the requirements of the first paragraph of this paper, our model also satisfies the AW theorem in our approximations: Eq. (7.13) of I shows that the AW theorem leads to a single condition for $A_1\pi$ scattering. In the approximation $m_\pi=0$, this condition is simply that the amplitude T_1 vanish at threshold for $I_s=1$. In the same approximation, the physical threshold is $(s,t,u)=(m_A^2,0,m_A^2)$. Assuming that the ρ trajectory satisfies $\alpha(m_\pi^2)=\frac{1}{2}$ and $\alpha(m_A^2)=\frac{3}{2}$, we find that $B_{12}(m_A^2,m_A^2)=B_{21}(m_A^2,m_A^2)=0$ and $B_{12}(m_A^2,0)=-B_{21}(m_A^2,0)$. Therefore, for the model of Eq. (2.14), $f_1^t(m_A^2,m_A^2)=0$ because of the vanishing of the beta functions, while $f_1^s(0,m_A^2)=0$ because the coefficients of B_{12} and B_{21} cancel. We therefore have a model of wider scope than that of Ref. 7, which satisfies the AW theorem, but does not require ρ universality (the parameters f_ρ and g_1 are unconstrained). Since we believe it likely that satisfactory Veneziano amplitudes (in the sense of paragraph one) can be constructed for any meson scattering without requiring special constraints on the low-spin couplings, we are reluctant to accept any coupling constant relation derived from special minimal Veneziano amplitudes. Our reluctance is only increased when the model in question fails to discuss all the amplitudes relevant to a full, crossing-symmetric, factorized Veneziano model which ensures that the leading trajectory is not parity doubled.

[27] McKay and Wada use the Pauli metric; their invariant amplitudes are related to ours by $A=T_2$, $B=2T_3$, $C=T_4$, and $D=T_1$, while their coupling constants are $f_{\rho\pi\pi}^{(\mathrm{MW})}=-f_\rho$, $f_{\rho AA}^{(\mathrm{MW})}=g_1$, $G_S^{A(\mathrm{MW})}=G_S$, and $G_D^{A(\mathrm{MW})}=-G_D$.

Other papers treating vector-scalar scattering, e.g., Ref. 15, treat only the single invariant amplitude (here T_1) appropriate to the Adler condition.

In the same spirit, one should refuse to consider, for example, $A\pi$ scattering by itself, but should instead study a set of coupled reactions (such as we have here), being careful to include the restrictions of factorization. There is, of course, no end to this process of expanding the scope of the treatment in order to achieve self-consistency. This is just another indication that a theory of the strong interactions must at once be a theory for all strongly interacting particles. Increasing calculational complexities dictate that, at present, one must stop somewhere, and we have restricted our attention to the three reactions treated above which form a relatively closed system.

A number of treatments of the reaction $A_1\pi \to \pi\pi$ in the Veneziano model[18–20,28,29] present model amplitudes for this process which may be written in the form (polynomial linear in s, t, and u)$\times B_{11}$. The present treatment shows that such amplitudes are not general enough to allow factorization between the Lovelace $\pi\pi \to \pi\pi$ amplitude and the $A_1\pi \to A_1\pi$ amplitude given here (except, perhaps, if stringent restrictions are imposed on the coupling constants—e.g., $G_S/G_D = \frac{1}{2}m_\rho^2$ as mentioned at the end of Sec. IV).

It is clear by now that the Veneziano model is sufficiently flexible to treat processes involving nonzero spins and isospins, and that the demands of crossing symmetry, proper asymptotic behavior, no parity doubling, etc., can be met by amplitudes constructed from relatively few terms. In light of our present work, it seems likely that, by adding a sufficient number of satellite terms, one could construct a Veneziano representation to produce any arbitrary (small) set of low-spin coupling constants while still meeting these demands. On the other hand, it is possible to construct "minimal" Veneziano amplitudes satisfying all other physical requirements and yet "predicting" relations between low-spin coupling constants which are in poor agreement with experiment or other theoretical calculations. We shall give a few examples of this latter statement.

In Secs. II–IV, we have found a set of Veneziano amplitudes which describe the contributions of the ρ-f trajectory to a set of three coupled reactions in terms of two arbitrary parameters and the coupling constants f_ρ, G_S, G_D, g_1, g_2, and g_3. These coupling constants are unrestricted except for the relation $G_D^2 = 4f_\rho g_3$. This restriction can undoubtedly be lifted by appending an appropriately chosen term to the $A_1\pi \to A_1\pi$ amplitude, and we emphasize that (in our opinion) any failure of this particular relation ought to be attributed to an overly restrictive Veneziano amplitude and any success understood as an accident.

Nevertheless, we find it instructive to compare various Veneziano models with the hard-pion calculations of Schnitzer and Weinberg.[6] SW present a one-parameter model for the $\rho\pi\pi$, $A\rho\pi$, and $AA\rho$ vertices. We can use their results to express the coupling constants of (2.8) in terms of their parameter δ:

$$\begin{aligned} f_\rho &= (1/\sqrt{2})F_\pi^{-1}m_\rho(\delta-3)/4\,,\\ g_1 &= (1/\sqrt{2})F_\pi^{-1}m_\rho\,,\\ G_S &= -\tfrac{1}{4}m_\rho^2F_\pi^{-1}(\delta+2)\,,\\ g_2 &= -(1/\sqrt{2})F_\pi^{-1}m_\rho(\delta+2)\,,\\ G_D &= \tfrac{1}{2}\delta F_\pi^{-1}\,,\\ g_3 &= 0\,. \end{aligned} \tag{5.3}$$

The quantity F_π here is $\frac{1}{2}$ times the corresponding constant in SW. The relation (2.10) then implies $\delta=0$, which corresponds to $\Gamma(A_1\to\rho\pi)=190$ MeV, $\Gamma(\rho\to\pi\pi)=79$ MeV, and $(g_T/g_L)^2=16/9$, in poor agreement with present experimental values.[30–32] [Perhaps it is worth noting that SW's work yields $g_3=0$ because g_3 corresponds to a term in the $AA\rho$ vertex cubic in the momentum and therefore excluded from SW's vertex by their smoothness assumption.[33] Thus, even if we accept the hard-pion analysis, it may be unfair to use it in its present form to interpret (2.10).] In any event, we get a Veneziano representation for arbitrary G_S, f_ρ, g_1, and g_2, including of course those values arising from (5.3) for some fixed δ, but also to others which do not correspond to a single value of δ. Thus, our Veneziano model neither predicts, nor contradicts [excepting relation (2.10)] the results of SW's current algegra, or its extension by other authors[33] who weaken the restrictions of Ref. 6.

As an example of a minimal model which makes "bad" predictions, consider the $A_1\pi\to A_1\pi$ model of I. (In retrospect, this is just a special case of the present model.) There we found the relations $G_S^2=\frac{4}{3}m_\rho^2f_\rho g_1$ and $G_D=0$. As noted previously, these arose from imposing signature relations more restrictive than the minimum ones needed to guarantee signature for the leading trajectory, but there is nothing in the Veneziano representation itself which prevents us from making this specialization. Comparing these with (5.3), we see that the former yields $\delta=-\frac{2}{3}(1\pm i\sqrt{26})$, which is certainly incompatible with Ref. 6 since all our coupling constants are real. (The latter, of course, yields $\delta=0$.) Perhaps more serious than this incompatibility with

[28] Fayyazuddin and Riazuddin, Phys. Letters **28B**, 561 (1969).

[29] C. J. Goebel, M. L. Blackmon, and K. C. Wali, Phys. Rev. **182**, 1487 (1969).

[30] g_T and g_L are the transverse and longitudinal couplings of F. Gilman and H. Harari, Phys. Rev. **165**, 1803 (1968).

[31] J. Ballam *et al.* [Phys. Rev. Letters **21**, 934 (1968)], as amended by the private communication of Dr. A. Brody quoted by P. Horwitz and P. Roy [Phys. Rev. **180**, 1430 (1969)], gives $(g_T/g_L)^2=0.64\pm0.25$ and $\Gamma_A=140\pm30$ MeV.

[32] J. Augustin *et al.*, Phys. Letters **28B**, 508 (1969) gives $\Gamma_\rho=112\pm11.5$ MeV.

[33] Later work, which replaces the smoothness assumption of SW by use of the Bjorken limit, also leads to vertices corresponding to $g_3=0$. See P. Horwitz and P. Roy, Phys. Rev. **180**, 1430 (1969); S. G. Brown and G. B. West, *ibid.* **180**, 1613 (1969).

hard-pion calculations is a violent conflict with ρ universality. Inspection of the effective Lagrangian densities (2.8) reveals that we must have $g_1=-f_\rho$ if the ρ is to couple universally to the isospin current densities $\epsilon_{abc}\pi^b(x)\partial_\mu\pi^c(x)$ and $-\epsilon_{abc}A_{1\nu}{}^b(x)\partial_\mu A_1{}^{c\nu}(x)$ of the π and A_1 mesons. In that case, $G_S{}^2=-\frac{4}{3}m_\rho{}^2f_\rho{}^2$. Since the two sides of this equation are opposite in sign and $f_\rho\neq 0$, we conclude that the minimal amplitude of I is incompatible with universality.

The new amplitude, which treats signature systematically to leading order, does not lead to the aforementioned conflicts with hard-pion results or with ρ universality. It is clear that to signature the amplitude to the next order would require many further terms to prevent overdetermination of the amplitude. But at this level (the first-daughter level) other trajectories probably enter in an important way, modifying the entire analysis.

We have also investigated the positivity of residues on the leading trajectory, and found no ghosts. Since the computation is lengthy and detailed, we only sketch the technique. First one uses the effective $AR\pi$ couplings defined in I to compute the pole structure due to a particle R. One notes that the amplitude T_1 (coefficient of $g_{\mu\nu}$) involves the square of the coupling $(a_J)^2$ and is independent of b_J. Next one verifies that the sign of this residue does not alternate with J by using explicit forms for the high-spin propagators. Next one compares with the Veneziano amplitude for T_1 by isolating the residue of the highest power of z. We thus verify that all residues have the same sign. In a similar way, $b_J{}^2$ appears as the coefficient of the leading asymptotic behavior of $T_{2,3,4}$. The positivity of daughter residues is very much in doubt. Our whole attitude is to ignore pathological properties of daughter singularities.

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Equal Strengths for "Second-" and "First-Order" Electromagnetic and Weak Processes

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The possibility is discussed of a "second-order" unitarity contribution which can be at least comparable to the conventional "first-order" term for high-energy, large-momentum-transfer lepton-hadron elastic scattering. A crucial ingredient is a recent diffraction model for virtual-current, hadron elastic scattering in which the diffraction peak broadens with increasing virtual mass.

I. INTRODUCTION

IN this paper we discuss the calculation of a certain "second-order" weak or electromagnetic contribution to elastic lepton-hadron scattering which, in a particular kinematic regime, can become comparable or exceed in strength the usual "first-order" contribution. This possibility is a direct consequence of a diffraction model[1] for the nonforward, small-angle scattering of a virtual current and spin-averaged hadron in which, for fixed momentum transfer t, the diffraction peak broadens with increasing (large, spacelike) virtual-current mass $|q^2|$. In particular, we shall consider the diffractive contribution of the two-virtual-current exchange, calculated from unitarity, to the absorptive parts $\mathrm{Im}F_{lp}(s,t)$ of elastic ep and νp amplitudes (see Fig. 1). It is this contribution, involving a sum over all allowed hadron intermediate states, whose strength will be compared with the corresponding single-current exchange. The applicable kinematic region will be high energy s and large momentum transfer $|t|$, but for small values of the ratio t/s.

Before relating the details, let us first give a qualitative description of the model comparison in, say, elastic ep scattering of this dominant second-order process and of the first-order one-photon exchange. Despite the fact that the former process is *a priori* smaller by a factor α, it turns out to be "*pointlike*" in t so that the latter process will be relatively damped for large enough $|t|$ (and energy s) by virtue of the proton form factor. In our model, which describes the current, spin-averaged proton component of the unitarity graph, the relevant invariant amplitudes (structure functions) are peaked functions of the ratio $R^2(q^2)|t|\sim|t|/|q^2|$, where $R(q^2)$ is the diffraction radius given for *large*, spacelike q^2. [For small $q^2\simeq 0$, $R(q^2)$ is given by a characteristic vector-meson–proton diffractive radius.] In deep-inelastic electroproduction[2] involving virtual photons (with helicity h), this same radius enters into cross sections in a way which is contrasted with the rapid decrease of form factors for excited resonances:

$$\sigma(\gamma_h(q^2)p\to\text{anything})\propto R^2(q^2)\sim 1/|q^2|\,.$$

[1] R. W. Griffith, Phys. Rev. 188, 2112 (1969).

[2] E. Bloom *et al.*, Phys. Rev. Letters 23, 930 (1969); 23, 935 (1969).